2-frugal Coloring of Planar Graphs with Maximum Degree at Most 6

Xiaoyuan Lou, Lei Sun* and Wei Zheng

***Abstract*—For a color c in a proper coloring of a graph G, let $n_c(v)$ denote the times of the color c be used in the neighbors $N(v)$ of the vertex v. A t-frugal k-coloring of G is a proper coloring $\varphi : V(G) \to \{1, 2, \ldots, k\}$ such that every vertex $v \in V(G)$ has $n_c(v) \leq t$ for each color c. The minimum number of colors required to ensure that a graph G has a t-frugal coloring is called t-frugal chromatic number of G, denoted by $\Phi_t(G)$. This paper proved that if G is a planar graph with $\Delta(G) \leq 6$, then $\Phi_2(G) \leq 17$. We also gave a linear time algorithm for producing a 2-frugal 17-coloring of a planar graph G without adjacent triangles.**

***Index Terms*—2-frugal coloring, maximum degree, discharging, algorithm.**

I. INTRODUCTION

IN this paper, the notations we adopt are standard and all graphs we use are simple, undirected and finite. A graph is planar if it can be drawn or embedded in the plane such that every two edges only intersect at their endpoints. A plane graph is a planar graph with a fixed planar embedding in the plane[12]. Considering a plane graph G, let $V(G)$ be the set of vertices of G, $E(G)$ be the set of edges of G and $F(G)$ be the set of faces of G. In addition, let $d(f)$ be the degree of face f, which is the number of edges on the boundary of f. We denote the minimum and maximum degree of vertices in G by $\delta(G)$ and $\Delta(G)$. For a color c in a proper coloring of a graph G, let $n_c(v)$ denote the times of the color c be used in the neighbors $N(v)$ of the vertex v. A t-frugal k-coloring of G is a proper coloring $\varphi : V(G) \to \{1, 2, \ldots, k\}$ such that every vertex $v \in V(G)$ has $n_c(v) \leq t$ for each color c. The minimum number of colors required to ensure that a graph G has a t-frugal coloring is called t-frugal chromatic number of G, denoted by $\Phi_t(G)$.

Hind, Molly and Reed [7] firstly introduced the concept of frugal coloring in 1997 to deeply study total chromatic number of graphs. They proved that every graph G with quite large maximum degree has a $\lceil \log^8 \Delta \rceil$-frugal $(\Delta + 1)$-coloring. In fact, t-frugal coloring has an application background. In the channel allocation problem, when an over-the-air-station covers a very large area, it is necessary to use the base station to enhance the signal. The prerequisite for the proper operation of each base station is that the adjacent base stations use different channels. Each channel base station is regarded as a vertex. If the two channel base stations interfere with each other, we give an edge between them. For each base station, it is not allowed to be interfered more than t times by a same channel. Then, what is the minimum number of channels to guarantee the normal operation of all base stations? Obviously, it can be converted into the t-frugal coloring problem. Amini, Esperet, and van den Heuvel [10] gave some results about t-frugal coloring of planar and outplanar graphs. They showed that $\Phi_t(G) \leq \lfloor \frac{2\Delta+19}{t} \rfloor + 6$, if G is a planar graph with $\Delta(G) \geq 12$. For a outerplanar graph with $\Delta(G) \geq 3$, they proved that $\Phi_t(G) \leq \lfloor \frac{\Delta-1}{t} \rfloor + 3$.

When $t = 1$, 1-frugal coloring is equivalent to 2-distance coloring. About 2-distance coloring, in 1977, Wegner [13] proposed a conjecture: if G is a planar graph, then $\chi_2(G) \leq 7$ if $\Delta \leq 3$, $\chi_2(G) \leq \Delta + 5$ if $4 \leq \Delta \leq 7$ and $\chi_2(G) \leq \lfloor \frac{3\Delta}{2} \rfloor + 1$ if $\Delta \geq 8$, where $\chi_2(G)$ is the minimum k so that G has a 2-distance k-coloring. Except the case when $\Delta \leq 3$, the conjecture is still open now. After this famous conjecture was proposed, many scholars conducted a lot of research on 2-distance coloring. With some special limitations on planar graph, Ming Chen, Lianying Miao, Shan Zhou[9] proved that if G is a planar graph with $\Delta \leq 5$, then $\chi_2(G) \leq 19$. There are many other research results, which readers can refer to[3], [4], [5], [6], [11], [14].

