Penalty Function Method for Solving Multiobjective Interval Bilevel Linear Programming Problem

Yiyuan Wang, Xianfeng Ding, Tingting Wei, Jiaxin Li and Chao Min

***Abstract*—In this paper, a penalty function method for multiobjective interval bilevel linear programming (MIBLP) problem is proposed. Firstly, interval order relation is used to transform objective functions. The possibility level based on interval reliability is designed to transform constraint inequalities. Then the MIBLP problem is transformed into multiobjective bilevel linear programming (MBLP) problem with coefficients determination. Secondly, the MBLP problem is converted to a nonlinear optimization problem by using the knowledge of the dual gap, the effective solution, and the linear theory. Then, for this nonlinear problem, we construct a second-order differentiable function. A penalty term is constructed, which combines this function with the dual gap. Consequently, a penalty function algorithm is proposed. Finally, two numerical examples are used to analyze and verify the feasibility of the model and algorithm.**

***Index Terms*—MIBLP, dual gap, differentiable function, penalty function method, possibility level based on interval reliability.**

I. Introduction

THE bilevel programming (BLP) is a kind of optimization problem of bilevel hierarchical structure, which can describe the hierarchical relationship problem in real life. It has strong practical application value in transportation, management, resource allocation and so on, and has been extensively studied by scholars. There are numerous literature on the theoretical knowledge and solving methods of the BLP problem with reference [1].

At present, the problem of single objective programming and multiobjective programming has been profoundly studied [2], [3], [4]. In the past few decades, various researchers have studied the BLP problem with a single objective [5]. Gradually, more and more scholars are interested in MBLP and begin to study it [6]. But all these studies are focused on the MBLP problem with coefficients determination. In fact, in real life, there are inevitably uncertainties in a large number of bilevel decision-making problems due to the inconsistent way of information collection and insufficient depth of understanding of information. There are currently two ways to deal with uncertain parameters. Stochastic programming uses probability distributions to quantify random parameters, and fuzzy programming uses membership functions to describe fuzzy uncertainties. These two methods are applied to the bilevel programming problem. Stochastic bilevel programming and fuzzy bilevel programming are generated, and a series of achievements have been made in these two topics [7]. However, determining the uncertain parameters using these two methods is often a complicated task.

Interval programming is one way to cope with uncertainties. Since the interval contains many numbers, it is advisable to unify the uncertainty coefficients in the bilevel programming problem into intervals. The coefficients are expressed by the maximum and minimum values of the intervals. There is no construction of membership functions and probability distributions. In this way, the interval bilevel programming problem is generated. For the BLP problem with interval coefficients (IBLP), Calvete and Galé [8] developed KBB and KBW algorithms based on the method of extreme point ranking to find the best and worst optimal solutions, respectively. Later, Nehi and Hamidi [9] proposed the RKBW algorithm, which solved the correctness problem of finding the worst optimal solution in [8]. In addition, Ren and Wang [10] proposed two cutting plane methods, and gave the best and worst optimal solutions of IBLP. Abass [11] adopted an order relation to deal with the uncertainty coefficients, and transformed the original problem into a bilevel problem with coefficients determination. Recently, Ren and Wang [12] proposed a method based on reliability-based possibility degree of interval to deal with interval constraints. By using a kind of interval order relation, they obtained the optimal solution of IBLP.

There has been limited work on general MIBLP. In this paper, we extend IBLP to MIBLP. Section II presents interval number and multiobjective bilevel programming. In Section III, the interval order relation and the possibility level based on interval reliability are used respectively to transform the objective functions and constraint conditions of the MIBLP problem. The MIBLP problem is transformed into the MBLP problem with definite coefficients. In Section IV, a second-order differentiable function is constructed. A penalty function method is proposed and the corresponding algorithm is presented. In Section V, the algorithm is explained by examples. In Section VI, the proposed penalty function

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algorithm is compared with the general penalty function algorithm. Finally, the work of this paper is summarized.

II. Preliminaries

In this article, the MIBLP problem is converted to the MBLP problem by a corresponding treatment. In this section, the concepts and results of interval number and multiobjective bilevel programming are introduced.

A. Related concepts of interval numbers

In this paper, some basic notions of interval numbers are important. We first recall the relevant preliminary results about interval numbers.

On the set of real number R, an interval number a^I [13] is defined as follows:

$$a^I=\left[a^L,a^R\right]=\left\{a \mid a^L\leq a\leq a^R, a\in R\right\},$$

where a^L and a^R are the minimum and maximum values of a^I, respectively. When $a^L=a^R$, a^I is reduced to a real number.

The midpoint and the radius of interval a^I are defined as $m\left(a^I\right)=\frac{a^L+a^R}{2}$ and $w\left(a^I\right)=\frac{a^R-a^L}{2}$. An arbitrary interval $a^I=\left[a^L,a^R\right]$ can be expressed as:

$$a^I=m\left(a^I\right)\pm w\left(a^I\right).$$

where $m\left(a^I\right)$ and $w\left(a^I\right)$ represent the midpoint and radius, respectively.

Let us use I to represent the set of all closed intervals in R. For any two intervals $a^I=\left[a^L,a^R\right], b^I=\left[b^L,b^R\right]\in I$, some arithmetic operations can be given as follows:

(i) $a^I+b^I=\left[a^L+b^L,a^R+b^R\right]$;

(ii) $a^I-b^I=\left[a^L-b^R,a^R-b^L\right]$;

(iii) $ka^I=\begin{cases}\left[ka^L,ka^R\right], & k\geq 0,\\ \left[ka^R,ka^L\right], & k<0.\end{cases}$

Since objective functions and inequality constraints are involved in the optimization problem, the order relation between any two intervals is particularly crucial. Ishibuchi and Tanaka [14] proposed an interval order relation to deal with the minimization problem.

Definition 1: [14] For a minimization problem, the order relations between interval numbers $a^I=\left[a^L,a^R\right]$ and $b^I=\left[b^L,b^R\right]$, denoted by symbols $\leq_{mr}$ and $<_{mr}$, are defined as follows.

(i) $a^I\leq_{mr}b^I$ if and only if $m\left(a^I\right)\leq m\left(b^I\right)$ and $a^R\leq b^R$;

(ii) $a^I<_{mr}b^I$ if and only if $a^I\leq_{mr}b^I$ and $a^I\neq b^I$.

For the MIBLP problem, the constraint coefficients are interval values, and the indefinite coefficients make the problem complicated. In reference [14], the probability degree of an interval indicates the degree to which an interval is better or worse than another interval. This approach can be used to deal with interval constraints.

Definition 2: [15] Let $a^I=\left[a^L,a^R\right]$ and $b^I=\left[b^L,b^R\right]$ be two intervals, then the probability level of reliability based on interval $a^I\leq b^I$ is defined as:

$$P_r\left(a^I\leq b^I\right)=\frac{b^R-a^L}{2w\left(a^I\right)+2w\left(b^I\right)},$$

where the value of $\Pr\left(a^I\leq b^I\right)$ is within the interval $[-\infty,+\infty]$ rather than [0,1]. In particular, when interval numbers a^I and b^I are degenerated into real numbers a and b, respectively. The possibility degrees of $a\leq b^I$ and $a^I\leq b$ are written as follows:

$$P_r\left(a\leq b^I\right)=\frac{b^R-a}{2w\left(b^I\right)},\quad P_r\left(a^I\leq b\right)=\frac{b-a^L}{2w\left(a^I\right)}.$$

Next, we introduce the basic definition on multiobjective bilevel linear programming problem.

B. Basic concepts of multiobjective bilevel linear programming problem

Since it is impossible to minimize or maximize the objectives of bilevel multiobjective programming simultaneously, optimistic optimization model or pessimistic optimization model can be used to deal with this situation. The optimistic MBLP considered is as follows:

$$\begin{aligned}&\min_{x,y}\left(F_1(x,y),F_2(x,y),\ldots,F_p(x,y)\right)^T\\&s.t. A_1x+B_1y\leq b_1,\\&\quad x\geq 0,\\&\quad \text{where } y \text{ solves}\\&\quad \min_{y}\quad Dy\\&\quad s.t. A_2x+B_2y\leq b_2,\\&\qquad y\geq 0,\end{aligned}\tag{1}$$

where $x\in X\subset R^n, y\in Y\subset R^m, A_1\in R^{r\times n}, B_1\in R^{r\times m}, b_1\in R^r, A_2\in R^{s\times n}, B_2\in R^{s\times m}$, $b_2\in R^s$. x and y are decision-making variables of each level, respectively. $F_i(x,y)$ and Dy are objective functions, where $i=1,2,\ldots,p, D\in R^{q\times m}$. Let both objective functions be continuous and differentiable.

