Outcomes and Relative Axiomatic Results under Multicriteria Management Models

Yan-An Hwang, Yu-Hsien Liao, Bo-Yao Wang and Ling-Shan Chou

***Abstract*—By evaluating on the maximal aggregate-marginal dedications among effect rank vectors, two outcomes are defined to deal with distribution mechANOsm under multicriteria management models. Further, some axiomatic results are provided to present the rationality for these two outcomes. In order to distinguish the differences among the operators and its effect ranks, several weighted generalizations and relative axiomatic results are also introduced.**

***Index Terms*—Outcome, effect rank, multicriteria management, weighted extension.**

I. Introduction

Reduced condition axiom (reduced game property) presents iMEYrtant characteristic under axiomatic procedures for traditional conditions. It expresses the independence of an outcome with regard to fixing some operators with its allotted payoffs. It has been applied in different shapes relying upon how the payoffs of the operators that "evacuate the bargaining" are determined. This axiom has been considered in numerous topics by pondering *reduced conditions*. Based on the conception of the marginal dedications, the *equal allocation of nonseparable costs* (EANSC, Ransmeier [15]) and the *normalized marginal index* have been considered respectively under traditional transferable-utility (TU) conditions. Moulin [13] adopted the *complement-reduced condition* to prove that the EANSC is a fair distributing rule.

Under traditional TU conditions, each operator is either totally involved or entirely out of participation with some other operators. Under *multi-choice TU conditions*, each operator could be allowed to operate with finite various effect ranks. By pondering overall outcomes for a given operator under multi-choice TU conditions, Hwang and Liao [4], Liao [8], [9] and Nouweland et al. [14] proposed respectively several extended allocating methods and relative results for the core, the EANSC and the Shapley value. Concerning results also could be studied in Chen et al. [1], Cheng et al. [2], Hwang and Liao [5], Huang et al. [6], Li et al. [7], Liao [10], Liao et al. [11], and so on. The above mentioned statements raise one thinking:

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- whether some outcomes might be generated by simultaneously combining multi-choice behavior and multicriteria circumstances.

This study is devoted to resolve above thinking. Two different outcomes, the *aggregate max-outcome* (AMO) and the *aggregate normalized outcome* (ANO), are defined respectively in Section 2. These two outcomes are multi-choice generalizations of the EANSC and the normalized marginal index throughout multicriteria circumstances. To evaluate the rationality of these two outcomes, an analogue reduction is introduced to propose some axiomatic results.

1) The AMO is the only outcome fitting *multicriteria standard for conditions* and *multicriteria reduced condition property*.
2) The AMO is the only outcome fitting *multicriteria efficiency*, *multicriteria equal state property*, *multicriteria covariance* and *multicriteria reduced condition property*.
3) Since the ANO violates multicriteria reduced condition property, the *revised reduced condition property* is defined to present that the ANO is the only outcome fitting *aggregate-marginal-standardness for conditions* and *revised reduced condition property*.
4) Under real-world circumstances, the operators and its effect ranks might be distinct. It is reasonable that weights might be alloted to the "operators" and the "ranks" for distinguishing respectively the discrepancy among the operators and its effect ranks. In Section 5, the *weight function for operators* and the *weighted function for ranks* are adopted to define weighted generalizations of the AMO and the ANO. Relative axiomatic results, numerical instances and comparisons are further offered throughout this paper.

II. Preliminaries

Let UM be the universe of operators. For $k \in UM$ and $r_k \in \mathbb{N}$, one could set $r_k = \{0, \cdots, r_k\}$ to be the rank space of operator i and $R_k^+ = r_k \setminus \{0\}$, where 0 means no participation. Let $R^M = \prod_{k\in M} R_k$ be the product set of the rank spaces for operators in M. An operator-coalition $H \subseteq M$ corresponds by a canonical way to the multi-choice coalition $e^H \in R^M$ with $e_k^H = 1$ for all $k \in T$, and $e_k^H = 0$ for all $k \in M \setminus H$. Denote 0_M the zero vector in $\mathbb{R}^M$. For $m \in \mathbb{N}$, let 0_m be the zero vector in $\mathbb{R}^m$ and $\mathbb{N}_m = \{1, 2, \cdots, m\}$.

A **multi-choice TU condition** is a triplet (M, r, o), where M with $0 < |M| < \infty$ is the set of operators, $r = (r_k)_{k\in M} \in \prod_{k\in M} R_k^+$ is the vector that presents the highest ranks among all operator, and $o : R^M \to \mathbb{R}$ is a mapping with $o(0_M) = 0$ which allots to each rank vector $\eta = (\eta_k)_{k\in M} \in R^M$ the value that the operators can get if each operator k operates with rank η_i. A **multicriteria multi-choice TU condition** is a triple (M, r, O^m), where $m \in \mathbb{N}$, $O^m = (o^t)_{t\in\mathbb{N}_m}$ and (M, r, o^t) is a multi-choice TU condition for all $t \in \mathbb{N}_m$. Denote the family of all multicriteria multi-choice TU conditions by $\mathbf{MTC}$.

An **outcome** is a mapping σ assigning to each $(M, r, O^m) \in \mathbf{MTC}$ an element

$$\sigma\big(M, r, O^m\big) = \big(\sigma^t\big(M, r, O^m\big)\big)_{t\in\mathbb{N}_m},$$

where $\sigma^t\big(M, r, O^m\big) = \big(\sigma^t_k\big(M, r, O^m\big)\big)_{k\in M} \in \mathbb{R}^M$ and $\sigma^t_k\big(M, r, O^m\big)$ is the payoff of the operator k if k participates in $\big(M, r, o^t\big)$. Let $(M, r, O^m) \in \mathbf{MTC}$, $H \subseteq M$ and $\eta \in \mathbb{R}^M$, we denote $L(\eta) = \{k \in M|\eta_k \neq 0\}$, and denote $\eta_H \in \mathbb{R}^H$ to be the restriction of η to H. Given $k \in M$, we introduce the substitution notation η_{-k} to stand for $\eta_{M\setminus\{k\}}$ and let $\tau = (\eta_{-k}, h) \in \mathbb{R}^M$ be defined by $\tau_{-k} = \eta_{-k}$ and $\tau_k = h$.

Here we define different generalizations of the EANSC and the normalized marginal index as follows.