In this paper, we proved that if G is a planar graph with $\Delta(G) \leq 6$, then $\Phi_2(G) \leq 17$. Besides, inspired by a linear time algorithm given by Baogang Xu and Haihui Zhang[1], we also gave a linear time algorithm for producing a 2-frugal 17-coloring of a planar graph G without adjacent triangles in the final section.

Theorem 1.1 If G is a planar graph with $\Delta(G) \leq 6$, then $\Phi_2(G) \leq 17$.

We will prove Theorem 1.1 by the method of contradiction. Firstly, suppose the theorem is not true. We pick a counterexample G with minimum $|V(G)| + |E(G)|$, which has no 2-frugal 17-coloring. Then, we will find a contradiction about G. Obviously, G is a connected planar graph by the minimality.

II. THEOREM PROVING

A. Forbidden configurations

Before giving the configurations, we introduce some notations. Let ϕ' be a legal partial coloring of G. For a vertex v in G, $f(v)$ is the number of the selectable colors. Let $c(G) = \{1, 2, \ldots, 17\}$ be the color set of G and $c(v)$ be the color of vertex v. For a 2-frugal coloring, the forbidden colors of v are the colors of $N(v)$ and the colors of $N_2(v)$ which occur 2 times, where $N_2(v)$ is the set of vertices at a distance of 2 from v in G. Hence, $f(v) \geq 17 - (|N(v)| + \lfloor \frac{|N_2(v)|}{2} \rfloor)$.

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An i-vertex (i^--vertex) is a vertex with degree i (at most i). An i-face (i^+-face) is a face with degree i (at least i). A 3-face is (i,j,k)-face if it is incident with a i-vertex, a j-vertex and a k-vertex. If $v_1, v_2, \ldots, v_n$ are all the vertices on the boundary of a face f in the order of given orientation, we denote the face f by $v_1 v_2 \ldots v_n$.

Let G be the counterexample with minimum $|V(G)| + |E(G)|$ of Theorem 1.1.

Lemma 2.1 G has no cut edges.

Proof: Assume that uv is a cut edge of G. The worst case is that the vertex u and the vertex v are both 6-vertex. Let $N(u) = \{v, u_1, u_2, u_3, u_4, u_5\}$ and $N(v) = \{u, v_1, v_2, v_3, v_4, v_5\}$. Now, delete the edge uv and get two components G_1 and G_2 which have a 2-frugal 17-coloring by the minimality, denoted by the ϕ_1 and ϕ_2 respecticely. Then, extend ϕ_1 and ϕ_2 to G. Firstly, we use the total permutation of color on ϕ_1 such that $\{u, u_1, u_2, u_3, u_4, u_5\}$ is colored by the color set$\{1,2,3,4,5,6\}$. Secondly, use the total permutation of color on ϕ_2 such that $\{v, v_1, v_2, v_3, v_4, v_5\}$ in ϕ_2 is colored by the color set $\{17,16,15,14,13,12\}$. Hence, G has a 2-frugal 17-coloring, a contradiction. □

Lemma 2.2 $\delta(G) \geq 5$.

Proof: Assume that u is a 4-vertex, $N(u) = \{u_1, u_2, u_3, u_4\}$ and all of the vertices u_1, u_2, u_3, u_4 are 6-vertex, which is the worst case. Now, delete the vertex u and get a graph G', which has a 2-frugal 17-coloring ϕ' by the minimality. Then, extend ϕ' to G. Except the vertex u, let the other vertices be colored the same as ϕ'. By the definition of 2-frugal coloring and the structure of G', we have

$$f(v) \geq 17 - (|N(v)| + \lfloor \tfrac{|N_2(v)|}{2} \rfloor)$$
$$= 17 - (4 + \lfloor \tfrac{4+4+4+4+4}{2} \rfloor) = 3 \geq 1,$$

a contradiction. □

Lemma 2.3 In the graph G, no 5-vertex is incident with more than three $(5,5,5)$-faces. (see Fig 1)

