Numerical Solution of Time Fractional Coupled Korteweg-de Vries Equation with a Caputo Fractional Derivative in Two Parameters

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Abstract—**This paper is devoted to constructing an analytical solution for a time-fractional coupled Korteweg-de Vries differential equations using a Caputo definition with two parameters, namely $D^{\alpha,\rho}$. The homotopy analysis method is utilized to construct a new framework that obtained an analytic solution for fractional coupled Korteweg-de Vries equation with two parameters. We study the effect of the new parameter ρ on the solution. The new parameter ρ can play a significant effect on the solution behaviors. The results are compared with the exact solution in the case of standard derivative, natural decomposition method, and spectral collection method in the case of the fractional derivative with $\rho = 1$. A comparison study is given to validate the efficiency and accuracy of the proposed algorithm.**

Index Terms—**Fractional coupled Korteweg de Vries, generalized Caputo fractional derivative, Homotopy Analysis method**

I. INTRODUCTION

FRACTIONAL differential equations are one of the important branches in the field differential equations, because of various phenomena in physics and chemistry, electronic and electrical, medicine and epidemiology spread, engineering, can be modeling into a fractional differential equation such as the coupled Korteweg-de Vries equation[1]:

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} &= \eta\frac{\partial^3 u(x,t)}{\partial x^3} + \gamma u(x,t)\frac{\partial u(x,t)}{\partial x} + \mu v(x,t)\frac{\partial v(x,t)}{\partial x}, \\ \frac{\partial v(x,t)}{\partial t} &= \lambda\frac{\partial^3 v(x,t)}{\partial x^3} - \nu u(x,t)\frac{\partial v(x,t)}{\partial x}. \end{aligned} \tag{1}$$

Different numerical method are used to solve different fractional differential equation, such as the residual power series method by H.M.Jaradat [4], and the hompotopy perturbation method by Sarmad A. Altaie [5], the projected differential transform Method by Sunday O. Edeki [6] The solution of coupled Korteweg-de Vries are introduced using different numerical methods, to obtain an accurate approximations and their properties, such as the Spectral collocation method used by Khader [7], and Hussain [8]. Homotopy-Sumudu transforms method by A.K alomari [9]. Bashan [10] used the Finite difference method and differential quadrature method. Bakodah applied the decomposition method [11], homotopy perturbation method investigated by Bothayna S.Kashkari [12], homotopy analysis method (HAM) in [15] used by Abbsbandy and the homotopy analysis transform method [16] by Saad.

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Several methods are used for solving differential equations, such as Laplace-Adomian Decomposition Method [17], Legendre Neural Network DDY2 Method [18], and Chebyshev Collocation Methods [19]. The homotopy analysis method is one of the most important approximate methods for solving the linear and nonlinear differential equations, including the fractional differential equations. HAM proposed in Ph.D. dissertation in 1992 of Shijun Liao [20],[21], and [22]. The method has many applications in various classes of famous differential equations.

Usually, the standard fractional differential equation contains one fractional parameter. Recently, Several definitions are introduced the fractional calculus derivative and integral with two or three parameters. Almeida et. al. [23] introduced the Caputo-Katugampola derivative which contains two parameters. Odibat and Baleanu [24] investigated the Caputo-Katugampola derivative for general cases. Abdeljawad [25] introduced the fractional operators with generalized Mittag-Leffler kernels which contain three fractional parameters. The effect of those new parameters can also reflect on the solution behaviors. For example, Alomari et. al. [26] presented the effect of the generalized Mittag-Leffler kernels for fractional parabolic equations. In this paper, we investigate the HAM for the following two parameter fractional differential equation [1]

$$\begin{aligned} {}^C D^{\alpha,\rho}_{a^+,t} u(x,t) &= \eta D_{xxx}u(x,t) + \gamma u(x,t)D_x u(x,t) \\ &\quad + \mu v(x,t)D_x v(x,t), \\ {}^C D^{\beta,\rho}_{a^+,t} v(x,t) &= \lambda D_{xxx}v(x,t) - \nu u(x,t)D_x v(x,t), \end{aligned} \tag{2}$$

where ${}^C D^{\alpha,\rho}_{a^+,t}$ is the generalized fractional derivative operator of order $0 < \alpha \leq 1$ and $\rho > 0$ in the Caputo sense.