Let $S=\{(x,y)\mid A_kx+B_ky|\leq b_k, k=1,2,x\in X^+,y\in Y^+\}$ be the constraint region of problem (1). For each given $x\in X^+$ denote the constraint region of the second level problem by $\bar{S}(x)=\{y\mid B_ky\leq b_k-A_kx, k=1,2,y\in Y^+\}$, and denote the projection of S in the first level decision space by $\Pi_y=\{x\in R^n\mid\exists y\in R^m, A_kx+B_ky\leq b_k, k=1,2,x\in X^+,y\in Y^+\}$. $\Psi_e(x)$ denote the set of the efficient solutions of the second level problem, and $IR=\{(x,y)\mid x\in\Pi_y, y\in\Psi_e(x)\}$ denote the feasible region. Then, problem (1) can be reformulated as follows:

$$\begin{aligned}&\min_{x,y}F(x,y)=\left(F_1(x,y),F_2(x,y),\ldots,F_p(x,y)\right)^T\\&s.t.\ y\in\Psi_e(x),\\&\qquad x\geq 0.\end{aligned}\tag{2}$$

Next, the assumptions required by the model are established.

(H1) S is nonempty and compact, and $\Psi_e(x)\neq\varnothing$ for all $x\geq 0$.

(H2) The set X^+ is a polytope.

Definition 3: (x,y) is called a feasible solution of problem (2) if $(x,y)\in IR$.

Definition 4: A feasible point (x^*,y^*) is called a Pareto optimal solution to problem (2), if there exists no other feasible point (x,y) such that $F(x,y)\leq F\left(x^*,y^*\right)$ and $F(x,y)\neq F\left(x^*,y^*\right)$ are satisfied for any $(x,y)\in IR$.

For the MBLP problem, Benson [16] defined a function to determine whether a point is an efficient solution. See [17], [18] for similar methods. Inspired by the above references, then we define $l(x,y) = e^T Dy - h(x,y)$ for all $x \geq 0$, where $e = (1, 1, \ldots, 1)^T \in R^q$. Since we cannot directly find the expression for $h(x,y)$, the following problem arises:

$$\begin{aligned} &\min e^T Dw \\ &s.t. w \in G(x,y) \end{aligned} \tag{3}$$

where $G(x,y) = \{w \mid Dw \leq Dy, w \in \bar{S}(x)\}$, $h(x,y)$ is the optimal value for problem (3). Then, we have the following results.

Lemma 1: For all $x \geq 0, y \in \bar{S}(x)$, the following assertions are satisfied.

(i) $l(x,y) \geq 0$.
(ii) $\Psi_e(x) = \{y \mid l(x,y) = 0, y \in \bar{S}(x)\}$.

Proof: See Lemma 2.3 in [19]. ■

Lemma 1 shows that for $l(x,y) = 0$, the solution is an efficient solution. Since $l(x,y)$ contains $h(x,y)$, we consider the dual of problem (3).

$$\begin{aligned} &\max_{u,v} - (b_2 - A_2 x)^T u - v^T Dy \\ &s.t. - D^T v - B_2^T u \leq D^T e \\ &\qquad u \geq 0, v \geq 0. \end{aligned} \tag{4}$$

From the duality theory, $h(x,y)$ is the optimal solution of both the original problem and the dual problem.

Let $Z = \left\{(u,v) \in R^s \times R^q \,|\, -D^T v - B_2^T u \leq D^T e, u \geq 0, v \geq 0\right\}$, and $\pi(x,y,u,v) = e^T Dy + v^T Dy + (b - Ax)^T u$ denote the duality gap of problem (3). According to the duality theory, $\pi(x,y,u,v) \geq 0$ for all $(x,y,u,v) \in S \times Z$.When $\pi(x,y,u,v) = 0$, problems (3) and (4) have a common optimal solution. We consider the following multiobjective problem with a bilinear constraint:

$$\begin{aligned} &\min_{x,y,u,v} F(x,y) = (F_1(x,y), F_2(x,y), \ldots, F_p(x,y))^T \\ &s.t. \pi(x,y,u,v) = 0, \\ &\qquad (x,y) \in S, \\ &\qquad (u,v) \in Z. \end{aligned} \tag{5}$$

Let $(\bar{x}, \bar{y}, \bar{u}, \bar{v})$ be any feasible point to problem (5). Then, we can obtain $\bar{y} \in \Psi_e(x)$. Therefore, $(\bar{x}, \bar{y}, \bar{u}, \bar{v})$ is also a feasible point to problem (2).

Lemma 2: If $(\tilde{x}, \tilde{y})$ is a Pareto optimal solution to problem (2), then there exists $(\tilde{u}, \tilde{v}) \in Z$, such that $(\tilde{x}, \tilde{y}, \tilde{u}, \tilde{v})$ is a Pareto optimal solution to problem (5). Vice versa.

Proof: See Lemma 2.4 in [19]. ■

III. Multiobjective interval bilevel programming problem

In general, there are multiple objective functions in the two-level problems. The objective functions and constraints are linear, and the coefficients are interval coefficients. This optimization problem is called the MIBLP problem. As follows:

$$\begin{aligned} &\min_x F^I(x,y) = \left(F_1^I(x,y), F_2^I(x,y), \ldots, F_p^I(x,y)\right)^T \\ &s.t. \left[a_{i1}^L, a_{i1}^R\right] x + \left[b_{i1}^L, b_{i1}^R\right] y \leq \left[h_i^L, h_i^R\right], i = 1, 2, \ldots, r, \\ &\qquad x \geq 0, \\ &\qquad \text{where } y \text{ solves} \\ &\quad \min_y f^I(x,y) = \left(f_1^I(x,y), f_2^I(x,y), \ldots, f_q^I(x,y)\right)^T \\ &\quad s.t. \left[a_{j2}^L, a_{j2}^R\right] x + \left[b_{j2}^L, b_{j2}^R\right] y \leq \left[h_j^L, h_j^R\right], j = 1, 2, \ldots, s, \\ &\qquad\quad y \geq 0, \end{aligned} \tag{6}$$

where $x \in X \subset R^n, y \in Y \subset R^m$. x and y are decision-making variables.

$$\begin{aligned} F^I(x,y) = &(\left[F_1^L(x,y), F_1^R(x,y)\right], \left[F_2^L(x,y), F_2^R(x,y)\right], \\ &\ldots, \left[F_p^L(x,y), F_p^R(x,y)\right])^T, \end{aligned}$$

$$\begin{aligned} f^I(x,y) = &(\left[f_1^L(x,y), f_1^R(x,y)\right], \left[f_2^L(x,y), f_2^R(x,y)\right], \\ &\ldots, \left[f_q^L(x,y), f_q^R(x,y)\right])^T, \end{aligned}$$

$F^I(x,y)$ and $f^I(x,y)$ are interval-valued objective functions. $\left[a_{i1}^L, a_{i1}^R\right]$ and $\left[a_{j2}^L, a_{j2}^R\right]$ are n-dimensional interval vectors whose components are all intervals. $\left[b_{i1}^L, b_{i1}^R\right]$ and $\left[b_{j2}^L, b_{j2}^R\right]$ are m-dimensional interval vectors. $\left[h_i^L, h_i^R\right]$ and $\left[h_j^L, h_j^R\right]$ are interval numbers. $i = 1, 2, \ldots, r, j = 1, 2, \ldots, s$.

In the MIBLP problem, the coefficients of objective functions and constraint conditions contain interval numbers, which is equivalent to an uncertain optimization problem. The general idea is to convert it into a programming problem with coefficients determination. In the following, how to transform multiobjective interval bilevel programming into multiobjective bilevel programming with coefficients determination will be discussed in detail.

A. The transformation of multiobjective interval objective functions

The relation $\leq_m$ is used to transform the interval objective functions of the first and second level problems, respectively. The corresponding interval objective functions become two objective functions with determined coefficients. Taking the middle position and upper bound of the original interval objective functions for each of these two functions, we obtain the following upper and lower level functions:

$$\begin{aligned} \min_{x \in X} &(m\left(F_1(x,y)\right), F_1^R(x,y), m\left(F_2(x,y)\right), F_2^R(x,y), \\ &\ldots, m\left(F_p(x,y)\right), F_p^R(x,y))^T, \end{aligned}$$

$$\begin{aligned} \min_{y \in Y} &(m\left(f_1(x,y)\right), f_1^R(x,y), m\left(f_2(x,y)\right), f_2^R(x,y), \\ &\ldots, m\left(f_q(x,y)\right), f_q^R(x,y))^T, \end{aligned}$$

where $m\left(F_k(x,y)\right) = \frac{F_k^L(x,y) + F_k^R(x,y)}{2}$, $m\left(f_l(x,y)\right) = \frac{f_l^L(x,y) + f_l^R(x,y)}{2}$, $k = 1, 2, \ldots, p$, $l = 1, 2, \ldots, q$. Hence, the number of objective functions at each level is doubled, $2p$ and $2q$, respectively. The interval-valued objective functions become the objective functions with coefficients determination.