Proof: Suppose there is a 5-vertex v with $N(v) = \{v_1, v_2, v_3, v_4, v_5\}$, incident with four 3-faces, respectively, $vv_1v_2, vv_2v_3, vv_3v_4$ and vv_4v_5. Now, delete the vertex v and get a graph G', which has a 2-frugal 17-coloring ϕ' by the minimality. Then, extend ϕ' to G. Except the vertex v, let the other vertices be colored the same as ϕ'. By the definition of 2-frugal coloring and the structure of G', we have

$$f(v) \geq 17 - (|N(v)| + \lfloor \tfrac{|N_2(v)|}{2} \rfloor)$$
$$= 17 - (5 + \lfloor \tfrac{3+2+2+2+3}{2} \rfloor) = 6 \geq 1,$$

a contradiction. □

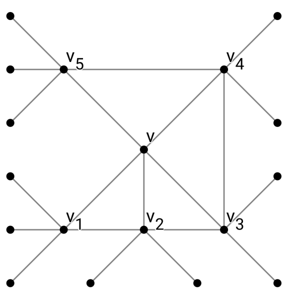


Fig 1 illustration of lemma 2.3

Lemma 2.4 In the graph G, when a 6-vertex is incident with three 3-faces, there is no more than one $(6,5,5)$-face. (see Fig 2)

Proof: Suppose there is a 6-vertex v with $N(v) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, incident with three 3-faces, including two $(6,5,5)$-faces. We have two cases. Case 1: the two $(6,5,5)$-faces are vv_1v_2 and vv_2v_3. Case 2: the two $(6,5,5)$-faces are vv_1v_2 and vv_3v_4. In the worst case, we let the rest 3-face be $(6,6,6)$-face.

Case 1: Now, delete the vertex v_2 and get a graph G', which has a 2-frugal 17-coloring ϕ' by the minimality. Then, extend ϕ' to G. Except the vertex v_2, let the other vertices be colored the same as ϕ'. By the definition of 2-frugal coloring and the structure of G', we have

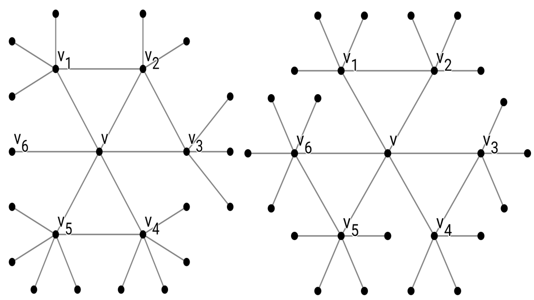
$$f(v) \geq 17 - (|N(v)| + \lfloor \tfrac{|N_2(v)|}{2} \rfloor)$$
$$= 17 - (5 + \lfloor \tfrac{5+5+3+3+3}{2} \rfloor) = 3 \geq 1,$$

a contradiction.

Case 2: Now, delete the vertex v and get a graph G', which has a 2-frugal 17-coloring ϕ' by the minimality. Then, extend ϕ' to G. Except the vertex v, let other the vertices be colored the same as ϕ'. By the definition of 2-frugal coloring and the structure of G', we have

$$f(v) \geq 17 - (|N(v)| + \lfloor \tfrac{|N_2(v)|}{2} \rfloor)$$
$$= 17 - (6 + \lfloor \tfrac{3\times 4+4\times 2}{2} \rfloor) = 1 \geq 1,$$

a contradiction. □



case1 case2

Fig 2 illustration of lemma 2.4

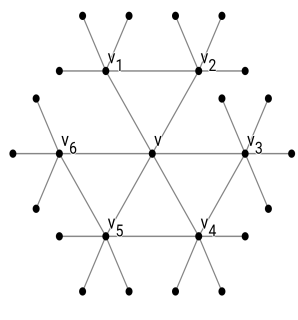


Fig 3 illustration of lemma 2.5

Lemma 2.5 In the graph G, when a 6-vertex is incident with four 3-faces, there is no $(6,5,5)$-face. (see Fig 3)

Proof: Assume there is a 6-vertex v with $N(v) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, incident with four 3-faces, including one $(6,5,5)$-face. In the worst case, we let the rest three 3-faces be $(6,6,6)$-faces. Now, delete the vertex v and get a graph G', which has a 2-frugal 17-coloring ϕ' by the minimality. Then, extend ϕ' to G. Except the vertex v, let the other vertices be colored the same as ϕ'. By the definition of 2-frugal coloring and the structure of G', we have

$$f(v) \geq 17 - (|N(v)| + \lfloor \tfrac{|N_2(v)|}{2} \rfloor)$$

$$= 17 - (6 + \lfloor \frac{3\times2+4+3\times3}{2} \rfloor) = 2 \geq 1,$$

a contradiction. □

Lemma 2.6 In the graph G, when a 6-vertex is incident with five 3-faces, there are at least three $(6, 6, 6)$-faces. (see Fig 4)

