Secant Method with Aitken Extrapolation Outperform Newton-Raphson Method in Estimating Stock Implied Volatility

S. Purwani, A. F. Ridwan, R. A. Hidayana, and S. Sukono

Abstract— **Stocks are one of the investment instruments in the form of securities that promise high returns. A very close relationship affects return and risk, the greater the profit, the greater the risk of loss. The volatility value can be used to estimate the level of risk of loss and even profit from decision making. The Black-Scholes model (BSM) has the form of a partial differential equation which defines the theoretical value of financial options based on certain parameters. This model can take into account implied volatility which tends to vary according to strike price and expiration time. This implied volatility is then solved using numerical methods, such as the Newton-Raphson Method (NRM) and Secant Method (SM). Based on the results of previous study, the convergence of NRM is always faster than that of SM. Therefore, this study aims to accelerate the convergence of the Secant Method using the Aitken Extrapolation (SMAE), which is then compared to the other two methods in estimating stock implied volatility. The results show that SMAE outperforms the other two methods in terms of accuracy and rate of convergence.**

Index Terms— **Aitken extrapolation, Newton-Raphson method, root finding, secant, stock, volatility.**

I. Introduction

INVESTMENT is the action of saving some funds that are expected to provide more profits in the future. Investment orders in the form of securities can generally be made available through the capital market and money market [1]. Stock investment through the capital market is considered as attractive. Having stocks from public companies is preferred by most investors rather than that from non-public companies because this promises high returns [2]. In general, risks and benefits have a very tight relationship, the greater the profit, the greater the risk of loss [3] [4]. Furthermore, risk can be thought as the chance of acquiring an unwanted outcome that occurs [5]. Financial institutions may need to manage the risk of reducing the cost of external capital. This is necessary to reduce the costs of financial distress and income volatility to avoid high tax obligations [6].

Volatility is a statistical measure of the fluctuation in the price of a stock or foreign exchange over a certain period. The volatility value can be used to estimate the level of risk of loss and even profit from decision-making [7]. The most popular and widely used model to determine volatility is the BSM [8]. BSM is defined as a partial differential equation to calculate the theoretical value of financial options based on certain parameters, namely stock price flow, expected volatility, expiration time, strike price, and expected dividends [9]. BSM is often criticized for assuming that the log-return of the underlying asset prices is normally distributed [10]. This model can take into account implied volatility which tends to vary concerning the strike price and expiration time [11]. As the basic principle behind BSM, it assumes that the volatility of assets is constant [12]. However, this assumption is not fully suitable for real market data [13]. This implied volatility model is then solved by using numerical methods, such as NRM and SM.

Nwry et al. [14] compared the use of the bisection, secant, and Newton methods in obtaining solutions to non-linear equations. They explained that the Newton and secant methods converge faster than the bisection method, with only a few iterations to converge and a very small error tolerance. In addition, Mahrudinda et al. [7] compared the use of the three methods to estimate implied volatility. They came to the same result as that in [14] in the case of estimating implied volatility. On the other hand, Sharma [15] compared the rate of convergence of linear and quadratic convergent approaches. The study explained that the latter has a rate of convergence higher than the former. However, the latter is computationally more expensive than the former. In general, the convergence rates of the bisection, secant, and Newton methods are linear, super linear, and quadratic respectively.

Given the previous results, we aim to accelerate the convergence of the secant method by using the Aitken extrapolation (SMAE) [16], [17]. The rate of SMAE's convergence is then compared to that of the original secant and Newton-Raphson method in the case of predicting implied volatility.

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II. Material and Methods

The data used is Microsoft Corporation (MSFT) stock data in the form of the current stock price (S_0), strike price (K), call option observation price (C_{obs}) with a maturity time (T) of 3 months or 0.25 year, and interest rates (r) obtained from https://www.investing.com/ on 16 November 2021. Programming done in research was done in Python.

