Optimal Eight Order Derivative-Free Family of Iterative Methods for Solving Nonlinear Equations

G Thangkhenpau and Sunil Panday

Abstract—**In this paper, we have developed a new derivative-free family of iterative methods of optimal order for finding the roots of nonlinear equations. The proposed family of methods has order eight and executes four function evaluations per iteration. In addition, it has the efficiency index of $8^{1/4} \approx 1.682$ and supports the Kung-Traub's conjecture. The theoretical convergence properties of new proposed family are studied in detail using the main theorem. Numerical experiments on some nonlinear functions are presented to demonstrate the effectiveness of the family of methods. Also, applications on some real world problems are included so as to validate its real-life applicability. Finally, the family of methods is found to be more efficient as compared to some standard iterative methods of similar nature.**

Index Terms—**Kung-Traub's conjecture, Efficiency index, Simple roots, Weight function, Engineering applications.**

I. Introduction

THE focus on finding the exact solution of nonlinear equations in various problems appearing in diverse fields of science and engineering has always been of much interest due to its vast applications. Determining solutions of such equations by analytical methods are scarce and almost non-existent. Numerical methods are the most widely used technique wherein approximate solutions are obtained using an iterative process. One such method for extracting the root of the nonlinear equation $\psi(s) = 0$, where $\psi : D \subseteq \mathbb{R} \to \mathbb{R}$ is a real function defined on D, is the widely known Newton method (NM) [1] which is given by

$$s_{n+1} = s_n - \frac{\psi(s_n)}{\psi'(s_n)}, n = 0, 1, 2, ... \tag{1}$$

If the first derivative $\psi'(s_n)$ in (1) is approximated as

$$\psi'(s_n) \approx \frac{\psi(s_n + \psi(s_n)) - \psi(s_n)}{\psi(s_n)},$$

then equation (1) becomes

$$s_{n+1} = s_n - \frac{\psi(s_n)^2}{\psi(s_n + \psi(s_n)) - \psi(s_n)} \tag{2}$$

This well-known equation (2), known as Steffensen method [2], is a tough competitor to (1). Both methods evaluate the functions twice at each iteration. They are both quadratically convergent and optimal as per Kung-Traub's conjecture [3] which states that any iterative method with k function evaluations per iteration is optimal if the order of convergence is 2^{k-1}. To determine the effectiveness of an iterative method, Ostrowski in [4] introduced the efficiency index $(EI) = p^{1/k}$, where k is the number of evaluations of function at each iteration and p is the order of convergence. Both methods (1) and (2) have the same efficiency index $2^{1/2} \approx 1.414$ for $k = 2$. But unlike Newton method, Steffensen method does not require any evaluation of the derivatives and is derivative-free. However, the application of one-point iterative methods is limited due to their low order of convergence. For improving the convergence order, many new and improved multipoint iterative methods based on Newton (1) and Steffensen (2) methods have been developed over the years with higher efficiency as compared to one-point iterative methods (see [5]- [12]). The weight function technique is one of the many different approaches employed in these multipoint methods to obtain the desired efficiency with higher order of convergence.

The authors in [13], [14] and [15] have proposed three-point optimal families of methods using the weight function technique to attain the optimal convergence order. These methods require three functions and one first derivative evaluations per iteration. However, the evaluation of the first derivative is sometimes complicated and also time consuming for some problems. To solve this complication, many new methods have been developed which are free from derivatives. Taher Lotfi and Elahe Tavakoli in [16], Xiaofeng Wang and Tie Zhang in [17] and Fazlollah Soleymani in [18] proposed new classes of eight order optimal methods which are completely derivative-free. These methods use the weight function technique to attain the desired convergence order.

In this paper, we develop a new three-step optimal derivative-free family of iterative methods having efficiency index $8^{1/4} \approx 1.682$ for determining simple roots of nonlinear equations. We employ the weight function approach to obtain the high efficiency and higher convergence order. The remaining section of the manuscript is arranged as follows. In section II development of the new optimal three-step optimal derivative-free family of methods using the weight function approach is discussed. The theoretical convergence analysis of new family are fully investigated in this section using the main theorem that proves the convergence order. Section III deals with the numerical experiments of the presented family on some smooth and non-smooth functions to illustrate its effectiveness. In section IV we study the applicability of the presented family of methods on some real world problems. Finally, section V presents the concluding remarks.

II. Development of Methods with Analysis of Convergence

A three-step seventh order with derivative method is developed in [6].