Definition 1:

1) The **aggregate max-outcome (AMO)**, $\overline{\phi}$, is defined by

$$\begin{aligned} &\overline{\phi^t_k}(M, r, O^m) \\ = \quad &\phi^t_k(M, r, O^m) + \tfrac{1}{|M|}\Big[o^t(r) - \sum_{k\in M}\phi^t_k(M, r, O^m)\Big] \end{aligned}$$

for all $(M, r, O^m) \in \mathbf{MTC}$, for all $t \in \mathbb{N}_m$ and for all $k \in M$. The value $\phi^t_k(M, r, O^m) = \max_{q\in R^+_k}\{o^t(r) - o^t(r_{-k}, q-1)\}$ is the **maximal aggregate-marginal dedication** of the operator k from rank q to r_k in (M, r, o^t). From now on one could restrict our attention to bounded multi-choice TU conditions, defined as those conditions (M, r, o^t) such that, there exists $B_o \in \mathbb{R}$ such that $o^t(\alpha) \leq B_o$ for all $\kappa \in R^M$. We adopt it to guarantee that $\phi_k(M, r, o^t)$ is well-defined.

2) The **aggregate normalized outcome (ANO)**, $\overline{\Delta}$, is defined by

$$\overline{\Delta^t_p}(M, r, O^m) = \frac{o^t(r)}{\sum\limits_{k\in M}\phi^t_k(M, r, O^m)} \cdot \phi^t_p(M, r, O^m)$$

for all $(M, r, O^m) \in \mathbf{MTC}^*$, for all $t \in \mathbb{N}_m$ and for all $p \in M$, where $\mathbf{MTC}^* = \{(M, r, O^m) \in \mathbf{MTC} \mid \sum_{i\in M}\phi^t_i(M, r, O^m) \neq 0$ for all $t \in \mathbb{N}_m\}$.

Remark 1: A brief application of multicriteria multi-choice TU conditions could be offered under the setting of "management models". This type of issue could be constructed as follows. Let M be a set of all operators of a grand management model (M, r, O^m). The mapping o^t could be regarded as a utility mapping which allots to each rank vector $\kappa = (\kappa_k)_{k\in M} \in R^M$ the rank that the operators could get if each operator k operates with strategy $\kappa_k \in R_k$ in the sub-management model (M, r, o^t). Modeled in this conception, the grand management model (M, r, O^m) could be regarded as a multicriteria multi-choice TU condition, with o^t being the characteristic mapping and R_k being the collection of all strategies of the operator k. In the following sections, one would like to prove that the AMO and the ANO might generate "*balanced and optimal distributing mechanisms*" among all operators under multi-choice behavior and multicriteria circumstances.

III. The AMO and relative axiomatic results

In this section, one would like to apply some properties to axiomatize the AMO. Therefore, some more properties are needed. An outcome σ fits **multicriteria efficiency (MEY)** if $\sum_{i\in M}\sigma^t_i(M, r, O^m) = o^t(r)$ for all $(M, r, O^m) \in \mathbf{MTC}$ and for all $t \in \mathbb{N}_m$. An outcome σ fits **multicriteria standardness for conditions (MSFC)** if $\sigma(M, r, O^m) = \overline{\phi}(M, r, O^m)$ for all $(M, r, O^m) \in \mathbf{MTC}$ with $|M| \leq 2$. An outcome σ fits **multicriteria equal state property (MESP)** if $\sigma_p(M, r, O^m) = \sigma_q(M, r, O^m)$ for all $(M, r, O^m) \in \mathbf{MTC}$ with $\phi^t_p(M, r, O^m) = \phi^t_q(M, r, O^m)$ for some $p, q \in M$ and for all $t \in \mathbb{N}_m$. An outcome σ fits **multicriteria covariance (MCO)** if $\sigma(M, r, O^m) = \sigma(M, r, W^m) + (f^t)_{t\in\mathbb{N}_m}$ for all $(M, r, O^m), (M, r, W^m) \in \mathbf{MTC}$ with $o^t(\kappa) = w^t(\kappa) + \sum_{k\in L(\kappa)} f^t_k$ for some $f^t \in \mathbb{R}^M$, for all $t \in \mathbb{N}_m$ and for all
$kappa \in R^M$.

MEY means that whole the utility should be distributed entirely. MSFC is a generalized analogue of the standardness for the axiomatic process of the Shapley value [16] due to Hart and Mas-Colell [3]. MESP means that the payoffs should be equal if the maximal aggregate-marginal dedications are coincident. MCO could be regarded as an extremely weak form of *additivity*. By Definition 1, it is clear that the AMO fits MEY, MSFC, MESP and MCO.

Moulin [13] introduced the reduced condition as that in which each coalition in the sub-condition could achieve payoffs to its operators only if they are compossible with the beginning payoffs to "whole" the operators outside of the sub-condition. A generalized Moulin-reduction under multi-choice TU conditions could be considered as follows. Let $(M, r, O^m) \in \mathbf{MTC}$, $H \subseteq M$ and σ be an outcome. The **reduced condition** $(H, r_H, o^m_{H,\sigma})$ is defined by $o^m_{H,\sigma} = (o^t_{H,\sigma})_{t\in\mathbb{N}_m}$ and

$$\begin{aligned} &o^t_{H,\sigma}(\kappa) \\ = \quad &\begin{cases} 0 & \text{if } \kappa = 0_H, \\ o^t\big(\kappa, r_{M\setminus H}\big) - \sum\limits_{k\in M\setminus H}\sigma^t_k(M, r, O^m) & \text{otherwise} \end{cases} \end{aligned}$$

for all $\kappa \in R^H$. An outcome σ fits **multicriteria reduced condition property (MRCP)** if $\sigma^t_k(H, r_H, o^m_{H,\sigma}) = \sigma^t_k(M, r, O^m)$ for all $(M, r, O^m) \in \mathbf{MTC}$, for all $t \in \mathbb{N}_m$, for all $H \subseteq M$ with $|H| \leq 2$ and for all $k \in H$.

Lemma 1: The AMO $\overline{\phi}$ fits MRCP.