B. The transformation of interval constraints

For the interval inequality constraints, in order to transform them into the general constraints, we can use the possibility degree method [20]. When the two intervals are completely separated, most of the existing probability degrees cannot reflect the reliability information. Recently, Jiang et al. [15] introduced a possibility level based on interval reliability to deal with interval constraints and effectively explained the reliability of constraints.

According to Definition 2, the two-level constraints can be transformed into the following forms, respectively.

$$\text{UC}: P_r\left(\left[a_{i1}^L, a_{i1}^R\right]x + \left[b_{i1}^L, b_{i1}^R\right]y \le \left[h_i^L, h_i^R\right]\right) = \frac{h_i^R - \left(a_{i1}^L x + b_{i1}^L y\right)}{2\left[\frac{\left(a_{i1}^R x + b_{i1}^R y\right) - \left(a_{i1}^L x + b_{i1}^L y\right)}{2} + \frac{\left(h_i^R - h_i^L\right)}{2}\right]} \ge \lambda_i,$$

$$\text{LC}: P_r\left(\left[a_{j2}^L, a_{j2}^R\right]x + \left[b_{j2}^L, b_{j2}^R\right]y \le \left[h_j^L, h_j^R\right]\right) = \frac{h_j^R - \left(a_{j2}^- x + b_{j2}^- y\right)}{2\left[\frac{\left(a_{j2}^+ x + b_{j2}^+ y\right) - \left(a_{j2}^- x + b_{j2}^- y\right)}{2} + \frac{\left(h_j^+ - h_j^-\right)}{2}\right]} \ge \lambda_j',$$

where λ_i and λ_j' are preset possibility levels based on interval reliability for the *i*th and *j*th inequality constraints of the two-level, respectively, $\lambda_i, \lambda_j' \in [-\infty, +\infty], i = 1, 2, \ldots, r, j = 1, 2, \ldots, s$. Moreover, the larger the values of λ_i and λ_j', the more reliable are the interval inequality constraints.

By simplifying the above formula, the linear constraints of the two-level problems with coefficients determination can be obtained, respectively.

$$\begin{aligned}\text{UC}: &\left[\lambda_i\left(a_{i1}^R - a_{i1}^L\right) + a_{i1}^L\right]x + \left[\lambda_i\left(b_{i1}^R - b_{i1}^L\right) + b_{i1}^L\right]y \\ &\le h_i^R - \lambda_i\left(h_i^R - h_i^L\right), i = 1, 2, \ldots, r,\end{aligned}$$

$$\begin{aligned}\text{LC}: &\left[\lambda_j'\left(a_{j2}^R - a_{j2}^L\right) + a_{j2}^L\right]x + \left[\lambda_j'\left(b_{j2}^R - b_{j2}^L\right) + b_{j2}^L\right]y \\ &\le h_j^R - \lambda_j'\left(h_j^R - h_j^L\right), j = 1, 2, \ldots, s,\end{aligned}$$

After processing the interval objective functions and the interval constraints inequality, the corresponding interval coefficients are converted into certain coefficients. Therefore, the MIBLP problem (6) is transformed into the multiobjective bilevel linear programming as follows:

$$\begin{aligned}&\min_x\left(m\left(F_1(x,y)\right), F_1^R(x,y), m\left(F_2(x,y)\right), F_2^R(x,y), \ldots,\right. \\ &\quad \left. m\left(F_p(x,y)\right), F_p^R(x,y)\right)^T \\ &s.t.\ \left[\lambda_i\left(a_{i1}^R - a_{i1}^L\right) + a_{i1}^L\right]x + \left[\lambda_i\left(b_{i1}^R - b_{i1}^L\right) + b_{i1}^L\right]y \\ &\quad \le h_i^R - \lambda_i\left(h_i^R - h_i^L\right), i = 1, 2, \ldots, r, \\ &\quad x \ge 0, \\ &\quad \text{where } y \text{ solves} \\ &\quad \min_y\left(m\left(f_1(x,y)\right), f_1^R(x,y), m\left(f_2(x,y)\right), f_2^R(x,y), \ldots,\right. \\ &\quad\quad \left. m\left(f_q(x,y)\right), f_q^R(x,y)\right)^T \\ &\quad s.t.\ \left[\lambda_j'\left(a_{j2}^R - a_{j2}^L\right) + a_{j2}^L\right]x + \left[\lambda_j'\left(b_{j2}^R - b_{j2}^L\right) + b_{j2}^L\right]y \\ &\quad\quad \le h_j^R - \lambda_j'\left(h_j^R - h_j^L\right), j = 1, 2, \ldots, s, \\ &\quad\quad y \ge 0.\end{aligned} \tag{7}$$

Now, problem (7) is further simplified. First, the two-level objective functions are reordered, respectively. The number of objective functions becomes t, where $t = 2p$. The second level objective functions can be simplified into a form containing only decision-making variable y when the first level decision-making variable x is given. The following optimistic multiobjective bilevel programming model is adopted in this paper.

$$\begin{aligned}&\min_{x,y}\left(F_1(x,y), F_2(x,y), \ldots, F_t(x,y)\right)^T \\ &s.t.\ \left[\lambda_i\left(a_{i1}^R - a_{i1}^L\right) + a_{i1}^L\right]x + \left[\lambda_i\left(b_{i1}^R - b_{i1}^L\right) + b_{i1}^L\right]y \\ &\quad \le h_i^R - \lambda_i\left(h_i^R - h_i^L\right), i = 1, 2, \ldots, r, \\ &\quad x \ge 0, \\ &\quad \text{where } y \text{ solves} \\ &\quad \min_y Dy \\ &\quad s.t.\ \left[\lambda_j'\left(a_{j2}^R - a_{j2}^L\right) + a_{j2}^L\right]x + \left[\lambda_j'\left(b_{j2}^R - b_{j2}^L\right) + b_{j2}^L\right]y \\ &\quad\quad \le h_j^R - \lambda_j'\left(h_j^R - h_j^L\right), j = 1, 2, \ldots, s, \\ &\quad\quad y \ge 0,\end{aligned} \tag{8}$$

where $D \in R^{2q \times m}$. Aiming at the problem of MBLP (8), in order to facilitate its solution and calculation, it is now expressed with more concise symbols.

$$\begin{aligned}&\min_{x,y}\left(F_1(x,y), F_2(x,y), \ldots, F_t(x,y)\right)^T \\ &s.t. A_1x + B_1y \le b_1, \\ &\quad x \ge 0, \\ &\quad \text{where } y \text{ solves} \\ &\quad \min_y \quad Dy \\ &\quad s.t. A_2x + B_2y \le b_2, \\ &\quad\quad y \ge 0,\end{aligned} \tag{9}$$

where $A_1 = \lambda_i\left(a_{i1}^R - a_{i1}^L\right) + a_{i1}^L, B_1 = \lambda_i\left(b_{i1}^R - b_{i1}^L\right) + b_{i1}^L$, $b_1 = h_i^R - \lambda_i\left(h_i^R - h_i^L\right), A_2 = \lambda_j'\left(a_{j2}^R - a_{j2}^L\right) + a_{j2}^L, B_2 = \lambda_j'\left(b_{j2}^R - b_{j2}^L\right) + b_{j2}^L, b_2 = h_j^R - \lambda_j'\left(h_j^R - h_j^L\right), i = 1, 2, \ldots, r, j = 1, 2, \ldots, s, A_1 \in R^{r \times n}, B_1 \in R^{r \times m}, b_1 \in R^r, A_2 \in R^{s \times n}, B_2 \in R^{s \times m}, b_2 \in R^s, D \in R^{2q \times m}$.

The optimal solution of problem (9) is equivalent to the optimal solution of problem (6) after the MIBLP problem (6) is processed and simplified. In Section II-B, we discuss in detail the related concepts of problem (9) and the basic concepts of optimal solutions. The solution process of problem (9) will be discussed in the following part.

IV. The Penalty Function Method and Algorithm

In this section, the corresponding penalty function method and algorithm are proposed, and the proof process of algorithm convergence is given.

A. The proposed penalty function method

According to the definition in Section II-B, the solution of problem (9) is obtained by solving problem (5). Notice that problem (5) contains bilinear constraints, in order to deal with nonlinear constraint term $\pi(x, y, u, v) = 0$, the penalty function approach is mostly used to solve them.

