Two New Concepts of Internal Direct Products of UP (BCC)-Algebras

C. Chanmanee, R. Chinram, R. Prasertpong, P. Julatha and A. Iampan

***Abstract*—Chanmanee et al. examined the idea of the external direct product of the infinite family of UP (BCC)-algebras, and the conclusion is reached for UP (BCC)-algebras. We apply the idea of the internal direct product of a groupoid to a UP (BCC)-algebra by introducing two new ideas for internal direct products of UP (BCC)-algebras: the internal and anti-internal direct products. This idea comes from the idea of the external direct products of UP (BCC)-algebras. We examine the attributes of both ideas and identify the crucial attributes for drawing the investigation to a conclusion. Finally, we establish the crucial statement that the internal and anti-internal direct products of a UP (BCC)-algebra may exist in only one form each.**

***Index Terms*—UP-algebra, external direct product, internal direct product, anti-internal direct product.**

I. Introduction and Preliminaries

BCK-algebras and BCI-algebras are two classes of abstract algebra that were developed by Imai and Iséki and have received a great deal of attention from academics. According to [11], [12], the class of BCK-algebras is a proper subclass of the class of BCI-algebras. A new algebraic structure was created in 2002 by Neggers and Kim [25]. They used a few characteristics from BCI and BCK-algebras to create the term "B-algebra". In addition, Kim and Kim presented a new concept known as a BG-algebra, which is an extension of B-algebra, on [18]. They discovered a number of BG-algebra isomorphism theorems and associated characteristics.

In 2017, Iampan [8] proposed the idea of UP-algebras, and it is well known that the class of KU-algebras [27] is a proper subclass of the class of UP-algebras. Many academics have looked at it, including Ansari et al. in 2018 [1], who explored graphs related to commutative UP-algebras and a graph of equivalence classes of commutative UP-algebras. The cubic set structure was applied to UP-algebras the same year by Senapati et al. [31], who also supported their findings with evidence. Satirad et al. demonstrated in 2019 that every nonempty set and every nonempty totally ordered set may be a UP-algebra by citing [29]. Romano and Jun [28] presented the idea of weak implicative UP-filters in UP-algebras in 2020. Jun and Iampan defined falling UP-filters and I-fuzzy filters in 2021 and established the relationship between falling UP-filters and falling UP-ideals in their paper cited as [15]. Bipolar fuzzy comparative UP-filters of UP-algebras were first developed in 2022 by Gaketem et al. [7]. The notion of UP-algebras (see [8]) and the concept of BCC-algebras (see [20]) are the same concept, as shown by Jun et al. [14] in 2022. In this publication and following investigations, our research team will refer to it as BCC rather than UP because of respect for Komori, who first characterized it in 1984.

A group is isomorphic to the direct product of two of its subgroups if it has an internal direct product, which is a form of direct product, according to [34]. It is often used with other algebraic structures. The internal direct product of two fuzzy subgroups is isomorphic to their external direct product, as shown by Makamba [23] in 1992. Pledger expanded the internal direct product from groups to all groupoids in 1999 [26]. After that, it creates what seems to be a logical fundamental description of the internal direct product and utilizes it as a baseline for contrasting more widely used limited variants. Jakubík and Csontóová [13] proposed two-factor internal direct product decompositions of a linked partially ordered set in 2000. Kamuti [16] proposed the semidirect product cycle index, or Frobenius groups, in 2012 and addressed internal direct products, a highly unique subset of semidirect products. A series of integral $\vee$-distributive binary aggregation functions' external direct product and internal direct product were presented in Karaçal and Khadjiev [17] 2015. The internal direct product of normal subalgebras was first developed by Lingcong [21] in 2017. The idea of a fuzzy internal direct product of fuzzy subgroups of group was first suggested by Nama [24] in the same year. Neutrosophic extended triplet internal direct product (NETIDP) and neutrosophic extended triplet external direct product (NETEDP) of the NET group were first presented by Shalla and Olgun [33] in 2019. Then, using Smarandache's idea of NET set theory, they defined NET group's internal and external semidirect products.

In this work, we offer and explore two unique ideas of internal direct products of BCC-algebras: the internal and anti-internal direct products. Finally, we arrived at the crucial conclusion that given a BCC-algebra, there can only be one occurrence of each of the internal and anti-internal direct products.

First, we begin with the definitions and examples of BCC-algebras (see [20]) as well as other definitions that are pertinent to the research in this work as follows:

Definition I.1. [26] An algebra $ß = (ß; \star, 0)$ of type $(2, 0)$

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is said to be a *BCC-algebra* if it adheres to the following axioms:

$$(\forall \omega, \kappa, \epsilon \in ß)((\kappa \star \epsilon) \star ((\omega \star \kappa) \star (\omega \star \epsilon)) = 0), \quad \text{(UP-1)}$$
$$(\forall \omega \in ß)(0 \star \omega = \omega), \quad \text{(UP-2)}$$
$$(\forall \omega \in ß)(\omega \star 0 = 0), \quad \text{(UP-3)}$$
$$(\forall \omega, \kappa \in ß)(\omega \star \kappa = 0, \kappa \star \omega = 0 \Rightarrow \omega = \kappa). \quad \text{(UP-4)}$$

Example I.2. Let $ß = \{0, 1, 2, 3, 4, 5, 6\}$ be a set with the Cayley table as follows:

$\star$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	0	0	3	3	4	1	1
2	0	0	0	0	0	0	0
3	0	0	3	0	0	0	0
4	0	0	3	3	0	0	0
5	0	0	3	3	4	0	1
6	0	0	3	3	4	0	0

Then $ß = (ß; \star, 0)$ is a BCC-algebra.

See [1], [2], [5], [6], [9], [10], [19], [29], [30], [31], [32] for further BCC-algebra studies and examples.

Consider the BCC-algebras $A = (A; \star_A, 0_A)$ and $B = (B; \star_B, 0_B)$. A map $\varphi : A \to B$ is called a *BCC-homomorphism* if

$$(\forall \omega, \kappa \in A)(\varphi(\omega \star_A \kappa) = \varphi(\omega) \star_B \varphi(\kappa))$$

and an *anti-BCC-homomorphism* if

$$(\forall \omega, \kappa \in A)(\varphi(\omega \star_A \kappa) = \varphi(\kappa) \star_B \varphi(\omega)).$$

The $\{\omega \in A \mid \varphi(\omega) = 0_B\}$ is defined as the *kernel* of φ, represented by $\ker \varphi$. The $\ker \varphi$ is a BCC-ideal of A, and $\ker \varphi = \{0_A\}$ if and only if φ is injective. A (anti-)BCC-homomorphism φ is called a (anti-)BCC-monomorphism, (anti-)BCC-epimorphism, or (anti-)BCC-isomorphism if φ is injective, surjective, or bijective, respectively.

In a BCC-algebra $ß = (ß; \star, 0)$, the following assertions are valid (see [8], [9]).

$$(\forall \omega \in ß)(\omega \star \omega = 0), \quad (1)$$
$$(\forall \omega, \kappa, \epsilon \in ß)(\omega \star \kappa = 0, \kappa \star \epsilon = 0 \Rightarrow \omega \star \epsilon = 0), \quad (2)$$
$$(\forall \omega, \kappa, \epsilon \in ß)(\omega \star \kappa = 0 \Rightarrow (\epsilon \star \omega) \star (\epsilon \star \kappa) = 0), \quad (3)$$
$$(\forall \omega, \kappa, \epsilon \in ß)(\omega \star \kappa = 0 \Rightarrow (\kappa \star \epsilon) \star (\omega \star \epsilon) = 0), \quad (4)$$
$$(\forall \omega, \kappa \in ß)(\omega \star (\kappa \star \omega) = 0), \quad (5)$$
$$(\forall \omega, \kappa \in ß)((\kappa \star \omega) \star \omega = 0 \Leftrightarrow \omega = \kappa \star \omega), \quad (6)$$
$$(\forall \omega, \kappa \in ß)(\omega \star (\kappa \star \kappa) = 0), \quad (7)$$
$$(\forall \nu, \omega, \kappa, \epsilon \in ß)((\omega \star (\kappa \star \epsilon)) \star (\omega \star ((\nu \star \kappa) \star (\nu \star \epsilon))) = 0), \quad (8)$$
$$(\forall \nu, \omega, \kappa, \epsilon \in ß)((((\nu \star \omega) \star (\nu \star \kappa)) \star \epsilon) \star ((\omega \star \kappa) \star \epsilon) = 0), \quad (9)$$
$$(\forall \omega, \kappa, \epsilon \in ß)(((\omega \star \kappa) \star \epsilon) \star (\kappa \star \epsilon) = 0), \quad (10)$$
$$(\forall \omega, \kappa, \epsilon \in ß)(\omega \star \kappa = 0 \Rightarrow \omega \star (\epsilon \star \kappa) = 0), \quad (11)$$
$$(\forall \omega, \kappa, \epsilon \in ß)(((\omega \star \kappa) \star \epsilon) \star (\omega \star (\kappa \star \epsilon)) = 0), \quad (12)$$
$$(\forall \nu, \omega, \kappa, \epsilon \in ß)(((\omega \star \kappa) \star \epsilon) \star (\kappa \star (\nu \star \epsilon)) = 0). \quad (13)$$