Proof: Let $(M, r, O^m) \in \mathbf{MTC}$, $H \subseteq M$ and $t \in \mathbb{N}_m$. Assume that $|M| \geq 2$ and $|H| \leq 2$. By definition of $\overline{\phi}$,

$$\begin{aligned} &\overline{\phi^t_p}(H, r_H, o^m_{H,\overline{\phi}}) \\ = \quad &\phi^t_p(H, r_H, o^m_{H,\overline{\phi}}) + \tfrac{1}{|H|}\Big[o^t_{H,\overline{\phi}}(r_H) - \sum_{k\in H}\phi^t_k(H, r_H, o^m_{H,\overline{\phi}})\Big] \end{aligned} \tag{1}$$

for all $p \in H$ and for all $t \in \mathbb{N}_m$. By definitions of ϕ^t and $o^t_{H,\overline{\phi}}$,

$$\begin{aligned} \phi^t_p(H, r_H, o^m_{H,\overline{\phi}}) &= \max_{q\in R^+_p}\{o^t_{H,\overline{\phi}}(r_H) - o^t_{H,\overline{\phi}}(r_{H\setminus\{p\}}, q-1)\} \\ &= \max_{q\in R^+_p}\{o^t(r) - o^t(r_{-p}, q-1)\} \\ &= \phi^t_p(M, r, O^m). \end{aligned} \tag{2}$$

By equations (1), (2) and definitions of $o^t_{H,\overline{\phi}}$ and $\overline{\phi}$,

$$\begin{aligned} & \overline{\phi^t_p}(H, r_H, o^m_{H,\overline{\phi}}) \\ = & \phi^t_p(M, r, O^m) + \tfrac{1}{|H|}\big[o^t_{H,\overline{\phi}}(r_H) - \sum_{k\in H} \phi^t_k(M, r, O^m)\big] \\ = & \phi^t_p(M, r, O^m) + \tfrac{1}{|H|}\big[o^t(r) - \sum_{k\in M\setminus H} \overline{\phi^t_k}(M, r, O^m) \\ & - \sum_{k\in H} \phi^t_k(M, r, O^m)\big] \\ = & \phi^t_p(M, r, O^m) + \tfrac{1}{|H|}\big[\sum_{k\in H} \overline{\phi^t_k}(M, r, O^m) \\ & - \sum_{k\in H} \phi^t_k(M, r, O^m)\big] \\ = & \phi^t_p(M, r, O^m) + \tfrac{1}{|H|}\Big[\tfrac{|H|}{|M|}\big[o^t(r) - \sum_{k\in M} \phi^t_k(M, r, O^m)\big]\Big] \\ = & \phi^t_p(M, r, O^m) + \tfrac{1}{|M|}\big[o^t(r) - \sum_{k\in M} \phi^t_k(M, r, O^m)\big] \\ = & \overline{\phi^t}_p(M, r, O^m) \end{aligned}$$

for all $p \in H$ and for all $t \in \mathbb{N}_m$, i.e., the AMO fits MRCP. ■

Inspired by Moulin [13], Hart and Mas-Colell [3] and Maschler and Owen [12], one would like to apply MRCP to axiomatize the AMO.

Theorem 1: On $\mathbf{MTC}$, the AMO is the only outcome fitting MSFC and MRCP.

Proof: $\overline{\phi}$ fits MRCP by Lemma 1. Clearly, $\overline{\phi}$ fits MSFC. To present uniqueness, assume that σ fits MSFC and MRCP. By MRCP and MSFC of σ, σ also fits MEY absolutely. Let $(M, r, O^m) \in \mathbf{MTC}$. By MSFC of σ, $\sigma(M, r, O^m) = \overline{\phi}(M, r, O^m)$ if $|M| \le 2$. The situation $|M| > 2$: Let $p \in M$, $t \in \mathbb{N}_m$ and $H = \{p, k\}$ for some $k \in M \setminus \{p\}$.

$$\begin{aligned} & \sigma^t_p(M, r, O^m) - \sigma^t_k(M, r, O^m) \\ = & \sigma^t_p(H, r_H, o^m_{H,\sigma}) - \sigma^t_k(H, r_H, o^m_{H,\sigma}) \\ & \textbf{(ry MRCP of } \sigma\textbf{)} \\ = & \overline{\phi^t_p}(H, r_H, o^m_{H,\sigma}) - \overline{\phi^t_k}(H, r_H, o^m_{H,\sigma}) \\ & \textbf{(By MSFC of } \sigma\textbf{)} \\ = & \phi^t_p(H, r_H, o^m_{H,\sigma}) - \phi^t_k(H, r_H, o^m_{H,\sigma}) \\ = & \max_{q\in R^+_p}\{o^t_{H,\sigma}(r_H) - o^t_{H,\sigma}(r_{H\setminus\{p\}}, q-1)\} \\ & - \max_{q\in R^+_k}\{o^t_{H,\sigma}(r_H) - o^t_{H,\sigma}(r_{H\setminus\{k\}}, q-1)\} \\ = & \max_{q\in R^+_p}\{o^t(r) - o^t(r_{-p}, q-1)\} \\ & - \max_{q\in R^+_k}\{o^t(r) - o^t(r_{-k}, q-1)\} \\ = & \phi^t_p(M, r, O^m) - \phi^t_k(M, r, O^m) \\ = & \overline{\phi^t_p}(M, r, O^m) - \overline{\phi^t_k}(M, r, O^m). \end{aligned}$$

Therefore,

$$\begin{aligned} & \sigma^t_p(M, r, O^m) - \sigma^t_k(M, r, O^m) \\ = & \overline{\phi^t_p}(M, r, O^m) - \overline{\phi^t_k}(M, r, O^m). \end{aligned}$$

By MEY of $\overline{\phi}$ and σ,

$$\begin{aligned} & |M| \cdot \sigma^t_p(M, r, O^m) - o^t(r) \\ = & \sum_{k\in M}[\sigma^t_p(M, r, O^m) - \sigma^t_k(M, r, O^m)] \\ = & \sum_{k\in M}[\overline{\phi^t_p}(M, r, O^m) - \overline{\phi^t_k}(M, r, O^m)] \\ = & |M| \cdot \overline{\phi^t_p}(M, r, O^m) - o^t(r). \end{aligned}$$

Thus, $\sigma^t_p(M, r, O^m) = \overline{\phi^t_p}(M, r, O^m)$ for all $p \in M$ and for all $t \in \mathbb{N}_m$. ■

Next, one would like to axiomatize the AMO by means of MEY, MESP, MCO and MRCP.

Lemma 2: If an outcome σ fits MEY, MESP and MCO, then σ fits MSFC.