In order to calculate the global optimal solution of problem (2), the Mangasarian-Fromowitz constraint [21] is assumed to hold. On the basis of this hypothesis, an accurate penalty function method for MBLP problem is proposed.

In fact, it is very difficult to obtain an exact optimal solution. Inspired by literature [22], we construct a second-order differentiable function, construct a penalty term by using second-order differentiability, and obtain an exact penalty function method. This method replaces the bilinear constrained programming problems with relaxation subproblems.

In real life, decision makers have a certain degree of preference for each objective function. The objective functions are combined with parameters, and the number of functions becomes one. Correspondingly, problem (5) can be transformed as:

$$\begin{aligned} &\min_{x,y,u,v} F(x,y) = \beta_1 F_1(x,y) + \beta_1 F_2(x,y) + \ldots + \beta_t F_t(x,y) \\ &s.t. \pi(x,y,u,v) \le \frac{\varepsilon}{\mu}, \\ &\quad (x,y) \in S, \\ &\quad (u,v) \in Z, \end{aligned} \tag{10}$$

where $\varepsilon > 0$ and $\varepsilon \to 0, \mu > 0$ and $\mu \to +\infty$, $\sum_{i=1}^{t} \beta_i = 1$.

Next, we construct a second-order differentiable approximation problem of the exact penalty function. The second-order differentiable function $p_{\varepsilon,\mu}(t)$ is designed as follows.

$$p_{\varepsilon,\mu}(t) = \begin{cases} 0 & t \le 0 \\ \frac{\mu^4}{10\varepsilon^4} t^5 & 0 < t < \frac{\varepsilon}{\mu}, \\ t + \frac{\varepsilon^4}{6\mu^4} \frac{1}{t^3} - \frac{16\varepsilon}{15\mu} & t \ge \frac{\varepsilon}{\mu} \end{cases}$$

where ε and μ are positive parameters. This function has the following lemmas.

Lemma 3: For any $\varepsilon > 0$ and a fixed $\mu > 0$, $p_{\varepsilon,\mu}(t)$ is twice continuously differentiable on R, where

$$p'_{\varepsilon,\mu}(t) = \begin{cases} \frac{\mu^4}{2\varepsilon^4} t^4 & 0 \le t < \frac{\varepsilon}{\mu} \\ 1 - \frac{\varepsilon^4}{2\mu^4} \frac{1}{t^4} & t \ge \frac{\varepsilon}{\mu} \end{cases},$$

$$p''_{\varepsilon,\mu}(t) = \begin{cases} \frac{2\mu^4}{\varepsilon^4} t^3 & 0 \le t < \frac{\varepsilon}{\mu} \\ \frac{2\varepsilon^4}{\mu^4} \frac{1}{t^5} & t \ge \frac{\varepsilon}{\mu} \end{cases}.$$

Lemma 4: For any $\varepsilon > 0$ and a fixed $\mu > 0$, $\lim_{\varepsilon \to 0} p_{\varepsilon,\mu}(t) = t$.

Lemma 4 means that $p_{\varepsilon,\mu}(\pi(x,y,u,v))$ can approximate $\pi(x,y,u,v)$ when the parameter ϵ is sufficiently small and a fixed $\mu > 0$.

We call $p_{\varepsilon,\mu}(\pi(x,y,u,v))$ the dual gap index. In fact, we construct a function $\alpha(t) = \max\{0, p_{\varepsilon,\mu}(t)\}$, where $t \le 0, p_{\varepsilon,\mu}(t) = 0, t > 0, p_{\varepsilon,\mu}(t) > 0$. Using $\alpha[\pi(x,y,u,v)]$ as a penalty term, we obtain the penalty problem:

$$\begin{aligned} &\min_{x,y,u,v} F(x,y,u,v) = F(x,y) + \mu\alpha[\pi(x,y,u,v)] \\ &s.t. \quad \sum_{i=1}^{t} \beta_i = 1, \\ &\qquad (x,y) \in S, \\ &\qquad (u,v) \in Z, \end{aligned} \tag{11}$$

where $\mu > 0$ is a penalty parameter. Denote the feasible region of problem (11) by W.

Remark 1: Since in the second-order differentiable function $p_{\varepsilon,\mu}(t)$, μ is a positive parameter with $\mu > 0$. In the penalty problem (11), μ is also a penalty parameter with $\mu > 0$. So in this case, these two parameters can take on the same value.

In fact, solving problem (11) gives the solution to problem (5). Under the assumptions of this article, there are the following results.

Lemma 5: [23] For any given $\varepsilon > 0$, let $\{(x_k, y_k, u_k, v_k)\}_{k \in N}$ be a sequence of optimal solutions to problem (10). The positive increasing sequence $\{\mu_k\}_{k \in N}$ such that $\mu_k \to +\infty$. Suppose that for each μ, there is a $(x_\mu, y_\mu, u_\mu, v_\mu)$ such that $\theta(\mu) = F(x_\mu, y_\mu) + \mu\alpha[\pi(x_\mu, y_\mu, u_\mu, v_\mu)]$. The sequences $\{F(x_k, y_k)\}_{k \in N}$ and $\alpha\{\pi(x_k, y_k, u_k, v_k)\}_{k \in N}$ are non-decreasing and non-increasing, respectively.

Since $F(x,y)$, $\alpha(t)$ and the constraint inequalities are continuous. In order to prove the Lemma 5, an auxiliary function $\theta(\mu)$ is introduced after referring to literature [24], where $\theta(\mu) = \inf\{F(x,y) + \mu\alpha(\pi(x,y,u,v)) : \sum_{i=1}^{t} \beta_i = 1, (x,y) \in S, (u,v) \in Z\}$ and $\mu > 0$.

Proof: Consider $t \le 0, \alpha(t) = 0$, let $\mu > 0$, then $F(x,y) = F(x,y) + \mu\alpha[\pi(x,y,u,v)] \ge \inf\{F(\hat{x},\hat{y}) + \mu\alpha(\pi(\hat{x},\hat{y},\hat{u},\hat{v}))\} = \theta(\mu)$.

Now set $\gamma < \mu$, by definition of $\theta(\gamma)$ and $\theta(\mu)$, the following two inequalites hold:

$F(x_\mu, y_\mu) + \gamma\alpha[\pi(x_\mu, y_\mu, u_\mu, v_\mu)] \ge F(x_\gamma, y_\gamma) + \gamma\alpha[\pi(x_\gamma, y_\gamma, u_\gamma, v_\gamma)]$,

$F(x_\gamma, y_\gamma) + \mu\alpha[\pi(x_\gamma, y_\gamma, u_\gamma, v_\gamma)] \ge F(x_\mu, y_\mu) + \mu\alpha[\pi(x_\mu, y_\mu, u_\mu, v_\mu)]$.

Adding these two inequalities and simplifying, according to the properties of the norm we can get

$$(\mu - \gamma)\{\alpha[\pi(x_\gamma, y_\gamma, u_\gamma, v_\gamma)] - \alpha[\pi(x_\mu, y_\mu, u_\mu, v_\mu)]\} \ge 0.$$

Since $\mu > \gamma$, then $\alpha[\pi(x_\gamma, y_\gamma, u_\gamma, v_\gamma)] \ge \alpha[\pi(x_\mu, y_\mu, u_\mu, v_\mu)]$. It then follows from $F(x_\mu, y_\mu) + \gamma\alpha[\pi(x_\mu, y_\mu, u_\mu, v_\mu)] \ge F(x_\gamma, y_\gamma) + \gamma\alpha[\pi(x_\gamma, y_\gamma, u_\gamma, v_\gamma)]$ that $F(x_\mu, y_\mu) > F(x_\gamma, y_\gamma)$ for $\gamma \ge 0$. This completes the proof. ■

According to [25], [26], there are the following theorems.

Theorem 1: For a given $\varepsilon > 0$, $\mu > 0$ and $\mu \to \infty$, let (x^*, y^*, u^*, v^*) be an optimal solution to problem (11), which is feasible for problem (10). Then, (x^*, y^*, u^*, v^*) is an optimal solution to problem (10).

Theorem 2: Let $\{(x_k, y_k, u_k, v_k)\}_{k \in N}$ be a sequence of optimal solutions for problem (11). $\{\varepsilon_k\} > 0$ with $\{\varepsilon_k\} \to 0$, and $\{\mu_k\} > 0$ with $\mu_k \to +\infty$. The sequence $\{(x_k, y_k, u_k, v_k)\}_{k \in N}$ has accumulation points, any one of them is a solution to problem (5).