II. External Direct Product of BCC-algebras

The concepts of the direct product of B-algebras, 0-commutative B-algebras, and B-homomorphisms were studied by Lingcong and Endam [22], who also discovered other related features, one of which is the direct product of two B-algebras, which is itself a B-algebra. The idea of the direct product of B-algebra was then expanded to include finite family B-algebra, and some of the associated characteristics were examined as follows:

Definition II.1. [22] For each $i \in \{1, 2, ..., k\}$, let $(ß_i; \star_i)$ be an algebra. Define the structure $(\prod_{i=1}^{k} ß_i; \otimes)$ as the *direct product* of the algebras $ß_1, ß_2, ..., ß_k$, where

$$\prod_{i=1}^{k} ß_i = ß_1 \times ß_2 \times ... \times ß_k = \{(\omega_1, \omega_2, ..., \omega_k) \mid \omega_i \in ß_i \ \forall i = 1, 2, ..., k\}$$

and whose operation $\otimes$ is given by

$$(\omega_1, \omega_2, ..., \omega_k) \otimes (\kappa_1, \kappa_2, ..., \kappa_k) = (\omega_1 \star_1 \kappa_1, \omega_2 \star_2 \kappa_2, ..., \omega_k \star_k \kappa_k)$$

for all $(\omega_1, \omega_2, ..., \omega_k), (\kappa_1, \kappa_2, ..., \kappa_k) \in \prod_{i=1}^{k} ß_i$.

We now give some of the direct product's features and extend the idea to the infinite family of BCC-algebras.

Definition II.2. [4] For each $i \in \Lambda$, let $ß_i$ be a nonempty set. Define the set $\prod_{i \in \Lambda} ß_i$ as the *external direct product* of sets $ß_i$ for all $i \in \Lambda$, where

$$\prod_{i \in \Lambda} ß_i = \{f : \Lambda \to \bigcup_{i \in \Lambda} ß_i \mid f(i) \in ß_i \ \forall i \in \Lambda\}.$$

For ease of use, we define an element of $\prod_{i \in \Lambda} ß_i$ with the function $(\omega_i)_{i \in \Lambda} : \Lambda \to \bigcup_{i \in \Lambda} ß_i$, where $i \mapsto \omega_i \in ß_i$ for all $i \in \Lambda$.

Remark II.3. [4] Let $ß_i$ be a nonempty set and $Ç_i$ a subset of $ß_i$ for all $i \in \Lambda$. Then $\prod_{i \in \Lambda} Ç_i$ is a nonempty subset of the external direct product $\prod_{i \in \Lambda} ß_i$ if and only if $Ç_i$ is a nonempty subset of $ß_i$ for all $i \in \Lambda$.

Definition II.4. [4] For any $i \in \Lambda$, let $ß_i = (ß_i; \star_i)$ be an algebra. Give the following definition for the binary operation $\otimes$ on the external direct product $\prod_{i \in \Lambda} ß_i = (\prod_{i \in \Lambda} ß_i; \otimes)$: $\forall (\omega_i)_{i \in \Lambda}, (\kappa_i)_{i \in \Lambda} \in \prod_{i \in \Lambda} ß_i$,

$$(\omega_i)_{i \in \Lambda} \otimes (\kappa_i)_{i \in \Lambda} = (\omega_i \star_i \kappa_i)_{i \in \Lambda}. \quad (14)$$

Theorem II.5. *[3] $ß_i = (ß_i; \star_i, 0_i)$ is a BCC-algebra for each $i \in \Lambda$ if and only if $\prod_{i \in \Lambda} ß_i = (\prod_{i \in \Lambda} ß_i; \otimes, (0_i)_{i \in \Lambda})$ is a BCC-algebra, where Definition II.4 defines the binary operation $\otimes$.*

III. Internal Direct Product of Algebras

In this part, we'll go over Pledger's (1999) definition of internal direct products of groupoids and the related theorems [26].

Definition III.1. An algebra $(ß; \star)$ is called the *internal direct product* of its subalgebras $ß_1$ and $ß_2$ if the mapping

$$\theta : (\omega_1, \omega_2) \mapsto \omega_1 \star \omega_2 \quad (15)$$

is an isomorphism from the algebra $(ß_1 \times ß_2; \otimes)$ surjective $ß$.

Then $\theta^{-1} : ß \to ß_1 \times ß_2$ is a BCC-isomorphism. Let $\alpha_1 : ß \to ß_1$ and $\alpha_2 : ß \to ß_2$ be such that

$$(\forall \omega \in ß)(\theta^{-1}(x) = (\alpha_1(x), \alpha_2(x))). \tag{16}$$

Lemma III.2. *Let an algebra $(ß;\star)$ is the internal direct product of its subalgebras $ß_1$ and $ß_2$. Then*

(i) $\alpha_1(ß) = ß_1$.
(ii) $\alpha_2(ß) = ß_2$.

Theorem III.3. *Let an algebra $(ß;\star)$ is the internal direct product of its subalgebras $ß_1$ and $ß_2$. Then $\forall \omega_1, \kappa_1 \in ß_1, \forall \omega_2, \kappa_2 \in ß_2, (\omega_1 \star \omega_2) \star (\kappa_1 \star \kappa_2) = (\omega_1 \star \kappa_1) \star (\omega_2 \star \kappa_2)$.*

Theorem III.4. *Let $(ß;\star,\alpha_1,\alpha_2)$ be an algebra of type $(2,1,1)$. Then the algebra $(ß;\star)$ is the internal direct product of $\alpha_1(ß)$ and $\alpha_2(ß)$ if and only if the algebra $(ß;\star,\alpha_1,\alpha_2)$ has the following properties:*

(i) $\forall \omega \in ß, \alpha_1(\omega) \star \alpha_2(\omega) = \omega$,
(ii) $\forall \omega_1, \omega_2 \in ß, \alpha_1(\omega_1) = \alpha_1(\alpha_1(\omega_1) \star \alpha_2(\omega_2))$ *and* $\alpha_2(\omega_2) = \alpha_2(\alpha_1(\omega_1) \star \alpha_2(\omega_2))$*, in particular,* $\alpha_1(\omega_1 \star \omega_2) = \omega_1$ *and* $\alpha_2(\omega_1 \star \omega_2) = \omega_2$ *for all* $\omega_1 \in ß_1, \omega_2 \in ß_2$,
(iii) α_1 *and* α_2 *are homomorphisms. Moreover,* $\alpha_1(ß_1), \alpha_1(ß_2), \alpha_2(ß_1)$ *and* $\alpha_2(ß_2)$ *are subalgebras of* $ß$.

Definition III.5. Let $(ß;\star)$ be an algebra. For any $a \in ß$,

$$(\forall \omega \in ß)(\rho_a(\omega) = a \star \omega), \tag{17}$$

and

$$(\forall \omega \in ß)(\lambda_a(\omega) = \omega \star a). \tag{18}$$

Theorem III.6. *In any internal direct product $(ß;\star,\alpha_1,\alpha_2)$ with $ß_1 = \alpha_1(ß)$ and $ß_2 = \alpha_2(ß), \alpha_1 |_{ß_1}$ and $\alpha_2 |_{ß_2}$ are injections.*

Corollary III.7. *For any $a \in ß_2, \rho_{\alpha_1(a)} |_{\alpha_1(ß_1)}$ is a left inverse of $\alpha_1 |_{ß_1}$. For any $a \in ß_1, \lambda_{\alpha_2(a)} |_{\alpha_2(ß_2)}$ is left inverse of $\alpha_2 |_{ß_2}$.*

Theorem III.8. *In any internal direct product $(ß;\star,\alpha_1,\alpha_2)$ with $ß_1 = \alpha_1(ß)$ and $ß_2 = \alpha_2(ß)$, the following conditions are equivalent.*

(i) $\alpha_1(ß_1) = ß_1$,
(ii) *there exists* $d_1 \in ß_1$ *such that* $\alpha_1(ß_2) = \{d_1\}$ *with* $\alpha_1(d_1) = d_1$,
(iii) $\alpha_1(ß_1) \cap \alpha_1(ß_2) = \{d_1\}$,
(iv) $\alpha_1(ß_1) \cap \alpha_1(ß_2) \neq \emptyset$.

Corollary III.9. *Under the conditions of Theorem III.8* (i)-(iv), *$\rho_{d_1} |_{ß_1}$ is the inverse automorphism of $\alpha_1 |_{ß_1}$. Under the corresponding conditions with transposed subscripts, $\lambda_{d_2} |_{ß_2}$ is the inverse automorphism of $\alpha_2 |_{ß_2}$.*

Theorem III.10. *In any internal direct product $(ß;\star,\alpha_1,\alpha_2)$ with $ß_1 = \alpha_1(ß)$ and $ß_2 = \alpha_2(ß)$, the following conditions are equivalent.*

(i) $\alpha_1(ß_1) = ß_1$ *and* $\alpha_2(ß_2) = ß_2$,
(ii) *there exists* $d \in ß$ *such that* $\alpha_1(ß_2) = \alpha_2(ß_1) = \{d\}$ *with* $\alpha_1(d) = \alpha_2(d) = d$ *and* $d \star d = d$,
(iii) $ß_1 \cap ß_2 = \{d\}$,
(iv) $ß_1 \cap ß_2 \neq \emptyset$.