Proof: Assume there is a 6-vertex v with $N(v) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, incident with five 3-faces, including only two $(6, 6, 6)$-faces. We have three cases. Case 1: the two $(6, 6, 6)$-faces are vv_1v_2 and vv_2v_3. Case 2: the two $(6, 6, 6)$-faces are vv_1v_2 and vv_5v_6. Case 3: the two $(6, 6, 6)$-faces are vv_1v_2 and vv_4v_5. In the worst case, we let the rest three 3-faces be $(6, 6, 5)$-faces.

Case 1: Now, delete the vertex v and get a graph $G^{'}$, which has a 2-frugal 17-coloring $\phi^{'}$ by the minimality. Then, extend $\phi^{'}$ to G. Except the vertex v, let the other vertices be colored the same as $\phi^{'}$. By the definition of 2-frugal coloring and the structure of $G^{'}$, we have

$$f(v) \geq 17 - (|N(v)| + \lfloor \frac{|N_2(v)|}{2} \rfloor)$$
$$= 17 - (6 + \lfloor \frac{4+3+3+2+3+3}{2} \rfloor) = 2 \geq 1,$$

a contradiction.

Case 2: Now, delete the vertex v and get a graph $G^{'}$, which has a 2-frugal 17-coloring $\phi^{'}$ by the minimality. Then, extend $\phi^{'}$ to G. Except the vertex v, let the other vertices be colored the same as $\phi^{'}$. By the definition of 2-frugal coloring and the structure of $G^{'}$, we have

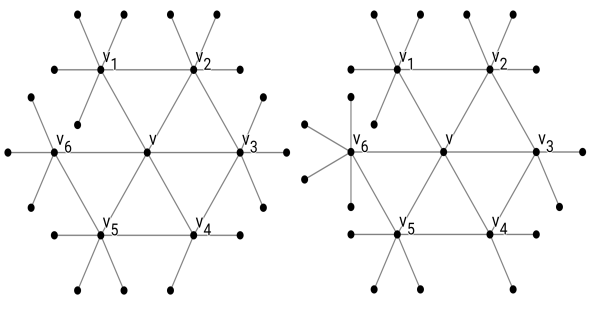
$$f(v) \geq 17 - (|N(v)| + \lfloor \frac{|N_2(v)|}{2} \rfloor)$$
$$= 17 - (6 + \lfloor \frac{4+3+2+2+3+4}{2} \rfloor) = 2 \geq 1,$$

a contradiction.

Case 3: Now, delete the vertex v and get a graph $G^{'}$, which has a 2-frugal 17-coloring $\phi^{'}$ by the minimality. Then, extend $\phi^{'}$ to G. Except the vertex v, let the other vertices be colored the same as $\phi^{'}$. By the definition of 2-frugal coloring and the structure of $G^{'}$, we have

$$f(v) \geq 17 - (|N(v)| + \lfloor \frac{|N_2(v)|}{2} \rfloor)$$
$$= 17 - (6 + \lfloor \frac{4+3+2+3+3+3}{2} \rfloor) = 2 \geq 1,$$

a contradiction. □



case1 case2

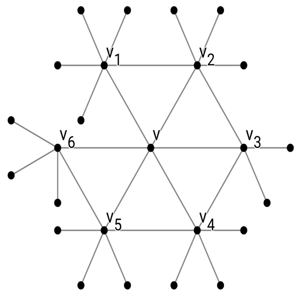
Fig 4 illustration of lemma 2.6

Lemma 2.7 In the graph G, when a 6-vertex is incident with six 3-faces, all 3-faces are $(6, 6, 6)$-faces. (see Fig 5)