According to Lemma 2 and the two theorems above, the approximate solution of problem (5) can be obtained through problem (11).

Remark 2: In theorem 2, $\{\varepsilon_k\}$ is a positive sequence with $\varepsilon_k \to 0$ and a positive sequence $\{\mu_k\}$ with $\mu_k \to +\infty$. Problem (11) have an optimal solution when $\left\{\frac{\varepsilon_k}{\mu_k}\right\}_{k \in N}$ is a very small constant (given precision).

B. The penalty function algorithm

Now, we propose the exact penalty function Algorithm 1 to solve problem (11).

Algorithm 1. The penalty function algorithm for the MIBLP problem.

Step 0. Choose $\varepsilon > 0, \mu_0 > 0, N \geq 1$ and set $k = 0$.

Step 1. Set $(x_k, y_k, u_k, v_k) \in S \times Z$ as the initial point of problem (11), and the obtained solution is set as $(x_{k+1}, y_{k+1}, u_{k+1}, v_{k+1})$.

Step 2. If $\pi(x_{k+1}, y_{k+1}, u_{k+1}, v_{k+1}) \leq \frac{\varepsilon}{\mu_k}$, stop, then (x_{k+1}, y_{k+1}) is an optimal solution

to problem (6). Else, go to Step 3.

Step 3. Set $\mu_{k+1} = N\mu_k, k = k + 1$, and go to Step 1.

In Step 1, trust-region algorithm or genetic algorithm can be used to solve the nonlinear programming problem.

In Step 2, since $\frac{\varepsilon}{\mu_k}$ is a very small constant, then $\pi(x_{k+1}, y_{k+1}, u_{k+1}, v_{k+1})$ is approximately equal to 0 for a given precision. In fact, the optimal solution obtained in this paper is approximate optimal solution.

The following is a convergence illustration of Algorithm 1.

Theorem 3: For a given $\varepsilon > 0$, if (x_k, y_k, u_k, v_k) is an optimal solution to problem (11), then there exists $\mu_k^* > 0$ such that (x_k, y_k, u_k, v_k) is a feasible solution to problem (10) for any $\mu_k > \mu_k^*$.

Proof: First, let: $W^0 = \left\{(x, y, u, v) \in W : \pi(x, y, u, v) \leqslant \frac{\varepsilon}{\mu}\right\}, W^1 = \left\{(x, y, u, v) \in W : \pi(x, y, u, v) > \frac{\varepsilon}{\mu}\right\}$.

Suppose $W^1 \neq \varnothing$. The minimum value of $F(x, y, u, v)$ on the set W^1 is given by $\inf_{x,y,u,v}\left\{F(x, y, u, v) : (x, y, u, v) \in W^1\right\}$. Clearly, the set W^1 is not closed (see [10]). Then the minimum function of $F(x, y, u, v)$ can be a finite value that is not available. So, instead of getting the exact solution, we can get the approximate solution. Now, suppose that $(x_k, y_k, u_k, v_k) \in W^1$ is an approximate Pareto optimal solution to problem (11) but not a feasible solution to problem (3) for a given $\varepsilon > 0$. Clearly, $\pi(x_k, y_k, u_k, v_k) > \frac{\varepsilon}{\mu_k}$.

Since the functions $F(x, y)$, $p_{\varepsilon,\mu}$ and $F(x, y, u, v)$ are continuous. Let $m^* = \min\left\{F(x, y, u, v) : (x, y, u, v) \in W^0\right\}$. Take $\mu_k^* > \frac{m^* - F(x_k, y_k)}{\varepsilon_\varepsilon(\pi(x_k, y_k, u_k, v_k))}$. Since (x_k, y_k, u_k, v_k) is an approximate Pareto optimal solution to problem (11), for all $(x, y, u, v) \in W^0$, we have

$F(x, y) + \mu_k\alpha[\pi(x, y, u, v)] \geqslant F(x_k, y_k) + \mu_k\alpha[\pi(x_k, y_k, u_k, v_k)]$,

specially, for any $\mu_k > \mu_k^*$, we have

$F(x, y) + \mu_k\alpha[\pi(x, y, u, v)] > F(x_k, y_k) + \mu_k^*\alpha[\pi(x_k, y_k, u_k, v_k)] > m^*$.

This contradicts the definition of m^*. Hence, for $\mu_k > \mu_k^*$, the approximate Pareto optimal solution (x_k, y_k, u_k, v_k) of problem (11) must belong to W^0. The theorem has been proved. ■

V. Numerical Examples

Two examples are presented below to illustrate the feasibility and effectiveness of Algorithm 1. The detailed iterative procedure of Algorithm 1 and the solution results are given below.

Example 1: Consider the following MIBLP problem. Example 1 adds an objective function to both levels, on the basis of reference [11].

$$\begin{aligned}
&\max_x F^I(x, y) = ([3, 6]x + [4, 9]y, [1, 2]x + [4, 5]y)^T \\
&\qquad \text{where } y \text{ solves} \\
&\max_y f^I(x, y) = ([11, 13]x + [7, 9]y, [3, 5]x + [6, 8]y)^T \\
&\textit{s.t. } [3, 5]x + [4, 6]y \geq 16, \\
&\qquad x + y \leq 6, \\
&\qquad x \geq 0, y \geq 0.
\end{aligned}$$

Firstly, according to the transformation method in Section III, the objective functions and constraint conditions of the MIBLP problem are processed, respectively. The MIBLP problem is transformed into the MBLP problem with determined coefficients, as shown below.

$$\begin{aligned}
&\min_x F^I(x, y) = (-4.5x - 6.5y, -3x - 4y, \\
&\qquad - 1.5x - 4.5y, -x - 4y) \\
&\min_y f^I(x, y) = (-12x - 8y, -11x - 7y, -4x - 7y, \\
&\qquad - 3x - 6y)^T \\
&\textit{s.t. } (2\lambda - 5)x + (2\lambda - 6)y \leq -16, \\
&\qquad x + y \leq 6, \\
&\qquad x \geq 0, y \geq 0.
\end{aligned}$$

Next, according to the dual theory and effective solution of the second level problem, the above problem is converted to a single level MBLP problem. We construct the following penalty problem:

$$\begin{aligned}
&\min_{x,y,u,v} F^I(x, y) = \beta_1(-4.5x - 6.5y) + \beta_2(-3x - 4y) \\
&\qquad + \beta_3(-1.5x - 4.5y) + \beta_4(-x - 4y) \\
&\qquad + \mu\alpha[-21y - 8v_1y - 7v_2y - 6v_3y \\
&\qquad + [-16 - (2\lambda - 5)x]u_1 + (6 - x)u_2] \\
&\qquad \text{where } y \text{ solves} \\
&\textit{s.t.} \quad (2\lambda - 5)x + (2\lambda - 6)y \leq -16, \\
&\qquad x + y \leq 6, \\
&\qquad 8v_1 + 7v_2 + 6v_3 - (2\lambda - 6)u_1 - u_2 \leq -21, \\
&\qquad \sum_{i=1}^{4} \beta_i = 1, \\
&\qquad x \geq 0, y \geq 0.
\end{aligned}$$

Example 1 is then discussed in detail using MATLAB programming and the proposed penalty algorithm.

Step 0: When $(\beta_1, \beta_2, \beta_3, \beta_4) = (0.25, 0.25, 0.25, 0.25)$, $\lambda = 1$, choose initial penalty parameters $\mu_0 = 1$, $N = 5$, $\varepsilon = 0.0001$ and $k = 0$.

Step 1: Select the initial point $\left(x^0, y^0, u_1^0, u_2^0, v_1^0, v_2^0, v_3^0\right) = (1, 5, 0, 23, 0, 0, 0)$, and use trust-region algorithm to solve the corresponding penalty problem.

Step 2: As $\left(x^1, y^1, u_1^1, u_2^1, v_1^1, v_2^1, v_3^1\right) = 1.1331\text{e} - 04 > 1.0000\text{e} - 04$, then go to step 3.

Step 3: $k = 1, \mu_1 = 5$, and go to Step 1.

The first iteration is completed in Step 3. After fourth iterations, we obtain $(x^*, y^*) = (0, 6)$, and the first level objective $F_1^I(x^*, y^*) = [24, 54]$, $F_2^I(x^*, y^*) = [24, 30]$.

TABLE I: Numerical results with $(\beta_1, \beta_2, \beta_3, \beta_4) = (0.25, 0.25, 0.25, 0.25), \mu_0 = 1$.