II. BASIC DEFINITIONS OF FRACTIONAL CALCULUS

The *Riemann-Liouville* fractional operator of order $\alpha \geq 0$, for a function h defined as

$$\begin{aligned} I^\alpha h(t) &= \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{(\alpha-1)}h(s)ds, \quad \mu > 0, t > 0 \\ I^0 h(t) &= h(t) \end{aligned}$$

where $\Gamma(.)$ is well-known Gamma function.
The Caputo fractional derivative of h given by

$$D^\alpha_C h(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\varsigma)^{(m-\alpha-1)}h^{(m)}(\varsigma)d\varsigma,$$

where $m-1 < \alpha < m, m \in \mathbb{N}$. The generalized fractional integral of the function h, $I^{\alpha,\rho}_{a^+}h(t)$, of order $\alpha > 0$, where $\rho > 0$ is given by[24]

$$\left(I^{\alpha,\rho}_{a^+}h\right)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha-1}f(s)ds \quad t > a.$$

Almeida et al. [23] defined the following Caputo-Katugampola derivative with two parameters, namely $0 < \alpha \leq 1$ and $\rho > 0$.

$$\left({}^{C}D_{a^+}^{\alpha,\rho}h\right)(t) = \frac{\rho^{\alpha}}{\Gamma(1-\alpha)}\int_a^t \left(t^{\rho}-s^{\rho}\right)^{-\alpha} f'(s)ds \quad 0<\alpha\leq 1, t>a\geq 0.$$

Recently, Odibat and Baleanu [24] modified the definition of two parameters for $m-1<\alpha\leq m$ in the form

Definition 2.1: The generalized Caputo derivative of the function $h : [0,\infty) \longrightarrow \mathbb{R}$, ${}^{C}D_{a^+}^{\alpha,\rho}h(t)$, of order $\alpha > 0$ is defined by:

$$\left({}^{C}D_{a^+}^{\alpha,\rho}h\right)(t) = \frac{\rho^{\alpha-m+1}}{\Gamma(m-\alpha)}\int_a^t s^{\rho-1}\left(t^{\rho}-s^{\rho}\right)^{m-\alpha+1} \left(s^{1-\rho}\frac{d}{ds}\right)^m f(s)ds, \quad m-1<\alpha\leq m, t>a\geq 0.$$

Whenever exist, where $m = \lceil\alpha\rceil$.

Theorem 2.2: [24]Let $m-1<\alpha\leq m, a\geq 0, \rho>0$ and $f\in C^m[a,b]$. Then, for $a<t\leq b$

$$I_{a^+}^{\alpha,\rho}D_{a^+}^{\alpha,\rho}f(t) = f(t) - \sum_{k=0}^{m-1}\frac{1}{\rho^k k!}\left(t^{\rho}-a^{\rho}\right)^k\left[\left(x^{1-\rho}\frac{d}{dx}\right)^k f(x)\right]_{x=a}, \tag{3}$$

Theorem 2.3: [24]Let $m-1<\alpha\leq m, a\geq 0, \rho>0$ and $f\in C^m[a,b]$. Then, for $a<t\leq b$

$$D_{a^+}^{\alpha,\rho}I_{a^+}^{\alpha,\rho}f(t) = f(t). \tag{4}$$

III. The Homotopy Analysis Method

Firstly, we present the general framework for solving two parameters fractional differential equation using the homotopy analysis method, so we consider the following form:

$${}^{C}D_{a^+}^{\alpha,\rho}h(x,t) + \Re h(x,t) + \mathfrak{N}h(x,t) = f(x,t), \quad 0<\alpha\leq 1, \rho>0 \tag{5}$$

where ${}^{C}D_{a^+}^{\alpha,\rho}h(x,t)$ is the generalized Caputo derivative of $h(x,t)$, $h_0(x,t)$ is the initial guess which satisfies the initial/ boundary conditions, $\Re$ and $\mathfrak{N}$ are the linear and nonlinear operator respectively, and $f(x,t)$ is the source term. By applying the homotopy analysis method shown in [14],[15] and [16], we define the non-linear operator

$$N[\phi(x,t,q)] = {}^{C}D_{a^+}^{\alpha,\rho}\phi(x,t,q) + \Re\phi(x,t,q) + \mathfrak{N}\phi(x,t,q) - f(x,t) \tag{6}$$

where $\phi(x,t,q)$ is a real-valued function of x,t and $q\in[0,1]$. The zeroth-order deformation constructed by Liao [14],[15] is

$$(1-q)\mathcal{L}[\phi(x,t,q)-h_0(x,t)] = \hbar qN[\phi(x,t,q)] \tag{7}$$

where $\hbar\neq 0$ is nonzero convergent control parameter, $h_0(x,t)$ is the initial guess, N is the nonlinear operator and $\mathcal{L}$ is an injective linear operator. Let $\mathcal{L} = D_{a^+}^{\alpha,\rho}$. Obviously $\phi(x,t,0) = h_0(x,t)$ and $\phi(x,t,1) = h(x,t)$. We expand $\phi(x,t,q)$ in Taylor series with respect q,

