Domination Number of Generalized Complements of a Graph

Harshitha A, Gowtham H. J, Sabitha D'Souza*, and Pradeep G. Bhat

***Abstract*—Let G be a graph with vertex set V. A set $D \subseteq V$ is a dominating set of G if each vertex of $V - D$ is adjacent to at least one vertex of D. The k $(k(i))-$ complement of G is obtained by partitioning V into k partites and removing the edges between the vertices of different (same) partites in G and adding the edges between the vertices of different (same) partites which are not in G. This paper studies different domination numbers of k and $k(i)$ complements of graphs.**

***Index Terms*—Dominating set; Domination number; $k-$ complement; $k(i)-$ complement.**

I. Introduction

CONSIDER a simple graph G with the vertex set V and edge set E. The distance from the vertex w_i to the vertex w_j of G is the number of edges in the shortest $w_i - w_j$ path. It is denoted by $d_G(w_i, w_j)$. The degree of a vertex w_i is the number of adjacent vertices of w_i, denoted by $\deg_G w_i$. The maximum (minimum) degree of G, denoted by $\Delta(G)$ $(\delta(G))$, is the maximum (minimum) among degrees of all the vertices of G. Two vertices w_i, w_j are called neighbors if $(w_i, w_j) \in E$. Two adjacent vertices w_i, w_j are represented by $w_i \sim w_j$. The set $N(w_i)$ $(N[w_i])$ is called open (closed) neighborhood of w_i which represents the set of all adjacent vertices of w_i excluding (including) w_i. A subset of vertices of a graph G is called an independent set, if no two vertices of it is adjacent. The complement of G, $\overline{G}$ is the graph obtained from G such that two vertices of $\overline{G}$ are adjacent if and only if they are non-adjacent in G. Let P be the partition of V consisting of k partites. To obtain $k-$complement of G, remove the edges between the partites in G and add the edges between the partites which are not in G [1]. To obtain the $k(i)-$complement of G, remove the edges inside each partite which are in G and add the edges between those vertices which are not in G [2]. The k, $k(i)$ complements of a graph G are represented by G_k^P, $G_{k(i)}^P$ respectively.

In the past, various domination numbers have been defined and studied by eminent researchers. A few basic domination numbers are presented in table I. Consider $G(V, E)$ with $D \subseteq V$. Cardinality of each minimum dominating set is the respective domination number.

TABLE I
VARIOUS DOMINATING SETS

Dominating sets	**Notation of domination number**
D is dominating set if $\forall$ $w_i \in V - D$ $\exists w_j \in D$ with $w_i \sim w_j$.	$\gamma(G)$
D is connected dominating set if $\langle D \rangle$ is connected [3].	$\gamma_c(G)$
D is total dominating set if $\forall$ $w_i \in V$ $\exists w_j \in D$ with $w_i \sim w_j$ [4].	$\gamma_t(G)$
D is global dominating set if it is a dominating set for both G and $\overline{G}$ [5].	$\gamma_g(G)$
D is $k-$ global dominating set if it is a dominating set for both G and G_k^P [6].	$\gamma_{kg}(G)$
D is $k(i)-$ global dominating set if it is a dominating set for both G and $G_{k(i)}^P$ [6].	$\gamma_{k(i)g}(G)$
D is a total global dominating set if it is total dominating set for G and a dominating set for $\overline{G}$ [7].	$\gamma_{tg}(G)$
D is a $k-$ total global dominating set if it is total dominating set for G and a dominating set for G_k^P [6].	$\gamma_{ktg}(G)$
D is a $k(i)-$ total global dominating set if it is total dominating set for G and a dominating set for $G_{k(i)}^P$ [6].	$\gamma_{k(i)tg}(G)$
D is an independent dominating set if it is both a dominating set and an independent set for G. [8].	$i(G)$

For more on domination of graphs one can refer [9], [10].

In this paper, the authors explore different domination numbers for the generalized complements of a graph. As the generalized complement of a graph depends on the partition of the vertex set, the proposed work examines different partitions of graph to obtain the minimum and maximum domination numbers. Some relationship between domination numbers in the context of generalized complements are achieved.