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$$
\begin{aligned}
y_n &= s_n - \frac{\psi(s_n)}{\psi'(s_n)} \\
z_n &= y_n - \frac{\psi(s_n)\psi(y_n)}{[\psi(s_n) - 2\psi(y_n)]\,\psi'(s_n)} \\
s_{n+1} &= z_n - \frac{\psi(s_n)\psi(z_n)}{[\psi(s_n) - 2\psi(y_n)]\,\psi'(s_n)}\Big[1 + \frac{\psi(z_n)}{\psi(s_n)} + \frac{\psi(z_n)}{\psi(y_n)} + \\
&\quad \frac{\psi(y_n)^2}{\psi(s_n)^2} + \frac{\psi(z_n)^2}{\psi(y_n)^2}\Big]
\end{aligned} \tag{3}
$$

The method given by (3) uses four function evaluations per iteration to achieve seventh order. Also, the calculation of derivative is cumbersome for most of the nonlinear equations. So, we will use weight function and approximation of derivative in terms of previously known function values to develop a new optimal derivative-free eight order method. We first approximate $\psi'(s_n)$ in (3) as follows

$$\psi'(s_n) \approx \frac{\psi(w_n) - \psi(s_n)}{w_n - s_n},$$

where $w_n = s_n + \gamma\psi(s_n)^3$, $\gamma \in \mathbb{R} - \{0\}$ ([19], [20]) and then use the weight function technique so as to attain the desired optimal eight order. Now, we propose the following new family of iterative methods:

$$
\begin{aligned}
y_n &= s_n - \frac{\gamma\psi(s_n)^4}{\psi(w_n) - \psi(s_n)},\ w_n = s_n + \gamma\psi(s_n)^3 \\
z_n &= y_n - T(t_1)\frac{\psi(s_n)\psi(y_n)(w_n - s_n)}{(\psi(s_n) - 2\psi(y_n))(\psi(w_n) - \psi(s_n))} \\
s_{n+1} &= z_n - [U(t_1) + V(t_2)]\,\frac{\psi(s_n)\psi(z_n)(w_n - s_n)}{(\psi(s_n) - 2\psi(y_n))(\psi(w_n) - \psi(s_n))}(A_n)
\end{aligned} \tag{4}
$$

where $A_n = 1 + \frac{\psi(z_n)}{\psi(s_n)} + \frac{\psi(z_n)}{\psi(y_n)} + \frac{\psi(y_n)^2}{\psi(s_n)^2} + \frac{\psi(z_n)^2}{\psi(y_n)^2}$,

$t_1 = \frac{\psi(y_n)}{\psi(s_n)},\ t_2 = \frac{\psi(z_n)}{\psi(s_n)}$.

The necessary conditions on weight functions $T(t_1), U(t_1)$ and $V(t_2)$ under which the proposed family of methods (4) has eighth-order convergence are shown in the following theorem.

Theorem: Let $s = \alpha \in D$ be a root of a sufficiently differentiable real function $\psi : D \subseteq \mathbb{R} \to \mathbb{R}$ in an open interval D. If an initial guess s_0 is close to the root α, then the family of iterative methods defined by equation (4) has eighth-order convergence for any $\gamma \in \mathbb{R}-\{0\}$ when it satisfies

$$
\begin{aligned}
&T(0) = 1;\ T'(0) = 0;\ V(0) = 1 - U(0);\ U'(0) = 0; \\
&U''(0) = T''(0); V'(0) = 1; U^{(3)}(0) = 12 + 6T''(0) + T^{(3)}(0); \\
&|T''(0)| < \infty; |T^{(3)}(0)| < \infty; |T^{(4)}(0)| < \infty; |U^{(4)}(0)| < \infty.
\end{aligned} \tag{5}
$$

and the family (4) has the error equation given by

$$
\begin{aligned}
\varepsilon_{n+1} =& \frac{1}{48d_1^7}d_2^2\Big(2d_1d_3 + d_2^2(-2 + T''(0)\Big)\Big(- 24\gamma d_1^5 d_2 + \\
& 120 d_1 d_2 d_3 - 24 d_1^2 d_4 + d_2^3\big(- 192 + 12T''(0) - 8T^{(3)}(0) \\
& - T^4(0) + U^{(4)}(0)\big)\Big)\Big)\varepsilon_n^8 + O\left(\varepsilon_n\right)^9
\end{aligned} \tag{6}
$$

where $d_j = \frac{\psi^{(j)}(\alpha)}{j!}, j = 1, 2, 3, ...,$ and $\varepsilon_n = s_n - \alpha$ is the error at n^{th} approximation.

Proof: Let $\varepsilon_n = s_n - \alpha$ be the error in n^{th} approximation. Then, using Taylor's series expansion near $s = \alpha$, we write

$$
\begin{aligned}
\psi(s_n) =& d_1\varepsilon_n + d_2\varepsilon_n^2 + d_3\varepsilon_n^3 + d_4\varepsilon_n^4 + d_5\varepsilon_n^5 + d_6\varepsilon_n^6 + \\
& d_7\varepsilon_n^7 + d_8\varepsilon_n^8 + O(\varepsilon_n)^9
\end{aligned} \tag{7}
$$

where $d_j = \frac{\psi^{(j)}(\alpha)}{j!}, j = 1, 2, 3, ...$

By using $w_n = s_n + \gamma\psi(s_n)^3$, the Taylor's series expansion of $\psi(w_n)$ gives

$$
\begin{aligned}
\psi(w_n) =& d_1\varepsilon_n + d_2\varepsilon_n^2 + (\gamma d_1^4 + d_3)\varepsilon_n^3 + \\
& (5\gamma d_1^3 d_2 + d_4)\varepsilon_n^4 + ... + O(\varepsilon_n)^9
\end{aligned} \tag{8}
$$