Corollary III.11. *Under the conditions of Theorem III.10* (i)-(iv), *$\rho_d |_{ß_1}$ is the inverse automorphisms of $\alpha_1 |_{ß_1}$ and $\lambda_d |_{ß_2}$ is the inverse automorphisms $\alpha_2 |_{ß_2}$ with $d \star d = d$.*

Corollary III.12. *In any internal direct product $(ß;\star,\alpha_1,\alpha_2)$ with $ß_1 = \alpha_1(ß)$ and $ß_2 = \alpha_2(ß)$, $\mid ß_1 \cap ß_2 \mid$ can only be 1 or 0.*

Theorem III.13. *In any internal direct product $(ß;\star,\alpha_1,\alpha_2)$ with $ß_1 = \alpha_1(ß)$ and $ß_2 = \alpha_2(ß)$, if $ß_1 \cap ß_2 = \{d\}$, and d commutes with every element of $ß$, then every element of $ß_1$ commutes with every element of $ß_2$.*

Theorem III.14. *In any internal direct product $(ß;\star,\alpha_1,\alpha_2)$ with $ß_1 = \alpha_1(ß)$, $ß_2 = \alpha_2(ß)$ and $ß_1 \cap ß_2 = \{d\}$, $\alpha_1 |_{ß_1} = \boldsymbol{I}_{ß_1}$ and $\alpha_2 |_{ß_2} = \boldsymbol{I}_{ß_2}$ (i.e., $\alpha_1^2 = \alpha_1$ and $\alpha_2^2 = \alpha_2$) if and only if d is both a right identity element for $ß_1$ and a left identity element for $ß_2$.*

Theorem III.15. *In any internal direct product $(ß;\star,\alpha_1,\alpha_2)$ with $ß_1 = \alpha_1(ß)$ and $ß_2 = \alpha_2(ß)$, if every element of $ß_1$ commutes with every element of $ß_2$, and $\alpha_1^2 = \alpha_1$ and $\alpha_2^2 = \alpha_2$, then $ß$ has an identity element.*

Theorem III.16. *In any internal direct product $(ß;\star,\alpha_1,\alpha_2)$ with $ß_1 = \alpha_1(ß)$ and $ß_2 = \alpha_2(ß)$, if $ß$ is finite, then $\mid ß_1 \cap ß_2 \mid = 1$.*

Theorem III.17. *In any internal direct product $(ß;\star,\alpha_1,\alpha_2)$ with $ß_1 = \alpha_1(ß)$ and $ß_2 = \alpha_2(ß)$, if $ß$ satisfies either the right or left cancellation law, then $\mid ß_1 \cap ß_2 \mid = 1$.*

Theorem III.18. *In any internal direct product $(ß;\star,\alpha_1,\alpha_2)$ with $ß_1 = \alpha_1(ß)$, $ß_2 = \alpha_2(ß)$ and $ß_1 \cap ß_2 = \{d\}$, $\alpha_1 |_{ß_1} = \boldsymbol{I}_{ß_1}$ and $\alpha_2 |_{ß_2} = \boldsymbol{I}_{ß_2}$ (i.e., $\alpha_1^2 = \alpha_1$ and $\alpha_2^2 = \alpha_2$) if and only if $\forall \omega_1 \in ß_1, \forall \kappa \in ß, \forall \epsilon_2 \in ß_2$,*

$$(\omega_1 \star \kappa) \star \epsilon_2 = \omega_1 \star (\kappa \star \epsilon_2).$$

Corollary III.19. *In any anti-internal direct product $(ß;\star,\alpha_1,\alpha_2)$ with $ß_1 = \alpha_1(ß), ß_2 = \alpha_2(ß)$, and $ß_1 \cap ß_2 = \{d\}$, if $ß$ is a semigroup, then $\alpha_1 |_{ß_1} = \boldsymbol{I}_{ß_1}$ and $\alpha_2 |_{ß_2} = \boldsymbol{I}_{ß_2}$ (i.e., $\alpha_1^2 = \alpha_1$ and $\alpha_2^2 = \alpha_2$) with d is the right identity element for $ß_1$ and the left identity element for $ß_2$.*

IV. Internal Direct Products of BCC-algebras

We apply the results from section III to the internal direct product of BCC-algebras and get the following results.

Using Theorem III.3, (UP-2), (UP-3), and (1), we get the following theorem.

Theorem IV.1. *Let a BCC-algebra $(ß;\star,0)$ is the internal direct product of its BCC-subalgebras $ß_1$ and $ß_2$. Then*

(i) $\forall \omega_1 \in ß_1, \forall \omega_2, \kappa_2 \in ß_2, (\omega_1 \star \omega_2) \star \kappa_2 = \omega_2 \star \kappa_2$,
(ii) $\forall \kappa_1 \in ß_1, \forall \omega_2, \kappa_2 \in ß_2, \omega_2 \star (\kappa_1 \star \kappa_2) = \kappa_1 \star (\omega_2 \star \kappa_2)$,
(iii) $\forall \omega_1, \kappa_1 \in ß_1, \forall \kappa_2 \in ß_2, \kappa_1 \star \kappa_2 = (\omega_1 \star \kappa_1) \star \kappa_2$,
(iv) $\forall \omega_1 \in ß_1, \forall \omega_2 \in ß_2, \forall \kappa \in ß_1 \cap ß_2, 0 = (\omega_1 \star \kappa) \star (\omega_2 \star \kappa)$,
(v) $\forall \kappa_1 \in ß_1, \forall \kappa_2 \in ß_2, \forall \omega \in ß_1 \cap ß_2, \kappa_1 \star \kappa_2 = (\omega \star \kappa_1) \star (\omega \star \kappa_2)$.

Theorem IV.2. *Let $(ß;\star,\alpha_1,\alpha_2,0)$ be a BCC-algebra and unary operations α_1 and α_2. The groupoid $(ß;\star)$ is the internal direct product of $\alpha_1(ß)$ and $\alpha_2(ß)$ if and only if*

(i) $\alpha_1 = \boldsymbol{0}_ß$ *is the zero function,*
(ii) $\alpha_2 = \boldsymbol{I}_ß$ *is the identity function.*

Proof: *Assume that* $(ß; \star, \alpha_1, \alpha_2, 0)$ *is a BCC-algebra and unary operations* α_1 *and* α_2. *The groupoid* $(ß; \star)$ *is the internal direct product of* $\alpha_1(ß)$ *and* $\alpha_2(ß)$. *Then*

$$\begin{aligned}\alpha_2(0) &= \alpha_2(0 \star 0) && \text{(by (1))}\\ &= \alpha_2(0) \star \alpha_2(0) && \text{(by Theorem III.4 (iii))}\\ &= 0 && \text{(by (1))}\end{aligned}$$

and

$$\begin{aligned}\alpha_1(0) &= \alpha_1(0 \star 0) && \text{(by (1))}\\ &= \alpha_1(0) \star \alpha_1(0) && \text{(by Theorem III.4 (iii))}\\ &= 0. && \text{(by (1))}\end{aligned}$$

(i) *Let* $x \in ß$. *Then*

$$\begin{aligned}\alpha_1(\omega) &= \alpha_1(\alpha_1(\omega) \star \alpha_2(0)) && \text{(by Theorem III.4 (ii))}\\ &= \alpha_1(\alpha_1(\omega) \star 0)\\ &= \alpha_1(0) && \text{(by (UP-3))}\\ &= 0.\end{aligned}$$

Hence, $\alpha_1 = \mathbf{0}_ß$.

(ii) *Let* $\omega \in ß$. *Then*

$$\begin{aligned}\omega &= \alpha_1(\omega) \star \alpha_2(\omega) && \text{(by Theorem III.4 (i))}\\ &= 0 \star \alpha_2(\omega) && \text{(by (i))}\\ &= \alpha_2(\omega). && \text{(by (UP-2))}\end{aligned}$$

Hence, $\alpha_2 = \mathbf{1}_ß$.

Conversely, assume that $\alpha_1 = \mathbf{0}_ß$ *and* $\alpha_2 = \mathbf{1}_ß$. *Then* (i)-(iii) *in Theorem III.4 hold. Hence,* $(ß; \star)$ *is the internal direct product of* $\alpha_1(ß) = \{0\}$ *and* $\alpha_2(ß) = ß$. ■

By Theorem IV.2, we have the following theorem.

Theorem IV.3. *Every BCC-algebra* $(ß; \star, 0)$ *is only the internal direct product of* $\{0\}$ *and* $ß$.

V. Anti-Internal Direct Products of Algebras

In this section, we introduce the concept of the anti-internal direct product of groupoids and find important theorems.

Definition V.1. An algebra $(ß; \star)$ is called the *anti-internal direct product* of its subalgebras $ß_1$ and $ß_2$ if the mapping

$$\phi : (\omega_1, \omega_2) \mapsto \omega_2 \star \omega_1 \tag{19}$$

is an isomorphism from the algebra $(ß_1 \times ß_2; \otimes)$ surjective $ß$.

