A Modified Generalized Relaxed Splitting Preconditioner for Generalized Saddle Point Problems

Jun He, Yanmin Liu, and Wei Lv,

Abstract—**In this paper, a modified generalized relaxed splitting (MGRS) preconditioner is established to accelerate the convergence of the associated Krylov subspace methods, which is often used to solve generalized saddle point problems. Some spectral characteristics of the preconditioned matrices are also studied. Finally, numerical experiments are also reported to show the efficiency of the proposed preconditioner.**

Index Terms—**saddle point problem, preconditioner, krylov subspace method, matrix splitting.**

I. Introduction

In the current study, we are interested in efficient solutions of the following generalized saddle point problems, which arise from the discretization of two-dimensional linearized Navier-Stokes equations [1], [2], [3], [17], [18]:

$$\mathcal{A}u \equiv \begin{bmatrix} A_1 & 0 & B_1^T \\ 0 & A_2 & B_2^T \\ -B_1 & -B_2 & C \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ p \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ -g \end{bmatrix} \equiv b, \quad (1)$$

where $A_1 \in R^{n_1 \times n_1}$, $A_2 \in R^{n_2 \times n_2}$ are nonsymmetric positive definite matrices, $B_1 \in R^{m \times n_1}$, $B_2 \in R^{m \times n_2}$ have full row rank, and $C \in R^{m \times m}$ is a symmetric positive semi-definite matrix, $u, b \in R^{n+m}$, $u_1, f_1 \in R^{n_1}$, $u_2, f_2 \in R^{n_2}$ and u_1, u_2, p are the unknown vectors. These assumptions guarantee the existence and uniqueness of the solution of the linear system (1).

The coefficient matrix $\mathcal{A}$ of the system (1) is often large-scale and sparse, the preconditioned Krylov subspace methods have been preferably considered to approximate the solution of generalized saddle point problems (1), e-specially the preconditioned GMRES [10] method. A lots of preconditioners are introduced in the past few years for generalized saddle point problems, such as the block diagonal preconditioners [4], constraint preconditioners [8], Hermitian and skew-Hermitian splitting (HSS) preconditioners [5], [6], [7], dimensional split preconditioners [1], [2], [3], and other matrix splitting preconditioners [12], [13], [14]. Moreover, the effectiveness of these block preconditioners has been also verified in aspects of both theoretical and numerical results. Each type of block preconditioners has its own advantages and disadvantages, and they might play out differently in accelerating Krylov subspace methods for solving various generalized saddle-point systems.

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Recently, when $C = 0$, based on the dimensional splitting (DS), Benzi and Guo [1] had proposed a DS preconditioner

$$P_{DS} = \frac{1}{2\alpha} \begin{bmatrix} \alpha I + A_1 & 0 & B_1^T \\ 0 & \alpha I & 0 \\ -B_1 & 0 & \alpha I \end{bmatrix} \begin{bmatrix} \alpha I & 0 & 0 \\ 0 & \alpha I + A_2 & B_2^T \\ 0 & -B_2 & \alpha I \end{bmatrix},$$

where α is a positive parameter and I is the identity matrix.

In order to get a better approximation of the coefficient matrix $\mathcal{A}$, Benzi et al. [2] proposed an improved variant of the DS preconditioner, which is called the relaxed dimensional factorization (RDF) preconditioner of the form:

$$P_{RDF} = \frac{1}{\alpha} \begin{bmatrix} A_1 & 0 & B_1^T \\ 0 & \alpha I & 0 \\ -B_1 & 0 & \alpha I \end{bmatrix} \begin{bmatrix} \alpha I & 0 & 0 \\ 0 & A_2 & B_2^T \\ 0 & -B_2 & \alpha I \end{bmatrix}.$$