II. Domination numbers of G and its $k/k(i)-$ complements

Theorem 2.1: Let G be a connected graph with vertex set V.

1) Let $P = \{P_1, P_2, \ldots, P_k\}$ be a partition of V with $|P_i| = n_i$, $1 \leq i \leq k$. If each $\langle P_i \rangle$ has a vertex of degree $n_i - 1$, $1 \leq i \leq k$ then $\gamma(G_k^P) \leq k$. The upper bound sharpness occurs if each vertex of any partite is adjacent to all the vertices of remaining partites.
2) If $P = \{P_1, P_2, \ldots, P_k\}$ is the partition of V such that for all $w_x, w_y \in P_i$, $d(w_x, w_y) \geq 2$, then $\gamma(G_{k(i)}^P) \leq k$. The upper bound sharpness occurs for a completely disconnected graph.

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Proof: Let G be a connected graph with vertex set V.

1) Suppose V is partitioned into $\{P_1, P_2, \ldots, P_k\}$ with $|P_i| = n_i$ and each $\langle P_i \rangle$ has a vertex of degree $n_i - 1,\ 1 \leq i \leq k$. In G_k^P, there exists at least one vertex in each $\langle P_i \rangle$ that dominates all other vertices of $\langle P_i \rangle$ for $1 \leq i \leq k$. Hence $\gamma(G_k^P) \leq k$.
   If each vertex of P_i is adjacent to all the vertices of $P_j,\ 1 \leq i, j \leq k,\ i \neq j$, then G_k^P is a disconnected graph with connected components $\langle P_i \rangle,\ 1 \leq i \leq k$ such that each $\langle P_i \rangle$ is of order n_i consisting a vertex of degree $n_i - 1$. Therefore $\gamma(G_k^P) = k$.
2) Suppose V is partitioned into $\{P_1, P_2, \ldots, P_k\}$ such that for all $w_x, w_y \in P_i,\ d(w_x, w_y) \geq 2$. In $G_{k(i)}^P$ the distance between any two vertices in each $\langle P_i \rangle$ is one. Therefore there exists at least one vertex in each $\langle P_i \rangle$ that dominates all the vertices of $\langle P_i \rangle$. Hence $\gamma(G_{k(i)}^P) \leq k$.
   For a completely disconnected graph, $G_{k(i)}^P$ is a graph with k connected components with each connected component is a complete subgraph. Hence $\gamma(G_{k(i)}^P) = k$.

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Theorem 2.2: Let G be a graph with $|V| = n$ having a vertex of degree $n - 1$.

1) If $\{P_1, P_2, \ldots, P_k\}$ is the partition of V such that for any vertex w_x of degree d, $P_1 = N[w_x]$, then $\gamma(G) = \gamma(G_k^P) = 1$ for $1 \leq k \leq n - d - 1$.
2) If $\{P_1, P_2, \ldots, P_k\}$ is the partition of V with $P_1 = V - N(w_x)$ for $w_x \in V$, then $\gamma(G) = \gamma(G_{k(i)}^P) = 1$ for $2 \leq k \leq n - 1$.

Proof: Let G be a graph with $|V| = n$ having a vertex of degree $n - 1$.

1) If $deg_G w_x = n - 1$ for $w_x \in V$, then $\gamma(G) = 1$. Suppose V is partitioned into $\{P_1, P_2, \ldots, P_k\}$ with $P_1 = N[w_x]$, where $\deg_G w_x = d$. Then in G_k^P, all the vertices are adjacent to w_x by definition of G_k^P [1]. Thus w_x dominates all the vertices of G_k^P and $\gamma(G_k^P) = 1$.
2) If $deg_G w_x = n - 1$ for $w_x \in V$, then $\gamma(G) = 1$. Suppose G is partitioned into $P = \{P_1, P_2, \ldots, P_k\}$ such that $P_1 = V - N(w_x)$ for $w_x \in V$. In $G_{k(i)}^P$, all the vertices are adjacent to w_x by definition of $G_{k(i)}^P$ [2]. Thus w_x dominates all the vertices of $G_{k(i)}^P$ and $\gamma(G_{k(i)}^P) = 1$.