$$\phi(x,t,q) = \sum_{i=0}^{\infty}h_i(x,t)q^i$$

where

$$h_i(x,t) = \frac{1}{i!}\frac{\partial^i\phi(x,t,q)}{\partial q^i}|_{q=0}, \tag{8}$$

by differentiating (7) m times with respect q, and setting $q=0$, we obtain the m-th order deformation equation

$$\mathcal{L}[h_m(x,t)-\ell_m h_{m-1}(x,t)] = \hbar H(x,t)R_m(\vec{h}_{m-1}(x,t)) \tag{9}$$

where the vector $\vec{h}_{m-1} = \{h_0,h_1,h_2,\cdots,h_{m-1}\}$. Applying $\mathcal{L}^{-1} = I_{a^+}^{\alpha,\rho}$ in (9) with $H(x,t)=1$, we get:

$$\begin{aligned} h_m(x,t) &= \ell_m h_{m-1}(x,t) + \hbar I_{a^+}^{\alpha,\rho}[R_m(h_{m-1}(x,t))] \\ &+ \sum_{k=0}^{\lceil\alpha\rceil-1}\frac{1}{\rho^k k!}\left(t^{\rho}-a^{\rho}\right)^k \times \\ &\left[\left(s^{1-\rho}\frac{d}{ds}\right)^k(h_m(x,s)-\ell_m h_{m-1}(x,s))\right]_{s=a} \end{aligned} \tag{10}$$

where

$$\ell_m = \begin{cases} 0, & m\leq 1 \\ 1, & m>1 \end{cases}$$

We note that, the initial conditions can be calculated by applying the homotopy series $\sum_{i=0}^{\infty}h_i(x,0)q^i = c(x)$ which lead to $h_0(x,0) = c(x)$ and $h_i(x,0) = 0$ for $i = 1,2,3,\cdots$. Now, for our equation (2) with initial conditions $u(x,0) = f_1(x), v(x,0) = f_2(x)$, we define the nonlinear operator as:

$$\begin{cases} N_1[\phi(x,t,q),\psi(x,t,q)] &= D_{a^+}^{\alpha,\rho}\phi - \left[\eta\frac{\partial^3\phi}{\partial x^3} + \gamma\phi\frac{\partial\phi}{\partial x} + \mu\psi\frac{\partial\psi}{\partial x}\right], \\ N_2[\phi(x,t,q),\psi(x,t,q)] &= D_{a^+}^{\beta,\rho}\psi - \left[\lambda\frac{\partial^3\psi}{\partial x^3} - \nu\phi\frac{\partial\psi}{\partial x}\right], \end{cases} \tag{11}$$

where

$$\begin{cases} \phi(x,t,q) = \sum_{m=0}^{\infty}u_m(x,t)q^m, \\ u_m(x,t) = \frac{1}{m!}\frac{\partial^m\phi(x,t,q)}{\partial q^m}|_{q=0}, \\ \psi(x,t,q) = \sum_{m=0}^{\infty}v_m(x,t)q^m, \\ v_m(x,t) = \frac{1}{m!}\frac{\partial^m\psi(x,t,q)}{\partial q^m}|_{q=0}. \end{cases} \tag{12}$$

Now, the m-th order deformation equations are

$$\begin{aligned} D_{a^+}^{\alpha,\rho}[u_m(x,t)-\ell_m u_{m-1}(x,t)] = \hbar(D_{a^+}^{\alpha,\rho}u_{m-1} \\ +K_m[\overrightarrow{u}_{m-1}(x,t),\overrightarrow{v}_{m-1}(x,t)]) \\ D_{a^+}^{\beta,\rho}[v_m(x,t)-\ell_m v_{m-1}(x,t)] = \hbar(D_{a^+}^{\alpha,\rho}v_{m-1} \\ +R_m[\overrightarrow{u}_{m-1}(x,t),\overrightarrow{v}_{m-1}(x,t)]) \end{aligned} \tag{13}$$

with,

$$\begin{aligned} K_m &= \eta\frac{\partial^3u_{m-1}}{\partial x^3} + \gamma\sum_{i=0}^{m-1}u_i(x,t)\frac{\partial u_{m-1-i}}{\partial x} \\ &+\mu\sum_{i=0}^{m-1}v_i\frac{\partial v_{m-1-i}}{\partial x}, \\ R_m &= \lambda\frac{\partial^3v_{m-1}}{\partial x^3} - \nu\sum_{i=0}^{m-1}u_i(x,t)\frac{\partial v_{m-1-i}}{\partial x}. \end{aligned}$$

subject to the initial conditions $u_m(x,0) = 0, v_m(x,0) = 0$ for $m = 1,2,3,\ldots$. At this line, we Apply the inverse operator $\mathcal{L}^{-1} = I_{0^+}^{\alpha,\rho}$ to have