Then, using (7) and (8), we have

$$
\begin{aligned}
y_n - \alpha =& \frac{d_2}{d_1}\varepsilon_n^2 + 2\frac{(-d_2^2 + d_1d_3)}{d_1^2}\varepsilon_n^3 + \Big(\frac{d_2}{d_1^3}(\gamma d_1^5 + 4d_2^2 \\
& - 7d_1d_3) + \frac{3d_4}{d_1}\Big)\varepsilon_n^4 + ... + O(\varepsilon_n)^9
\end{aligned} \tag{9}
$$

The expansion of $\psi(y_n)$ using (9) gives

$$
\begin{aligned}
\psi(y_n) =& d_2\varepsilon_n^2 + \Big(-\frac{2d_2^2}{d_1} + 2d_3\Big)\varepsilon_n^3 + \Big(\frac{d_2}{d_1^2}(\gamma d_1^5 + 5d_2^2 - 7d_1d_3) \\
& + 3d_4\Big)\varepsilon_n^4 + ... + O(\varepsilon_n)^9
\end{aligned} \tag{10}
$$

Using (7), (8), (9) and (10), we have

$$z_n - \alpha = \frac{K_1}{d_1}\varepsilon_n^2 + \frac{K_2}{d_1^2}\varepsilon_n^3 + \frac{K_3}{2d_1^3}\varepsilon_n^4 + ... + O(\varepsilon_n)^9 \tag{11}$$

where

$$
\begin{aligned}
K_1 =& -\big(- 1 + T(0)\big)d_2 \\
K_2 =& -2\big(- 1 + T(0)\big)d_1d_3 + d_2^2\big(- 2 + 2T(0) - T'(0)\big) \\
K_3 =& -\Big(2\gamma\big(- 1 + T(0)\big)d_1^5 d_2 + 6\big(- 1 + T(0)\big)d_1^2 d_4 \\
& + 2d_1d_2d_3\big(7 - 6T(0) + 4T'(0)\big) + d_2^3\big(- 8 + \\
& 6T(0) - 10T'(0) + T''(0)\big)\Big)
\end{aligned}
$$

Using the conditions $T(0) = 1, T'(0) = 0$ and substituting in the above equation (11), we get

$$\psi(z_n) = \frac{L_1}{2d_1^2}\varepsilon_n^4 + \frac{L_2}{6d_1^3}\varepsilon_n^5 + \frac{L_3}{24d_1^4}\varepsilon_n^6 + ... + O(\varepsilon_n)^9 \tag{12}$$

where

$$
\begin{aligned}
L_1 =& -\big(2d_1d_2d_3 + d_2^3(-2 + T''(0))\big) \\
L_2 =& -6\gamma d_1^5 d_2^2 - 12d_1^2\big(d_3^2 + d_2d_4\big) + 6d_1d_2^2d_3\big(8- \\
& 3T''(0)\big) + d_2^4\big(- 24 + 24T''(0) - T^{(3)}(0)\big) \\
L_3 =& -\Big(144\gamma d_1^6 d_2 d_3 + 24d_1^3\big(7d_3d_4 + 3d_2d_5\big) + \\
& 12\gamma d_1^5 d_2^3\big(- 2 + 3T''(0)\big) + 36d_1^2 d_2\big(4d_3^2(-3 + \\
& T''(0)) + d_2d_4(-8 + 3T''(0))\big) + 8d_1d_2^3d_3(90- \\
& 87T''(0) + 4T^{(3)}(0)) + d_2^5\big(- 240 + 480T''(0) \\
& - 44T^{(3)}(0) + T^{(4)}(0)\big)\Big)
\end{aligned}
$$

Now, substituting the values of equations (7), (8), (10), (11) and (12) in the final step of equation (4), the error equation is obtained as

$$\varepsilon_{n+1} = \frac{M_1}{2d_1^3}\varepsilon_n^4 + M_2\varepsilon_n^5 + M_3\varepsilon_n^6 + ... + O(\varepsilon_n)^9 \tag{13}$$

where

$$
\begin{aligned}
M_1 =& \big(- 1 + U(0) + V(0)\big)d_2\big(2d_1d_3 + d_2^2(-2 + T''(0))\big) \\
M_2 =& \gamma\big(- 1 + U(0) + V(0)\big)d_1d_2^2 + \frac{2}{d_1^2}\big(- 1 + U(0) + \\
& V(0)\big)\big(d_3^2 + d_2d_4\big) + \frac{d_2^2 d_3}{d_1^3}\big(8 - 8U(0) - 8V(0) + \\
& U'(0) + 3\big(- 1 + U(0) + V(0)\big)T''(0)\big) +
\end{aligned}
$$

$$\frac{d_2^4}{6d_1^4}\big(24\big(-1+U(0)+V(0)\big)-6U'(0)+3\big(-8\big(-1+U(0)+V(0)\big)+U'(0)\big)T''(0)+\big(-1+U(0)+V(0)\big)T^{(3)}(0)\big)$$