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Corollary 2.1: 1) For the graphs G and G_k^P, $\gamma(G) = \gamma(G_k^P) = n$ if and only if $G \cong \overline{K_n}$ and $k = 1$.
2) For the graphs G and $G_{k(i)}^P$, $\gamma(G) = \gamma(G_{k(i)}^P) = n$ if and only if $G \cong \overline{K_n}$ and $k = n$.

Theorem 2.3: Let D be a minimum dominating set of G and let $P = \{P_1, P_2, \ldots, P_k\}$ be the partition of V.

1) If each P_i has exactly one vertex of D and its neighbors, then $\gamma_{kg}(G) = k$.
2) If each P_i has exactly one vertex of D and all its non-neighbors, then $\gamma_{k(i)g}(G) = k$.

Proof: Let $D = \{w_1, w_2, ..., w_k\}$ be a minimum dominating set of G and let $P = \{P_1, P_2, \ldots, P_k\}$ be the partition of V.

1) Suppose each P_i contains exactly one vertex of D and all the vertices which are adjacent to that vertex. In G_k^P, each vertex is adjacent to exactly one vertex of D. Therefore D is also a dominating set for G_k^P.
2) Suppose each P_i contains exactly one vertex of D and all the vertices which are non-adjacent to that vertex. In $G_{k(i)}^P$, each vertex is adjacent to exactly one vertex of D. Therefore D is also a dominating set for $G_{k(i)}^P$.

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Theorem 2.4: 1) Let $\gamma(G) \geq 2$ be the domination number of G with dominating set D such that all vertices of D are non-adjacent. If $P = \{P_1, P_2\}$ is a partition of V with $\langle P_1 \rangle = \langle D \rangle$, $\langle P_2 \rangle = \langle V - D \rangle$ and no vertex of $V - D$ is adjacent to all the vertices of D, then D is a $k-$ global dominating set of G.
2) Let $\gamma(G) \geq 2$ be the domination number of G with dominating set D such that $\langle D \rangle$ be a complete subgraph of G. If $P = \{P_1, P_2\}$ with $\langle P_1 \rangle = \langle D \rangle$, then D is a $k(i)-$ global dominating set of G.

Proof:

1) Let $\gamma(G) \geq 2$ be the domination number of G with dominating set D and all vertices of D be non-adjacent. Suppose $P = \{P_1, P_2\}$ is a partition of G with $\langle P_1 \rangle = \langle D \rangle$ and no vertex of P_2 is adjacent to all the vertices of D. In G_k^P, at least all the vertices of D must be in dominating set since all the vertices of D are non-adjacent in G. No vertex of G_k^P is an isolated vertex since no vertex of G is adjacent to all the vertices of D. The set D is a dominating set of G_k^P since each vertex of G_k^P is adjacent to at least one vertex of D.
2) Let $\gamma(G) \geq 2$ be the domination number of G with dominating set D such that $\langle D \rangle$ be a complete subgraph of G. Suppose $P = \{P_1, P_2\}$ with $\langle P_1 \rangle = \langle D \rangle$, $\langle P_2 \rangle = \langle V - D \rangle$. Then in $G_{k(i)}^P$, at least all the vertices of D must be in the dominating set of $G_{k(i)}^P$ since no two vertices of D are adjacent in $G_{k(i)}^P$ and the adjacency between the vertices of $V - D$ and D remains same in both G and $G_{k(i)}^P$. The set D is a dominating set of $G_{k(i)}^P$ since all the vertices of $G_{k(i)}^P$ is adjacent to at least one vertex of D.