$$\begin{aligned} u_m(x,t) &= (\ell_m+\hbar)u_{m-1} \\ &+\hbar I_{0^+}^{\alpha,\rho}K_m[\overrightarrow{u}_{m-1}(x,t),\overrightarrow{v}_{m-1}(x,t)] \\ &+u_m(x,0)-(1+\ell_m)u_{m-1}(x,0), \\ v_m(x,t) &= (\ell_m+\hbar) \\ &+\hbar I_{0^+}^{\beta,\rho}R_m[\overrightarrow{u}_{m-1}(x,t),\overrightarrow{v}_{m-1}(x,t)] \\ &+v_{m-1}+(v_m(x,0)-(1+\ell_m)v_{m-1}(x,0)), \end{aligned}$$

for $m = 1, 2, 3, \ldots$. Finally, the $M-$th order of series solutions are

$$\begin{aligned} u(x,t) &= u_0(x,t) + \textstyle\sum_{i=1}^{M} u_i(x,t), \\ v(x,t) &= v_0(x,t) + \textstyle\sum_{i=1}^{M} v_i(x,t). \end{aligned} \quad (14)$$

As $M \to \infty$, we have the exact solution.

IV. Numerical experiment

In this section, we evaluate the fractional coupled Korteweg-de Vries equation (2) using HAM with initial conditions. For that, we consider (2) with $\eta = -a, \gamma = -6a, \mu = 2b, \lambda = -r$ and $\nu = 3r$ subject to the initial conditions

$$\begin{aligned} u(x,0) &= \tfrac{\zeta}{a}\left(sech(\tfrac{1}{2}\sqrt{\tfrac{\zeta}{a}}x)\right)^2 \\ v(x,0) &= \tfrac{\zeta}{\sqrt{2a}}\left(sech(\tfrac{1}{2}\sqrt{\tfrac{\zeta}{a}}x)\right)^2. \end{aligned} \quad (15)$$

Apply the HAM algorithm in section III, and we obtained the following first few terms of the approximation:

$$\begin{aligned} u_0(x,t) &= sech(\tfrac{x}{2})^2, \\ u_1(x,t) &= -\frac{8\hbar t^{\alpha\rho}\rho^{-\alpha}\sinh^4\left(\frac{x}{2}\right)csch^3(x)}{\alpha\Gamma(\alpha)}, \\ u_2(x,t) &= -\frac{\hbar t^{\alpha}(sech[\frac{x}{2}]^4\tanh[\frac{x}{2}]^2+sech[\frac{x}{2}]^2\tanh[\frac{x}{2}]^3)}{\Gamma[\alpha+1]} + \\ &= \frac{\hbar sech^6\left(\frac{x}{2}\right)\rho^{-2\alpha-\beta}t^{\alpha\rho}\left(\beta\rho^{\beta}\Gamma(\beta)\Gamma(\alpha+\beta+1)\right)}{8\alpha\beta\Gamma(\alpha)\Gamma(2\alpha+1)} \\ & \frac{(\hbar\Gamma(\alpha+1)(22\cosh(x)+\cosh(2x)-39)t^{\alpha\rho})}{\Gamma(\beta)\Gamma(\alpha+\beta+1)} \\ & \frac{-12\alpha\hbar\rho^{\alpha}\Gamma(\alpha)\Gamma(2\alpha+1)\Gamma(\beta+1)(2\cosh(x)-3)t^{\beta\rho}}{8\alpha\beta\Gamma(\alpha)\Gamma(2\alpha+1)} \\ & \frac{-2(\hbar+1)\rho^{\alpha}\Gamma(2\alpha+1)\sinh(x)(\cosh(x)+1)}{\Gamma(\beta)\Gamma(\alpha+\beta+1)} \end{aligned}$$

for $v(x,t)$, we have:

$$\begin{aligned} v_0(x,t) &= \frac{sech^2\left(\frac{x}{2}\right)}{\sqrt{2}}, \\ v_1(x,t) &= -\frac{4\sqrt{2}\hbar\rho^{-\beta}\sinh^4\left(\frac{x}{2}\right)csch^3(x)t^{\beta\rho}}{\beta\Gamma(\beta)}, \\ v_2(x,t) &= -\frac{\hbar sech^6\left(\frac{x}{2}\right)\rho^{-\alpha-2\beta}t^{\beta\rho}}{8\sqrt{2}\alpha\beta\Gamma(\alpha)\Gamma(\beta)\Gamma(2\beta+1)} \\ & \frac{\left(\alpha\rho^{\alpha}\Gamma(\alpha)\Gamma(\alpha+\beta+1)\left(2(\hbar+1)\rho^{\beta}\Gamma(2\beta+1)\sinh(x)(\cosh(x)+1)\right.\right.}{\Gamma(\alpha+\beta+1)} \\ & \frac{\left.\left.-\hbar\Gamma(\beta+1)(-14\cosh(x)+\cosh(2x)+9)t^{\beta\rho}\right)\right)}{8\sqrt{2}\alpha\beta\Gamma(\alpha)\Gamma(\beta)} \\ & \frac{-12\beta\hbar\Gamma(\alpha+1)\rho^{\beta}\Gamma(\beta)\Gamma(2\beta+1)(\cosh(x)-1)t^{\alpha\rho}}{\Gamma(2\beta+1)\Gamma(\alpha+\beta+1)} \end{aligned}$$