Proof: Suppose that an outcome σ fits MEY, MESP and MCO. Let $(M, r, O^m) \in \mathbf{MTC}$. The proof is finished by MEY of σ if $|M| = 1$. Let $(M, r, O^m) \in \mathbf{MTC}$ with $M = \{p, k\}$ for some $p \ne k$. We define a condition (M, r, W^m) to be that $w^t(\kappa) = o^t(\kappa) - \sum_{s\in L(\kappa)} \phi^t_s(M, r, O^m)$ for all $\kappa \in R^M$ and for all $t \in \mathbb{N}_m$. By definition of W^m,

$$\begin{aligned} & \phi^t_p(M, r, W^m) \\ = & \max_{q\in R^+_p}\{w^t(r_p, r_k) - w^t(q-1, r_k)\} \\ = & \max_{q\in R^+_p}\{o^t(r_p, r_k) - o^t(q-1, r_k) - \phi^t_p(M, r, O^m)\} \\ = & \max_{q\in R^+_p}\{o^t(r_p, r_k) - o^t(q-1, r_k)\} - \phi^t_p(M, r, O^m) \\ = & \phi^t_p(M, r, O^m) - \phi^t_p(M, r, O^m) \\ = & 0. \end{aligned}$$

Similarly, $\phi^t_k(M, r, W^m) = 0$. Thus, $\phi^t_p(M, r, W^m) = \phi^t_k(M, r, W^m)$. By MESP of σ, $\sigma^t_p(M, r, W^m) = \sigma^t_k(M, r, W^m)$. By MEY of σ, $w^t(r) = \sigma^t_p(M, r, W^m) + \sigma^t_k(M, r, W^m) = 2 \cdot \sigma^t_p(M, r, W^m)$, i.e., $\sigma^t_p(M, r, W^m) = \frac{w^t(r)}{2} = \frac{1}{2} \cdot \big[o^t(r) - \phi^t_p(M, r, O^m) - \phi^t_k(M, r, O^m)\big]$. By MCO of σ,

$$\begin{aligned} & \sigma^t_p(M, r, O^m) \\ = & \phi^t_p(M, r, O^m) + \tfrac{1}{2} \cdot [o^t(r) - \phi^t_p(M, r, O^m) - \phi^t_k(M, r, O^m)] \\ = & \overline{\phi^t_p}(M, r, O^m). \end{aligned}$$

Similarly, $\sigma^t_k(M, r, O^m) = \overline{\phi^t_k}(M, r, O^m)$, i.e., σ fits MSFC. ■

Theorem 2: On $\mathbf{MTC}$, the AMO is the only outcome fitting MEY, MESP, MCO and MRCP.

Proof: $\overline{\phi}$ fits MEY, MESP and MRCP by Definition 1. The rest of proofs could be finished by Lemmas 1, 2 and Theorem 1. ■

Based on the following instances, one would like to show that each of the properties appeared in Theorems 1 and 2 is logically independent of the rest of properties.

Example 1: For all $(M, r, O^m) \in \mathbf{MTC}$, for all $t \in \mathbb{N}_m$ and for all $p \in M$, one would define the outcome σ as

$$\sigma^t_p(M, r, O^m) = \begin{cases} \overline{\phi^t_p}(M, r, O^m) & \text{if } |M| \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

σ fits MSFC, but it violates MRCP.

Example 2: For all $(M, r, O^m) \in \mathbf{MTC}$, for all $t \in \mathbb{N}_m$ and for all $p \in M$, one would define the outcome σ as $\sigma^t_p(M, r, O^m) = \phi^t_p(M, r, O^m)$. σ fits MESP, MCO and MRCP, but it violates MEY and MSFC.

Example 3: One would define the outcome σ as $\sigma^t_p(M, r, O^m) = \frac{o^t(r)}{|M|}$ for all $(M, r, O^m) \in \mathbf{MTC}$, for all $t \in \mathbb{N}_m$ and for all $p \in M$. σ fits MEY, MESP and MRCP, but it violates MCO.

Example 4: For all $(M, r, O^m) \in \mathbf{MTC}$, for all $t \in \mathbb{N}_m$ and for all $p \in M$, one would define the outcome σ as

$$\begin{aligned} & \sigma^t_p(M, r, O^m) \\ = & \big[o^t(r) - o^t(r_{-p}, 0)\big] + \tfrac{1}{|M|} \cdot \Big[o^t(r) - \sum_{k\in M}\big[o^t(r) - o^t(r_{-k}, 0)\big]\Big]. \end{aligned}$$

σ fits MEY, MCO and MRCP, but it violates MESP.

Example 5: For all $(M, r, O^m) \in \mathbf{MTC}$, for all $t \in \mathbb{N}_m$ and for all $p \in M$, one would define the outcome σ as

$$\begin{aligned} & \sigma^t_p(M, r, O^m) \\ = & \phi^t_p(M, r, O^m) + \frac{d^t(p)}{\sum_{k\in M} d^t(k)} \cdot \big[o^t(r) - \sum_{k\in M} \phi^t_k(M, r, O^m)\big], \end{aligned}$$

where for all $(M,r,O^m)\in \mathbf{MTC}$, $d^t: M\to \mathbb{R}^+$ is defined as $d^t(p)=d^t(k)$ if $\phi^t_p(M,r,O^m)=\phi^t_k(M,r,O^m)$. Define an outcome β as for all $(M,r,O^m)\in \mathbf{MTC}$, for all $t\in\mathbb{N}_m$ and for all $p\in M$,

$$\beta^t_p(M,r,O^m)=\begin{cases}\overline{\phi^t_p}(M,r,O^m) & \text{if } |M|\le 2,\\ \sigma^t_p(M,r,O^m) & \text{otherwise.}\end{cases}$$

β fits MEY, MESP and MCO, but it violates MRCP.

IV. The ANO and relative axiomatic results

Clearly, by Definition 1, the ANO fits MEY and MESP, but it violates MCO. Similar to above section, one would like to axiomatize the ANO by means of reduced condition property. Unfortunately, $(H,r_H,o^m_{H,\overline{\Delta}})$ does not exist if $\sum_{p\in H}\phi^t_p(M,r,O^m)=0$. One would consider the **revised reduced condition property** as follows. An outcome σ fits **revised reduced condition property (RRCP)** if $(H,r_H,o^m_{H,\sigma})\in \mathbf{MTC}^*$ for some $(M,r,O^m)\in\mathbf{MTC}$ and for some $H\subseteq M$, it holds that $\sigma^t_p(H,r_H,O^m_{H,\sigma})=\sigma^t_p(M,r,O^m)$ for all $t\in\mathbb{N}_m$ and for all $p\in H$. σ fits **aggregate-marginal-standard for conditions (AMSC)** if $\sigma(M,r,O^m)=\overline{\Delta}(M,r,O^m)$ for all $(M,r,O^m)\in\mathbf{MTC}$, $|M|\le 2$. Clearly, the ANO fits AMSC.