Then $\phi^{-1} : ß \to ß_1 \times ß_2$ is a BCC-isomorphism. Let $\beta_1 : ß \to ß_1$ and $\beta_2 : ß \to ß_2$ be such that

$$(\forall \omega \in ß)(\phi^{-1}(x) = (\beta_1(x), \beta_2(x))). \tag{20}$$

Lemma V.2. *Let an algebra* $(ß; \star)$ *is the anti-internal direct product of its subalgebras* $ß_1$ *and* $ß_2$. *Then*

(i) $\beta_1(ß) = ß_1$.
(ii) $\beta_2(ß) = ß_2$.

We conclude that β_1 *and* β_2 *are surjective.*

Proof: (i) Clearly, $\beta_1(ß) \subseteq ß_1$. Let $\omega_1 \in ß_1$ and choose $\omega_2 \in ß_2$. Since ϕ^{-1} is surjective, there exists $\omega \in ß$ such that

$$\begin{aligned}(\omega_1, \omega_2) &= \phi^{-1}(\omega)\\ &= (\beta_1(\omega), \beta_2(\omega)). && \text{(by (20))}\end{aligned}$$

Thus $\omega_1 = \beta_1(\omega) \in \beta_1(ß)$, that is, $ß_1 \subseteq \beta_1(ß)$. Hence, $\beta_1(ß) = ß_1$.

(ii) Clearly, $\beta_2(ß) \subseteq ß_2$. Let $\omega_2 \in ß_2$ and choose $\omega_1 \in ß_1$. Since ϕ^{-1} is surjective, there exists $\omega \in ß$ such that

$$\begin{aligned}(\omega_1, \omega_2) &= \phi^{-1}(\omega)\\ &= (\beta_1(\omega), \beta_2(\omega)). && \text{(by (20))}\end{aligned}$$

Thus $\omega_2 = \beta_2(\omega) \in \beta_2(ß)$, that is, $ß_2 \subseteq \beta_2(ß)$. Hence, $\beta_2(ß) = ß_2$. ■

Theorem V.3. *Let an algebra* $(ß; \star)$ *is the anti-internal direct product of its subalgebras* $ß_1$ *and* $ß_2$. *Then* $\forall \omega_1, \kappa_1 \in ß_1, \forall \omega_2, \kappa_2 \in ß_2, (\omega_2 \star \omega_1) \star (\kappa_2 \star \kappa_1) = (\omega_2 \star \kappa_2) \star (\omega_1 \star \kappa_1)$.

Proof: Let $\omega_1, \kappa_1 \in ß_1$ and $\omega_2, \kappa_2 \in ß_2$. Then

$$\begin{aligned}&(\omega_2 \star \omega_1) \star (\kappa_2 \star \kappa_1)\\ &= \phi(\omega_1, \omega_2) \star \phi(\kappa_1, \kappa_2) && \text{(by (19))}\\ &= \phi((\omega_1, \omega_2) \otimes (\kappa_1, \kappa_2)) && \text{(by homomorphism)}\\ &= \phi(\omega_1 \star \kappa_1, \omega_2 \star \kappa_2) && \text{(by Definition II.1)}\\ &= (\omega_2 \star \kappa_2) \star (\omega_1 \star \kappa_1). && \text{(by (19))}\end{aligned}$$

■

Theorem V.4. *Let* $(ß; \star, \beta_1, \beta_2)$ *be an algebra of type* $(2, 1, 1)$. *Then the algebra* $(ß; \star)$ *is the anti-internal direct product of* $\beta_1(ß)$ *and* $\beta_2(ß)$ *if and only if the algebra* $(ß; \star, \beta_1, \beta_2)$ *has the following properties:*

(i) $\forall \omega \in ß, \beta_2(\omega) \star \beta_1(\omega) = \omega$,
(ii) $\forall \omega_1, \omega_2 \in ß, \beta_1(\omega_1) = \beta_1(\beta_2(\omega_2) \star \beta_1(\omega_1))$ *and* $\beta_2(\omega_2) = \beta_2(\beta_2(\omega_2) \star \beta_1(\omega_1))$, *in particular,* $\beta_1(\omega_2 \star \omega_1) = \omega_1$ *and* $\beta_2(\omega_2 \star \omega_1) = \omega_2$ *for all* $\omega_1 \in ß_1, \omega_2 \in ß_2$,
(iii) β_1 *and* β_2 *are homomorphisms. Moreover,* $\beta_1(ß_1), \beta_1(ß_2), \beta_2(ß_1)$ *and* $\beta_2(ß_2)$ *are subalgebras of* $ß$.

Proof: Write $\beta_1(ß) = ß_1$ and $\beta_2(ß) = ß_2$. First, assume $(ß; \star)$ is the anti-internal direct product of $ß_1$ and $ß_2$.

(i) $\forall \omega \in ß$,

$$\begin{aligned}\omega &= \phi(\phi^{-1}(\omega))\\ &= \phi(\beta_1(x), \beta_2(x)) && \text{(by (20))}\\ &= \beta_2(x) \star \beta_1(x). && \text{(by (19))}\end{aligned}$$

(ii) Let $\omega_1, \omega_2 \in ß$. Then there exist $\kappa_1 \in ß_1, \kappa_2 \in ß_2$ such that $\beta_1(\omega_1) = \kappa_1$ and $\beta_2(\omega_2) = \kappa_2$. Thus

$$\begin{aligned}&(\beta_1(\omega_1), \beta_2(\omega_2))\\ &= (\kappa_1, \kappa_2)\\ &= \phi^{-1}(\phi(\kappa_1, \kappa_2)) && \text{(by (20))}\\ &= \phi^{-1}(\kappa_2 \star \kappa_1) && \text{(by (19))}\\ &= (\beta_1(\kappa_2 \star \kappa_1), \beta_2(\kappa_2 \star \kappa_1)) && \text{(by (20))}\\ &= (\beta_1(\beta_2(\omega_2) \star \beta_1(\omega_1)), \beta_2(\beta_2(\omega_2) \star \beta_1(\omega_1))).\end{aligned}$$

Hence, $\beta_1(\omega_1) = \beta_1(\beta_2(\omega_2) \star \beta_1(\omega_1))$ and $\beta_2(\omega_2) = \beta_2(\beta_2(\omega_2) \star \beta_1(\omega_1))$.

(iii) Let $\omega, \kappa \in ß$. Then

$$\begin{aligned}
&(\beta_1(\omega \star \kappa), \beta_2(\omega \star \kappa)) \\
&= \phi^{-1}(\omega \star \kappa) && \text{(by (20))} \\
&= \phi^{-1}(\omega) \otimes \phi^{-1}(\kappa) && \text{(by homomorphism)} \\
&= (\beta_1(\omega), \beta_2(\omega)) \otimes (\beta_1(\kappa), \beta_2(\kappa)) && \text{(by (20))} \\
&= (\beta_1(\omega) \star \beta_1(\kappa), \beta_2(\omega) \star \beta_2(\kappa)). && \text{(by Definition II.1)}
\end{aligned}$$

Hence, $\beta_1(\omega \star \kappa) = \beta_1(\omega) \star \beta_1(\kappa)$ and $\beta_2(\omega \star \kappa) = \beta_2(\omega) \star \beta_2(\kappa)$. Therefore β_1 and β_2 are homomorphisms.

Conversely, assume (i)-(iii) are satisfied. By (iii), we have $\beta_1(ß) = ß_1$ and $\beta_2(ß) = ß_2$ are subalgebras of ß. Define the function $\eta : ß \to ß_1 \times ß_2$ by

$$\eta(\omega) = (\beta_1(x), \beta_2(\omega)) \tag{21}$$

for all $\omega \in ß$. Let $\omega, \kappa \in ß$ be such that $\eta(\omega) = \eta(\kappa)$. Then

$$\begin{aligned}
\eta(\omega) = \eta(\kappa) &\Rightarrow \beta_1(\omega) = \beta_1(\kappa) \text{ and } \beta_2(\omega) = \beta_2(\kappa) \\
&\Rightarrow \beta_2(\omega) \star \beta_1(\omega) = \beta_2(\kappa) \star \beta_1(\kappa) \\
&\Rightarrow \omega = \kappa. && \text{(by (i))}
\end{aligned}$$

So, η is injective.

Also, let $(\omega_1, \omega_2) \in ß_1 \times ß_2$. Then there exists $\omega_2 \star \omega_1 \in ß$ such that

$$\begin{aligned}
\eta(\omega_2 \star \omega_1) &= (\beta_1(\omega_2 \star \omega_1), \beta_2(\omega_2 \star \omega_1)) && \text{(by (21))} \\
&= (\omega_1, \omega_2). && \text{(by (ii))}
\end{aligned}$$

So, η is surjective.

Also, let $\omega, \kappa \in ß$. Then

$$\begin{aligned}
&\eta(\omega) \otimes \eta(\kappa) \\
&= (\beta_1(\omega), \beta_2(\omega)) \otimes (\beta_1(\kappa), \beta_2(\kappa)) && \text{(by (21))} \\
&= (\beta_1(\omega) \star \beta_1(\kappa), \beta_2(\omega) \star \beta_2(\kappa)) && \text{(by Definition II.1)} \\
&= (\beta_1(\omega \star \kappa), \beta_2(\omega \star \kappa)) && \text{(by (iii))} \\
&= \eta(\omega \star \kappa). && \text{(by (21))}
\end{aligned}$$

So, η is a homomorphism. Hence, η is an isomorphism and so η^{-1} is an isomorphism.