λ	μ_0	n	(x^*, y^*)	$F_1^I(x^*, y^*)$	$F_2^I(x^*, y^*)$
-3	1	1	$(0, 6)$	$[24, 54]$	$[24, 30]$
-2	1	1	$(0, 6)$	$[24, 54]$	$[24, 30]$
-1	1	2	$(0, 6)$	$[24, 54]$	$[24, 30]$
0	1	5	$(0, 6)$	$[24, 54]$	$[24, 30]$
0.5	1	3	$(0, 6)$	$[24, 54]$	$[24, 30]$
1	1	4	$(0, 6)$	$[24, 54]$	$[24, 30]$
1.5	1	5	$(0, 6)$	$[24, 54]$	$[24, 30]$

Since decision-makers have different preferences for each objective function, this paper considers adjusting the parameters β_i of linear combination. In the same linear combination, in order to explain the influence of different λ_i on the calculation results, λ_i can be adjusted, where $\lambda_i \in [-\infty, +\infty]$. Now different parameters β_i and λ_i are selected and the algorithm above is used to solve Example 1.

When Algorithm 1 is used to solve the problem, different λ_i can be adjusted. The initial value of this problem is chosen to be $(x^0,y^0,u^0,v^0) = (1, 5, 0, 23, 0, 0, 0)$. The optimal solution and the first level objective function values are obtained and shown in Table I, n indicates the number of iterations.

From table I, when λ goes from -3 to 1.5, neither of the first level two objective functions change. The higher the value of λ is, the higher the reliability of the constraint inequality is. Therefore, $\lambda = 1.5$ can be selected to obtain the optimal solution (0,6). In this case, the constraint inequality is more reliable. Due to $\lambda \in [-\infty, +\infty]$, only a few values are listed here, this paper considers the extreme case of λ. When $\lambda > 1.6$, the feasible region is empty. When λ is negative, the optimal solution can be obtained through different iterations, but the constraint is not reliable at this time. In real life, a larger value of λ should be selected as far as possible.

Notice that the initial penalty parameters are all $\mu_0 = 1$, but different possibility level based on interval reliability have different iterations. Now consider increasing the initial parameter values. When $n > 1$, the initial parameter values are changed, and the results obtained are shown in Table II.

TABLE II: Numerical results with $(\beta_1, \beta_2, \beta_3, \beta_4) = (0.25, 0.25, 0.25, 0.25)$ and different initial parameter values of $\boldsymbol{u}_0$.

λ	μ_0	n	(x^*, y^*)	$F_1^I(x^*, y^*)$	$F_2^I(x^*, y^*)$
0	10	2	$(0, 6)$	$[24, 54]$	$[24, 30]$
0.5	10	3	$(0.0051, 5.9949)$	$[23.9949, 53.9848]$	$[23.9848, 29.9848]$
0.5	50	2	$(0, 6)$	$[24, 54]$	$[24, 30]$
0.5	70	1	$(0, 6)$	$[24, 54]$	$[24, 30]$
1	10	2	$(0, 6)$	$[24, 54]$	$[24, 30]$
1.5	10	2	$(0, 6)$	$[24, 54]$	$[24, 30]$
1.5	40	1	$(0, 6)$	$[24, 54]$	$[24, 30]$

From Table II, in general, when the initial penalty parameters increase, n decreases. However, when $\lambda = 0.5$, the penalty parameter is increased to 10. The number of iterations does not decrease, which is the same as $\mu_0 = 1$ and iterated three times. This is because in this paper the penalty function is constructed using $p_{\varepsilon,\mu}(\pi(x, y, u, v))$. The parameter μ of second-order differentiable function $p_{\varepsilon,\mu}(t)$ is the same as those of penalty problem (11). $p_{\varepsilon,\mu}(t)$ will select different function according to the size relationship between t and $\frac{\varepsilon}{\mu}$.

Now adjust the parameters of the linear combination. When $(\beta_1, \beta_2, \beta_3, \beta_4) = (0.3, 0.2, 0.3, 0.2)$, the results are shown in Table III. As for $n > 1$, we consider changing the initial parameter values to reduce n, and the results are shown in Table IV.

TABLE III: Numerical results with $(\beta_1, \beta_2, \beta_3, \beta_4) = (0.3, 0.2, 0.3, 0.2), \mu_0 = 1$.

λ	μ_0	n	(x^*, y^*)	$F_1^I(x^*, y^*)$	$F_2^I(x^*, y^*)$
-3	1	1	$(0, 6)$	$[24, 54]$	$[24, 30]$
-2	1	1	$(0, 6)$	$[24, 54]$	$[24, 30]$
-1	1	2	$(0, 6)$	$[24, 54]$	$[24, 30]$
0	1	3	$(0, 6)$	$[24, 54]$	$[24, 30]$
0.5	1	7	$(0, 6)$	$[24, 54]$	$[24, 30]$
1	1	4	$(0, 6)$	$[24, 54]$	$[24, 30]$
1.5	1	3	$(0, 6)$	$[24, 54]$	$[24, 30]$
1.8	1	$-$	infeasible	$-$	$-$

TABLE IV: Numerical results with $(\beta_1, \beta_2, \beta_3, \beta_4) = (0.3, 0.2, 0.3, 0.2)$ and different initial parameter values of μ_0.

λ	μ_0	n	(x^*, y^*)	$F_1^I(x^*, y^*)$	$F_2^I(x^*, y^*)$
0	30	2	$(0, 6)$	$[24, 54]$	$[24, 30]$
0	40	1	$(0, 6)$	$[24, 54]$	$[24, 30]$
0.5	30	3	$(0, 6)$	$[24, 54]$	$[24, 30]$
0.5	60	1	$(0, 6)$	$[24, 54]$	$[24, 30]$
1	30	3	$(0, 6)$	$[24, 54]$	$[24, 30]$
1	80	1	$(0, 6)$	$[24, 54]$	$[24, 30]$
1.5	0.4	1	$(0, 6)$	$[24, 54]$	$[24, 30]$
1.5	10	3	$(0, 6)$	$[24, 54]$	$[24, 30]$
1.5	40	1	$(0, 6)$	$[24, 54]$	$[24, 30]$

From Table III, we adjust the linear combination mode. For λ which is the same as that in Table I, although n is different, the optimal solution is the same. When $\lambda = 1$ is selected, the optimal solution can be obtained under the high λ. When $\lambda = 1.8$, no solution exists.

From Table IV, in general, increasing the initial penalty parameter decreases n and the obtained optimal solution remains unchanged. But we notice that when $\mu_0 = 10$, n does not decrease, it's still three. At this point, this paper considers reducing the value of μ_0. When $\mu_0 = 0.4$, $n = 1$. When $\mu_0 = 40$, $n = 1$. This is because the penalty function constructed from $p_{\varepsilon,\mu}(t)$ has a local extreme value. In this case, this paper attempts to modify the initial values of the penalty parameters several times. The observed solution is found to be stable. The occurrence of local extreme values of the penalty function are shown not to affect the optimal solution.

For extreme value of linear combination parameter, when $(\beta_1, \beta_2, \beta_3, \beta_4) = (0.1, 0.05, 0.05, 0.8)$, the results are shown in Table V. When $n > 1$, the initial value of penalty parameter is tried to be changed, and the test is performed in Table VI.

Table V shows that extreme values of parameters of linear combination does not affect the optimal value and upper level optimal solution of Example 1. When $\lambda = 0$ and $\lambda = 1.5$,

TABLE V: Numerical results with $(\beta_1, \beta_2, \beta_3, \beta_4) = (0.1, 0.05, 0.05, 0.8), \mu_0 = 1$.

λ	μ_0	n	(x^*, y^*)	$F_1^I(x^*, y^*)$	$F_2^I(x^*, y^*)$
-3	1	2	$(0, 6)$	$[24, 54]$	$[24, 30]$
-2	1	2	$(0, 6)$	$[24, 54]$	$[24, 30]$
-1	1	6	$(0, 6)$	$[24, 54]$	$[24, 30]$
0	1	4	$(0.0001, 5.9999)$	$[23.9999, 53.9997]$	$[23.9997, 29.9997]$
0.5	1	6	$(0.00215.9979)$	$[23.9979, 53.9938]$	$[23.9938, 29.9938]$
1	1	2	$(0, 6)$	$[24, 54]$	$[24, 30]$
1.5	1	3	$(0, 6)$	$[24, 54]$	$[24, 30]$

TABLE VI: Numerical results with $(\beta_1, \beta_2, \beta_3, \beta_4) = (0.1, 0.05, 0.05, 0.8)$ and different initial parameter values of μ_0.