A. Black Scholes Model

The BSM structure is used to model the option price related to the current time, the stock price, the stock's volatility, the interest rate, the expiration date, and the strike price. Generally, the model assumes that volatility is a constant function throughout the option's life. The option pricing formula can be used to display volatility stated by market price [12]. BSM can be defined as follows,

$$C_{obs} = C_{BS}(\sigma)$$

where C_{obs} represents the call option observation price obtained from the actual market price and $C_{BS}(\sigma)$ represents the theoretical option price from BSM which is defined by,

$$C_{BS}(\sigma) = S_0 N(d_1) - Ke^{-r(T)} N(d_2)$$

with $$d_1 = \frac{\ln \frac{S_0}{K} + \left(r + \frac{1}{2}\sigma^2\right)(T)}{\sigma\sqrt{T-t}}$$

and $$d_2 = d_1 - \sigma\sqrt{T}$$

where S_0 is the current stock price, K is the strike price, r is the interest rate, σ is the volatility, and $N(d_i)$ is the standard cumulative normal distribution function. Thus, the volatility function related to BSM can be written as follows,

$$f(\sigma) = C_{obs} - \left(S_0 N(d_1) - Ke^{-r(T)} N(d_2)\right) = 0 \qquad (1)$$

The root of (1) is then estimated by using the NRM, SM, and SMAE methods.

B. Newton-Raphson Method

NRM is one of the most well-known methods to find the root of a function. To obtain the root α of $y = f(\sigma)$ we generally provide an estimate of α, e. g. stated by σ_0. An improvement on this estimate is then performed by replacing the root-finding of $y = f(\sigma)$ with that of the straight line tangent to the graph $y = f(\sigma)$ at $(\sigma_0, f(\sigma_0))$. It is simply a graph for a linear Taylor polynomial $y = p_1(\sigma)$ defined as,

$$p_1(\sigma) = f(\sigma_0) + f'(\sigma_0)(\sigma - \sigma_0)$$

Suppose that σ_1 is the root of $p_1(\sigma) = 0$, then we have

$$f(\sigma_0) + f'(\sigma_0)(\sigma - \sigma_0) = 0.$$

This leads to

$$\sigma_1 = \sigma_0 - \frac{f(\sigma_0)}{f'(\sigma_0)}.$$

We hope σ_1 to be a better estimate than σ_0, an approximate of α, the same way can be performed with σ_1 as an initial prediction. By repeating this method, we will get a numerical sequence $\sigma_1, \sigma_2, \sigma_3, \ldots$ They are expected to approach the root α, and the general iteration formula defines them recursively as follows,

$$\sigma_{n+1} = \sigma_n - \frac{f(\sigma_n)}{f'(\sigma_n)}, \quad n = 0,1,2,\ldots \qquad (2)$$

This is called NRM which will converge if $f'(\alpha) \neq 0$, that has a simple root. It is not only used for linear equations but also for non-linear equations [14]. Using the following Taylor expansion of $f(\alpha)$ around σ_n gives,

$$f(\alpha) = f(\sigma_n) + (\alpha - \sigma_n) f'(\sigma_n) + \frac{1}{2}(\alpha - \sigma_n)^2 f''(c_n) \quad (3)$$

where c_n is unknown located between α and σ_n. By having $f(\alpha) = 0$ and then dividing (3) by $f'(\sigma_n)$, we obtain

$$0 = \frac{f(\sigma_n)}{f'(\sigma_n)} + (\alpha - \sigma_n) + (\alpha - \sigma_n)^2 \frac{f''(c_n)}{2f'(\sigma_n)} \qquad (4)$$

With (2), we have the first term of (4) to be equal to $\sigma_n - \sigma_{n+1}$, and we then solve for $\alpha - \sigma_{n+1}$ giving

$$\alpha - \sigma_{n+1} = (\alpha - \sigma_n)^2 \left(\frac{-f''(c_n)}{2f'(\sigma_n)}\right). \qquad (5)$$

This shows that the iterate σ_{n+1} has an error that is almost proportional to the square of the error of the previous iterate σ_n. As long as the initial error is small enough, formula (5) shows that the error in the next iteration will decrease very rapidly. The convergence of NRM will be discussed in the following section.