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Theorem 2.5: 1) If $G \cong \frac{n}{2} K_2$, and G is partitioned into $\frac{n}{2}$ partites such that each $\langle P_i \rangle = K_2$, then $\gamma_t(G) = n$ and $\gamma_t(G_k^P) = 2$.
2) If $G \cong \frac{n}{2} K_2$ is partitioned into P_1 and P_2, each consisting of independent set of vertices, then $\gamma_t(G) = n$ and $\gamma_t(G_{k(i)}^P) = 2$.

Proof:

1) Let $G \cong \frac{n}{2} K_2$ and V be partitioned into $\{P_1, P_2, \ldots, P_{\frac{n}{2}}\}$ with each $\langle P_i \rangle = K_2$. Obviously, total domination number of G is n. The graph G_k^P is a complete graph with n vertices and hence total domination number of G_k^P is 2.
2) Let $G \cong \frac{n}{2} K_2$ be partitioned into P_1 and P_2, each consisting of independent set of vertices. Obviously, total domination number of G is n. In $G_{k(i)}^P$, $\langle P_1 \rangle \cong \langle P_2 \rangle \cong K_{\frac{n}{2}}$ and hence $\gamma_t(G_{k(i)}^P) = 2$.

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Remark 2.1: 1) Suppose $G \cong K_n$, where n is even and is partitioned into $\frac{n}{2}$ partites P_i such that each $\langle P_i \rangle = K_2$. Then $\gamma_t(G_k^P) = n$.

2) Suppose G is the graph $G_{k(i)}^P$ obtained in (2) of Theorem 2.5. If G is partitioned into two partites P_1 and P_2 such that the graphs induced by both P_1 and P_2 is complete graphs with $\frac{n}{2}$ vertices. Then, total domination number of $G_{k(i)}^P$ is n.

Theorem 2.6: 1) $\gamma_t(G) = \gamma_t(G_k^P) = n$ if and only if $G \cong \frac{n}{2}K_2$ and $k = 1$.

2) $\gamma_t(G) = \gamma_t(G_{k(i)}^P) = n$ if and only if $G \cong \frac{n}{2}K_2$ and $k = n$.

Proof:

1) Suppose $\gamma_t(G) = \gamma_t(G_k^P) = n$. If G consists of an isolated vertex, then $\gamma_t(G)$ is undefined. If G is a graph without isolates, then $\gamma_t(G) \leq n - \Delta(G) + 1$[4] and $\gamma_t(G) = n$ if and only if $\Delta(G) = 1$. This is possible if and only if $G \cong \frac{n}{2}K_2$. Suppose $k > 1$, then $\Delta(G_k^P) \geq 2$ and hence $\gamma_t(G_k^P) < n$. Therefore, $G \cong \frac{n}{2}K_2$ and $k = 1$. The converse part of the proof is trivial since G_k^P is the graph G itself.
2) Suppose $\gamma_t(G) = \gamma_t(G_{k(i)}^P) = n$. The first part of the proof is same as the proof of (1) in Theorem 2.6. Suppose $k = n$, then either $\Delta(G_{k(i)}^P) \geq 2$ or $G_{k(i)}^P$ has an isolate. Thus $\gamma_t(G_k^P)$ is either less than n or undefined. Therefore $G \cong \frac{n}{2}K_2$ and $k = n$. The converse part of the proof is trivial since $G_{k(i)}^P$ is the graph G itself.

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Theorem 2.7: 1) $\gamma_{ktg}(G) = 2$ with total dominating set $D = \{w_x, w_y\}$ if and only if $\gamma_t(G) = 2$ with dominating set D and G is partitioned into $P = \{P_1, P_2, \ldots, P_k\}$, $1 \leq k \leq |N(w_x, w_y)| + 2$ such that $P_1 = N[w_x] \cup N[w_y]$.

2) $\gamma_{k(i)tg}(G) = 2$ with total dominating set $D = \{w_x, w_y\}$ if and only if $\gamma_t(G) = 2$ with dominating set D and G is partitioned into $P = \{P_1, P_2, \ldots, P_k\}$, $2 \leq k \leq n$ such that $w_x \in P_1$ and $(V - N(w_x)) \not\subseteq P_1$, $w_y \in P_2$ and $(V - N(w_y)) \not\subseteq P_2$.