When $C \neq 0$, Cao et al. [3] introduced a modified dimensional splitting (MDS) preconditioner as follows:

$$P_{MDS} = \frac{1}{\alpha} \begin{bmatrix} \alpha I + A_1 & 0 & B_1^T \\ 0 & \alpha I & 0 \\ -B_1 & 0 & \alpha I \end{bmatrix} \times \begin{bmatrix} \alpha I & 0 & 0 \\ 0 & \alpha I + A_2 & B_2^T \\ 0 & -B_2 & \alpha I + C \end{bmatrix}.$$

Therefore, we can get the difference between the MDS preconditioner P_{MDS} and the generalized saddle point matrix $\mathcal{A}$

$$P_{MDS} - \mathcal{A} = \begin{bmatrix} \alpha I & -\frac{1}{\alpha} B_1^T B_2 & \frac{1}{\alpha} B_1^T C \\ 0 & \alpha I & 0 \\ 0 & 0 & \alpha I \end{bmatrix}. \quad (2)$$

In this paper, a variant of the MDS preconditioner is proposed for the generalized saddle point problems (1) in Section II, which is much better approximation to the coefficient matrix $\mathcal{A}$ of the generalized saddle point problems than the MDS preconditioner. In Section III, some properties of the preconditioned matrix are established. In Section IV, numerical experiments are given to show the effectiveness of the new preconditioner. Finally, the paper closes with conclusions in Section V.

II. The modified generalized relaxed splitting (MGRS) preconditioner

In this section, based on the following modified dimensional splitting iteration method

$$\begin{cases} (\alpha I+\mathcal{A}_1)u^{(k+\frac{1}{2})}=(\alpha I-\mathcal{A}_2)u^{(k)}+b, \\ (\alpha I+\mathcal{A}_2)u^{(k+1)}=(\alpha I-\mathcal{A}_1)u^{(k+\frac{1}{2})}+b, \end{cases}$$

where

$$\mathcal{A}_1=\begin{bmatrix} A_1 & 0 & B_1^T \\ 0 & 0 & 0 \\ -B_1 & 0 & 0 \end{bmatrix}, \quad \mathcal{A}_2=\begin{bmatrix} 0 & 0 & 0 \\ 0 & A_2 & B_2^T \\ 0 & -B_2 & C \end{bmatrix},$$

and $\mathcal{A}=\mathcal{A}_1+\mathcal{A}_2$ [3], we can establish a variant of the generalized relaxed splitting preconditioner

$$P_{MGRS}=\frac{1}{\alpha}\begin{bmatrix} A_1 & 0 & B_1^T \\ 0 & \alpha I & 0 \\ -B_1 & 0 & \alpha I \end{bmatrix}\begin{bmatrix} \alpha I & 0 & 0 \\ 0 & A_2 & B_2^T \\ 0 & -B_2 & C \end{bmatrix}. \quad (3)$$

By direct calculation, the preconditioner P_{MGRS} has the following block structure

$$P_{MGRS}=\begin{bmatrix} A_1 & -\frac{1}{\alpha}B_1^TB_2 & \frac{1}{\alpha}B_1^TC \\ 0 & A_2 & B_2^T \\ -B_1 & -B_2 & C \end{bmatrix},$$

and the difference between the preconditioner P_{MGRS} and the generalized saddle point matrix $\mathcal{A}$ is given by

$$P_{MGRS}-\mathcal{A}=\begin{bmatrix} 0 & -\frac{1}{\alpha}B_1^TB_2 & \frac{1}{\alpha}B_1^T(C-\alpha I) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (4)$$

Compared with (2), the block diagonal matrices in (4) vanish. Therefore, we expect that, the preconditioner P_{MGRS} is much better approximation to the coefficient matrix $\mathcal{A}$ of the generalized saddle point problems than the MDS preconditioner and can perform better than the MDS preconditioner P_{MDS} in the preconditioned Krylov subspace methods.

First, we need the following lemma.