$$\begin{aligned}M_3 =&6\gamma\big(-1+U(0)+V(0)\big)d_1d_2d_3+\frac{1}{d_1^2}\big(-1+U(0)+V(0)\big)\big(7d_3d_4+3d_2d_5\big)+\frac{\gamma d_2^3}{2}\big(-2\big(-1+U(0)+V(0)-U'(0)\big)+3\big(-1+U(0)+V(0)\big)T''(0)\big)+\\&\frac{d_2}{2d_1^3}\big(4d_3^2\big(9-9U(0)-9V(0)+2U'(0)+3\big(-1+U(0)+V(0)\big)T''(0)\big)+d_2d_4\big(4\big(-6\big(-1+U(0)+V(0)\big)+U'(0)\big)+9\big(-1+U(0)+V(0)\big)T''(0)\big)\big)+\\&\frac{d_2^3d_3}{6d_1^4}\big(-3\big(60-60U(0)-60V(0)+26U'(0)-58T''(0)+\big(59\big(U(0)+V(0)\big)-8U'(0)\big)T''(0)-U''(0)\big)+8\big(-1+U(0)+V(0)\big)T^{(3)}\big)+\frac{d_2^5}{24d_1^5}\big(6\big(-40\big(-1+U(0)+V(0)\big)+28U'(0)+2\big(-40+41U(0)+41V(0)-11U'(0)\big)T''(0)-\big(U(0)+V(0)\big)T''(0)^2+\big(-2+T''(0)\big)U''(0)\big)+4\big(-11\big(-1+U(0)+V(0)\big)+U'(0)\big)T^{(3)}(0)+\big(-1+U(0)+V(0)\big)T^{(4)}(0)\big)\end{aligned}$$

Finally, putting the conditions $V(0)=1-U(0), U'(0)=0, U''(0)=T''(0), V'(0)=1, U^{(3)}(0)=12+6T''(0)+T^{(3)}(0)$ in the above equation (13), the error equation becomes

$$\varepsilon_{n+1}=\frac{1}{48d_1^7}d_2^2\Big(2d_1d_3+d_2^2(-2+T''(0)\big)\Big)\Big(-24\gamma d_1^5d_2+120d_1d_2d_3-24d_1^2d_4+d_2^3\big(-192+12T''(0)-8T^{(3)}(0)-T^4(0)+U^{(4)}(0)\big)\Big)\varepsilon_n^8+O\left(\varepsilon_n\right)^9 \quad (14)$$

which shows that the new family of iterative methods is of order eight. This completes the proof. ■

It is possible that different values for the weight functions $T(t_1)$, $U(t_1)$ and $V(t_2)$ may be chosen which satisfy the conditions (5). Depending on these choices, a number of derivative-free methods of eighth-order for determining simple roots of nonlinear equations can be obtained.

Particular Case: For the three weight functions $T(t_1)$, $U(t_1)$ and $V(t_2)$ satisfying the conditions (5), let us consider the following case:

$$\begin{aligned}&T(t_1)=1+t_1^4\\&U(t_1)=2t_1^3\\&V(t_2)=1+t_2, \text{where } t_1=\tfrac{\psi(y_n)}{\psi(s_n)}, t_2=\tfrac{\psi(z_n)}{\psi(s_n)}\end{aligned} \quad (15)$$

Then, after substituting these values, the new family of methods (4) becomes

$$\begin{aligned}y_n&=s_n-\frac{\gamma\psi(s_n)^4}{\psi(w_n)-\psi(s_n)},\ w_n=s_n+\gamma\psi(s_n)^3\\z_n&=y_n-\left[1+t_1^4\right]\frac{\psi(s_n)\psi(y_n)(w_n-s_n)}{(\psi(s_n)-2\psi(y_n))(\psi(w_n)-\psi(s_n))}\\s_{n+1}&=z_n-\left[2t_1^3+1+t_2\right]\frac{\psi(s_n)\psi(z_n)(w_n-s_n)}{(\psi(s_n)-2\psi(y_n))(\psi(w_n)-\psi(s_n))}(A_n)\end{aligned} \quad (16)$$

where $A_n=1+\frac{\psi(z_n)}{\psi(s_n)}+\frac{\psi(z_n)}{\psi(y_n)}+\frac{\psi(y_n)^2}{\psi(s_n)^2}+\frac{\psi(z_n)^2}{\psi(y_n)^2}$, and the error equation has the following expression

$$\varepsilon_{n+1}=\frac{d_2^2}{d_1^7}\left(d_2^2-d_1d_3\right)\Big(d_2\big(\gamma d_1^5+9d_2^2-5d_1d_3\big)+d_1^2d_4\Big)\varepsilon_n^8+O\left(\varepsilon_n\right)^9 \quad (17)$$

III. Numerical Results

In this section, we analyze the effectiveness and the computational efficiency of newly proposed family of methods and compare with some well-known methods available in literature. We have considered, in particular, the following eighth-order methods for comparing with our proposed family of methods (PFM) given in (16) with $\gamma=1$:

The derivative-free Kung-Traub's method [3] (KTM):

$$\begin{aligned}y_n=&s_n-\frac{\gamma\psi(s_n)^2}{\psi(s_n+\gamma\psi(s_n))-\psi(s_n)}, n=0,1,2,...\\z_n=&y_n-\frac{\psi(y_n)\psi(s_n+\gamma\psi(s_n))}{[\psi(s_n+\gamma\psi(s_n))-\psi(y_n)]\psi[s_n,y_n]}\\s_{n+1}=&z_n-\frac{\psi(y_n)\psi(s_n+\gamma\psi(s_n))(y_n-s_n+\frac{\psi(s_n)}{\psi[s_n,z_n]})}{[\psi(y_n)-\psi(z_n)][\psi(s_n+\gamma\psi(s_n))-\psi(z_n)]}\\&+\frac{\psi(y_n)}{\psi[y_n,z_n]}\end{aligned} \quad (18)$$

where $\gamma=1$ and $\psi[s_n,y_n]=\frac{\psi(s_n)-\psi(y_n)}{s_n-y_n}$.