C. Convergence of Newton-Raphson Method

NRM can be restated, in terms of fixed point iteration, as follows,

$$\sigma_{n+1} = g(\sigma_n),\ g(\sigma) = \sigma - \frac{f(\sigma)}{f'(\sigma)}$$

We then have

$$g'(\sigma) = \frac{f(\sigma)\, f''(\sigma)}{\left[f'(\sigma)\right]^2}$$

and if $f'(\alpha) \neq 0$, then $g'(\alpha) = 0\ \left(f(\alpha) = 0\right)$. We obtain

$$g''(\sigma) = \frac{f''(\sigma)}{f'(\sigma)}$$

We can also show that $g''(\alpha) \neq 0$ if furthermore, $f''(\alpha) \neq 0$. Thus convergence of NRM is of order 2 provided $f'(\alpha) \neq 0$ and $f''(\alpha) \neq 0$. This is shown by taking a Taylor expansion of $g(\sigma_n)$ about α, and by assuming that $g(\sigma)$ is twice continuously differentiable,

$$g(\sigma_n) = g(\alpha) + (\sigma_n - \alpha) g'(\alpha) + \frac{1}{2}(\sigma_n - \alpha)^2 g''(c_n)$$

with c_n between σ_n and α.

Applying $\sigma_{n+1} = g(\sigma_n),\ \alpha = g(\alpha)$ and the previous result $g'(\alpha) = 0$, we have

$$\sigma_{n+1} = \alpha + \frac{1}{2}(\sigma_n - \alpha)^2 g''(c_n)$$

Taking the error form and the limit as $n \to \infty$ gives,

$$\alpha - \sigma_{n+1} = -\frac{1}{2}(\sigma_n - \alpha)^2 g''(c_n)$$

$$\lim_{n\to\infty} \frac{\alpha - \sigma_{n+1}}{(\sigma_n - \alpha)^2} = -\frac{1}{2} g''(\alpha) \tag{6}$$

If $g''(\alpha) \neq 0$, then (6) implies that the convergence of $\sigma_{n+1} = g(\sigma_n)$ (2) is of order 2. This means that NRM converges quadratically which follows the explanation of (5).

D. Secant Method

SM is one of the methods that approximate the root of $y = f(\sigma)$ by taking that of a linear approximation to $y = f(\sigma)$. This hence includes SM to methods for obtaining a root of $f(\sigma)$. The SM requires 2 initial estimates to the root to start the iteration, denoted here by σ_0 and σ_1. The linear approximation is then given by,

$$y = p(\sigma) \equiv f(\sigma_1) + (\sigma - \sigma_1)\frac{f(\sigma_1) - f(\sigma_0)}{\sigma_1 - \sigma_0}$$

Solving for $p(\sigma_2) = 0$ gives

$$\sigma_2 = \sigma_1 - f(\sigma_1)\frac{\sigma_1 - \sigma_0}{f(\sigma_1) - f(\sigma_0)}$$

By obtaining σ_2 we can omit σ_0 and have a new pair of estimates σ_1 and σ_2 to the root α. Repeating the iteration with the new initial estimates results in an improved value σ_3, and we can state the iteration as follows,

$$\sigma_{n+1} = \sigma_n - f(\sigma_n)\frac{\sigma_n - \sigma_{n-1}}{f(\sigma_n) - f(\sigma_{n-1})},\quad n \geq 1 \tag{7}$$

This is the secant method (SM) which requires two initial values to start the iteration. The order of convergence is

$$r = \frac{1+\sqrt{5}}{2} \approx 1.618 \tag{8}$$

which is super linear [14]. Its derivation will be given in the next section.

E. Convergence of Secant Method

Restating the secant method (7) for solving $f(\sigma) = 0$ gives,

$$\sigma_{n+1} = \sigma_n - f(\sigma_n)\frac{\sigma_n - \sigma_{n-1}}{f(\sigma_n) - f(\sigma_{n-1})},\quad n = 1, 2, \ldots$$