Lemma 2.1: Let

$$M_1=\begin{bmatrix} A_1 & 0 & B_1^T \\ 0 & \alpha I & 0 \\ -B_1 & 0 & \alpha I \end{bmatrix}$$

and

$$M_2=\begin{bmatrix} \alpha I & 0 & 0 \\ 0 & A_2 & B_2^T \\ 0 & -B_2 & C \end{bmatrix}.$$

Then

$$M_1^{-1}=\begin{bmatrix} \hat{A}_1^{-1} & 0 & -\frac{1}{\alpha}\hat{A}_1^{-1}B_1^T \\ 0 & \frac{1}{\alpha}I & 0 \\ \frac{1}{\alpha}B_1\hat{A}_1^{-1} & 0 & \frac{1}{\alpha^2}I-\hat{A}_1^{-1}S_1 \end{bmatrix}$$

$$M_2^{-1}=\begin{bmatrix} \frac{1}{\alpha}I & 0 & 0 \\ 0 & A_2^{-1}-S_2 & -A_2^{-1}B_2^T\hat{C}^{-1} \\ 0 & \hat{C}^{-1}B_2A_2^{-1} & \hat{C}^{-1} \end{bmatrix}.$$

where $\hat{A}_1=A_1+\frac{1}{\alpha}B_1^TB_1$, $S_1=B_1\hat{A}_1^{-1}B_1^T$, $\hat{C}=C+B_2A_2^{-1}B_2^T$, and $S_2=A_2^{-1}B_2^T\hat{C}^{-1}B_2A_2^{-1}$.

Proof. We can factorize M_1 as

$$M_1=\begin{bmatrix} I & 0 & \frac{1}{\alpha}B_1^T \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}\begin{bmatrix} A_1+\frac{1}{\alpha}B_1^TB_1 & 0 & 0 \\ 0 & \alpha I & 0 \\ 0 & 0 & \alpha I \end{bmatrix} \times\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -\frac{1}{\alpha}B_1 & 0 & I \end{bmatrix}.$$

Then

$$M_1^{-1}=\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ \frac{1}{\alpha}B_1 & 0 & I \end{bmatrix}\begin{bmatrix} \hat{A}_1^{-1} & 0 & 0 \\ 0 & \frac{1}{\alpha}I & 0 \\ 0 & 0 & \frac{1}{\alpha}I \end{bmatrix} \times\begin{bmatrix} I & 0 & -\frac{1}{\alpha}B_1^T \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} =\begin{bmatrix} \hat{A}_1^{-1} & 0 & -\frac{1}{\alpha}\hat{A}_1^{-1}B_1^T \\ 0 & \frac{1}{\alpha}I & 0 \\ \frac{1}{\alpha}B_1\hat{A}_1^{-1} & 0 & \frac{1}{\alpha^2}I-\hat{A}_1^{-1}S_1 \end{bmatrix},$$

where $\hat{A}_1=A_1+\frac{1}{\alpha}B_1^TB_1$, and $S_1=B_1\hat{A}_1^{-1}B_1^T$.

We can factorize M_2 as

$$M_2=\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & -B_2A_2^{-1} & I \end{bmatrix}\begin{bmatrix} \alpha I & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & \hat{C} \end{bmatrix} \times\begin{bmatrix} I & 0 & 0 \\ 0 & I & A_2^{-1}B_2^T \\ 0 & 0 & I \end{bmatrix}.$$

where $\hat{C}=C+B_2A_2^{-1}B_2^T$. Then

$$M_2^{-1}=\begin{bmatrix} I & 0 & 0 \\ 0 & I & -A_2^{-1}B_2^T \\ 0 & 0 & I \end{bmatrix}\begin{bmatrix} \frac{1}{\alpha}I & 0 & 0 \\ 0 & A_2^{-1} & 0 \\ 0 & 0 & \hat{C}^{-1} \end{bmatrix} \times\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & B_2A_2^{-1} & I \end{bmatrix} =\begin{bmatrix} \frac{1}{\alpha}I & 0 & 0 \\ 0 & A_2^{-1}-S_2 & -A_2^{-1}B_2^T\hat{C}^{-1} \\ 0 & \hat{C}^{-1}B_2A_2^{-1} & \hat{C}^{-1} \end{bmatrix},$$

where $S_2=A_2^{-1}B_2^T\hat{C}^{-1}B_2A_2^{-1}$.