Proof: Assume there is a 6-vertex v with $N(v) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, incident with six 3-faces, including two $(6, 6, 5)$-faces. Let the rest four 3-faces be $(6, 6, 6)$-faces. This is the worst case. Now, delete the vertex v and get a graph $G^{'}$, which has a 2-frugal 17-coloring $\phi^{'}$ by the minimality. Then, extend $\phi^{'}$ to G. Except the vertex v, let the other vertices be colored the same as $\phi^{'}$. By the definition of 2-frugal coloring and the structure of $G^{'}$, we have

$$f(v) \geq 17 - (|N(v)| + \lfloor \frac{|N_2(v)|}{2} \rfloor)$$
$$= 17 - (6 + \lfloor \frac{2+3+2+3+3+3}{2} \rfloor) = 3 \geq 1,$$

a contradiction. □



case3

Fig 4 illustration of lemma 2.6

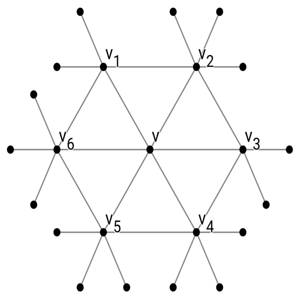


Fig 5 illustration of lemma 2.7

B. Discharging

In the following part, we get a contradiction by the discharging method and complete the proof. By the Euler's formula $|V| + |F| - |E| = 2$ and the Handshake lemma $\sum_{v\in V(G)} d(v) = \sum_{f\in F(G)} d(f) = 2|E|$, we get:

$$\sum_{v\in V(G)}(2d(v) - 6) + \sum_{f\in F(G)}(d(f) - 6) = -12.$$

Let the initial charge of each vertex v be $2d(v) - 6$ and each face f be $d(f) - 6$. Then, we design some appropriate discharging rules and redistribute charges among vertices and faces, such that the final sum of charges of all vertices and faces is nonnegative, a contradiction.

The discharging rules as follows:

$R1$. Every 5-vertex gives 1 to each of its incident $(5, 5, 5)$-face.

$R2$. Every 5-vertex gives 1/2 to each of its incident face, except $(5, 5, 5)$-face.

$R3$. Every 6-vertex gives 1 to each of its incident $(6, 6, 6)$-face.

$R4$. Every 6-vertex gives 5/4 to each of its incident $(6, 6, 5)$-face.

$R5$. Every 6-vertex gives 2 to each of its incident $(6, 5, 5)$-face.

$R6$. Every 6-vertex gives 1/2 to each of its incident face, except $(6, 6, 5)$-face, $(6, 6, 5)$-face and $(6, 6, 6)$-face.

Now, we check the final charge of vertices and faces.

Firstly, according to the previous proof, we show that the final charge of each face is nonnegative.

3-face: By lemma 2.2, there only exists 5-vertex and 6-vertex. Hence, there are four cases of 3-face. They are

$(5,5,5)$-face, $(6,5,5)$-face, $(6,6,5)$-face and $(6,6,6)$-face. If the 3-face is a $(5,5,5)$-face, by $R1$, the final charge is $3 \times 1 - 3 = 0$. If the 3-face is a $(6,5,5)$-face, by $R2$ and $R5$, the final charge is $2 \times 1/2 + 2 - 3 = 0$. If the 3-face is a $(6,6,5)$-face, by $R2$ and $R4$, the final charge is $1/2 + 2 \times 5/4 - 3 = 0$. If the 3-face is a $(6,6,6)$-face, by $R3$, the final charge is $3 \times 1 - 3 = 0$.

4-face: By $R2$ and $R6$, the final charge is $4 \times 1/2 - 2 = 0$.

5-face: By $R2$ and $R6$, the final charge is $5 \times 1/2 - 1 = 3/2$.

6^+-face: There is no change in its initial charge $d(f) - 6$, obviously, the final charge is nonnegative.

Secondly, we show that the final charge of every vertex is nonnegative. According to the lemma 2.2, we check the final charge of 5-vertex and 6-vertex:

5-vertex: By Lemma 2.3, there is no 5-vertex incident with more than three $(5,5,5)$-faces in the graph G. By $R1$ and $R2$, the final charge of 5-vertex is $4 - 3 \times 1 - 2 \times 1/2 = 0$.