The derivative-free methods proposed by R. Behl et al. in [9] (RBM):

$$\begin{aligned}y_n&=s_n-\frac{\psi(s_n)}{\psi[w_n,s_n]}, w_n=s_n+\gamma\psi(s_n)\\z_n&=y_n-\frac{\psi(w_n)\psi(y_n)(y_n-s_n)}{[\psi(w_n)-\psi(y_n)][\psi(y_n)-\psi(s_n)]}\\s_{n+1}&=z_n-\frac{\psi(z_n)(w_n-s_n)(w_n-y_n)(s_n-y_n)}{\psi[y_n,z_n](w_n-s_n)(w_n-z_n)(s_n-z_n)-D_n(y_n-z_n)}\end{aligned} \quad (19)$$

where $\gamma=1$, $D_n=\psi[s_n,z_n](w_n-y_n)(w_n-z_n)-\psi[w_n,z_n](s_n-y_n)(s_n-z_n)$.

The efficient Steffensen-like methods by Taher Lotfi and Elahe Tavakoli in [16] with $\gamma=1$ (TEM):

$$\begin{aligned}y_n&=s_n-\frac{\psi(s_n)}{\psi[s_n,w_n]}, w_n=s_n+\gamma\psi(s_n)\\z_n&=y_n-\Big[1+\frac{\psi(y_n)}{\psi(s_n)}\Big]\frac{\psi(y_n)}{\psi[y_n,w_n]}\\s_{n+1}&=z_n-\Big[1+\frac{\psi(y_n)}{\psi(s_n)}+\frac{\psi(z_n)}{\psi(y_n)}+2\frac{\psi(y_n)}{\psi(s_n)}\frac{\psi(z_n)}{\psi(y_n)}+\Big(-1-\frac{1}{1+\gamma\psi[s_n,w_n]}\Big)\Big(\frac{\psi(y_n)}{\psi(s_n)}\Big)^3\Big]\Big[1+\Big(\frac{\psi(z_n)}{\psi(y_n)}\Big)^2+\Big(\frac{\psi(z_n)}{\psi(s_n)}\Big)^2\Big]\frac{\psi(z_n)}{\psi[z_n,w_n]}\end{aligned} \quad (20)$$

And, the optimal eighth-order Steffensen type methods by Wang in [17] (WM):

$$\begin{aligned}y_n&=s_n-\frac{\psi(s_n)}{\psi[s_n,w_n]}, w_n=s_n+\gamma\psi(s_n)\\z_n&=y_n-(1+F_n+F_n^2-2G_n)\frac{\psi(y_n)}{\psi[s_n,w_n]}\\s_{n+1}&=z_n-\Big((1+F_n+F_n^2-2G_n)+\frac{\psi(z_n)}{\psi(y_n)}(1+2F_n)+G_n\Big)\frac{\psi(z_n)}{\psi[s_n,w_n]},\end{aligned} \quad (21)$$

TABLE I
TEST FUNCTIONS WITH THE ROOTS AND INITIAL GUESSES.

Test functions $\psi(s)$	Roots (α)	Initial guesses (s_0)
$\psi_1(s) = \begin{cases} s(s-1), & s \leq 0 \\ -2s(s+1), & s \geq 0 \end{cases}$	$\alpha = 0$	0.5
$\psi_2(s) = \lvert s^2 - 2 \rvert$	$\alpha \approx 1.4142135623730950$	1.3
$\psi_3(s) = \sin s + \cos s + s$	$\alpha \approx -0.45662470456763082$	-0.6
$\psi_4(s) = \log_e s - s^3 + 2\sin s$	$\alpha \approx 1.2979977432803718$	1.4
$\psi_5(s) = \sin^2 s + s$	$\alpha = 0$	0.5
$\psi_6(s) = \sin(2\cos s) - 1 - s^2 + e^{\sin(s^3)}$	$\alpha \approx -0.78489598766121254$	-1

where $\gamma = 1, F_n = \frac{\psi(y_n)}{\psi(s_n)} + \frac{\psi(y_n)}{\psi(z_n)}$ and $G_n = \frac{\psi(y_n)}{\psi(s_n)} \frac{\psi(y_n)}{\psi(z_n)}$.