Finally let $\phi = \eta^{-1}$. Then ß is an anti-internal direct product of $ß_1$ and $ß_2$. ■

Theorem V.5. *In any anti-internal direct product $(ß; \star, \beta_1, \beta_2)$ with $ß_1 = \beta_1(ß)$ and $ß_2 = \beta_2(ß)$, $\beta_1 |_{ß_1}$ and $\beta_2 |_{ß_2}$ are injective.*

Proof: Choose any $a \in ß_2$. Let $\omega \in ß_1$. Then

$$\begin{aligned}
(\rho_{\beta_1(a)} \circ \beta_1 |_{ß_1})(\omega) &= \rho_{\beta_1(a)}(\beta_1 |_{ß_1} (\omega)) \\
&= \rho_{\beta_1(a)}(\beta_1(\omega)) \\
&= \beta_1(a) \star \beta_1(\omega) && \text{(by (17))} \\
&= \beta_1(a \star \omega) && \text{(by Theorem V.4 (iii))} \\
&= \omega && \text{(by Theorem V.4 (ii))} \\
&= \mathbf{1}_{ß_1}(\omega).
\end{aligned}$$

Thus $\beta_1 |_{ß_1}$ is injective.

And, choose any $a \in ß_1$. Let $\omega \in ß_2$. Then

$$\begin{aligned}
(\lambda_{\beta_2(a)} \circ \beta_2 |_{ß_2})(\omega) &= \lambda_{\beta_2(a)}(\beta_2 |_{ß_2} (\omega)) \\
&= \lambda_{\beta_2(a)}(\beta_2(\omega)) \\
&= \beta_2(\omega) \star \beta_2(a) && \text{(by (18))} \\
&= \beta_2(\omega \star a) && \text{(by Theorem V.4 (iii))} \\
&= \omega && \text{(by Theorem V.4 (ii))} \\
&= \mathbf{1}_{ß_2}(\omega).
\end{aligned}$$

Thus $\beta_2 |_{ß_2}$ is injective. ■

Corollary V.6. *For any $a \in ß_2, \rho_{\beta_1(a)} |_{\beta_1(ß_1)}$ is a left inverse of $\beta_1 |_{ß_1}$. For any $a \in ß_1, \lambda_{\beta_2(a)} |_{\beta_2(ß_2)}$ is a left inverse of $\beta_2 |_{ß_2}$.*

Theorem V.7. *In any anti-internal direct product $(ß; \star, \beta_1, \beta_2)$ with $ß_1 = \beta_1(ß)$ and $ß_2 = \beta_2(ß)$, the following conditions are equivalent.*

(i) $\beta_1(ß_1) = ß_1$,
(ii) there exists $d_1 \in ß_1$ such that $\beta_1(ß_2) = \{d_1\}$ with $\beta_1(d_1) = d_1$,
(iii) $\beta_1(ß_1) \cap \beta_1(ß_2) = \{d_1\}$,
(iv) $\beta_1(ß_1) \cap \beta_1(ß_2) \neq \emptyset$.

Proof: (i)⇒(ii) Choose any $c_1 \in ß_1$. Then $\beta_1(\beta_2(c_1)) \in \beta_1(ß) = ß_1 = \beta_1(ß_1)$ from (i), so there exists $d_1 \in ß_1$ such that $\beta_1(\beta_2(c_1)) = \beta_1(d_1)$. So, let $\omega_2 \in ß_2$. Then

$$\begin{aligned}
\beta_1(\omega_2) &= \beta_1(\beta_2(\omega_2 \star c_1)) && \text{(by Theorem V.4 (ii))} \\
&= \beta_1(\beta_2(\omega_2) \star \beta_2(c_1)) && \text{(by Theorem V.4 (iii))} \\
&= \beta_1(\beta_2(\omega_2)) \star \beta_1(\beta_2(c_1)) && \text{(by Theorem V.4 (iii))} \\
&= \beta_1(\beta_2(\omega_2)) \star \beta_1(d_1) && \text{(by } \beta_1(\beta_2(c_1)) = \beta_1(d_1)\text{)} \\
&= \beta_1(\beta_2(\omega_2) \star d_1) && \text{(by Theorem V.4 (iii))} \\
&= d_1. && \text{(by Theorem V.4 (ii))}
\end{aligned}$$

So, $\beta_1(ß_2) \subseteq \{d_1\}$. Since $\beta_1(ß_2)$ is nonempty, we have $\beta_1(ß_2) = \{d_1\}$. Also $\beta_1(d_1) = \beta_1(\beta_2(c_1)) \in \beta_1(ß_2) = \{d_1\}$, so $\beta_1(d_1) = d_1$. Hence, $\beta_1(ß_2)$ is a singleton $\{d_1\}$ with $\beta_1(d_1) = d_1$.

(ii)⇒(iii) From (ii), we have $\beta_1(ß_2) = \{d_1\} = \{\beta_1(d_1)\} \subseteq \beta_1(ß_1)$. Hence, $\beta_1(ß_1) \cap \beta_1(ß_2)$ is the singleton $\{d_1\}$.

(iii)⇒(iv) Obviously.

(iv)⇒(i) From (iv), there exist $a_1 \in ß_1$ and $a_2 \in ß_2$ such that $\beta_1(a_1) = \beta_1(a_2)$. Clearly, $\beta_1(ß_1) \subseteq ß_1$. So, let $\omega \in ß_1$. Then

$$\begin{aligned}
\omega &= \beta_1(a_2 \star \omega) && \text{(by Theorem V.4 (ii))} \\
&= \beta_1(a_2) \star \beta_1(\omega) && \text{(by Theorem V.4 (iii))} \\
&= \beta_1(a_1) \star \beta_1(\omega) && \text{(by } \beta_1(a_1) = \beta_1(a_2)\text{)} \\
&= \beta_1(a_1 \star \omega) && \text{(by Theorem V.4 (iii))} \\
&\in \beta_1(ß_1).
\end{aligned}$$

So, $ß_1 \subseteq \beta_1(ß_1)$. Hence, $\beta_1(ß_1) = ß_1$. ■

Theorem V.8. *In any anti-internal direct product $(ß; \star, \beta_1, \beta_2)$ with $ß_1 = \beta_1(ß)$ and $ß_2 = \beta_2(ß)$, the following conditions are equivalent.*

(i) $\beta_2(ß_2) = ß_2$,
(ii) there exists $d_2 \in ß_2$ such that $\beta_2(ß_1) = \{d_2\}$ with $\beta_2(d_2) = d_2$,
(iii) $\beta_2(ß_1) \cap \beta_2(ß_2) = \{d_2\}$,

(iv) $\beta_2(ß_1) \cap \beta_2(ß_2) \neq \emptyset$.

Proof: Prove it using the same principle as Theorem V.7, replacing the subscript 1 to 2 and 2 to 1. ■

Corollary V.9. *Under the conditions of Theorem V.7* (i)-(iv), $\rho_{d_1} |_{ß_1}$ *is the inverse automorphism of* $\beta_1 |_{ß_1}$. *Moreover,* $\rho_{d_1} |_{ß_1} = \rho_a |_{ß_1}$ *for all* $a \in ß_2$.

Proof: We shall prove that $\rho_{d_1} |_{ß_1}$ is the left inverse automorphism of $\beta_1 |_{ß_1}$. Choose any $a \in ß_2$. Let $\omega \in ß_1$. Then

$$\begin{aligned}
&(\rho_{d_1} |_{ß_1} \circ \beta_1 |_{ß_1})(x) \\
&= \rho_{d_1} |_{ß_1} (\beta_1 |_{ß_1} (\omega)) \\
&= \rho_{d_1}(\beta_1(\omega)) \\
&= \rho_{\beta_1(a)}(\beta_1(\omega)) && \text{(by Theorem V.7 (ii))} \\
&= \beta_1(a) \star \beta_1(\omega) && \text{(by (17))} \\
&= \beta_1(a \star \omega) && \text{(by Theorem V.4 (iii))} \\
&= \omega. && \text{(by Theorem V.4 (ii))}
\end{aligned}$$

Thus $\beta_1 |_{ß_1}$ is injective.

Next, we shall prove that $\rho_a |_{ß_1}$ is the right inverse automorphism of $\beta_1 |_{ß_1}$. Choose any $a \in ß_2$. Let $\omega \in ß_1$. Then

$$\begin{aligned}
(\beta_1 |_{ß_1} \circ \rho_a |_{ß_1})(x) &= \beta_1 |_{ß_1} (\rho_a(\omega)) \\
&= \beta_1(\rho_a(\omega)) \\
&= \beta_1(a \star \omega) && \text{(by (17))} \\
&= \omega. && \text{(by Theorem V.4 (ii))}
\end{aligned}$$

Thus $\beta_1 |_{ß_1}$ is surjective, so $\beta_1 |_{ß_1}$ is bijective. Hence, $\rho_{d_1} |_{ß_1} = \rho_a |_{ß_1}$ is the inverse of $\beta_1 |_{ß_1}$. By Theorem V.4 (iii), we have $\beta_1 |_{ß_1}$ is an automorphism. Therefore, $\rho_{d_1} |_{ß_1} = \rho_a |_{ß_1}$ is the inverse automorphism of $\beta_1 |_{ß_1}$. ■

Corollary V.10. *Under the conditions of Theorem V.8* (i)-(iv), $\lambda_{d_2} |_{ß_2}$ *is the inverse automorphism of* $\beta_2 |_{ß_2}$. *Moreover,* $\lambda_{d_2} |_{ß_2} = \lambda_a |_{ß_2}$ *for all* $a \in ß_1$.