Proof:

1) Let G be a graph with total dominating set $D = \{w_x, w_y\}$. Suppose $P = \{P_1, P_2, \ldots, P_k\}$ is the partition of G such that $P_1 = N[w_x] \cup N[w_y]$. In G_k^P, the common neighbors of w_x and w_y remain same. The vertices which are adjacent to the vertex w_x and non-adjacent to the vertex w_y in G, are adjacent to the vertex w_y and non-adjacent to the vertex w_x in G_k^P and vice versa. Hence all the vertices are dominated by either w_x or w_y and also w_x is adjacent to w_y in G_k^P. Therefore, $\gamma_{ktg}(G) = 2$.
   Conversely, suppose $\gamma_t(G) = 2$ and the vertices w_x and w_y are in different partites. In G_k^P, w_x and w_y are non-adjacent. Hence $D = \{w_x, w_y\}$ can not be a total dominating set. Suppose $P = \{P_1, P_2, \ldots, P_k\}$ is the partition of V such that P_1 consists of the vertices w_x and w_y. If there exists a vertex w_z which is adjacent to both w_x and w_y in G, but it belongs to the partite other than P_1, then in G_k^P, the vertex w_z is non-adjacent to w_x or w_y. Hence $D = \{w_x, w_y\}$ is not a dominating set.
2) Let G be a graph with total dominating set $D = \{w_x, w_y\}$. Suppose $P = \{P_1, P_2, \ldots, P_k\}$ is the partition of V such that $w_x \in P_1$ and $(V - N(w_x)) \not\subseteq P_1$, $w_y \in P_2$ and $(V - N(w_y)) \not\subseteq P_2$. In $G_{k(i)}^P$, the vertices w_x and w_y are adjacent since they are in the different partites. All the vertices which are adjacent to w_x and w_y are adjacent in $G_{k(i)}^P$. Thus, the vertices w_x and w_y dominate all the remaining vertices. Hence D is the total dominating set of $G_{k(i)}^P$.
   Conversely, suppose the vertices w_x and w_y are in the same partite, D can not be a total dominating set as w_x and w_y are non-adjacent in $G_{k(i)}^P$. Suppose $P = \{P_1, P_2, \ldots, P_k\}$ is the partition of V such that $w_x \in P_1$. If there exists a vertex $w_z \in P_1$ with $w_z \sim w_x$ and $w_z \not\sim w_y$ in G, then $w_z \not\sim w_x$ and $w_z \not\sim w_y$ in $G_{k(i)}^P$. Hence D can not be a total dominating set.

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Theorem 2.8: Let D be a total dominating set of G.

1) If V is partitioned into $\left\{P_1, P_2, \ldots, P_{\frac{\gamma_t}{2}}\right\}$ with each P_i has exactly two adjacent vertices of D and their neighbors, then D is a $k-$ total global dominating set of G.
2) If V is partitioned into $\left\{P_1, P_2, \ldots, P_{\frac{\gamma_t}{2}}\right\}$ with each P_i has exactly two non-adjacent vertices w_x, w_y of D and the vertices which are non-adjacent to both w_i, w_j, then D is a $k(i)-$ total global dominating set of G.

Proof: Let D be a total dominating set of G.

1) Let $P = \left\{P_1, P_2, \ldots, P_{\frac{\gamma_t}{2}}\right\}$ be the partition V having P_i has exactly two adjacent vertices of D and their neighbors. Then the vertices adjacent to vertices of D in G are adjacent in $G_{\frac{\gamma_t}{2}}^P$ also. Hence D is also a total dominating set of $G_{\frac{\gamma_t}{2}}^P$. Thus $\gamma_t(G) = \gamma_{ktg}(G)$ with dominating set D.
2) Let $P = \left\{P_1, P_2, \ldots, P_{\frac{\gamma_t}{2}}\right\}$ be the partition of V having P_i has exactly two non-adjacent vertices w_x, w_y of D and non-adjacent vertices to both w_x, w_y. Then the vertices adjacent to vertices of D in G are adjacent in $G_{\frac{\gamma_t}{2}(i)}^P$ also. Hence D is also a total dominating set of $G_{\frac{\gamma_t}{2}(i)}^P$. Thus $\gamma_t(G) = \gamma_{k(i)tg}(G)$ with dominating set D.