Lemma 3: The ANO fits RRCP on $\mathbf{MTC}^*$.

Proof: Let $(M,r,O^m)\in\mathbf{MTC}^*$. The proof is finished if $|M|\le 2$. Suppose that $|M|\ge 3$ and $H\subseteq M$ with $|H|\le 2$. Similar to equation (2),

$$\phi^t_p(H,r_H,o^m_{H,\overline{\Delta}})=\phi^t_p(M,r,O^m). \tag{3}$$

for all $p\in H$ and for all $t\in\mathbb{N}_m$. Define that $\alpha_t=\frac{o^t(r)}{\sum_{k\in M}\phi^t_k(M,r,O^m)}$. For all $p\in H$ and for all $t\in\mathbb{N}_m$,

$$\begin{aligned}
&\overline{\Delta^t_p}(H,r_H,o^m_{H,\overline{\Delta}})\\
=\;&\frac{o^t_{H,\overline{\Delta}}(r_H)}{\sum_{k\in H}\phi^t_k(H,r_H,o^m_{H,\overline{\Delta}})}\cdot\phi^t_p(H,r_H,o^m_{H,\overline{\Delta}})\\
=\;&\frac{o^t(r)-\sum_{k\in H^c}\overline{\Delta^t_k}(M,r,O^m)}{\sum_{k\in H}\phi^t_k(M,r,O^m)}\cdot\phi^t_p(M,r,O^m)\\
&\textbf{(By equation (3) and definition of } o^m_{H,\overline{\Delta}}\textbf{)}\\
=\;&\frac{\sum_{k\in H}\overline{\Delta^t_k}(M,r,O^m)}{\sum_{k\in H}\phi^t_k(M,r,O^m)}\cdot\phi^t_p(M,r,O^m)\\
&\textbf{(By MEY of } \overline{\Delta}\textbf{)}\\
=\;&\alpha_t\cdot\phi^t_p(M,r,O^m)\\
&\textbf{(By Definition 1)}\\
=\;&\overline{\Delta^t_p}(M,r,O^m).\\
&\textbf{(By Definition 1)}
\end{aligned}\tag{4}$$

By equations (3) and (4), the outcome $\overline{\Delta}$ fits RRCP. ■

Theorem 3: On $\mathbf{MTC}^*$, the outcome $\overline{\Delta}$ is the only outcome fitting AMSC and RRCP.

Proof: $\overline{\Delta}$ fits RRCP by Lemma 3. Clearly, $\overline{\Delta}$ fits AMSC. To present uniqueness, assume that σ fits RRCP and AMSC on $\mathbf{MTC}^*$. By AMSC and RRCP of σ, σ also fits MEY absolutely. Let $(M,r,O^m)\in\mathbf{MTC}^*$. One would finish the proof by induction on $|M|$. It is trivial that $\sigma(M,r,O^m)=\overline{\Delta}(M,r,O^m)$ by AMSC if $|M|\le 2$. Suppose that it holds if $|M|\le c-1$, $c\le 3$. The situation $|M|=c$: Let $i,j\in M$ with $i\neq j$ and $t\in\mathbb{N}_m$. By Definition 1, $\overline{\Delta^t_k}(M,r,O^m)=\frac{o^t(r)}{\sum_{h\in M}\phi^t_h(M,r,O^m)}\cdot\phi^t_k(M,r,O^m)$ for all $k\in M$. Assume that $\lambda^t_k=\frac{\phi^t_k(M,r,O^m)}{\sum_{h\in M}\phi^t_h(M,r,O^m)}$ for all $k\in M$. Thus,

$$\begin{aligned}
&\sigma^t_i(M,r,O^m)\\
=\;&\sigma^t_i\big(M\setminus\{j\},r_{M\setminus\{j\}},O^m_{M\setminus\{j\},\sigma}\big)\\
&\textbf{(By RRCP of } \sigma\textbf{)}\\
=\;&\overline{\Delta^t_i}\big(M\setminus\{j\},r_{M\setminus\{j\}},O^m_{M\setminus\{j\},\sigma}\big)\\
&\textbf{(By AMSC of } \sigma\textbf{)}\\
=\;&\frac{o^t_{M\setminus\{j\},\sigma}(r_{M\setminus\{j\}})\cdot\phi^t_i\big(M\setminus\{j\},r_{M\setminus\{j\}},O^m_{M\setminus\{j\},\sigma}\big)}{\sum_{k\in M\setminus\{j\}}\phi^t_k\big(M\setminus\{j\},r_{M\setminus\{j\}},O^m_{M\setminus\{j\},\sigma}\big)}\\
=\;&\frac{o^t(r)-\sigma^t_i(M,r,O^m)\cdot\phi^t_i(M,r,O^m)}{\sum_{k\in M\setminus\{j\}}\phi^t_k(M,r,O^m)}\\
&\textbf{(By equation (2))}\\
=\;&\frac{o^t(r)-\sigma^t_i(M,r,O^m)\cdot\phi^t_i(M,r,O^m)}{-\phi^t_j(M,r,O^m)+\sum_{k\in M}\phi^t_k(M,r,O^m)}.
\end{aligned}\tag{5}$$

By equation (5),

$$\begin{aligned}
&\sigma^t_i(M,r,O^m)\cdot[1-\lambda^t_j]=[o^t(r)-\sigma^t_j(M,r,O^m)]\cdot\lambda^t_i\\
\Longrightarrow\;&[1-\lambda^t_j]\sum_{i\in M}\sigma^t_i(M,r,O^m)=[o^t(r)-\sigma^t_j(M,r,O^m)]\cdot\sum_{i\in M}\lambda^t_i\\
\Longrightarrow\;&[1-\lambda^t_j]\cdot o^t(r)=[o^t(r)-\sigma^t_j(M,r,O^m)]\cdot 1\\
&\textbf{(By MEY of } \sigma\textbf{)}\\
\Longrightarrow\;&o^t(r)-o^t(r)\cdot\lambda^t_j=o^t(r)-\sigma^t_j(M,r,O^m)\\
\Longrightarrow\;&\overline{\Delta^t_j}(M,r,O^m)=\sigma^t_j(M,r,O^m).
\end{aligned}$$

The proof is completed. ■

Based on the following instances, one would like to show that each of the properties appeared in Theorem 3 is logically independent of the rest of properties.