Assume the iterations converge and obtain a solution α, that is $f(\alpha) = 0$, as $n \to \infty$. Based on the rule as follows,

$$\left|\sigma_{n+1} - \alpha\right| \approx K\left|\sigma_n - \alpha\right|^l$$

we consider the convergence of the secant method. Let

$$\sigma_n = \alpha + \beta_n \tag{9}$$

Since $\sigma_n \to \alpha$ then $\beta_n \to 0$, as $n \to \infty$. The iterations (7) hence become

$$\beta_{n+1} = \beta_n - \frac{f(\alpha + \beta_n)(\beta_n - \beta_{n-1})}{f(\alpha + \beta_n) - f(\alpha + \beta_{n-1})}) \tag{10}$$

Suppose that $f(\sigma)$ has two derivatives with $f'(\alpha) \neq 0$ and $f''(\alpha) \neq 0$. The Taylor expansion of $f(\sigma)$ around α can be written as follows,

$$f(\alpha + \beta) = f(\alpha) + f'(\alpha)\beta + \frac{f''(\alpha)}{2}\beta^2 + S_2(\beta).$$

We have $f(\alpha) = 0$, β being small, and $S_2(\beta)$ being the remainder term. Neglecting the terms of order higher than 2 in β gives the estimate,

$$f(\alpha + \beta) \approx f'(\alpha)\beta + \frac{f''(\alpha)}{2}\beta^2.$$

For simplicity, denote

$$N=\frac{f''(\alpha)}{2f'(\alpha)} \tag{11}$$

and apply the estimate equalities

$$f(\alpha+\beta_n)\approx\beta_n f'(\alpha)(1+N\beta_n)$$

$$f(\alpha+\beta_n)-f(\alpha+\beta_{n-1})\approx f'(\alpha)(\beta_n-\beta_{n-1})(1+N(\beta_n+\beta_{n-1}))$$

Equation (10) becomes

$$\beta_{n+1}\approx\beta_n-\frac{\beta_n f'(\alpha)(1+N\beta_n)(\beta_n-\beta_{n-1})}{f'(\alpha)(\beta_n-\beta_{n-1})(1+N(\beta_n+\beta_{n-1}))}$$

$$=\beta_n-\frac{\beta_n(1+N\beta_n)}{(1+N(\beta_n+\beta_{n-1}))}$$

$$=\frac{\beta_n\beta_{n-1}N}{(1+N(\beta_n+\beta_{n-1}))}$$

$$\approx\beta_n\beta_{n-1}N$$

Using (11) this gives an error relation

$$\beta_{n+1}\approx\frac{f''(\alpha)}{2f'(\alpha)}\beta_n\beta_{n-1} \tag{12}$$

where the terms of order higher than 2 in β are omitted. Compare (12) with (5), the error for Newton, that is rewritten as follows,

$$\beta_{n+1}\approx\frac{f''(\alpha)}{2f'(\alpha)}\beta_n{}^2$$

Equation (12) shows that the error tends to zero faster than a linear function as $n\to\infty$, and yet not quadratically. Finding out the exponent l in the rule

$$|\beta_{n+1}|\approx K|\beta_n|^l$$

defined by (12), leads us to

$$K|\beta_n|^l\approx|N||\beta_n||\beta_{n-1}|$$

$$|\beta_n|^{l-1}\approx\frac{|N|}{K}|\beta_{n-1}|$$

$$|\beta_n|\approx\left(\frac{|N|}{K}\right)^{\frac{1}{l-1}}|\beta_{n-1}|^{\frac{1}{l-1}}$$

Hence, $K=\left(\frac{|N|}{K}\right)^{\frac{1}{l-1}}$ and $l=\frac{1}{l-1}$. Since $l>0$ the condition on l results in $l=\frac{1+\sqrt{5}}{2}=1.618$ the convergence order of SM (8)

Whereas the condition on K implies that

$$K^l=|N|\,or\,K=|N|^{\frac{1}{l}}=\left|\frac{f''(\alpha)}{2f'(\alpha)}\right|^{l-1}$$

This comes to a conclusion that for the secant method

$$|\sigma_{n+1}-\alpha|\approx\left|\frac{f''(\alpha)}{2f'(\alpha)}\right|^{\frac{\sqrt{5}-1}{2}}|\sigma_n-\alpha|^{\frac{\sqrt{5}+1}{2}} \tag{13}$$

Even though the order of convergence for the secant method (8) is commonly lower than the Newton method, however, the former has the advantage of not computing $f'(\sigma)$. This is a definite computational advantage.