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Theorem 2.9: Let D be a connected dominating set of a graph G on n vertices.

1) If V is partitioned into $P = \{P_1, P_2, \ldots, P_k\}$ with $P_1 = D$ and no vertex of D is adjacent to all the vertices of P_i, $2 \leq i \leq k$, then $\gamma_c(G_k^P) \leq \gamma_c(G)$. Equality holds if G has a vertex of degree $n - 1$.
2) If V is partitioned into $P = \{P_1, P_2, \ldots, P_k\}$ such that $P_1 = V - D$ and each P_i, $2 \leq i \leq k$ has exactly one vertex of D, then $\gamma_c(G_{k(i)}^P) \leq \gamma_c(G)$. Equality holds if G is a connected graph with $\Delta(G) \leq 2$.

Proof: Let G be a graph with connected dominating set D.

1) Suppose V is partitioned into $P = \{P_1, P_2, \ldots, P_k\}$ with $P_1 = D$ and no vertex of D is adjacent to all the vertices of P_i, $2 \leq i \leq k$. In G_k^P, each vertex of P_1 is adjacent to at least one vertex of any P_i, $2 \leq i \leq k$. Therefore $\gamma_c(G_k^P) \leq \gamma_c(G)$.

If G has a vertex w_x of degree $n-1$, then $D = P_1 = \{w_x\}$ and $P_i, 2 \leq i \leq k$ consists of remaining vertices. But $\deg_{G_k^P} w_x = 1$ and therefore $\gamma_c(G_k^P) = \gamma_c(G) = 1$.

2) Suppose V is partitioned into $P = \{P_1, P_2, \ldots, P_k\}$ with $P_1 = V - D$, and each P_i, $2 \leq i \leq k$ has exactly one vertex of D. In $G_{k(i)}^P$, the adjacency between the vertices inside each P_i changes. The vertices of D dominates all the vertices of $G_{k(i)}^P$ and D induces a connected subgraph of $G_{k(i)}^P$. Therefore $\gamma_c(G_{k(i)}^P) \leq \gamma_c(G)$.
If G is a connected graph with $\Delta(G) \leq 2$. Then P_1 consists of two vertices say w_x and w_y so that each P_i, $2 \leq i \leq k$ consists of exactly one vertex of the set D. The adjacency between all the pairs of vertices remain same in $G_{k(i)}^P$ except between w_x and w_y. Therefore $\gamma_c(G_{k(i)}^P) = \gamma_c(G) = n-2$. ∎

Theorem 2.10: Let G be a graph with n vertices.

1) $\gamma_c(G) = \gamma_c(G_k^P) = 1$ if and only if G has a vertex of degree $n-1$ and it is partitioned into $P = \{P_1, P_2, \ldots, P_k\}$ with $P_1 = N[w_x],\ w_x \in V$.
2) $\gamma_c(G) = \gamma_c(G_{k(i)}^P) = 1$ if and only if G has a vertex of degree $n-1$ and it is partitioned into $P = \{P_1, P_2, \ldots, P_k\}$ with $P_1 = \{w_x\} \cup (V - N(w_x))$.

Proof: Let G be a graph with $|V| = n$.