Now, we study the effect of the two parameter fractional derivative on the Korteweg-de Vries equation. Firstly, we optimally choose the convergent control parameter $\hbar$ by defining the residual error

$$\begin{aligned} Res1 &= {}^{C}D^{\alpha,\rho}_{a^+,t}u(x,t) - (\eta D_{3x}u(x,t) \\ &\quad +\gamma u(x,t)D_x u(x,t) + \mu v(x,t)D_x v(x,t)) \\ Res2 &= {}^{C}D^{\alpha,\rho}_{a^+,t}v(x,t) \\ &\quad -\lambda D_{3x}v(x,t) + \nu u(x,t)D_x v(x,t), \end{aligned} \quad (16)$$

where $u(x,t)$, and $v(x,t)$ are the HAM solutions. Now, via the least square error we have

$$\Delta(\hbar) = \frac{1}{(M+1)(N+1)}\sum_{i=0}^{M}\sum_{j=0}^{N}\left(Res\left(\frac{i}{M},\frac{j}{N}\right)\right)^2. \quad (17)$$

By minimize the function $\Delta(\hbar)$, we can determine the optimal value of $\hbar$. In figures 1,4 we plot $\hbar-$ curve with fixed $\alpha = \beta = \rho = 1$. The optimal values of $\hbar$ when $\alpha = \beta = 0.9$

TABLE I
Comparison of HAM results of $u(x,t)$, and $v(x,t)$ with other result obtained by other numerical method for different values of $\alpha = \beta = 1$.

x	t	Exact($u(x,t)$)	HAM
-10	0.1	0.000164304	0.000164304
-10	0.2	0.000148670	0.000148670
-5	0.1	0.024092321	0.024092321
-5	0.2	0.021824797	0.021824797
5	0.1	0.029347625	0.029347625
5	0.2	0.032383774	0.032383773
10	0.1	0.000200678	0.000200678
10	0.2	0.000221781	0.000221781
x	t	Exact($v(x,t)$)	HAM
-10	0.1	0.000116180	0.000116180
-10	0.2	0.000105125	0.000105125
-5	0.1	0.017035843	0.017035843
-5	0.2	0.015432462	0.015432462
5	0.1	0.020751905	0.020751905
5	0.2	0.022898786	0.022848786
10	0.1	0.000141901	0.000141901
10	0.2	0.000156823	0.000156823

Error (HAM)	Error(NDM)[1]	Error[2]
7.86041×10^{-19}	2.95039×10^{-8}	2.99039×10^{-8}
2.57498×10^{-18}	2.30335×10^{-7}	2.33335×10^{-7}
1.11022×10^{-16}	3.93592×10^{-6}	3.96592×10^{-6}
3.43475×10^{-16}	3.08049×10^{-5}	3.38049×10^{-5}
4.71428×10^{-14}	2.02966×10^{-4}	3.97592×10^{-6}
5.97775×10^{-13}	5.15558×10^{-4}	3.78049×10^{-5}
4.24086×10^{-16}	2.11254×10^{-8}	2.96039×10^{-8}
8.61846×10^{-15}	2.28173×10^{-7}	2.37335×10^{-7}
Error (HAM)	Error(NDM)[1]	Error[2]
1.19262×10^{-18}	2.08624×10^{-8}	2.18624×10^{-8}
2.43945×10^{-19}	1.62872×10^{-7}	1.64872×10^{-7}
1.76942×10^{-16}	2.78312×10^{-6}	2.88312×10^{-6}
1.73472×10^{-17}	2.21782×10^{-5}	2.87824×10^{-5}
3.33934×10^{-14}	2.90940×10^{-6}	2.98312×10^{-6}
4.22783×10^{-13}	2.23803×10^{-5}	2.47824×10^{-5}
3.00297×10^{-16}	2.19313×10^{-8}	2.09624×10^{-8}
6.09484×10^{-15}	1.79991×10^{-7}	1.72872×10^{-7}

are $-0.08729, -0.07339, -0.05500$ for $\rho = 1.2, 1, 0.75$ respectively.

