Cubic B-spline Technique for Numerical Solution of Singularly Perturbed Convection-Diffusion Equations with Discontinuous Source Term

Shilpkala Mane, Ram Kishun Lodhi

Abstract—In this paper, we have developed a cubic B-spline technique to obtain numerical solutions for second-order linear singularly perturbed boundary value problems of convection-diffusion type equations with discontinuous source terms. The solution to such types of problems contains boundaries and an interior layer. The error analysis of the proposed method has been studied and also proved the convergence of the method. The cubic B-spline technique has been implemented on two numerical examples which shows the efficiency and accuracy of the method. Obtained numerical results of the proposed method have compared with other existing methods and found that it gives better numerical solutions at the same number of mesh points.

Index Terms—Singularly perturbed boundary value problems, convection-diffusion equation, interior layer, discontinuous source term, cubic B-spline method, convergence.

I. INTRODUCTION

SINGULARLY perturbed problems (SPPs) with discontinuous source terms are differential equations (DE) where the highest derivative term of differential equations is multiplied by a small positive parameter and contains a discontinuity term. These types of problems occur in various engineering and applied science branches, such as fluid mechanics, modeling of semiconductor devices, mechanical and electrical frameworks, quantum mechanics, Navier–Stokes equation at high Reynolds number, etc. [1], [2]. The numerical solution of SPPs with discontinuous source terms present some major computational difficulties due to the presence of perturbation parameter ϵ, interior layer and discontinuous term. Farrell et al. [3], [4] have discussed the numerical solution of convection–diffusion problems with discontinuous source term.

We propose a cubic B-spline method (CBSM) to find the numerical solution of second order linear singularly perturbed convection-diffusion equations with discontinuous source term. It is appropriate to set up the notation for convection-diffusion equations with discontinuous source terms. The solution to such types of problems contains boundaries and an interior layer. The error analysis of the proposed method has been studied and also proved the convergence of the method. The cubic B-spline technique has been implemented on two numerical examples which shows the efficiency and accuracy of the method. Obtained numerical results of the proposed method have compared with other existing methods and found that it gives better numerical solutions at the same number of mesh points.

II. CUBIC B-SPACE METHOD

In this section, the CBSM is described to determine the approximate solution of equation (1)-(2). We divide the domain Ψ = (0, 1) into two subdomains Ψ− and Ψ+.

\[ v(0) = v_0, \ v(1) = v_1 \]
Each subdomains are divided into $N/2$ equal points and piecewise uniform mesh points are generated by $y_i = i h_1$ in the subdomain $\Psi^-$, where $h_1 = 2 b / N$, for $i = 0, 1, 2, \ldots, N/2$ and $y_i = i h_2$ in the subdomain $\Psi^+$, where $h_2 = 2 (1 - b) / N$, for $i = N/2 + 1, N/2 + 2, \ldots, N$. The set of partition of subdomains $\Psi^-$ and $\Psi^+$ are defined as: $\pi_1 = \{ 0 = y_0, y_1, \ldots, y_{N/2} \}$, $\pi_2 = \{ y_{N/2+1}, y_{N/2+2}, \ldots, y_N = 1 \}$. We define $L_2 [0,1]$ as a vector space of all square integrable functions on $\Psi$ and $X$ be the linear subspace of $L_2 [0,1]$. We introduce two additional mesh points on each side of the partition $\pi = \pi_1 \cup \pi_2$ as $y_{-2} < y_1 < y_0$ and $y_N < y_{N+1} < y_{N+2}$. Now the cubic B-spline basis functions are defined [26], [27] for $i = -1, 0, \ldots, N + 1$, as:

\[
\varpi_i (y) = \begin{cases} 
\frac{(y - y_i)^3}{h^3}, & \text{if } y \in [y_{i-1}, y_i] \\
-h^3 + 3h^2 (y - y_i) - 3h (y - y_i)^2 + (y - y_i - y_{i-1})^3, & \text{if } y \in [y_{i-1}, y_i] \\
-h^3 + 3h^2 (y_{i+1} - y) - 3h (y_{i+1} - y)^2 + (y_{i+1} - y - y_{i+2})^3, & \text{if } y \in [y_{i+1}, y_i + 1] \\
0, & \text{otherwise}
\end{cases}
\]

(3)

The functions $\varpi_i (y)$ given by equation (3) is a piecewise cubic polynomial with mesh points $y_i$ for $0 \leq i \leq N$ and $\varpi_i (y) \in X$, also, each $\varpi_i (y)$ is twice continuously differentiable functions on the whole interval. The values of basis functions $\varpi_i (y)$, $\varpi_i' (y)$, $\varpi_i'' (y)$ at the mesh points $y_i$ are given in Table I.

| TABLE I: BASIS FUNCTIONS AND THEIR DERIVATIVES VALUES AT MESH POINTS |
|-----------------|-----|-----|
| $y_{i-1}$ | $y_i$ | $y_{i+1}$ |
| $\varpi_i (y)$ | 1 | 4 | 4 |
| $\varpi_i' (y)$ | -3/h | 0 | -3/h |
| $\varpi_i'' (y)$ | 6/h^2 | -12/h^2 | 6/h^2 |