1) If $\deg_G w_y = n-1$ for some $w_y \in V$ and it is partitioned into $P = \{P_1, P_2, \ldots, P_k\}$ with $P_1 = \{N[w_x] : w_x \in V\}$. Then vertex w_y dominates every vertex of G and vertex w_x dominates every vertex of G_k^P. Thus $\gamma_c(G) = \gamma_c(G_k^P) = 1$.
Conversely, if $\deg_G w_y \neq n-1$ for all $w_y \in V$, then $\gamma_c(G) \geq 2$. If G has a vertex of degree $n-1$ and it is partitioned into $P = \{P_1, P_2, \ldots, P_k\}$ with P_1 has a vertex w_y of G and there exists a vertex $w_x \in P_j$, $2 \leq j \leq k$ with $w_x \sim w_y$. Then $\deg_{G_k^P} w_x \leq n-2$ and hence $\gamma_c(G_k^P) \geq 2$.
2) If $\deg_G w_y = n-1$ for some $w_y \in V$ and it is partitioned into $P = \{P_1, P_2, \ldots, P_k\}$ with $P_1 = \{w_x\} \cup (V - N(w_x))$ for $w_x \in V$. Then vertex w_y dominates every vertex of G and vertex w_x dominates every vertex of $G_{k(i)}^P$. Thus $\gamma_c(G) = \gamma_c(G_{k(i)}^P) = 1$.
Conversely, if $\deg_G w_y \neq n-1$ for all $w_y \in V$, then $\gamma_c(G) \geq 2$. If G has a vertex of degree $n-1$ and it is partitioned into $P = \{P_1, P_2, \ldots, P_k\}$ such that P_1 has a vertex w_y and there exists a vertex $w_x \in P_j$, $2 \leq j \leq k$ with $w_x \not\sim w_y$. Then $\deg_{G_{k(i)}^P} w_y \leq n-2$ and hence $\gamma_c(G_{k(i)}^P) \geq 2$. ∎

Theorem 2.11: Let G be a connected graph with $|V| = n$.

1) $\gamma_c(G) = \gamma_c(G_k^P) = n-2$ if and only if $\Delta(G) \leq 2$ and $k = 1$.
2) $\gamma_c(G) = \gamma_c(G_{k(i)}^P) = n-2$ if and only if $\Delta(G) \leq 2$ and $k = n$.

Proof: Let G be a connected graph on n vertices with $\Delta(G) \leq 2$.

1) Let D be the minimum connected dominating set of G. Since $\langle D \rangle$ is a connected subgraph, there must be a path between every pair of vertices of D. Since each vertex can dominate at most two vertices, the graph induced by the vertices of D must be a path on $n-2$ vertices. If $k = 1$, then $G_k^P \cong G$ and hence $\gamma_c(G) = \gamma_c(G_k^P) = n-2$.
Conversely, suppose G is a connected graph on n vertices with $\Delta(G) \geq 3$, then there exists a vertex that dominates at least three vertices. Therefore $\gamma_c(G) \leq n-3$. Suppose $\Delta(G) \leq 2$ and G is partitioned into more than one partites, then there exists a vertex whose degree is at least three in G_k^P. Therefore $\gamma_c(G_k^P) \leq n-3$.
2) Let D be the minimum connected dominating set of G. Since $\langle D \rangle$ is a connected subgraph, there must be a path between every pair of vertices of D. Since each vertex can dominate at most two vertices, the graph induced by the vertices of D must be a path on $n-2$ vertices. If $k = n$, then $G_{k(i)}^P \cong G$ and hence $\gamma_c(G) = \gamma_c(G_{k(i)}^P) = n-2$.
Conversely, suppose G is a connected graph on n vertices with $\Delta(G) \geq 3$, then there exists a vertex that dominates at least three vertices. Therefore $\gamma_c(G) \leq n-3$. Suppose $\Delta(G) \leq 2$ and G is partitioned into less than n partites, then there exists a vertex whose degree is at least three in $G_{k(i)}^P$. Therefore $\gamma_c(G_{k(i)}^P) \leq n-3$. ∎

Theorem 2.12: An independent dominating set D of a graph G is also an independent dominating set of G_k^P if and only if

1) For every vertex w_x in D, there exists a vertex w_y in $V - D$ such that $w_x \not\sim w_y$ and
2) V is partitioned into $P = \{P_1, P_2, \ldots, P_k\}$ such that $P_1 = D$.