The efficient derivative-free optimal methods of Soleymani in [18] (SM):

$$
\begin{aligned}
y_n &= s_n - \frac{\psi(s_n)}{\psi[s_n, w_n]}, \ w_n = s_n + \gamma\psi(s_n) \\
z_n &= y_n - \frac{\psi(y_n)}{\psi[s_n, w_n]}(B_n) \\
s_{n+1} &= z_n - \frac{\psi(z_n)}{\psi[s_n, w_n]}(B_n)(C_n)
\end{aligned} \tag{22}
$$

where $\gamma = 1$, $B_n = \frac{1}{1 - \frac{\psi(y_n)}{\psi(s_n)} - \frac{\psi(y_n)}{\psi(w_n)}}$, $C_n = 1 + \frac{1}{1+\psi[s_n,w_n]}\left(\frac{\psi(y_n)}{\psi(s_n)}\right)^2 + \big(1 + \psi[s_n, w_n]\big)\big(2 + \psi[s_n, w_n]\big)\left(\frac{\psi(y_n)}{\psi(w_n)}\right)^3 + \frac{\psi(z_n)}{\psi(y_n)} + \frac{\psi(z_n)}{\psi(s_n)} + \frac{\psi(z_n)}{\psi(w_n)}$.

The derivative-free method by Solaiman et al. in [21] (OSM):

$$
\begin{aligned}
y_n &= s_n - \frac{\psi(s_n)}{\psi[w_n, s_n]}, w_n = s_n + \psi(s_n) \\
z_n &= y_n - \frac{\psi(y_n)}{H(s_n)} \frac{\psi(s_n) + \beta\psi(y_n)}{\psi(s_n) + (\beta - 2)\psi(y_n)} \\
s_{n+1} &= z_n - \frac{\psi(s_n)(E_1 + E_2 + E_3)}{E_1\psi[w_n, s_n] + E_2\psi[y_n, s_n] + E_3\psi[z_n, s_n]},
\end{aligned} \tag{23}
$$

where $\beta = 2$, $H(s_n) = \psi[w_n, s_n] + 2(w_n - s_n)\psi[w_n, s_n, y_n] - \psi[y_n, w_n] + \psi[s_n, y_n]$, $E_1 = \psi(y_n)\psi(z_n)(z_n - y_n)$, $E_2 = \psi(w_n)\psi(z_n)(w_n - z_n)$, $E_3 = \psi(w_n)\psi(y_n)(y_n - w_n)$.

Also, like our proposed family of methods all the above methods are derivative-free, so it is quite reasonable for the choice of these methods for the comparison and they should be easily comparable with our proposed family of methods. Some numerical test functions along with their simple roots (α) and initial guesses are given in Table I, where the first two functions are non-smooth functions and the remaining four functions are smooth functions.

For obtaining better accuracy and to minimize the loss of significant digits, all numerical tests have been executed using 4000 significant digits in the programming software Mathematica 12.2. For analysis of the convergence of each method to their simple roots, the following condition

$$
|s_n - s_{n-1}| + |\psi(s_n)| < 10^{-65} \tag{24}
$$

has been used as the stopping criterion. In Table II to Table X, we have displayed the number of iterations (n) required (NIR) by the methods to satisfy the stopping criterion given in (24), the corresponding absolute residual error for each test function i.e., $|\psi(s_n)|$ and the errors in consecutive iterations, $|s_n - s_{n-1}|$. We have also given the computational order of convergence (COC) of each method satisfying the stopping criterion (24) in Table II to Table X. The COC is represented as (ρ) and is calculated by the following formula [22]:

$$
\text{COC} = \frac{\log|\psi(s_n)/\psi(s_{n-1})|}{\log|\psi(s_{n-1})/\psi(s_{n-2})|} \tag{25}
$$

Also, we provide the CPU time (in seconds) utilized by each method at the last columns of the Tables. Here, we point out that the CPU time is not unique as it largely depends on the computer's specifications. So, to ensure the robustness of each compared method we take the average running time utilized by the CPU when the methods are executed four times on each test function. The CPU time is computed by taking $|\psi(s_n)| \leq 10^{-1500}$ as the stopping criterion using Mathematica 12.2 software on a system running Windows 11 with Intel(R) Core(TM) i5-10210U CPU @ 1.60GHz 2.11 GHz and 8GB of RAM.

From all the numerical results in Table II to Table X, it is easy to conclude that the proposed family of methods (4) is highly competitive and possesses fast convergence towards the root in minimum number of iterations (n) consuming lesser CPU time. The absolute residual error value and error values in consecutive iterations for the new proposed family of methods are also minimal as compared to the other existing methods. And, we can also observe from the numerical test results that the COC (ρ) supports the theoretical convergence order of the new presented family of methods in the test functions.

IV. Applications to Real World Problems

Here, we discuss the applicability of the presented family of methods on two particular real world problems.

A. *Problem on Planck's Radiation Law*

Let us consider the Planck's radiation law problem (for details see [23], [24]) which is used to compute the energy density within an isothermal black body. It can be expressed by the following nonlinear equation:

$$
\psi_7(s) = e^{-s} + \frac{s}{5} - 1 = 0 \tag{26}
$$

The approximate root of (26) is found to be $\alpha \approx 4.9651142317442763$. Using the initial guess $s_0 = 6$, the results satisfying the stopping criterion (24) are displayed in Table VIII.

TABLE II
COMPARISON ON TEST FUNCTION $\psi_1(s)$.