The implementation of the preconditioner P_{MGRS} also needs to solve a system of linear equations at each step of the generalized minimum residual (GMRES) method. The system of linear equations is of the form

$$P_{MGRS}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}=\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}. \quad (5)$$

The preconditioner P_{MGRS} can be factorized as

$$P_{MGRS}=\begin{bmatrix} I & 0 & \frac{1}{\alpha}B_1^T \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}\begin{bmatrix} A_1+\frac{1}{\alpha}B_1^TB_1 & 0 & 0 \\ 0 & I & 0 \\ -B_1 & 0 & I \end{bmatrix}$$
$$\times\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & -B_2A_2^{-1} & I \end{bmatrix}\begin{bmatrix} I & 0 & 0 \\ 0 & A_2 & B_2^T \\ 0 & 0 & C+B_2A_2^{-1}B_2^T \end{bmatrix}.$$

Then, by Lemma 2.1, we have

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}=\begin{bmatrix} I & 0 & 0 \\ 0 & A_2^{-1} & -A_2^{-1}B_2^T(C+B_2A_2^{-1}B_2^T)^{-1} \\ 0 & 0 & (C+B_2A_2^{-1}B_2^T)^{-1} \end{bmatrix}\times$$
$$\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & B_2A_2^{-1} & I \end{bmatrix}\begin{bmatrix} (A_1+\frac{1}{\alpha}B_1^TB_1)^{-1} & 0 & 0 \\ 0 & I & 0 \\ \frac{1}{\alpha}B_1(A_1+\frac{1}{\alpha}B_1^TB_1)^{-1} & 0 & I \end{bmatrix}$$
$$\times\begin{bmatrix} I & 0 & -\frac{1}{\alpha}B_1^T \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}.$$

III. Spectral property of the preconditioned matrix

The following theorem provides the spectrum results of the preconditioned matrix $P_{MGRS}^{-1}A$.

Theorem 3.1: Let the preconditioner P_{MGRS} be defined in (3), then the preconditioned matrix $P_{MGRS}^{-1}\mathcal{A}$ has an eigenvalue 1 with multiplicity at least n, and the remaining eigenvalues are $1-\lambda$, where λ satisfy the following generalized eigenvalue problem

$$S_1A_2^{-1}B_2^T+S_1(C-\alpha I)=\alpha\lambda\hat{C}x.$$

Proof. Using the equalities (3) and (4), we have

$$P_{MGRS}^{-1}\mathcal{A}=I-P_{MGRS}^{-1}\left(P_{MGRS}-\mathcal{A}\right)$$
$$=I-\begin{bmatrix} \frac{1}{\alpha}I & 0 & 0 \\ 0 & A_2^{-1}-S_2 & -A_2^{-1}B_2^T\hat{C}^{-1} \\ 0 & \hat{C}^{-1}B_2A_2^{-1} & \hat{C}^{-1} \end{bmatrix}$$
$$\times\begin{bmatrix} 0 & -\hat{A_1}^{-1}B_1^TB_2 & \hat{A}_1B_1^T(C-\alpha I) \\ 0 & 0 & 0 \\ 0 & -\frac{1}{\alpha}S_1 & \frac{1}{\alpha}S_1(C-\alpha I) \end{bmatrix}$$
$$=I-\begin{bmatrix} 0 & M_{12} \\ 0 & M_{22} \end{bmatrix},$$

where

$$M_{22}=XY,$$

with

$$X=\begin{bmatrix} -A_2^{-1}B_2^T\hat{C}^{-1} \\ \hat{C}^{-1} \end{bmatrix}\in R^{(n_2+m)\times m},$$

$$Y=\begin{bmatrix} -\frac{1}{\alpha}S_1 & \frac{1}{\alpha}S_1(C-\alpha I) \end{bmatrix}\in R^{m\times(n_2+m)}.$$