Proof: Let G be a graph with independent dominating set D such that for every vertex w_x in D, there exists a vertex w_y in $V - D$ such that $w_x \not\sim w_y$. Suppose V is partitioned into $P = \{P_1, P_2, \ldots, P_k\}$ such that $P_1 = D$. In the graph G_k^P, all the vertices of D are non-adjacent. Also each w_x in D is adjacent to at least one vertex w_y since $w_x \not\sim w_y$ and they belong to different partites. Therefore D is an independent dominating set for G_k^P.

Conversely, Suppose V is partitioned into $P = \{P_1, P_2, \ldots, P_k\}$ such that $P_1 = D$. If there exists a vertex w_x in D which is adjacent to all the vertices of $V - D$, then in G_k^P it is non-adjacent to any of the vertex of $V - D$. Hence D is not a dominating set. Suppose for every vertex w_x in D, there exists a vertex w_y in $V - D$ such that $w_x \not\sim w_y$. Let w_x and w_y be any two vertices of D. If V is partitioned into $P = \{P_1, P_2, \ldots, P_k\}$ such that $w_x \in P_i$ and $w_y \in P_j$ for $i \neq j$, then in G_k^P, $w_x \sim w_y$ and hence D is not an independent set of G_k^P. ∎

Theorem 2.13: An independent dominating set D of a graph G is also an independent dominating set of $G_{k(i)}^P$ if and only if V is partitioned into $P = \{P_1, P_2, \ldots, P_k\}$ such that

1) no two vertices of D are in same partite and
2) if w_x is a vertex of G with w_x is adjacent to only one vertex w_y of D, then w_x and w_y are in different partites.

Proof: Let G be a graph with independent dominating set D.

1) Suppose V is partitioned into $P = \{P_1, P_2, \ldots, P_k\}$ such that no two vertices of D are in the same partite, then in $G^P_{k(i)}$, no two vertices of D are adjacent. Hence the vertices of D are independent.
2) Let $w_x \in V(G)$. There are two possibilities. Either w_x is adjacent to only one vertex of D or w_x is adjacent to more than one vertex of D. If w_x is adjacent to only vertex w_y of D, then w_x and w_y are in different partites and hence they are adjacent in $G^P_{k(i)}$. If w_x is adjacent to more than one vertex of D, say $u_1, u_2, \ldots, u_l$ and w_x and any one of u_i, $1 \leq i \leq l$, are in same partite, then in $G^P_{k(i)}$, w_x is adjacent to remaining u_j, $1 \leq j \leq l$, $i \neq j$. If w_x is adjacent to more than one vertex of D, say $u_1, u_2, \ldots, u_l$ and w_x and u_i, $1 \leq i \leq l$, are in different partites, then in $G^P_{k(i)}$, w_x is adjacent to all u_i, $1 \leq i \leq l$.

Hence, all the vertices of $V - D$ in $G^P_{k(i)}$ is adjacent to at least one vertex of D and D is an independent set. Therefore, D is an independent dominating set of $G^P_{k(i)}$.

Conversely, suppose D is an independent dominating set of both G and $G^P_{k(i)}$.

1) If any two vertices $w_x,\ w_y$ of D are in same partite, then in $G^P_{k(i)}$, w_x and w_y are adjacent and hence D is not an independent set of $G^P_{k(i)}$.
2) If w_x is a vertex of G with w_x is adjacent to only one vertex w_y of D and w_x and w_y are in same partite. Then in $G^P_{k(i)}$, w_x and w_y are non-adjacent. Also, w_x is non-adjacent to all the vertices of D since it is adjacent to only w_y of D and $w_x,\ w_y$ are in same partite. Hence D is not a dominating set of $G^P_{k(i)}$.

■

III. Conclusion

Though generalized complements of a graph is introduced in the year 1998, there are less studies have been done on the topic. In this paper, the authors considered various domination numbers and obtained relationship between the domination number of graph and its $k/k(i)$ complements.

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