Methods	NIR(n)	$\lvert s_n - s_{n-1}\rvert$	$\lvert\psi_1(s)\rvert$	COC(ρ)	CPU Time
RBM	7	1.3170×10^{-98}	1.1563×10^{-197}	2.0000	0.02543
OSM	7	3.9450×10^{-99}	1.0375×10^{-198}	2.0000	0.03190
SM	7	8.6016×10^{-88}	5.4806×10^{-176}	2.0000	0.02615
TEM	7	1.3037×10^{-101}	8.6930×10^{-204}	2.0000	0.02559
KTM	7	1.1090×10^{-218}	5.8562×10^{-214}	2.0000	0.02166
WM	7	3.6832×10^{-90}	8.4787×10^{-181}	2.0000	0.03275
PFM	4	4.4595×10^{-210}	3.1282×10^{-1675}	8.0000	0.01486

TABLE III
COMPARISON ON TEST FUNCTION $\psi_2(s)$.

Methods	NIR(n)	$\lvert s_n - s_{n-1}\rvert$	$\lvert\psi_2(s)\rvert$	COC(ρ)	CPU Time
RBM	6	5.2366×10^{-104}	9.4900×10^{-827}	8.0000	0.02417
OSM	-	$Divergent$	$Divergent$	-	-
SM	6	4.9562×10^{-173}	1.6939×10^{-1378}	8.0000	0.01985
TEM	-	$Divergent$	$Divergent$	-	-
KTM	7	1.2726×10^{-82}	2.8865×10^{-655}	8.0000	0.02488
WM	-	$Divergent$	$Divergent$	-	-
PFM	5	3.3720×10^{-69}	6.2033×10^{-549}	8.0000	0.01670

TABLE IV
COMPARISON ON TEST FUNCTION $\psi_3(s)$.

Methods	NIR(n)	$\lvert s_n - s_{n-1}\rvert$	$\lvert\psi_3(s)\rvert$	COC(ρ)	CPU Time
RBM	3	4.7850×10^{-91}	2.6469×10^{-726}	8.0000	0.08802
OSM	3	3.8689×10^{-88}	2.9788×10^{-702}	8.0000	0.04416
SM	3	6.9841×10^{-75}	2.6343×10^{-595}	8.0000	0.04998
TEM	3	2.6172×10^{-82}	5.8951×10^{-655}	8.0000	0.04953
KTM	3	3.3195×10^{-84}	2.0131×10^{-670}	8.0000	0.04788
WM	3	5.4874×10^{-74}	8.4137×10^{-588}	8.0000	0.08753
PFM	3	5.8931×10^{-95}	4.4069×10^{-757}	8.0000	0.03733

TABLE V
COMPARISON ON TEST FUNCTION $\psi_4(s)$.

Methods	NIR(n)	$\lvert s_n - s_{n-1}\rvert$	$\lvert\psi_4(s)\rvert$	COC(ρ)	CPU Time
RBM	4	1.0059×10^{-192}	7.2733×10^{-1533}	8.0000	0.07157
OSM	4	3.8201×10^{-186}	1.7408×10^{-1480}	8.0000	0.05424
SM	4	2.9615×10^{-203}	3.0456×10^{-1617}	8.0000	0.06238
TEM	4	3.5765×10^{-239}	4.1929×10^{-1906}	8.0000	0.07267
KTM	4	1.6950×10^{-160}	1.0978×10^{-1274}	8.0000	0.05521
WM	4	8.5014×10^{-154}	7.9220×10^{-1221}	8.0000	0.05605
PFM	3	3.0702×10^{-66}	4.6521×10^{-522}	8.0000	0.03550

TABLE VI
COMPARISON ON TEST FUNCTION $\psi_5(s)$.

Methods	NIR(n)	$\lvert s_n - s_{n-1}\rvert$	$\lvert\psi_5(s)\rvert$	COC(ρ)	CPU Time
RBM	4	6.4695×10^{-200}	1.6367×10^{-1592}	8.0000	0.03185
OSM	4	6.8189×10^{-190}	3.4903×10^{-1512}	8.0000	0.03304
SM	4	2.7981×10^{-147}	5.5121×10^{-1171}	8.0000	0.04165
TEM	4	3.0383×10^{-155}	2.1689×10^{-1234}	8.0000	0.03099
KTM	4	7.1643×10^{-168}	1.0365×10^{-1335}	8.0000	0.03867
WM	4	2.4886×10^{-130}	1.9005×10^{-1034}	8.0000	0.03757
PFM	4	4.0261×10^{-215}	6.6739×10^{-1715}	8.0000	0.02396

TABLE VII
COMPARISON ON TEST FUNCTION $\psi_6(s)$.

Methods	NIR(n)	$\lvert s_n - s_{n-1}\rvert$	$\lvert\psi_6(s)\rvert$	COC(ρ)	CPU Time
RBM	4	1.4022×10^{-77}	4.8779×10^{-613}	8.0000	0.11714
OSM	5	3.5559×10^{-76}	3.0626×10^{-601}	8.0000	0.13993
SM	6	4.9016×10^{-375}	6.9908×10^{-2992}	8.0000	0.13222
TEM	-	$Divergent$	$Divergent$	-	-
KTM	4	5.3033×10^{-105}	9.8504×10^{-832}	8.0000	0.12037
WM	-	$Divergent$	$Divergent$	-	-
PFM	4	1.4231×10^{-389}	5.3757×10^{-3110}	8.0000	0.06394

TABLE VIII
COMPARISON RESULTS OF PLANCK'S RADIATION LAW PROBLEM, $\psi_7(s)$.