In Table I, we can see that the approximation solution obtained by HAM, is agreed and appropriate with the exact solution more than the NDM and spectral collection methods. Moreover, Table II present the HAM solution for various values of fractional order α, β and ρ.

Figures 1,2,3 and 4 present the 10-th order of the HAM solution with the exact solution.

$$u(x,t) = \frac{\zeta}{a}\left(sech\left(\frac{1}{2\sqrt{\frac{\zeta}{a}}(x-\zeta t)}\right)\right)^2$$

$$v(x,t) = \frac{\zeta}{\sqrt{2a}}\left(sech\left(\frac{1}{2\sqrt{\frac{\zeta}{a}}(x-\zeta t)}\right)\right)^2.$$

The HAM solution and the exact solution have coincided. Now, we study the effect of the new parameter ρ on the solution. For that, we plot the HAM solution with $\alpha = 0.9$ and several values of ρ is presented in figure 7 and 8 for $u(x,t)$ and $v(x,t)$ respectively. We can observe the effect of the new parameter ρ on the solution of the coupled Korteweg-de Vries equation. Finally, we study the effect of α on the solution with fixed $\rho = 0.9$ in figures 9 and 10. It is worth

TABLE II
10−TH ORDER APPROXIMATIONS OF $u(x,t)$, AND $v(x,t)$ FOR DIFFERENT VALUES OF α, β, AND ρ.

x	t	Exact $u(x,t)$	$\alpha=\beta=0.9, \rho=1$
-10	0.1	0.000164304	0.000169184
-10	0.2	0.000148670	0.000158888
-5	0.1	0.024092321	0.024799633
-5	0.2	0.021824797	0.023309349
5	0.1	0.029347625	0.028463892
5	0.2	0.032383774	0.030150474
10	0.1	0.000200678	0.000194543
10	0.2	0.000221781	0.000206235
x	t	Exact $v(x,t)$	$\alpha=\beta=0.9, \rho=1$
-10	0.1	0.000116180	0.000119631
-10	0.2	0.000105125	0.000112351
-5	0.1	0.017035843	0.017535989
-5	0.2	0.015432462	0.016482199
5	0.1	0.020751905	0.020127011
5	0.2	0.022898786	0.021319605
10	0.1	0.000141901	0.000137563
10	0.2	0.000156823	0.000145830

x	t	$\alpha=\beta=0.9$ $\rho=1.2$	$\alpha=\beta=0.9$ $\rho=0.75$
-10	0.1	0.000173718	0.000160061
-10	0.2	0.000165224	0.000147973
-5	0.1	0.025455300	0.023479293
-5	0.2	0.024226448	0.021728234
5	0.1	0.027761809	0.029936917
5	0.2	0.029103962	0.032047081
10	0.1	0.000189680	0.000204753
10	0.2	0.000198980	0.000219394
x	t	$\alpha=\beta=0.9,$ $\rho=1.2$	$\alpha=\beta=0.9,$ $\rho=0.75$
-10	0.1	0.000122837	0.000113180
-10	0.2	0.000116831	0.000104633
-5	0.1	0.017999615	0.016602367
-5	0.2	0.017130685	0.015364181
5	0.1	0.019630563	0.021168597
5	0.2	0.020579609	0.022660708
10	0.1	0.000134124	0.000144782
10	0.2	0.000140700	0.000155135

x	t	Exact $u(x,t)$	$\alpha=\beta=0.9, \rho=1$
-10	0.1	0.000164304	0.000169184
-10	0.2	0.000148670	0.000158888
-5	0.1	0.024092321	0.024799633
-5	0.2	0.021824797	0.023309349
5	0.1	0.029347625	0.028463892
5	0.2	0.032383774	0.030150474
10	0.1	0.000200678	0.000194543
10	0.2	0.000221781	0.000206235
x	t	Exact $v(x,t)$	$\alpha=\beta=0.9, \rho=1$
-10	0.1	0.000116180	0.000119631
-10	0.2	0.000105125	0.000112351
-5	0.1	0.017035843	0.017535989
-5	0.2	0.015432462	0.016482199
5	0.1	0.020751905	0.020127011
5	0.2	0.022898786	0.021319605
10	0.1	0.000141901	0.000137563
10	0.2	0.000156823	0.000145830

mentioning that the new parameter ρ can highly affect the solution of the coupled Korteweg-de Vries equation.

V. CONCLUSION

In this work, we successfully construct the HAM solution for a two parameter fractional coupled Korteweg-de Vries equation, and we show that this method is efficient and applicable for this type of fractional differential equation. We compared our results with others obtained by different numerical methods. The algorithm is robust for solving this kind of equations.