Proof: The proof is in the same way as Corollary V.9. ■

Theorem V.11. *In any anti-internal direct product* $(ß; \star, \beta_1, \beta_2)$ *with* $ß_1 = \beta_1(ß)$ *and* $ß_2 = \beta_2(ß)$, *the following conditions are equivalent.*

(i) $\beta_1(ß_1) = ß_1$ *and* $\beta_2(ß_2) = ß_2$,
(ii) there exists $d \in ß$ *such that* $\beta_1(ß_2) = \beta_2(ß_1) = \{d\}$ *with* $\beta_1(d) = \beta_2(d) = d$ *and* $d \star d = d$,
(iii) $ß_1 \cap ß_2 = \{d\}$,
(iv) $ß_1 \cap ß_2 \neq \emptyset$.

Proof: (i)⇒(ii) From (i), Theorem V.7 gives $\beta_1(ß_2) = \{d_1\}$ with $\beta_1(d_1) = d_1$ for some $d_1 \in ß_1$ and Theorem V.8 gives $\beta_2(ß_1) = \{d_2\}$ with $\beta_2(d_2) = d_2$ for some $d_2 \in ß_2$. Also, $\beta_1(d_2) \in \beta_1(ß_2) = \{d_1\}$ and $\beta_2(d_1) \in \beta_2(ß_1) = \{d_2\}$, so $\beta_1(d_2) = d_1$ and $\beta_2(d_1) = d_2$. Thus

$$\begin{aligned}
d_1 &= \beta_2(d_1) \star \beta_1(d_1) && \text{(by Theorem V.4 (i))} \\
&= d_2 \star d_1 && (\text{by } \beta_2(d_1) = d_2, \beta_1(d_1) = d_1) \\
&= \beta_2(d_2) \star \beta_1(d_2) && (\text{by } \beta_2(d_2) = d_2, \beta_1(d_2) = d_1) \\
&= d_2. && \text{(by Theorem V.4 (i))}
\end{aligned}$$

Choose $d = d_1$. Hence, $\beta_1(ß_2) = \beta_2(ß_1) = \{d\}$ with $\beta_1(d) = \beta_2(d) = d$.

(ii)⇒(iii) From (ii), $d \in \beta_1(ß_2) \cap \beta_2(ß_1) \subseteq ß_1 \cap ß_2$. Thus $\{d\} \subseteq ß_1 \cap ß_2$. Let $\omega \in ß_1 \cap ß_2$. Then $\beta_1(\omega) \in \beta_1(ß_1 \cap ß_2) \subseteq \beta_1(ß_2) = \{d\}$ and $\beta_2(\omega) \in \beta_2(ß_1 \cap ß_2) \subseteq \beta_2(ß_1) = \{d\}$. Thus

$$\begin{aligned}
\omega &= \beta_2(\omega) \star \beta_1(\omega) && \text{(by Theorem V.4 (i))} \\
&= d \star d \\
&= d. && \text{(by assumption)}
\end{aligned}$$

Thus, $ß_1 \cap ß_2 \subseteq \{d\}$. Hence, $ß_1 \cap ß_2 = \{d\}$.

(iii)⇒(iv) Obviously.

(iv)⇒(i) We see that $\beta_1(ß_1 \cap ß_2) \subseteq \beta_1(ß_1) \cap \beta_1(ß_2)$. Thus

$$\begin{aligned}
&ß_1 \cap ß_2 \neq \emptyset \\
&\Rightarrow \beta_1(ß_1) \cap \beta_2(ß_2) \neq \emptyset \\
&\Rightarrow \beta_1(ß_1) = ß_1 \text{ and } \beta_2(ß_2) = ß_2. \\
&\qquad \text{(by Theorems V.7 (i) and V.8 (i))}
\end{aligned}$$

Hence, $\beta_1(ß_1) = ß_1$ and $\beta_2(ß_2) = ß_2$. ■

Corollary V.12. *Under the conditions of Theorem V.11 i-iv,* $\rho_d |_{ß_1}$ *is the inverse automorphism of* $\beta_1 |_{ß_1}$ *and* $\lambda_d |_{ß_2}$ *is the inverse automorphism of* $\beta_2 |_{ß_2}$. *Moreover,* $\rho_d |_{ß_1} = \rho_a |_{ß_1}$ *and* $\lambda_d |_{ß_2} = \lambda_b |_{ß_2}$ *for all* $a \in ß_2$ *and* $b \in ß_1$.

Proof: It is a direct result of Corollaries V.9 and V.10. ■

Corollary V.13. *In any anti-internal direct product* $(ß; \star, \beta_1, \beta_2)$ *with* $ß_1 = \beta_1(ß)$ *and* $ß_2 = \beta_2(ß)$, $|ß_1 \cap ß_2|$ *can only be* 1 *or* 0.

Proof: It follows from Theorem V.11 (iii) and (iv). ■

Theorem V.14. *In any anti-internal direct product* $(ß; \star, \beta_1, \beta_2)$ *with* $ß_1 = \beta_1(ß)$ *and* $ß_2 = \beta_2(ß)$, *if* $ß_1 \cap ß_2 = \{d\}$, *and* d *commutes with every element of* $ß$, *then every element of* $ß_1$ *commutes with every element of* $ß_2$.

Proof: Let $\omega_1 \in ß_1$ and $\omega_2 \in ß_2$. Then

$$\begin{aligned}
&\omega_1 \star \omega_2 \\
&= (\beta_2(\omega_1) \star \beta_1(\omega_1)) \star (\beta_2(\omega_2) \star \beta_1(\omega_2)) \\
&\qquad \text{(by Theorem V.4 (i))} \\
&= (d \star \beta_1(\omega_1)) \star (\beta_2(\omega_2) \star d) && \text{(by Theorem V.11 (ii))} \\
&= (d \star \beta_2(\omega_2)) \star (\beta_1(\omega_1) \star d) && \text{(by Theorem V.3)} \\
&= (\beta_2(\omega_2) \star d) \star (d \star \beta_1(\omega_1)) \\
&= (\beta_2(\omega_2) \star \beta_1(\omega_2)) \star (\beta_2(\omega_1) \star \beta_1(\omega_1)) \\
&\qquad \text{(by Theorem V.11 (ii))} \\
&= \omega_2 \star \omega_1. && \text{(by Theorem V.4 (i))}
\end{aligned}$$

The proof is completed. ■

Theorem V.15. *In any anti-internal direct product* $(ß; \star, \beta_1, \beta_2)$ *with* $ß_1 = \beta_1(ß)$, $ß_2 = \beta_2(ß)$ *and* $ß_1 \cap ß_2 = \{d\}$, $\beta_1 |_{ß_1} = \mathbf{1}_{ß_1}$ *and* $\beta_2 |_{ß_2} = \mathbf{1}_{ß_2}$ *(i.e.,* $\beta_1^2 = \beta_1$ *and* $\beta_2^2 = \beta_2$*) if and only if* d *is both the left identity element for* $ß_1$ *and the right identity element for* $ß_2$.

Proof: Assume both $\beta_1 |_{ß_1} = \mathbf{1}_{ß_1}$ and $\beta_2 |_{ß_2} = \mathbf{1}_{ß_2}$. Then Corollary V.12 gives $\rho_d |_{ß_1} = \mathbf{1}_{ß_1}$ and $\lambda_d |_{ß_2} = \mathbf{1}_{ß_2}$. Let $\omega_1 \in ß_1$. Then $d \star \omega_1 = \rho_d(\omega_1) = \rho_d |_{ß_1} (\omega_1) = \mathbf{1}_{ß_1}(\omega_1) = \omega_1$, that is, d is the left identity element for $ß_1$. Let $\omega_2 \in ß_2$.

Then $\omega_2 \star d = \lambda_d(\omega_2) = \lambda_d \mid_{ß_2} (\omega_2) = \mathbf{1}_{ß_2}(\omega_2) = \omega_2$, that is, d is the right identity element for $ß_2$. Hence, d is both the left identity element for $ß_1$ and the right identity element for $ß_2$.