Example 6: For all $(M,r,O^m)\in\mathbf{MTC}^*$, for all $t\in\mathbb{N}_m$ and for all $p\in M$, one would define the outcome σ as $\sigma^t_p(M,r,O^m)=0$. σ fits RRCP, but it violates AMSC.

Example 7: For all $(M,r,O^m)\in\mathbf{MTC}^*$, for all $t\in\mathbb{N}_m$ and for all $p\in M$, one would define the outcome σ as

$$\sigma^t_p(M,r,O^m)=\begin{cases}\overline{\Delta^t_p}(M,r,O^m) & \text{, if } |M|\le 2,\\ 0 & \text{, otherwise.}\end{cases}$$

σ fits AMSC, but it violates RRCP.

V. Weighs and relative generalizations

In this section, one would like to apply the *weight mapping for operators* and the *weighted mapping for ranks* to define several weighted generalizations and relative axiomatic results for the AMO and the ANO.

A. Weighted generalizations for the AMO

If $s: UM\to\mathbb{R}^+$ be a positive mapping, then s is called a **weight mapping for operators**. If $\xi: R^U\to\mathbb{R}^+$ be a positive mapping, then ξ is called a **weight mapping for ranks**. By these two kinds of the weight mapping, two weighted generalizations of the AMO are generated as follows.

Definition 2:

- The **P-weighted aggregate max-outcome (P-WAMO)**, ϕ^s, is defined as for all $(M,r,O^m)\in\mathbf{MTC}$, for all weight mapping for operators s, for all $t\in\mathbb{N}_m$ and for all operator $p\in M$,

$$\begin{aligned}
&\phi^{s,t}_p(M,r,O^m)\\
=\;&\phi^t_p(M,r,O^m)+\frac{s(p)}{\sum_{k\in M}s(k)}\cdot\Big[o^t(r)-\sum_{k\in M}\phi^t_k(M,r,O^m)\Big].
\end{aligned}\tag{6}$$

- The **L-weighted aggregate max-outcome (L-WAMO)**, ϕ^ξ, is defined as for all $(M,r,O^m)\in\mathbf{MTC}$, for all weight mapping for ranks ξ, for all $t \in \mathbb{N}_m$ and for all operator $p \in M$,

$$\begin{aligned} &\phi_p^{\xi,t}(M,r,O^m) \\ = \ &\tau_p^{\xi,t}(M,r,O^m) + \frac{1}{|M|}\cdot\Big[o^t(r) - \sum_{k\in M}\tau_k^{\xi,t}(M,r,O^m)\Big], \end{aligned} \tag{7}$$

where

$$\tau_p^{\xi,t}(M,r,O^m) = \max_{q\in R_p^+}\{\xi(q)\cdot[o^t(r) - o^t(r_{-p}, j-1)]\}.$$

Remark 2: Clearly, based on Definition 2, the P-WAMO fits MEY and MCO, but it violates MESP. Similarly, the L-WAMO fits MEY, but it violates MESP and MCO.

An outcome σ fits **P-weighted standard for conditions (PWSC)** if $\sigma(M,r,O^m) = \phi^s(M,r,O^m)$ for all $(M,r,O^m) \in \mathbf{MTC}$ with $|M| \le 2$ and for all weight mapping for operators s. σ fits **L-weighted standard for conditions (LWSC)** if $\sigma(M,r,O^m) = \phi^\xi(M,r,O^m)$ for all $(M,r,O^m) \in \mathbf{MTC}$ with $|M| \le 2$ and for all weight mapping for ranks ξ. Similar to the proofs of Lemma 1 and Theorem 1, one would like to propose the analogies of Lemma 1 and Theorem 1.

Theorem 4:

- The P-WAMO ϕ^s and the L-WAMO ϕ^ξ fit MRCP.
- On $\mathbf{MTC}$, the P-WAMO ϕ^s is the only outcome fitting PWSC and MRCP.
- On $\mathbf{MTC}$, the L-WAMO ϕ^ξ is the only outcome fitting LWSC and MRCP.

Based on the following instances, one would like to show that each of the properties appeared in above axiomatizations is logically independent of the rest of properties.

Example 8: For all $(M,r,O^m) \in \mathbf{MTC}$, for all $t \in \mathbb{N}_m$, for all weight mappings s and ξ and for all $p \in M$, one define the outcome σ as $\sigma_p^t(M,r,O^m) = 0$. σ fits MRCP, but it violates PWSC and LWSC.

Example 9: For all $(M,r,O^m) \in \mathbf{MTC}$, for all $t \in \mathbb{N}_m$, for all weight mapping for operators s and for all $p \in M$, one define the outcome σ as

$$\sigma_p^t(M,r,O^m) = \begin{cases} \phi_p^{s,t}(M,r,O^m) & \text{if } |M| \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

σ fits PWSC, but it violates MRCP.

Example 10: For all $(M,r,O^m) \in \mathbf{MTC}$, for all $t \in \mathbb{N}_m$, for all weight mapping for ranks ξ and for all $p \in M$, one define the outcome σ as

$$\sigma_p^t(M,r,O^m) = \begin{cases} \phi_p^{\xi,t}(M,r,O^m) & \text{if } |M| \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

σ fits LWSC, but it violates MRCP.

B. Weighted generalizations for the ANO

By these two kinds of the weight mapping, two weighted generalizations of the ANO are generated as follows.