6-vertex: By Lemma 2.4, in the graph G, when a 6-vertex is incident with three 3-faces, there is no more than one $(6,5,5)$-face. By Lemma 2.5, in the graph G, when a 6-vertex is incident with four 3-faces, there is no $(6,5,5)$-face. By Lemma 2.6, in the graph G, when a 6-vertex is incident with five 3-faces, there are at least three $(6,6,6)$-faces. By Lemma 2.7, in the graph G, when a 6-vertex is incident with six 3-faces, all 3-faces are $(6,6,6)$-faces. For convenience, we make a table of all the situations about a 6-vertex, according to the above lemmas. We denote the number of 3-faces incident with a 6-vertex by $Num(3)$ and the final charge of the 6-vertex by $FC(6)$. In this table we write (i,j,k)-face as ijk-face for short. (see Table I)

Table I all the situations about a 6-vertex

$Num(3)$	Other face	666-face	665-face	655-face	$FC(6)$
0	6	0	0	0	3
1	5	1	0	0	2.5
	5	0	1	0	2.25
	5	0	0	1	1.5
2	4	1	1	0	1.75
	4	1	0	1	1
	4	0	1	1	0.75
	4	2	0	0	2
	4	0	2	0	1.5
	4	0	0	2	0
3	3	3	0	0	1.5
	3	2	1	0	1.25
	3	2	0	1	0.5
	3	1	2	0	1
	3	1	1	1	0.25
	3	0	3	0	0.75
	3	0	2	1	0
4	2	4	0	0	1
	2	3	1	0	0.75
	2	3	0	1	0
	2	2	2	0	0.5
	2	1	3	0	0.25
	2	0	4	0	0
5	1	5	0	0	0.5
	1	4	1	0	0.25
	1	3	2	0	0
6	0	6	0	0	0

From the table above, the final charge of each 6-vertex is nonnegative. By the Euler's formula, the contradiction is obvious:

$$0 \leq \sum_{v \in V(G)} (2d(v) - 6) + \sum_{f \in F(G)} (d(f) - 6) = -12.$$

That means that such a counterexample G to Theorem 1.1 does not exist. We complete the proof. □

III. Algorithm

About the algorithm for frugal coloring, McCormick and Thomas [8] determined the complexity of deciding whether a given graph has a 1-frugal k-coloring. By generalizing the result from [8], Stefan, Gary MacGillivray and Shayla Redlin [2] proved that if $k = 3$ and $t \geq 2$, or $k \geq 4$ and $t \geq 1$, then the problem of deciding whether a given graph has a t-frugal k-coloring is NP-complete.

Before giving the algorithm, we will give a corollary as follow.

Corollary 1.2 If G is a planar graph without adjacent triangles, $\Delta(G) \leq 6$, then, one of the following holds:

(1) G contains a cut edge.

(2) $\delta(G) < 5$.

Proof: By Theorem 1.1, it is easy to prove Corollary 1.2 and we omit it here. □

From the proof of Theorem 1.1 and Corollary 1.2, we give a linear time algorithm, inspired by the algorithm given by Baogang Xu and Haihui Zhang[1]. For arbitrary planar graph G without adjacent triangles, $\Delta(G) \leq 6$, there exists a 2-frugal 17-coloring of G. We denote the number of connected components of graph G by $\omega(G)$ and the color set of G by $c(G)$.

Algorithm:

Input: A planar graph G without adjacent triangles, $\Delta(G) \leq 6$, and $c(G) = \{1, 2, \ldots, 17\}$.

Output: A 2-frugal coloring ϕ of G with $\Phi_2(G) = 17$.

Step 0: Set $i = 0$, $G_0 = G$, $V_0 = \{v \mid d(v) \leq 4\}$ and $E_0 = \{uv \mid u, v \notin V_0 \text{ and } \omega(G - uv) > \omega(G)\}$.

Step 1: If $\Delta(G_i) \leq 2$, color G_i with a proper coloring greedily, and go to step 3.

Step 2: If $V_0 \neq \emptyset$, choose $v \in V_0$, set $S_i := \{v\}$ and reset $V_0 := V_0 \setminus \{v\}$;

Else, choose an $uv \in E_0$, set $S_i := \{u, v\}$ and reset $E_0 := E_0 \setminus \{uv\}$.

Reset $G_i := G_i - S_i$, $i = i + 1$, and add the new 4^--vertices and cut edges of G_i into V_0 and E_0. Go to step 1.

Step 3: If $i = 0$, output ϕ.

Step 4: If $S_{i-1} = \{v\}$, color v by one of $f(v)$ in the proof of Lemma 2.2.

Else, $S_{i-1} = \{u, v\}$, color u, v by the method in the proof of Lemma 2.1.

Reset $i = i - 1$ and go to step 3.

From the algorithm complexity's proof by Baogang Xu and Haihui Zhang[1], it is obvious to know that the algorithm given above is a linear time algorithm.

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