λ	μ_0	n	(x^*, y^*)	$F_1^I(x^*, y^*)$	$F_2^I(x^*, y^*)$
-3	0.2	1	$(0, 6)$	$[24, 54]$	$[24, 30]$
-3	30	4	$(0, 6)$	$[24, 54]$	$[24, 30]$
-3	50	1	$(0, 6)$	$[24, 54]$	$[24, 30]$
-2	20	1	$(0, 6)$	$[24, 54]$	$[24, 30]$
-1	20	3	$(0.0017, 5.9983)$	$[23.9983, 53.9948]$	$[23.9948, 29.9948]$
-1	70	1	$(0, 6)$	$[24, 54]$	$[24, 30]$
0	50	2	$(0.0050, 5.9950)$	$[23.9950, 53.9851]$	$[23.9851, 29.9851]$
0	100	1	$(0.0001, 5.9999)$	$[23.9999, 53.9998]$	$[23.9998, 29.9998]$
0.5	30	1	$(0, 6)$	$[24, 54]$	$[24, 30]$
1	0.3	1	$(0, 6)$	$[24, 54]$	$[24, 30]$
1	10	3	$(0.0056, 5.9944)$	$[23.9944, 53.9831]$	$[23.9831, 29.9831]$
1	60	1	$(0, 6)$	$[24, 54]$	$[24, 30]$
1.5	50	1	$(0, 6)$	$[24, 54]$	$[24, 30]$

the optimal solution is (0.0001,5.9999) and (0.0021 5.9979), respectively. The optimal solution is close to (0,6).

From table VI, when $\lambda = -3$ and $\lambda = 1$, the initial penalty parameter are $\mu_0 = 30$ and $\mu_0 = 10$, respectively. n increases as μ_0 increases. However, when the value of μ_0 decreases or increases more, n decreases, and the optimal solution is (0,6). This is because the penalty function constructed by $p_{\varepsilon,\mu}(t)$ has local extreme values, but it does not affect the optimal solution, which will not be described here.

In summary, the penalty function algorithm can solve the multiobjective interval programming problem efficiently and quickly. The algorithm can stably obtain the optimal solution according to different possibility level based on interval reliability and linear combination parameters.

Example 2: Consider the following general MIBLP problem. The Example 2 adds an objective function to both levels, on the basis of reference [11].

$$\begin{aligned}
&\min_x F^I(x,y) = ([-1,-0.5]y, [1,2]x + [-2.5,-2]y)^T \\
&\quad \text{where } y \text{ sollves} \\
&\min_y f^I(x,y) = ([1,2]y, [3,4]y)^T \\
&\textit{s.t.}\ [0.5,1]x + [1.9,2]y \geq [10,10.5], \\
&\quad [-2,-1]x + [1,2]y \geq [-6,-5], \\
&\quad [-3,-2]x + [0.5,1]y \geq [-21,-20], \\
&\quad [-2,-1]x + [-3,-2]y \geq [-38,-37], \\
&\quad [0.5,1]x + [-3,-2]y \geq [-18,-17], \\
&\quad x \geq 0, y \geq 0.
\end{aligned}$$

Firstly, the objective functions and constraint conditions of Example 2 are transformed by the model in Section III to obtain the MBLP problem with definite coefficients.

$$\begin{aligned}
&\min_x F^I(x,y) = (-0.75y, -0.5y, 1.5x - 2.25y, \\
&\quad 2x - 2y)^T \\
&\min_y \text{ where } y \text{ sollves} \\
&\textit{s. t.}\ f^I(x,y) = (1.5y, 2y, 3.5y, 4y)^T \\
&\quad (\lambda_2+1)\,x + (\lambda_2-2)\,y \leq -\lambda_2 + 6, \\
&\quad (\lambda_3+2)\,x + (0.5\lambda_3-1)\,y \leq -\lambda_3 + 21, \\
&\quad (\lambda_4+1)\,x + (\lambda_4+2)\,y \leq -\lambda_4 + 38, \\
&\quad (0.5\lambda_5-1)\,x + (\lambda_5+2)\,y \leq -\lambda_5 + 18, \\
&\quad x \geq 0, y \geq 0.
\end{aligned}$$

Next, similar to Example 1, we obtain the penalty function as follows:

$$\begin{aligned}
&\min_{x,y,u,v} F^I(x,y) = \beta_1(-0.75y) + \beta_2(-0.5y) \\
&\quad + \beta_3(1.5x - 2.25y) + \beta_4(2x - 2y) \\
&\quad + \mu\alpha\{11y + 1.5v_1y + 2v_2y + 3.5v_3y + 4v_4y + \\
&\quad [-0.5\lambda_1 - 10 - (0.5\lambda_1 - 1)x]u_1 \\
&\quad + [-\lambda_2 + 6 - (\lambda_2+1)x]u_2 \\
&\quad + [-\lambda_3 + 21 - (\lambda_3+2)x]u_3 \\
&\quad + [-\lambda_4 + 38 - (\lambda_4+1)x]u_4 \\
&\quad + [-\lambda_5 + 18 - (0.5\lambda_5-1)x]u_5\} \\
&\quad \text{where } y \text{ sollves} \\
&\quad (0.5\lambda_1-1)\,x + (0.1\lambda_1-2)\,y \leq -0.5\lambda_1 - 10, \\
&\quad (\lambda_2+1)\,x + (\lambda_2-2)\,y \leq -\lambda_2 + 6, \\
&\quad (\lambda_3+2)\,x + (0.5\lambda_3-1)\,y \leq -\lambda_3 + 21, \\
&\quad (\lambda_4+1)\,x + (\lambda_4+2)\,y \leq -\lambda_4 + 38, \\
&\quad (0.5\lambda_5-1)\,x + (\lambda_5+2)\,y \leq -\lambda_5 + 18, \\
&\quad -1.5v_1 - 2v_2 - 3.5v_3 - 4v_4 - (0.1\lambda_1-2)\,u_1 \\
&\quad - (\lambda_2-2)\,u_2 - (0.5\lambda_3-1)\,u_3 \\
&\quad - (\lambda_4+2)\,u_4 - (\lambda_5+2)\,u_5 \leq 11, \\
&\quad x, y, v_1, v_2, v_3, v_4, u_1, u_2, u_3, u_4, u_5 \geq 0.
\end{aligned}$$

There are five interval inequality constraints in Example 2, but the solving process is similar to that in Example 1. When the initial point is $(x^0, y^0, u^0, v^0) = (3, 11, 1, 1, 2, 1, 1, 1, 0, 0, 1)$ and the linear combination parameter is $(\beta_1, \beta_2, \beta_3, \beta_4) = (0.4, 0.1, 0.4, 0.1)$. In order to discuss the algorithm more concisely. We only discuss the effect of different λ on the optimal solution and the first level objective function values. The influence of different initial penalty parameters on the optimal solution and the first level objective function values.

When the value of λ for the five constraint inequalities are equal. The results of the algorithm are shown in Table VII. When $n > 1$, μ_0 is changed, and the results of the algorithm are shown in Table VIII.

From table VII, when λ increase, the value of the first level objective functions increase. For example, when $\lambda = 0.8$ and $\lambda = 1$, the interval of the value of the second objective function changes from [-13.5417, -10.8333] to [-6.1047, 0.6977]. But what we calculate is the minimum value of the objective function. This is because λ becomes larger,

TABLE VII: Numerical results with $(\beta_1, \beta_2, \beta_3, \beta_4) = (0.4, 0.1, 0.4, 0.1), \mu_0 = 1$.

λ	μ_0	n	(x^*, y^*)	$F_1^I(x^*, y^*)$	$F_2^I(x^*, y^*)$
-0.1	1	1	$(0.0000, 4.9502)$	$[-4.9502, -2.4751]$	$[-12.3756, -9.9005]$
0	1	1	$(0.0000, 5.0000)$	$[-5.0000, -2.5000]$	$[-12.5000, -10.0000]$
0.2	1	1	$(0.0000, 5.1010)$	$[-5.1010, -2.5505]$	$[-12.7525, -10.2020]$
0.4	1	2	$(0.0000, 5.2041)$	$[-5.2041, -2.6020]$	$[-13.0102, -10.4082]$
0.8	1	2	$(0.0000, 5.4167)$	$[-5.4167, -2.7083]$	$[-13.5417, -10.8333]$
1	1	1	$(4.6512, 4.3023)$	$[-4.3023, -2.1512]$	$[-6.1047, 0.6977]$
1.3	1	1	$(3.5733, 5.0264)$	$[-5.0264, -2.5132]$	$[-8.9928, -2.9063]$

TABLE VIII: Numerical results with $(\beta_1, \beta_2, \beta_3, \beta_4) = (0.4, 0.1, 0.4, 0.1)$ and different initial parameter values of μ_0.