F. *Aitken Extrapolation*

The Aitken Extrapolation is a method used to speed up the convergence of iteration methods. The explicit assumption in deriving the Aitken Extrapolation and assigning acceleration is that successive iterations of the error or its approximation have the same sign or have alternating sign patterns [18].

We assume that $g(\sigma)$ is continuous on an interval $[a,b]$ and satisfies

$$a\le\sigma\le b\ \Rightarrow\ a\le g(\sigma)\le b \tag{14}$$

This assumption guaranties the existence of at least one solution α for the iteration $\sigma=g(\sigma)$ in the interval $[a,b]$. Then, assume that $g'(\sigma)$ is continuous for $a\le\sigma\le b$, and g follows (14). Furthermore, assume that

$$\mu\equiv\max_{a\le\sigma\le b}|g'(\sigma)|<1$$

then taking any initial value σ_0 in $[a,b]$, results in the sequences σ_n that will converge to α and satisfy [19].

$$\lim_{n\to\infty}\frac{\alpha-\sigma_{n+1}}{\alpha-\sigma_n}=g'(\alpha)$$

Thus, for σ_n close to α,

$$\alpha-\sigma_{n+1}\approx g'(\alpha)(\alpha-\sigma_n)$$

Let $g'(\alpha)$ be denoted by λ, and write the equation with n replaced by $n-1$ then solve for α so that

$$\alpha-\sigma_n\approx\lambda(\alpha-\sigma_{n-1})$$

$$\alpha\approx\sigma_n+\frac{\lambda}{1-\lambda}(\sigma_n-\sigma_{n-1}) \tag{15}$$

To estimate λ we use the ratios

$$\lambda_n = \frac{\sigma_n - \sigma_{n-1}}{\sigma_{n-1} - \sigma_{n-2}}, \quad n \geq 2 \tag{16}$$

Furthermore, combining (15) and (16) gives,

$$\alpha \approx \sigma_n + \frac{\lambda_n}{1-\lambda_n}(\sigma_n - \sigma_{n-1}) \tag{17}$$

This is called the Aitken Extrapolation Formula which can be used to create a more rapidly convergent sequence. Modifying (17) leads to the error in σ_n.

$$\alpha - \sigma_n \approx \frac{\lambda_n}{1-\lambda_n}(\sigma_n - \sigma_{n-1}) \tag{18}$$

Equation (18) gives a very good error estimate in σ_n. The convergence will be discussed in the following section.

G. Convergence of Aitken Extrapolation

Restating (17) and taking $\sigma_{n+1} \approx \alpha$ gives,

$$\sigma_{n+1} \approx \sigma_n + \frac{\lambda_n}{1-\lambda_n}(\sigma_n - \sigma_{n-1}) \tag{19}$$

Using (9) leads (19) to,

$$\beta_{n+1} \approx \beta_n + \frac{\lambda_n}{1-\lambda_n}(\beta_n - \beta_{n-1}) \tag{20}$$

Using (16), leads (20) to the error in σ_{n+1}

$$\beta_{n+1} \approx \beta_n + \frac{(\beta_n - \beta_{n-1})^2}{2\beta_{n-1} - \beta_{n-2} - \beta_n}$$

$$\beta_{n+1} = \frac{\beta_{n-1}{}^2 - \beta_n \beta_{n-2}}{2\beta_{n-1} - \beta_{n-2} - \beta_n} \tag{21}$$

This equation was also used to transform sequences that converge slowly and diverge [20]. It is not clearly shown that the iterates σ_n converge quadratically or even more (see (21)). Therefore instead of using (21), we will approximate the convergence of the Aitken Extrapolation empirically using the following equation,

$$r_i \approx \frac{\log|\sigma_{i+2} - \sigma_{i+1}|}{\log|\sigma_{i+1} - \sigma_i|} \tag{22}$$

This will also be applied to the other methods discussed in the following section.