Methods	NIR(n)	$\lvert s_n - s_{n-1}\rvert$	$\lvert\psi_7(s)\rvert$	COC(ρ)	CPU Time
RBM	3	7.4189×10^{-82}	1.2733×10^{-658}	8.0000	0.03659
OSM	3	1.9066×10^{-81}	3.0027×10^{-655}	8.0000	0.03183
SM	3	3.3690×10^{-78}	5.8295×10^{-629}	8.0000	0.03673
TEM	3	4.1321×10^{-81}	1.6084×10^{-652}	8.0000	0.03852
KTM	3	2.7843×10^{-81}	6.4078×10^{-654}	8.0000	0.04053
WM	3	1.5405×10^{-77}	1.5695×10^{-623}	8.0000	0.04283
PFM	3	3.2923×10^{-83}	1.2348×10^{-669}	8.0000	0.02484

TABLE IX
COMPARISON RESULTS OF VAN DER WAALS EQUATION PROBLEM, $\psi_8(s)$.

Methods	NIR(n)	$\lvert s_n - s_{n-1}\rvert$	$\lvert\psi_8(s)\rvert$	COC(ρ)	CPU Time
RBM	5	3.2148×10^{-203}	1.2428×10^{-1615}	8.0000	0.02279
OSM	5	8.6739×10^{-240}	2.2938×10^{-1908}	8.0000	0.02610
SM	5	1.4332×10^{-171}	3.2612×10^{-1362}	8.0000	0.02349
TEM	5	5.0912×10^{-132}	1.2049×10^{-1045}	8.0000	0.02185
KTM	5	3.5440×10^{-167}	5.5510×10^{-1327}	8.0000	0.02381
WM	5	1.3079×10^{-84}	1.9165×10^{-665}	8.0000	0.03215
PFM	5	2.2341×10^{-260}	8.7632×10^{-2073}	8.0000	0.01757

TABLE X
COMPARISON RESULTS OF THE MULTIPACTOR EFFECT PROBLEM, $\psi_9(s)$.

Methods	NIR(n)	$\lvert s_n - s_{n-1}\rvert$	$\lvert\psi_9(s)\rvert$	COC(ρ)	CPU Time
RBM	4	2.5333×10^{-492}	2.1170×10^{-3936}	8.0000	0.03192
OSM	4	1.7195×10^{-453}	1.0336×10^{-2659}	5.5651	0.03241
SM	4	3.4696×10^{-433}	1.0355×10^{-3462}	8.0000	0.03533
TEM	4	5.4780×10^{-410}	6.0418×10^{-3277}	8.0000	0.03267
KTM	4	5.7853×10^{-437}	5.6819×10^{-3493}	8.0000	0.03542
WM	4	2.2686×10^{-362}	4.9114×10^{-2895}	8.0000	0.03314
PFM	3	1.8501×10^{-66}	1.3729×10^{-529}	8.0000	0.02648

B. Van Der Waals Equation

Let us consider the well-known Van der Waals equation from chemical engineering problem (for details see [25], [26]). The equation has been used to examine the behaviour of real and ideal gases . It is represented by the following expression:

$$\psi_8(s) = 0.986s^3 - 5.181s^2 + 9.067s - 5.289 \tag{27}$$

where the variable s represents the volume of the gas to be determined. It has only one feasible positive real root $\alpha \approx 1.9298462428478622$. We use $s_0 = 2.4$ as the initial guess and the results satisfying the stopping criterion (24) are displayed in Table IX.

C. Study of The Multipactor Effect

Let us consider the analysis of the multipactor effect (for details see [27], [28]). The trajectory of an electron in the air gap between two parallel plates is given by the following nonlinear function:

$$\psi_9(s) = s - \frac{1}{2}\cos(s) + \frac{\pi}{4} \tag{28}$$

The nonlinear equation $\psi_9(s) = 0$ has a simple root at $\alpha \approx -0.30909327154179495$. We take $s_0 = 0$ as the initial guess and the results satisfying the stopping criterion (24) are displayed in Table X.

From Table VIII to Table X, we can observe that the numerical applications of the proposed family of methods on some real world problems have illustrated the efficiency and applicability of the presented family of methods. Moreover, our presented family of methods is found to have better performance as compared to the existing methods in comparison.

V. Concluding Remarks

We have presented in this paper a new optimal order derivative-free family of iterative methods for determining the solutions of nonlinear equations. The procedure uses the Steffensen-like approach and the weight function technique to achieve derivative-free optimal order convergence method. Analysis of the numerical test results have shown the robust performance of our family of methods. It is found to be more efficient as compared to the other well-known methods in terms of minimal residual errors, errors in consecutive iterations and smaller number of iterations required with minimal CPU time consumed for convergence to the roots. Also, the study on some real world problems has demonstrated the applicability and validity of the presented family of methods. Moreover, we can conclude that the overall performance of the presented family of methods is really good with fast convergence speed and will be a good alternative for solving nonlinear equations.

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