λ	μ	n	(x^*, y^*)	$F_1^I(x^*, y^*)$	$F_2^I(x^*, y^*)$
0.4	10	1	$(0.0000, 5.2041)$	$[-5.2041, -2.6020]$	$[-13.0102, -10.4082]$
0.8	0.3	1	$(0.00005.4167)$	$[-5.4167, -2.7083]$	$[-13.5417, -10.8333]$
0.8	10	5	$(8.1481, 7.8889)$	$[-7.8889, -3.9444]$	$[-11.5741, 0.5185]$
0.8	70	1	$(0.00005.4167)$	$[-5.4167, -2.7083]$	$[-13.5417, -10.8333]$

which reduces the feasible region of the penalty problem. However, a larger λ means that the interval constraint has higher reliability.

From Table VIII, local extreme values of the penalty function also appear, and the final optimal solution is stable at (0.0000 5.4167). To sum up, we choose the higher possibility degree level $\lambda = 0.8$ and apply penalty function algorithm to obtain the optimal solution (0.0000 5.4167). The upper level objective function values are [-5.4167, -2.7083] and [-13.5417, -10.8333], respectively.

When the value of λ for the five constraint inequalities are unequal. The initial point is $(x^0, y^0, u^0, v^0) = (3, 11, 1, 1, 2, 1, 1, 1, 0, 0, 1)$, the linear combination form is $(\beta_1, \beta_2, \beta_3, \beta_4) = (0.4, 0.1, 0.4, 0.1)$, and $\mu_0 = 1$, $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) = (0.8, 0.4, 0.6, -0.2, 0.7)$. The penalty function algorithm only goes through one iteration and also obtains the optimal solution (0.0000,5.4167). $F_1^I(x^*, y^*) = [-5.4167, -2.7083]$, $F_2^I(x^*, y^*) = [-13.5417, -10.8333]$.

According to the above discussion, different λ and linear combination modes have an impact on the optimal solution of the problem. In real life, the preferences of the decision-makers should be taken into account and different preference values should be set for the objective function. In the constraint inequalities, the value of λ with high reliability should be selected as far as possible.

VI. Comparison analysis

In order to study the efficiency of Algorithm 1, a comparative analysis with the general penalty function algorithm is considered below. If the penalty function is constructed without the second-order differentiable function, the dual gap is directly added to the first level objective function. The general penalty problem is obtained as follows:

$$\begin{aligned} \min_{x,y,u,v} \; & F(x,y,u,v) = F(x,y) + \mu\pi(x,y,u,v) \\ s.t. \; & \sum_{i=1}^{t} \beta_i = 1, \\ & (x,y) \in S, \\ & (u,v) \in Z, \end{aligned} \tag{12}$$

The corresponding general penalty function algorithm is similar to the one presented in Section IV, with the following modifications in Step 2.

Step 2. If $\pi(x_{k+1}, y_{k+1}, u_{k+1}, v_{k+1}) > 0$, then $\mu_{k+1} = N\mu_k, k = k+1$, and go to Step 1. If $\pi(x_{k+1}, y_{k+1}, u_{k+1}, \mathcal{V}_{k+1}) < \varepsilon$, stop, then (x_{k+1}, y_{k+1}) is an optimal solution to problem (6).

When the linear combination parameters is $(\beta_1, \beta_2, \beta_3, \beta_4) = (0.25, 0.25, 0.25, 0.25)$ and the initial penalty parameter is $\mu_0 = 10$. The general penalty function algorithm is used to solve Example 1. Table IX and Table X show the solution results of Algorithm 1 and the general penalty function algorithm, respectively.

TABLE IX: Result of algorithm 1 with $(\beta_1, \beta_2, \beta_3, \beta_4) = (0.25, 0.25, 0.25, 0.25), \mu_0 = 10$.

λ	n	(x^*, y^*, u^*, v^*)	$\pi(x^*, y^*, u^*, v^*)$	Computation time (sec)
-3	1	(0.0000,6.0000,0.0000,22.5219, 0.0480,0.1575,0.0059)	4.3250e − 07	0.243114
-2	1	(0.0000,6.0000,0.0000,23.3331, 0.1101,0.0870,0.1405)	6.0901e − 05	0.187693
-1	2	(0.0000,6.0000,0.0000,23.3134, 0.0858,0.1142,0.1379)	5.9543e − 07	0.175272
0	2	(0.0000,6.0000,0.0000,21.0028, 0.0000,0.0000,0.0005)	6.1834e − 08	0.216107
0.5	3	(0.0051,5.9949,0.0000,24.0504, 0.0005,0.0007,0.5069)	4.7667e − 08	0.505102
1	2	(0.0000,6.0000,0.0000,21.0045, 0.0000,0.0000,0.0008)	1.0824e − 07	0.329742
1.5	2	(0.0000,6.0000,0.0000,21.0000, 0.0000,0.0000,0.0000)	1.8729e − 06	0.261642

TABLE X: Result of general algorithm with $(\beta_1, \beta_2, \beta_3, \beta_4) = (0.25, 0.25, 0.25, 0.25), \mu_0 = 10$.

λ	n	(x^*, y^*, u^*, v^*)	$\pi(x^*, y^*, u^*, v^*)$	Computation time (sec)
-3	1	(0.0000,6.0000,0.0000,82.4335, 3.1064,2.9027,2.7105)	1.1977e − 07	0.405990
-2	2	(0.0000,6.0000,0.0000,43.2191, 0.7910,1.0845,1.3833)	2.9729e − 07	0.503517
-1	3	(0.0000,6.0000,0.0000,58.2170, 1.8369,1.7542,1.7071)	1.1968e − 07	0.279900
0	3	(0.0000,6.0000,0.0000,98.9738, 4.0325,3.6532,3.3568)	5.9953e − 08	0.428077
0.5	3	(0.0000,6.0000,0.0000,21.3903, 0.0162,0.0184,0.0219)	1.9194e − 08	0.539147
1	3	(0.0000,6.0000,0.0000,21.0021, 0.0001,0.0001,0.0001)	6.0732e − 09	0.447953
1.5	2	(0.0000,6.0000,0.0000,35.5659, 0.1563,0.7959,1.2907)	2.3792e − 07	0.363594

From Table IX and Table X, when $\lambda = -3, \lambda = 0.5$ and $\lambda = 1.5$, the iteration times of Algorithm 1 are the same as those of the general penalty function algorithm. Algorithm 1 takes less time, but it also has some drawbacks. Its accuracy is not as high as that of the general penalty function algorithm. However, in general, $\pi(x^*, y^*, u^*, v^*)$ is close to 0. It does not affect the optimal solution. The small defect can be ignored. When $\lambda = -2, \lambda = -1, \lambda = 0, \lambda = 1$, the Algorithm 1 has fewer iterations and shorter time than the general penalty function algorithm. Obviously, the optimal solution for both algorithms is (0,6).

To some extent, the proposed algorithm can reduce the memory occupation, improve the solving efficiency, and achieve a given accuracy. For the MIBLP problem, the proposed method and algorithm in this paper can adequately consider the preferences of decision makers. The uncertain information contained in the interval coefficients is well used. This is essential for guiding practical problems.

VII. Conclusion

In this paper, a penalty function method is proposed for a class of MIBLP problem. Firstly, the related knowledge of interval numbers and the multiobjective bilevel programming is reviewed in this paper. Secondly, for the MIBLP problem, the interval order relation and the possibility level based on interval reliability are used respectively to deal with the objective functions and constraints. In this way, the MIBLP problem is converted to the MBLP problem with coefficients determination. Next, linear programming, dual gap and efficient solution of the lower programming problem are used. The MBLP problem is formulated as a general single objective optimization problem with dual gap. Here, the dual gap is a bilinear constraint term. Then, to deal with the bilinear constraint, a second-order differentiable function is constructed. The penalty term is constructed by combining the second-order differentiable function with the dual gap, and the penalty term is added to the objective function. The corresponding penalty problem is obtained and the corresponding penalty function algorithm is given. Finally, two examples are used to analyze and discuss Algorithm 1. It can be concluded that Algorithm 1 has fewer iterations, shorter time and higher accuracy than the general penalty function algorithm. Moreover, the proposed MIBLP model and the solution algorithm can deal with the uncertainty information in real life more accurately. The preferences for the decision-makers are fully considered to help them make decisions on different levels of reliability.

In the past, there have been limited studies of the MIBLP problem. This paper provides an idea about this direction. In real life, this model can be applied to the management of the bilevel supply chain and the design of the bilevel logistics network. This is of great importance for solving practical problems. How to effectively transform multiobjective problems into singleton objective problems, the treatment of interval inequality constraints and the applications to practical problems are directions for future research.

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