Therefore, M_{22} has an eigenvalue 0 of multiplicity at least n_2 and the remaining eigenvalues are the eigenvalues of the matrix YX, where

$$YX=\frac{1}{\alpha}S_1A_2^{-1}B_2^T\hat{C}^{-1}+\frac{1}{\alpha}S_1(C-\alpha I)\hat{C}^{-1}.$$

Then the preconditioned matrix $P_{MGRS}^{-1}\mathcal{A}$ has an eigenvalue 1 with multiplicity at least n, and the remaining eigenvalues are $1-\lambda$, where λ satisfy the following generalized eigenvalue problem

$$S_1A_2^{-1}B_2^T+S_1(C-\alpha I)=\alpha\lambda\hat{C}x.$$

Remark 3.1: By Theorem 3.1, we can get, the degree of the minimal polynomial of the preconditioned matrix $P_{MGRS}^{-1}\mathcal{A}$ is at most $m+1$. Therefore, the dimension of the Krylov subspace $K(P_{MGRS}^{-1}\mathcal{A},b)$ is at most $m+1$.

IV. Numerical examples

All the numerical experiments were performed with MATLAB 2014a under the Windows 7 operating system. In all of our runs we used a zero initial guess. The stopping criterion is $||r^{(k)}||_2/||r^{(0)}||_2\le 10^{-6}$, where $r^{(k)}$ is the residual vector after k-th iteration. The right-hand side vectors b and q are taken such that the exact solutions x and y are both vectors with all components being 1. The initial guess is chosen to be zero vector.

Consider the following Stokes type problem:

$$\begin{cases} -\nu\Delta\mathbf{u}+(\mathbf{u}.\nabla)\mathbf{u}+\nabla\mathbf{p}=f, & \text{in} \quad \Omega, \\ -div\mathbf{u}=0, & \text{in} \quad \Omega, \\ \mathbf{u}=g, & \text{on} \quad \Gamma. \end{cases} \tag{6}$$

Here u denotes the velocity vector field, p is the pressure, Ω is a bounded domain. A stable finite element or finite difference method applied to discretize the problem leads to the solution of the generalized saddle point linear system.

The generated test problems are leaky two-dimensional lid-driven cavity problems in square domain $(-1,1)\times(-1,1)$ with the lid flowing from the left to right. A Dirichlet no-flow condition is applied on the side and bottom boundaries. The nonzero horizontal velocity on the lid is chosen to be $\{y=1;-1\le x\le 1\,|u_x=1\}$. Using the IFISS software written by Silvester, Elman and Ramage [11] to discretize (1), we take a finite element subdivision based on uniform grids of square elements. The mixed finite element used is the bilinear-constant velocity u pressure: Q_1-P_0 pair.

TABLE I
GMRES(30) numerical results for Oseen problem with $\nu=1$.

Grid		16×16	32×32	64×64
MGRS	IT	4	4	4
	CPU	0.0866	0.4835	4.5181
MDS	IT	31	36	45
	CPU	0.2598	1.2995	8.3888
HSS	IT	43	61	105
	CPU	0.8962	11.0854	39.8236

The eigenvalue distributions of the original and the preconditioned matrices $P_{MGRS}^{-1}\mathcal{A}$ are given in Figures 1 and 2, we can observe that, the eigenvalues of preconditioned matrix $P_{MGRS}^{-1}\mathcal{A}$ are much more clustered than the eigenvalues of the original matrix. From Table 1, we can see that the modified generalized relaxed splitting preconditioner P_{MGRS} for generalized saddle point problems can accelerate the convergence rate of the GMRES method efficiently, including the CPU time and iteration steps.