III. Result and Discussion

We form the volatility function using BSM on the data obtained which is shown as follows,

$$f(\sigma) = 24.59 - 339.51 N(d_1) - 325.00 e^{-0.023(0.25)} N(d_2) \tag{23}$$

with $$d_1 = \frac{\ln\frac{339.51}{325.00} + \left(0.023 + \frac{1}{2}\sigma^2\right)(0.25)}{\sigma\sqrt{0.25}} \tag{24}$$

and $$d_2 = d_1 - \sigma\sqrt{0.25} \tag{25}$$

Using (23), (24), and (25), gives the volatility value which is the root of $f(\sigma) = 0$. The plot of the function is shown in Fig. 1.

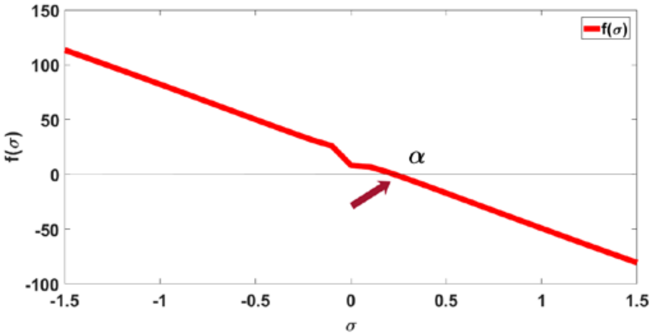

Fig. 1. The plot of the implied volatility function.

The root α is the intersection of the curve with the sigma axis which is located in the interval $0 < \sigma < 0.5$ (see Fig. 1). The volatility value is then estimated using the Newton-Raphson Method (NRM), Secant Method (SM), and Secant Method accelerated by the Aitken Extrapolation (SMAE). The iteration results using NRM and SM are shown in Tables I and II respectively. The initial value for NRM is $\sigma_0 = 0.5$, whereas those for SM are $\sigma_0 = 10^{-12}$ and $\sigma_1 = 0.5$ with the same error tolerance $\text{Tol} = 10^{-12}$, as well as the same number of digit accuracy (double precision).

TABLE I
Result of NRM Iteration

i	σ_i	$f(\sigma_i)$	$\lvert\sigma_i - \sigma_{i-1}\rvert$
0	0.5	-17.1941572976078	-
1	0.23253759195479934	-0.321853505543185	0.2674
2	0.22719641341702468	-0.00067039971379401	0.0053
3	0.22718524102320506	-3.0677078655116e-09	1.11e-05
4	0.2271852409720804	1.77635683940025e-14	5.11e-11
5	0.2271852409720807	-1.06581410364015e-14	3.05e-16

It can be seen in Table I, with $\text{Tol} = 10^{-12}$ NRM gives an estimated volatility value of 0.2271852409720807 with the function value of -1.06581410364015e-14 reached in 5 iterations. Performing (22) to the last column of Table I gives some empirical prediction of the convergence order of NRM with the successive values of $r_1 \approx 3.97$, $r_2 \approx 2.18$, and $r_3 \approx 2.07$. The last value confirms the convergence rate of NRM that equals 2. The greater values may reflect the value of N (11) being involved. Furthermore, the way NRM iterates converge is shown graphically in Fig. 2.

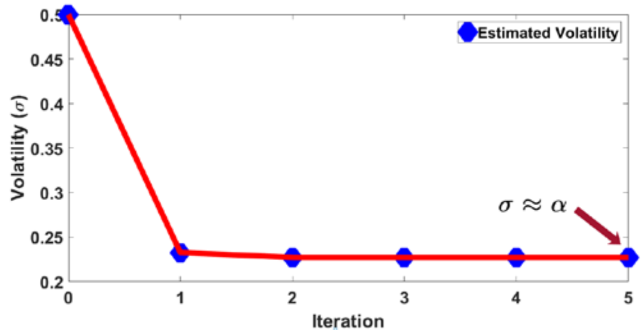

Fig. 2. The plot of NRM iteration.

It can be seen in Fig. 2 that iterates quickly converge to the estimated volatility value. The first iteration almost reaches the estimated value, however since it has not met the given error tolerance, it continues until the fifth iteration.