Conversely, assume that d is both the left identity element for $ß_1$ and the right identity element for $ß_2$. By the assumption, Theorem V.11 (iii) holds. Thus by Theorem V.11, we have the conditions (i)-(iv). It follows from Corollary V.12 that $\rho_d \mid_{ß_1} \circ \beta_1 \mid_{ß_1} = \mathbf{1}_{ß_1}$ and $\beta_2 \mid_{ß_2} \circ \lambda_d \mid_{ß_2} = \mathbf{1}_{ß_2}$. Let $\omega_1 \in ß_1$. Since d is the left identity element for $ß_1$, we have $\mathbf{1}_{ß_1}(\omega_1) = (\rho_d \mid_{ß_1} \circ \beta_1 \mid_{ß_1})(\omega_1) = \rho_d \mid_{ß_1} (\beta_1 \mid_{ß_1} (\omega_1)) = \rho_d(\beta_1 \mid_{ß_1} (\omega_1)) = d \star \beta_1 \mid_{ß_1} (\omega_1) = \beta_1 \mid_{ß_1} (\omega_1)$. Thus $\beta_1 \mid_{ß_1} = \mathbf{1}_{ß_1}$. And let $\omega_2 \in ß_2$. Since d is the right identity element for $ß_2$, we have $\mathbf{1}_{ß_2}(\omega_2) = (\beta_2 \mid_{ß_2} \circ \lambda_d \mid_{ß_2})(\omega_2) = \beta_2 \mid_{ß_2} (\lambda_d \mid_{ß_2} (\omega_2)) = \beta_2 \mid_{ß_2} (\lambda_d(\omega_2)) = \beta_2 \mid_{ß_2} (\omega_2 \star d) = \beta_2 \mid_{ß_2} (\omega_2)$. Thus $\beta_2 \mid_{ß_2} = \mathbf{1}_{ß_2}$. ■

Theorem V.16. *In any anti-internal direct product* $(ß; \star, \beta_1, \beta_2)$ *with* $ß_1 = \beta_1(ß)$, $ß_2 = \beta_2(ß)$ *and* $ß_1 \cap ß_2 = \{d\}$, *if every element of* $ß_1$ *commutes with every element of* $ß_2$, *and* $\beta_1^2 = \beta_1$ *and* $\beta_2^2 = \beta_2$, *then* $ß$ *has the identity element.*

Proof: Assume that every element of $ß_1$ commutes with every element of $ß_2$, and $\beta_1^2 = \beta_1$ and $\beta_2^2 = \beta_2$. By Theorem V.15, we have d is the left identity element for $ß_1$ and d is the right identity element for $ß_2$. Let $\omega \in ß$. Then

$$\begin{aligned}
&\omega \star d \\
&= (\beta_2(\omega) \star \beta_1(\omega)) \star (d \star d) && \text{(by Theorem V.4 (i))} \\
&= (\beta_2(\omega) \star d) \star (\beta_1(\omega) \star d) && \text{(by Theorem V.3)} \\
&= \beta_2(\omega) \star (\beta_1(\omega) \star d) && \text{(by } d \text{ is the right identity for } ß_2) \\
&= \beta_2(\omega) \star (d \star \beta_1(\omega)) && \text{(by assumption)} \\
&= \beta_2(\omega) \star \beta_1(\omega) && \text{(by } d \text{ is the left identity for } ß_1) \\
&= \omega && \text{(by Theorem V.4 (i))}
\end{aligned}$$

and

$$\begin{aligned}
&d \star \omega \\
&= (d \star d) \star (\beta_2(\omega) \star \beta_1(\omega)) && \text{(by Theorem V.4 (i))} \\
&= (d \star \beta_2(\omega)) \star (d \star \beta_1(\omega)) && \text{(by Theorem V.3)} \\
&= (d \star \beta_2(\omega)) \star \beta_1(\omega) && \text{(by } d \text{ is the left identity for } ß_1) \\
&= (\beta_2(\omega) \star d) \star \beta_1(\omega) && \text{(by assumption)} \\
&= \beta_2(\omega) \star \beta_1(\omega) && \text{(by } d \text{ is the right identity for } ß_2) \\
&= \omega. && \text{(by Theorem V.4 (i))}
\end{aligned}$$

Hence, d is the identity element of $ß$. ■

Theorem V.17. *In any anti-internal direct product* $(ß; \star, \beta_1, \beta_2)$ *with* $ß_1 = \beta_1(ß)$ *and* $ß_2 = \beta_2(ß)$, *if* $ß_1$ *and* $ß_2$ *are finite, then* $\mid ß_1 \cap ß_2 \mid = 1$.

Proof: Assume that $ß_1$ and $ß_2$ are finite. By Theorem V.5, we have $\beta_1 \mid_{ß_1}$ and $\beta_2 \mid_{ß_2}$ are injective. By the assumption, we have $\beta_1(ß_1) = \beta_1 \mid_{ß_1} (ß_1) = ß_1$ and $\beta_2(ß_2) = \beta_2 \mid_{ß_2} (ß_2) = ß_2$. It follows from Theorem V.11 that $\mid ß_1 \cap ß_2 \mid = 1$. ■

Theorem V.18. *In any anti-internal direct product* $(ß; \star, \beta_1, \beta_2)$ *with* $ß_1 = \beta_1(ß)$ *and* $ß_2 = \beta_2(ß)$, *if* $ß$ *satisfies either the right or left cancellation law, then* $\mid ß_1 \cap ß_2 \mid = 1$.

Proof: Assume that $ß$ satisfies the right cancellation law. Choose any $a_1 \in ß_1$ and $a_2 \in ß_2$. Then

$$\begin{aligned}
&\beta_2(a_1) \star \beta_1(a_1) \\
&= a_1 && \text{(by Theorem V.4 (i))} \\
&= \beta_1(a_2 \star a_1) && \text{(by Theorem V.4 (ii))} \\
&= \beta_1(a_2) \star \beta_1(a_1). && \text{(by Theorem V.4 (iii))}
\end{aligned}$$

Cancel $\beta_1(a_1)$ from the right so $\beta_2(a_1) = \beta_1(a_2)$, therefore $ß_1 \cap ß_2 \neq \emptyset$. It follows from Theorem V.11 that $\mid ß_1 \cap ß_2 \mid = 1$.

Next, assume that $ß$ satisfies the left cancellation law. Choose any $a_1 \in ß_1$ and $a_2 \in ß_2$. Then

$$\begin{aligned}
&\beta_2(a_2) \star \beta_1(a_2) \\
&= a_2 && \text{(by Theorem V.4 (i))} \\
&= \beta_2(a_2 \star a_1) && \text{(by Theorem V.4 (ii))} \\
&= \beta_2(a_2) \star \beta_2(a_1). && \text{(by Theorem V.4 (iii))}
\end{aligned}$$

Cancel $\beta_2(a_2)$ from the left so $\beta_1(a_2) = \beta_2(a_1)$, therefore $ß_1 \cap ß_2 \neq \emptyset$. It follows from Theorem V.11 that $\mid ß_1 \cap ß_2 \mid = 1$. ■

Theorem V.19. *In any anti-internal direct product* $(ß; \star, \beta_1, \beta_2)$ *with* $ß_1 = \beta_1(ß), ß_2 = \beta_2(ß)$ *and* $ß_1 \cap ß_2 = \{d\}$, $\beta_1 \mid_{ß_1} = \mathbf{1}_{ß_1}$ *and* $\beta_2 \mid_{ß_2} = \mathbf{1}_{ß_2}$ *(i.e.,* $\beta_1^2 = \beta_1$ *and* $\beta_2^2 = \beta_2$*) if and only if* $\forall \omega_1 \in ß_1, \forall \omega_2 \in ß_2, \forall \kappa \in ß$,

$$(\omega_2 \star \kappa) \star \omega_1 = \omega_2 \star (\kappa \star \omega_1).$$

Proof: Assume that $(\omega_2 \star \kappa) \star \omega_1 = \omega_2 \star (\kappa \star \omega_1)$ for all $\omega_1 \in ß_1, \omega_2 \in ß_2$, and $\kappa \in ß$. Choose $\omega_2 \in ß_2$. Let $\omega_1 \in ß_1$. Then

$$\begin{aligned}
&\beta_1(\omega_1) \\
&= \beta_1((\omega_2 \star \omega_2) \star \beta_1(\omega_1)) && \text{(by Theorem V.4 (ii))} \\
&= \beta_1(\omega_2 \star (\omega_2 \star \beta_1(\omega_1))) && \text{(by assumption)} \\
&= \beta_1(\omega_2) \star \beta_1(\omega_2 \star \beta_1(\omega_1)) && \text{(by Theorem V.4 (iii))} \\
&= \beta_1(\omega_2) \star \beta_1(\omega_1) && \text{(by Theorem V.4 (ii))} \\
&= \beta_1(\omega_2 \star \omega_1) && \text{(by Theorem V.4 (iii))} \\
&= \omega_1. && \text{(by Theorem V.4 (ii))}
\end{aligned}$$

Hence, $\beta_1 \mid_{ß_1} = \mathbf{1}_{ß_1}$. Next, choose $\omega_1 \in ß_1$. Let $\omega_2 \in ß_2$. Then

$$\begin{aligned}
&\beta_2(\omega_2) \\
&= \beta_2(\beta_2(\omega_2) \star (\omega_1 \star \omega_1)) && \text{(by Theorem V.4 (ii))} \\
&= \beta_2((\beta_2(\omega_2) \star \omega_1) \star \omega_1) && \text{(by assumption)} \\
&= \beta_2(\beta_2(\omega_2) \star \omega_1) \star \beta_2(\omega_1) && \text{(by Theorem V.4 (iii))} \\
&= \beta_2(\omega_2) \star \beta_2(\omega_1) && \text{(by Theorem V.4 (ii))} \\
&= \beta_2(\omega_2 \star \omega_1) && \text{(by Theorem V.4 (iii))} \\
&= \omega_2. && \text{(by Theorem V.4 (ii))}
\end{aligned}$$

Hence, $\beta_2 \mid_{ß_2} = \mathbf{1}_{ß_2}$.