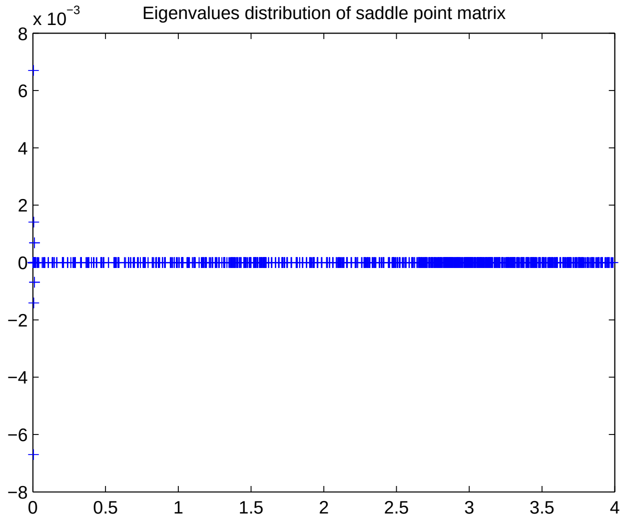

Fig. 1. Eigenvalues distribution of the original matrix with grid 16×16 with $\nu = 1$.

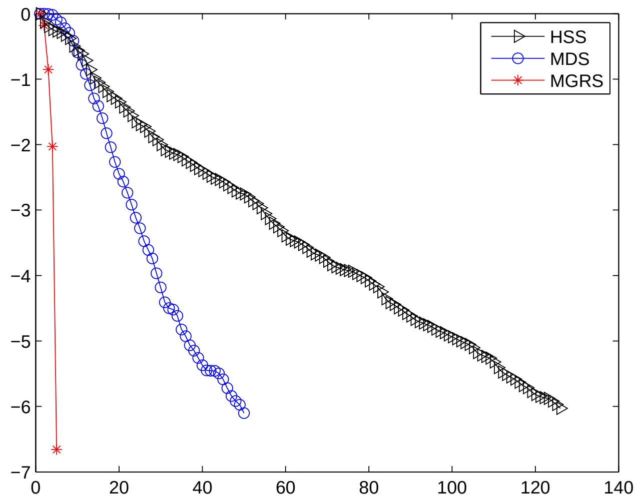

Fig. 3. Convergence curve and total numbers of GMRES(30) iterations with grid 16×16 with $\nu = 1$.

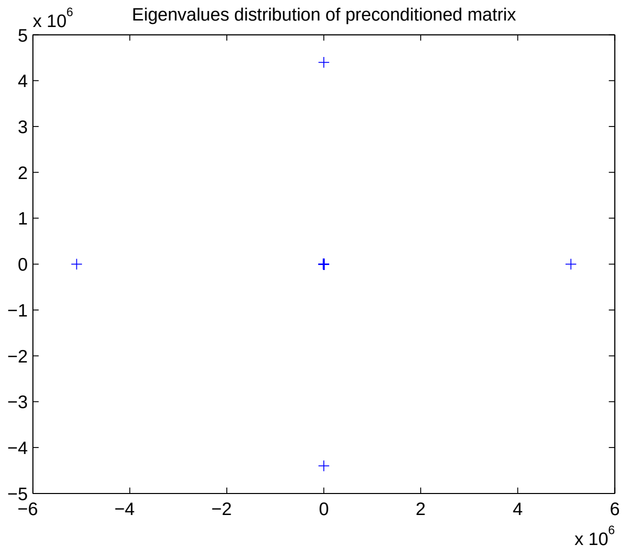

Fig. 2. Eigenvalues distribution of the preconditioned matrix with grid 16×16 with $\nu = 1$.

V. Conclusion

In this paper, a modified generalized relaxed splitting (MGRS) preconditioner has been established. The preconditioner P_{MGRS} can be also employed to accelerate the Krylov subspace method with inexact inner solves. Numerical experiments have shown the effectiveness of the new preconditioner. Admittedly, the optimal parameters α_* are crucial for guaranteeing fast convergence speeds of these parameter-dependent iteration methods, but they are generally very difficult to be determined, refer, e.g., to the recently related work [15], [16], this topic will be the subject of our future research.

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