TABLE II
RESULT OF SM ITERATION

i	σ_i	$f(\sigma_i)$	$\lvert\sigma_i - \sigma_{i-1}\rvert$
0	0.5	-17.1941572976078	0.9999
1	0.161675787074624	3.7857963446334	0.3383
2	0.22272580608014933	0.26708785395261	0.0611
3	0.22735981273433428	-0.0104758346961908	0.0046
4	0.2271849156217955	1.95224690564543e-05	0.0002
5	0.22718524094883427	1.39488420813904e-09	3.25e-07
6	0.22718524097208062	1.06581410364015e-14	2.32e-11
7	0.2271852409720808	-1.77635683940025e-14	1.67e-16

In addition, Table II represents those results produced by SM, with the volatility value of 0.2271852409720808 and the function value of -1.77635683940025e-14 attained with 7 iterations. Performing (22) to the last column of Table II gives a few successive values of $r_1 \approx 1.92$, $r_2 \approx 1.61$, $r_3 \approx 1.73$ $r_4 \approx 1.64$ and $r_5 \approx 1.48$, which are some empirical prediction of convergence order of SM. These values reflect that SM converges more than linear. The various values around 1.618 (8) confirm that the convergence of SM is super linear. The greater or smaller values may ascribe the multiplicity value of (13) being involved.

These results also imply that SM converges slower than NRM. With the error tolerance of 10^{-12} SM needs 7 iterates, whereas NRM requires 5 iterates. However, both methods provide almost the same values of the function at the estimated volatility value. Furthermore, the way SM iterates converge is shown graphically in Fig. 3.

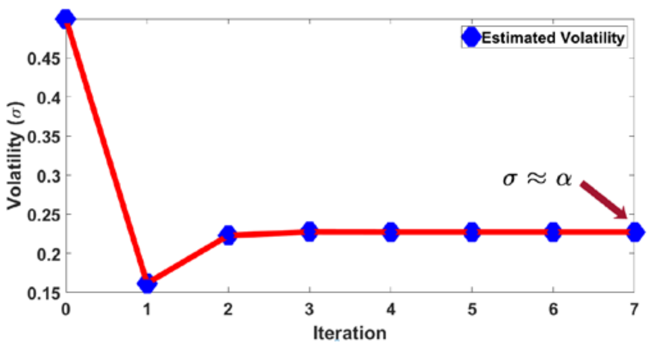

Fig. 3. The plot of SM iteration.

Fig. 3 shows that iterates fluctuate at the start and then seem to converge monotonically to the estimated volatility value. This reflects that sign changes of errors occur at the beginning which might be due to the initial values trapping the root. However, further iterates seem to have the same signs of errors or to have very tiny differences of errors with alternating signs, hence they visually seem to converge monotonically. The iterate data in Table II can provide this information.

NRM outperforms SM theoretically and empirically, we hence intend to accelerate the convergence of SM. We then apply the Aitken Extrapolation to SM method to improve its performance. The so called SMAE requires 3 initial values to start the iteration which include $\sigma_0 = 0.5$, $\sigma_1 = 0.161675787074624$, $\sigma_2 = 0.22272580608014933$, as well as the same error tolerance of 10^{-12}. The SMAE iteration results are given in Table III,

TABLE III
RESULT OF SMAE ITERATION

i	σ_i	$f(\sigma_i)$	$\lvert\sigma_i - \sigma_{i-1}\rvert$
0	0.5	-17.1941572976078	-
1	0.21339344429367368	0.82260548020850	0.2866
2	0.22720351671147748	-0.00109663427894091	0.0138
3	0.22718524097113688	5.66267033264011e-11	1.83e-05
4	0.22718524097208054	-3.55271367880050e-15	9.44e-13

Table III shows those results produced by SMAE with the volatility value of 0.22718524097208054 and the function value of -3.55271367880050e-15 attained with 4 iterations. Applying (22) to the last column of Table III gives a few successive values of $r_1 \approx 3.43$, $r_2 \approx 2.55$ and $r_3 \approx 2.53$, some empirical prediction of convergence order of SMAE. These values confirm that SMAE has higher convergence order than 2, and the value taken near the root confirms its convergence order. Therefore, with the same error tolerance SMAE only requires 4 iterations to converge. The way SMAE iterates converge can then be seen in Fig. 4.