Definition 3:

- The **P-weighted aggregate normalized outcome (P-WANO)**, Θ^s, is defined as

$$\begin{aligned} &\Theta_p^{s,t}(M,r,O^m) \\ = \ &\frac{o^t(r)}{\sum_{k\in M} s(k)\phi_k^t(M,r,O^m)}\cdot s(p)\phi_p^t(M,r,O^m) \end{aligned} \tag{8}$$

  for all $(M,r,O^m) \in \mathbf{MTC}^{**}$, for all weight mapping for operators s, for all $t \in \mathbb{N}_m$ and for all $p \in M$, where $\mathbf{MTC}^{**} = \{(M,r,O^m) \in \mathbf{MTC} \mid \sum_{k\in M} s(k)\phi_i^t(M,r,O^m) \neq 0 \text{ for all } t \in \mathbb{N}_m\}$.
- The **L-weighted aggregate normalized outcome (L-WANO)**, Φ^ξ, is defined as for all $(M,r,O^m) \in \mathbf{MTC}$, for all weight mapping for ranks ξ, for all $t \in \mathbb{N}_m$ and for all operator $p \in M$,

$$\Phi_p^{\xi,t}(M,r,O^m) = \frac{o^t(r)}{\sum_{k\in M}\tau_k^{\xi,t}(M,r,O^m)}\cdot\tau_p^{\xi,t}(M,r,O^m) \tag{9}$$

  for all $(M,r,O^m) \in \mathbf{MTC}^{***}$, for all $t \in \mathbb{N}_m$ and for all $p \in M$, where $\mathbf{MTC}^{***} = \{(M,r,O^m) \in \mathbf{MTC} \mid \sum_{k\in M}\tau_k^{\xi,t}(M,r,O^m) \neq 0 \text{ for all } t \in \mathbb{N}_m\}$.

Remark 3: Clearly, based on Definition 3, the P-WANO fits MEY, but it violates MESP and MCO. Similarly, the L-WANO fits MEY, but it violates MESP and MCO.

Similar to Theorem 3, one would like to axiomatize the P-WANO and L-WANO by means of RRCP. Unfortunately, $(H, r_H, o^m_{H,\Theta^s})$ does not exist if $\sum_{k\in M} s(k)\phi_k^t(M,r,O^m) = 0$. Therefore, one would consider the **1-revised reduced condition property** as follows. An outcome σ fits **1-revised reduced condition property (1RRCP)** if $(H, r_H, o^m_{H,\sigma}) \in \mathbf{MTC}^{**}$ for some $(M,r,O^m) \in \mathbf{MTC}$ and for some $H \subseteq M$, it holds that $\sigma_p^t(H, r_H, O^m_{H,\sigma}) = \sigma_p^t(M,r,O^m)$ for all $t \in \mathbb{N}_m$ and for all $p \in H$. Similarly, $(H, r_H, o^m_{H,\sigma^\xi})$ does not exist if $\sum_{k\in M}\tau_k^{w,t}(M,r,O^m) = 0$. Therefore, one would consider the **2-revised reduced condition property** as follows. An outcome σ fits **2-revised reduced condition property (2RRCP)** if $(H, r_H, o^m_{H,\sigma}) \in \mathbf{MTC}^{***}$ for some $(M,r,O^m) \in \mathbf{MTC}$ and for some $H \subseteq M$, it holds that $\sigma_p^t(H, r_H, O^m_{H,\sigma}) = \sigma_p^t(M,r,O^m)$ for all $t \in \mathbb{N}_m$ and for all $p \in H$. An outcome σ fits **P-weighted normalization for conditions (PWNC)** if $\sigma(M,r,O^m) = \Theta^s(M,r,O^m)$ for all $(M,r,O^m) \in \mathbf{MTC}$ with $|M| \le 2$ and for all weight mapping for operators s. σ fits **L-weighted normalization for conditions (LWNC)** if $\sigma(M,r,O^m) = \Phi^\xi(M,r,O^m)$ for all $(M,r,O^m) \in \mathbf{MTC}$ with $|M| \le 2$ and for all weight mapping for ranks ξ. Similar to the proofs of Lemma 3 and Theorem 3, one would like to propose the analogies of Lemma 3 and Theorem 3.

Theorem 5:

- The P-WANO Θ^s and the L-WANO Φ^ξ fit 1RRCP and 2RRCP respectively.
- On $\mathbf{MTC}^{**}$, the P-WANO Θ^s is the only outcome fitting PWNC and RRCP.
- On $\mathbf{MTC}^{***}$, the L-WANO Φ^ξ is the only outcome fitting LWNC and RRCP.

Based on the following instances, one would like to show that each of the properties appeared in above axiomatizations is logically independent of the rest of properties.

Example 11: For all $(M,r,O^m) \in \mathbf{MTC}^{**}$, for all $t \in \mathbb{N}_m$, for all weight mappings s and ξ and for all $p \in M$, one would define the outcome σ as $\sigma_p^t(M,r,O^m) = 0$. σ fits RRCP, but it violates PWSC.

Example 12: For all $(M,r,O^m) \in \mathbf{MTC}^{***}$, for all $t \in \mathbb{N}_m$, for all weight mappings s and ξ and for all $p \in M$, one would define the outcome σ as $\sigma_p^t(M,r,O^m) = 0$. σ fits RRCP, but it violates LWSC.

Example 13: For all $(M,r,O^m) \in \mathbf{MTC}^{**}$, for all $t \in \mathbb{N}_m$, for all weight mapping for operators s and for all $p \in M$, one define the outcome σ as

$$\sigma_p^t(M,r,O^m) = \begin{cases} \Theta_p^{s,t}(M,r,O^m) & \text{if } |M| \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

σ fits PWSC, but it violates 1RRCP.

Example 14: For all $(M,r,O^m) \in \mathbf{MTC}^{***}$, for all $t \in \mathbb{N}_m$, for all weight mapping for ranks ξ and for all $p \in M$, one define the outcome σ as

$$\sigma_p^t(M,r,O^m) = \begin{cases} \Phi_p^{\xi,t}(M,r,O^m) & \text{if } |M| \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

σ fits LWSC, but it violates 2RRCP.

C. Natural weights and relative generalization

Clearly, weighted outcomes are defined by applying two types of weight mappings. However, these two weight mappings might be artificial. It is rational that the weights could be replaced by marginal dedications. Therefore, a specific generalization of the AMO could be generated as follows.

Definition 4: The **N-weighted aggregate max-outcome (N-WAMO)**, ϕ^s, is defined as for all $(M,r,O^m) \in \mathbf{MTC}^*$, for all $t \in \mathbb{N}_m$ and for all operator $p \in M$,

$$\begin{aligned} &\Gamma_p^t(M,r,O^m) \\ = \;& \phi_p^t(M,r,O^m) + \frac{\phi_p^t(M,r,O^m)}{\sum\limits_{k\in M}\phi_k^t(M,r,O^m)}\Big[o^t(r) - \sum_{k\in M}\phi_k^t(M,r,O^m)\Big]. \end{aligned} \tag{10}$$

Remark 4: Clearly, based on Definition 4, the N-WAMO fits MEY, but it violates MESP and MCO.