Conversely, assume that $\beta_1 \mid_{ß_1} = \mathbf{1}_{ß_1}$ and $\beta_2 \mid_{ß_2} = \mathbf{1}_{ß_2}$. By Theorem V.15, we have d is both the left identity element for $ß_1$ and the right identity element for $ß_2$. Let $\omega_1 \in ß_1$,

$\omega_2 \in ß_2$, and $\kappa \in ß$. Then

$$\begin{aligned}
&\beta_1((\omega_2 \star \kappa) \star \omega_1) \\
&= \beta_1(\omega_2 \star \kappa) \star \beta_1(\omega_1) && \text{(by Theorem V.4 (iii))} \\
&= (\beta_1(\omega_2) \star \beta_1(\kappa)) \star \beta_1(\omega_1) && \text{(by Theorem V.4 (iii))} \\
&= \beta_1(\kappa) \star \beta_1(\omega_1) && \text{(by Theorems V.15 and V.11 (ii))} \\
&= \beta_1(\kappa \star \omega_1) && \text{(by Theorem V.4 (iii))} \\
&= \beta_1(\omega_2) \star \beta_1(\kappa \star \omega_1) && \text{(by Theorems V.15 and V.11 (ii))} \\
&= \beta_1(\omega_2 \star (\kappa \star \omega_1)) && \text{(by Theorem V.4 (iii))}
\end{aligned}$$

and

$$\begin{aligned}
&\beta_2((\omega_2 \star \kappa) \star \omega_1) \\
&= \beta_2(\omega_2 \star \kappa) \star \beta_2(\omega_1) && \text{(by Theorem V.4 (iii))} \\
&= (\beta_2(\omega_2) \star \beta_2(\kappa)) \star \beta_2(\omega_1) && \text{(by Theorem V.4 (iii))} \\
&= \beta_2(\omega_2) \star \beta_2(\kappa) && \text{(by Theorems V.15 and V.11 (ii))} \\
&= \beta_2(\omega_2) \star (\beta_2(\kappa) \star \beta_2(\omega_1)) \\
& && \text{(by Theorems V.15 and V.11 (ii))} \\
&= \beta_2(\omega_2) \star \beta_2(\kappa \star \omega_1) && \text{(by Theorem V.4 (iii))} \\
&= \beta_2(x_2 \star (\kappa \star \omega_1)). && \text{(by Theorem V.4 (iii))}
\end{aligned}$$

Therefore,

$$\begin{aligned}
(\omega_2 \star \kappa) \star \omega_1 &= \beta_2((\omega_2 \star \kappa) \star \omega_1) \star \beta_1((\omega_2 \star \kappa) \star \omega_1) \\
& \qquad \text{(by Theorem V.4 (i))} \\
&= \beta_2(\omega_2 \star (\kappa \star \omega_1)) \star \beta_1(\omega_2 \star (\kappa \star \omega_1)) \\
&= \omega_2 \star (\kappa \star \omega_1). \qquad \text{(by Theorem V.4 (i))}
\end{aligned}$$

■

Corollary V.20. *In any anti-internal direct product* $(ß; \star, \beta_1, \beta_2)$ *with* $ß_1 = \beta_1(ß), ß_2 = \beta_2(ß)$*, and* $ß_1 \cap ß_2 = \{d\}$*, if* $ß$ *is a semigroup, then* $\beta_1 |_{ß_1} = \boldsymbol{1}_{ß_1}$ *and* $\beta_2 |_{ß_2} = \boldsymbol{1}_{ß_2}$ *(i.e.,* $\beta_1^2 = \beta_1$ *and* $\beta_2^2 = \beta_2$*) with* d *is the left identity element for* $ß_1$ *and the right identity element for* $ß_2$*.*

Proof: Assume that $ß$ is a semigroup. By Theorems V.19 and V.15, we have $\beta_1 |_{ß_1} = \mathbf{1}_{ß_1}$ and $\beta_2 |_{ß_2} = \mathbf{1}_{ß_2}$ with d is the left identity element for $ß_1$ and the right identity element for $ß_2$. ■

VI. Anti-Internal Direct Products of BCC-Algebras

Using Theorem V.3, (UP-2), (UP-3), and (1), we get the following theorem.

Theorem VI.1. *Let a BCC-algebra* $(ß; \star, 0)$ *is the anti-internal direct product of its BCC-subalgebras* $ß_1$ *and* $ß_2$*. Then*

(i) $\forall \kappa_1 \in ß_1, \forall \omega_2, \kappa_2 \in ß_2, \kappa_2 \star \kappa_1 = (\omega_2 \star \kappa_2) \star \kappa_1$,
(ii) $\forall \omega_1, \kappa_1 \in ß_1, \forall \kappa_2 \in ß_2, \omega_1 \star (\kappa_2 \star \kappa_1) = \kappa_2 \star (\omega_1 \star \kappa_1)$,
(iii) $\forall \omega_1, \kappa_1 \in ß_1, \forall \omega_2 \in ß_2, (\omega_2 \star \omega_1) \star \kappa_1 = \omega_1 \star \kappa_1$,
(iv) $\forall \omega_1 \in ß_1, \forall \omega_2 \in ß_2, \forall \kappa \in ß_1 \cap ß_2, 0 = (\omega_2 \star \kappa) \star (\omega_1 \star \kappa)$,
(v) $\forall \kappa_1 \in ß_1, \forall \kappa_2 \in ß_2, \forall \omega \in ß_1 \cap ß_2, \kappa_2 \star \kappa_1 = (\omega \star \kappa_2) \star (\omega \star \kappa_1)$.

Theorem VI.2. *Let* $(ß; \star, \beta_1, \beta_2, 0)$ *be a BCC-algebra and unary operations* β_1 *and* β_2*. The groupoid* $(ß; \star)$ *is the anti-internal direct product of* $\beta_1(ß)$ *and* $\beta_2(ß)$ *if and only if the algebra* $(ß; \star, \beta_1, \beta_2, 0)$ *has the following properties:*

(i) $\beta_1 = \boldsymbol{1}_ß$ *is the identity function,*
(ii) $\beta_2 = \boldsymbol{0}_ß$ *is the zero function.*

Proof: Assume that $(ß; \star, \beta_1, \beta_2, 0)$ is a BCC-algebra and unary operations β_1 and β_2. The groupoid $(ß; \star)$ is the anti-internal direct product of $\beta_1(ß)$ and $\beta_2(ß)$. Then

$$\begin{aligned}
\beta_2(0) &= \beta_2(0 \star 0) && \text{(by (1))} \\
&= \beta_2(0) \star \beta_2(0) && \text{(by Theorem V.4 (iii))} \\
&= 0 && \text{(by (1))}
\end{aligned}$$

and

$$\begin{aligned}
\beta_1(0) &= \beta_1(0 \star 0) && \text{(by (1))} \\
&= \beta_1(0) \star \beta_1(0) && \text{(by Theorem V.4 (iii))} \\
&= 0. && \text{(by (1))}
\end{aligned}$$

(ii) Let $x \in ß$. Then

$$\begin{aligned}
\beta_2(\omega) &= \beta_2(\beta_2(\omega) \star \beta_1(0)) && \text{(by Theorem V.4 (ii))} \\
&= \beta_2(\beta_2(\omega) \star 0) \\
&= \beta_2(0) && \text{(by (UP-3))} \\
&= 0.
\end{aligned}$$

Hence, $\beta_2 = \mathbf{0}_ß$.

(i) Let $\omega \in ß$. Then

$$\begin{aligned}
\omega &= \beta_2(\omega) \star \beta_1(\omega) && \text{(by Theorem V.4 (i))} \\
&= 0 \star \beta_1(\omega) && \text{(by (i))} \\
&= \beta_1(\omega). && \text{(by (UP-2))}
\end{aligned}$$

Hence, $\beta_1 = \mathbf{1}_ß$.

Conversely, assume that $\beta_1 = \mathbf{1}_ß$ and $\beta_2 = \mathbf{0}_ß$. Then (i)-(iii) in Theorem V.4 hold. Hence, $(ß; \star)$ is the anti-internal direct product of $\beta_1(ß) = ß$ and $\beta_2(ß) = \{0\}$. ■

By Theorem VI.2, we have the following theorem.

Theorem VI.3. *Every BCC-algebra* $(ß; \star, 0)$ *is only the anti-internal direct product of* $ß$ *and* $\{0\}$*.*

VII. Conclusion and Future Work

The internal and anti-internal direct products of BCC-algebras are two novel notions for internal direct products of BCC-algebras that we present in this study. We have investigated the characteristics of the internal and anti-internal direct products of BCC-algebras.

Finally, we can conclude that for a BCC-algebra $(ß; \star, 0)$ there is only one form of the internal direct product is $\{0\} \times ß$ refer to Theorem IV.3, and there is only one form of the anti-internal direct product is $ß \times \{0\}$ refer to Theorem VI.3.

It is possible to analyze the internal and anti-internal direct products in various algebraic systems using the idea of the internal and anti-internal direct products of BCC-algebras presented in this article. The two new concepts of internal direct products of BCC-algebras in this article will be developed into new concepts for future studies: the internal and anti-internal direct products of type 2.

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