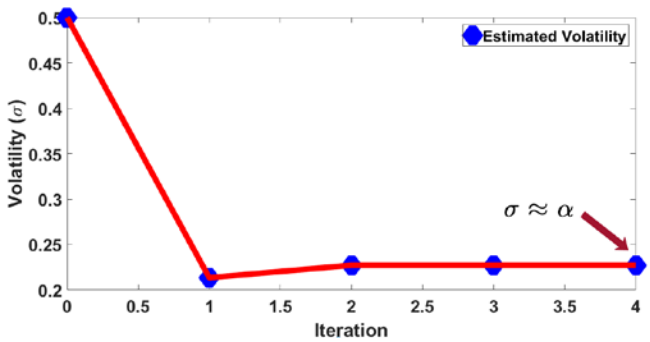

Fig. 4. The plot of SMAE iteration.

It can be seen in Fig. 4 that SMAE convergence pattern is almost similar to that of SM, where the second iterate decreases away from the initial value and then moves up to converge to the estimated value. Another result is the number of iterations required for convergence by SM reduced by SMAE by almost 50%, that is, from 7 iterations to 4 iterations. Finally, the convergence of SMAE is more than quadratic, an improvement over SM which is super linear. This is also an improvement over NRM which is quadratic. Having the empirical convergence rates for each iteration of each method calculated, we can plot them as shown in Fig. 5.

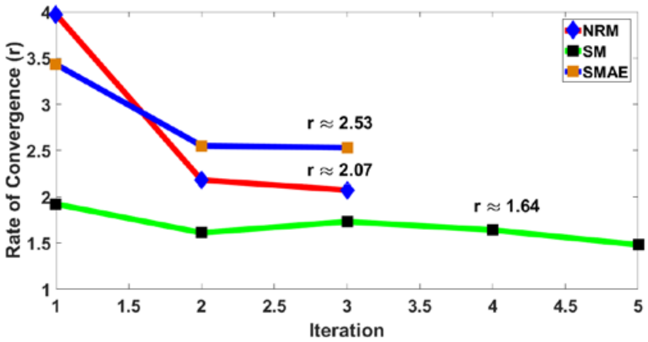

Fig. 5. The plot of rate of convergence estimates.

We can see that each method has a different range of convergence rates. However, theoretically this is considered as that near the root, as $n \to \infty$ or $\sigma_n \to \alpha$ (6). Therefore, according to the result displayed in Fig. 5, the convergence rate of NRM is 2.07, which is almost equal to the theoretical result being quadratic. Those of SM are 1.48 and 1.64 however the latter is almost equal to the theoretical result (8). Finally, that of SMAE is 2.53 which confirms its rate of convergence be more than quadratic.

We then perform a comparison of the simulation results in estimating implied volatility by using NRM, SM, and SMAE. This is concisely presented in Table IV.

TABLE IV
COMPARISON OF THE THREE METHODS

Method	Total Iteration	σ	$f(\sigma)$
NRM	5	0.2271852409720807	-1.06581410e-14
SM	7	0.2271852409720808	-1.77635684e-14
SMAE	4	0.22718524097208054	-3.55271368e-15

The superiority of SMAE can be explained by the correction factor made to the SM's estimates that is by adding its error term to the estimates (17). This is one of the characteristics of extrapolation methods. This hence results in a better estimate produced by SMAE than SM or even NRM. This infers that SMAE is the best among the three methods, in terms of order of convergence and accuracy.

IV. CONCLUSION

The simulation results in estimating the value of implied volatility using the Newton-Raphson method, secant and accelerated secant using the Aitken extrapolation yield the conclusion that the secant method can produce implied volatility at a slower convergence rate than the Newton-Raphson method. However, the secant method accelerated with the Aitken Extrapolation outperforms the Newton-Raphson method in terms of accuracy and convergence rate. Therefore, the secant method accelerated using the Aitken extrapolation can be a recommendation in determining a root of equations, especially in determining implied volatility in the black-schooles model.

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