Similar to above sections, one would like to axiomatize the N-WAMO by means of reduced condition property. Similar to relative results for the ANO, $(H, r_H, o^m_{H,\Gamma})$ does not exist if $\sum_{p\in H}\phi_p^t(M,r,O^m) = 0$. One would adopt the **revised reduced condition property** introduced in Section IV to axiomatize N-WAMO. An outcome σ fits **revised reduced condition property (RRCP)** if $(H, r_H, o^m_{H,\sigma}) \in \mathbf{MTC}^*$ for some $(M,r,O^m) \in \mathbf{MTC}$ and for some $H \subseteq M$, it holds that $\sigma_p^t(H, r_H, O^m_{H,\sigma}) = \sigma_p^t(M,r,O^m)$ for all $t \in \mathbb{N}_m$ and for all $p \in H$. An outcome σ fits **N-weighted standard for conditions (NWSC)** if $\sigma(M,r,O^m) = \phi^s(M,r,O^m)$ for all $(M,r,O^m) \in \mathbf{MTC}$ with $|M| \leq 2$. Similar to the proofs of Lemma 3 and Theorem 1, one would like to propose the analogies of Lemma 3 and Theorem 1.

Theorem 6:

- The N-WAMO Γ fits RRCP on $\mathbf{MTC}^*$.
- On $\mathbf{MTC}^*$, the N-WAMO Γ is the only outcome fitting NWSC and RRCP.

Based on the following instances, one would like to show that each of the properties appeared in above axiomatizations is logically independent of the rest of properties.

Example 15: For all $(M,r,O^m) \in \mathbf{MTC}^*$, for all $t \in \mathbb{N}_m$, for all weight mappings s and ξ and for all $p \in M$, one define the outcome σ as $\sigma_p^t(M,r,O^m) = 0$. σ fits RRCP, but it violates NWSC.

Example 16: For all $(M,r,O^m) \in \mathbf{MTC}^*$, for all $t \in \mathbb{N}_m$ and for all $p \in M$, one define the outcome σ as

$$\sigma_p^t(M,r,O^m) = \begin{cases} \Gamma_p^t(M,r,O^m) & \text{if } |M| \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

σ fits NWSC, but it violates RRCP.

D. Numerical instances

Here one would like to provide numerical instances and relative comparisons follows. Let $(M,r,O^m) \in \mathbf{MTC}$ with $M = \{p,q\}$, $m = 1$ and $r = (2,1)$, i.e, $(M,r,O^m) = (\{p,q\},(2,1),o^1)$. Let $s(p) = 2$, $s(q) = 3$, $\xi(2_p) = 1$, $\xi(1_p) = 5$ and $\xi(1_q) = 2$, where j_i means rank j of operator i, $i \in \{p,q\}$. For convenience, one would denote that $(M,r,o^1) = (M,r,o)$, $\phi^1 = \phi$, $\overline{\phi^1} = \overline{\phi}$, $\overline{\Delta^1} = \overline{\Delta}$, $\phi^{s,1}(M,r,o) = \phi^s(M,r,o)$, $\tau^{\xi,1}(M,r,o) = \tau^{\xi}(M,r,o)$, $\phi^{\xi,1}(M,r,o) = \phi^{\xi}(M,r,o)$, $\Theta^{s,1}(M,r,o) = \Theta^s(M,r,o)$ and $\Phi^{\xi,1}(M,r,o) = \Phi^{\xi}(M,r,o)$. Further, let $o(2,1) = 6$, $o(2,0) = 5$, $o(1,1) = 3$, $o(0,1) = 4$, $o(1,0) = 8$ and $o(0,0) = 0$. By Definitions 1, 2 and 3,

$$\begin{aligned}
\phi_p(M,r,o) &= 3, & \phi_q(M,r,o) &= 1,\\
\overline{\phi_p}(M,r,o) &= 4, & \overline{\phi_q}(M,r,o) &= 2,\\
\overline{\Delta_p}(M,r,o) &= \tfrac{9}{2}, & \overline{\Delta_q}(M,r,o) &= \tfrac{3}{2},\\
\phi_p^s(M,r,o) &= \tfrac{19}{5}, & \phi_q^s(M,r,o) &= \tfrac{11}{5},\\
\tau_p^{\xi}(M,r,o) &= 10, & \tau_q^{\xi}(M,r,o) &= 2,\\
\phi_p^{\xi}(M,r,o) &= 7, & \phi_q^{\xi}(M,r,o) &= -1\\
\Theta_p^s(M,r,o) &= 4, & \Theta_q^s(M,r,o) &= 2,\\
\Phi_p^{\xi}(M,r,o) &= 5, & \Phi_q^{\xi}(M,r,o) &= 1,\\
\Gamma_p(M,r,o) &= \tfrac{9}{2}, & \Gamma_q(M,r,o) &= \tfrac{3}{2}.
\end{aligned}$$

Based on above numerical instances, relative comparisons could be generated from the values determined by these outcomes.

VI. Conclusions

1) Differing from existing researches under multi-choice TU conditions, some results of this study are presented as follows.
   - Based on relative considerations of multicriteria circumstances and multi-choice behavior, the notion of multicriteria multi-choice TU conditions has been applied throughout this study.
   - By simultaneously applying the maximal aggregate-marginal dedications under multicriteria circumstances and multi-choice behavior, the AMO, the ANO and relative axiomatic results are proposed.
   - To distinguish the differences among the operators and its effect ranks respectively, several weighted generalizations of the AMO, the ANO and relative axiomatic results are proposed by applying .
   - The weighted outcomes introduced in Sections V-A and V-B are defined by respectively applying weight mappings for operators and weight mappings for ranks. However, these two weight mappings might be artificial under some real-world situations. It is rational that the weights could be replaced to be marginal dedications. Therefore, a specific generalization of the AMO and relative

axiomatic results are proposed to distinguish the differences among the operators and its effect ranks simultaneously.
- The outcomes and relative axiomatic results proposed in this study do not appear in existing researches.

2) Based on the main results of this study, two reasonable motivations could be considered as follows.
   - Whether some traditional outcomes could be generalized by simultaneously applying the maximal aggregate-marginal dedications and weights under multicriteria circumstances and multi-choice behavior.
   - Whether weight mappings could be considered by naturally applying different notions under multicriteria circumstances and multi-choice behavior.

These are left to the researchers.

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