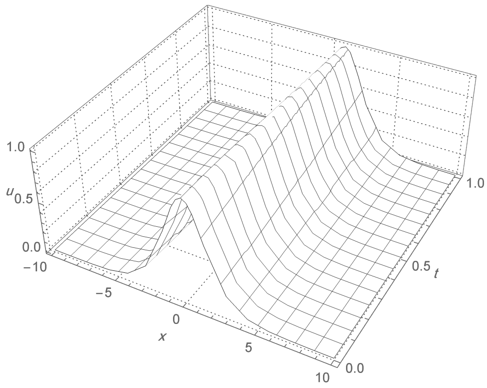

Fig. 1. The exact solution of $u(x,t)$

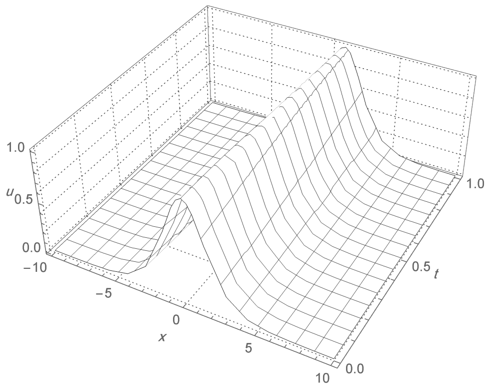

Fig. 2. The HAM solution of $u(x,t)$

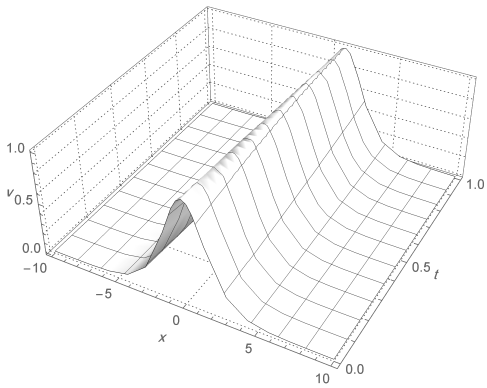

Fig. 3. The exact solution of $v(x,t)$

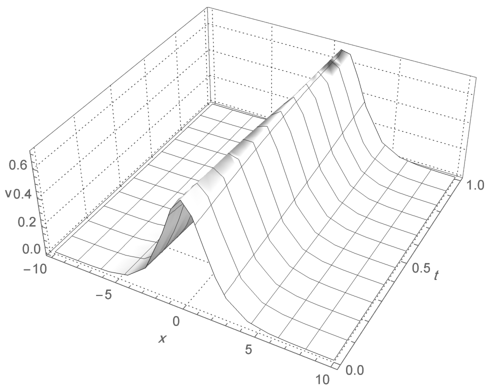

Fig. 4. The HAM solution of $v(x,t)$

x	t	$\alpha=\beta=0.8$ $\rho=0.75$	$\alpha=\beta=0.8$ $\rho=1$	$\alpha=\beta=0.8$ $\rho=1.2$
-10	0.1	0.000159982	0.000168279	0.000172611
-10	0.2	0.000149504	0.000158844	0.000164411
-5	0.1	0.023467727	0.024668556	0.025295201
-5	0.2	0.021949788	0.023302797	0.024108744
5	0.1	0.029965984	0.028611447	0.027933303
5	0.2	0.031807379	0.030171331	0.029242420
10	0.1	0.000204956	0.000195566	0.000190868
10	0.2	0.000217735	0.000206382	0.000199940
x	t	$\alpha=\beta=0.8$ $\rho=0.75$	$\alpha=\beta=0.8$ $\rho=1$	$\alpha=\beta=0.8$ $\rho=1.2$
-10	0.1	0.000113124	0.000118991	0.000122054
-10	0.2	0.000105715	0.000112319	0.000116256
-5	0.1	0.016594189	0.017443303	0.017886408
-5	0.2	0.015520844	0.016477565	0.017047457
5	0.1	0.021189150	0.020231348	0.019751828
5	0.2	0.022491213	0.021334353	0.020677513
10	0.1	0.000144926	0.000138286	0.000134964
10	0.2	0.000153961	0.000145934	0.000141379

x	t	$\alpha=\beta=0.7$ $\rho=0.75$	$\alpha=\beta=0.7$ $\rho=1$	$\alpha=\beta=0.7$ $\rho=1$
-10	0.1	0.000159964	0.000167315	0.000171392
-10	0.2	0.000151048	0.000158811	0.000163633
-5	0.1	0.023465027	0.024529012	0.025118849
-5	0.2	0.022173228	0.023297819	0.023996069
5	0.1	0.029987148	0.028772119	0.028125439
5	0.2	0.031565605	0.030194573	0.029380122
10	0.1	0.000205105	0.000196680	0.000192199
10	0.2	0.000216061	0.000206545	0.000200896
x	t	$\alpha=\beta=0.7$ $\rho=0.75$	$\alpha=\beta=0.7$ $\rho=1$	$\alpha=\beta=0.7$ $\rho=1.2$
-10	0.1	0.000113112	0.000118309	0.000121192
-10	0.2	0.000106807	0.000112296	0.000115706
-5	0.1	0.016592279	0.017344631	0.017761708
-5	0.2	0.015678840	0.016474046	0.016967783
5	0.1	0.021204115	0.020344961	0.019887688
5	0.2	0.022320253	0.021350787	0.020774883
10	0.1	0.000145031	0.020344961	0.000135905
10	0.2	0.000152778	0.021350787	0.000142055

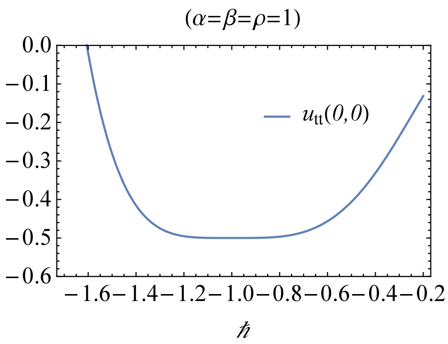

Fig. 5. The $\hbar-$curve using 10-order of approximation

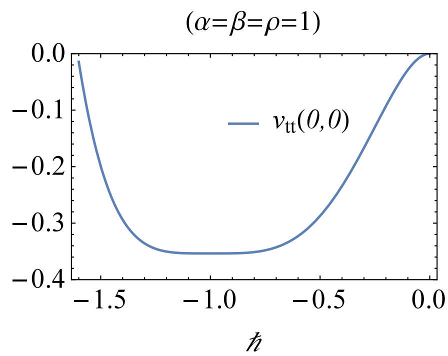

Fig. 6. The $\hbar-$curve using 10-order of approximation

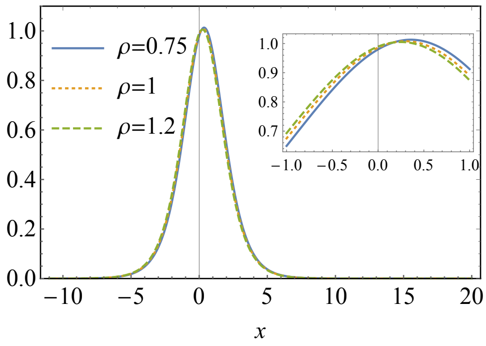

Fig. 7. The HAM solution for $u(x,0.5)$ with $\alpha=0.9$ and several values of ρ.

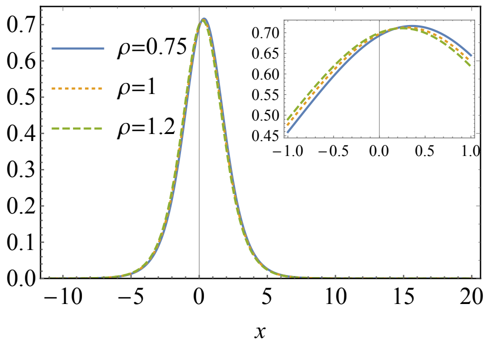

Fig. 8. The HAM solution for $v(x,0.5)$ with $\alpha=0.9$ and several values of ρ.

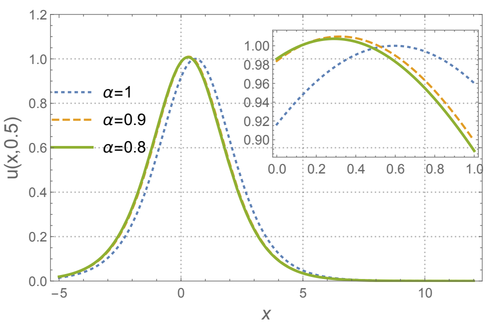

Fig. 9. The HAM solution for $u(x,0.5)$ with $\rho=0.9$ and several values of α.

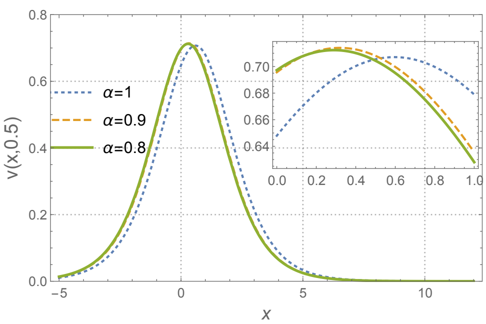

Fig. 10. The HAM solution for $v(x,0.5)$ with $\rho=0.9$ and several values of α.

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