Let $\chi = \{ \varpi_1, \varpi_0, \varpi_1, \ldots, \varpi_N \}$ and $\xi_3 (\pi) = \text{span } \chi$. The function $\chi$ is linearly independent on $\Psi$ and hence $\xi_3 (\pi)$ is $N+3$ dimensional. It is observed that $\xi_3 (\pi)$ is a linear subspace of $X$. Let $L \in L_2 [0,1]$ be a linear operator whose domain and range are $X$. Let $S (y)$ be an approximate solution of (y) defined as $S (y) = \sum_{i=1}^{N+1} \alpha_i \varpi_i (y)$, where $\alpha_i$ are unknowns real coefficients. Hence

\[
L_N^\infty (S (y)) = g(y), \ y_i \in \Psi^- \cup \Psi^+ \quad (4)
\]

$S (y_0) = v_0$, $S (y_N) = v_N \quad (5)$

At the point of discontinuity $y_N = b$, we shall use the hybrid difference operator:

\[
L_N^\infty (y_N) = \begin{cases} 
-\frac{y_N^2 + 4y_N y_N - 3y_N}{2h} - \frac{y_N^2 - 4y_N y_N + 3y_N}{2h} = 0 
\end{cases}
\]

To find the approximate solution $S (y)$ by using CBSM, we calculate the spline functions at the mesh points $y = y_i$. From Table I and assumed cubic B-spline solution $S (y)$, we can obtain the following relationships in the domain $\Psi^- \cup \Psi^+$:

\[
S (y_i) = \alpha_i + 4 \alpha_{i+1} + \alpha_{i+1} \quad (6)
\]

\[
S' (y_i) = \frac{3}{h} (-\alpha_i + \alpha_{i+1}) \quad (7)
\]

\[
S'' (y_i) = \frac{6}{h^2} (\alpha_i - 2 \alpha_{i+1} + \alpha_{i+1}) \quad (8)
\]

Hence, we get

\[
\varepsilon S'' (y) + a (y) S' (y) = g(y), \ y \in \Psi^- \cup \Psi^+ \quad (9)
\]

Discretizing equation (9) at the mesh points $y_i$ for $0, 1, \ldots, N-1, N, 1, \ldots, N$, we obtain

\[
\varepsilon S'' (y_i) + a (y_i) S' (y_i) = g(y_i) \quad (10)
\]

Using equations (7) and (8) in equation (10) and simplifying, we get

\[
A_i \alpha_{i-1} + B_i \alpha_i + C_i \alpha_{i+1} = D_i, \quad (11)
\]

where, $A_i = (6 \varepsilon - 3 a_i h), \ B_i = -12 \varepsilon, \ C_i = (6 \varepsilon + 3 a_i h), \ D_i = g_i h^2, \ for \ i = 0, 1, \ldots, N$ and hence (9)

\[
\alpha_0 + 4 \alpha_1 + \alpha_1 = v_0 \quad (13)
\]

\[
\alpha_{N-1} + 4 \alpha_N + \alpha_{N+1} = v_N \quad (14)
\]

Hence in the interval $\Psi$ there are $N+3$ equations and $N+3$ unknowns, including discontinuity condition at point $b$ and boundary conditions. Now eliminating $\alpha_{i-1}$ from the first equation of equation (11) and (13) and $\alpha_{N+1}$ from the last equation of equation (11) and (14), we have

\[
(-36 \varepsilon + 12 a_0 h) \alpha_0 + (6 a_0 h) \alpha_1 = g_0 h^2 - v_0 (6 \varepsilon - 3 a_0 h) \quad (15)
\]

\[
(-6 a_0 h) \alpha_{N-1} + (-36 \varepsilon - 12 a_N h) \alpha_N = g_N h^2 - v_1 (6 \varepsilon + 3 a_N h) \quad (16)
\]

Coupling equation (15) and (16) with the second ($N - 2$) equations of (11) including discontinuity condition at point $b$, this lead to $N + 1$ linear equations in $N + 1$ unknowns. These system of equations can be expressed in matrix form $Ax = t_n$, where a matrix of unknown $x_n = (0_0, \ldots, 0_N)$, right-hand column matrix $t_n$ is given by

\[
t_n = (g_N h^2 - v_1 (6 \varepsilon + 3 a_N h), \ h_i g_i (y_{i+1}), \ldots, h_1 g_1 (y_N) - v_1, 0, \ldots, 0)
\]

Here matrix $A$ is nonsingular, we can solve the system $Ax = t_n$ for $0_0, \ldots, 0_N$ and substitute in the boundary conditions (13) and (14) to obtain $\alpha_{N+1}$. 

\[
A = \begin{bmatrix}
R_0 & 0 & \cdots & 0 & 0 \\
A_1 & B_1 & C_1 & \cdots & 0 \\
0 & A_2 & B_2 & \cdots & 0 \\
0 & 0 & A_3 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & A_{N-1} & B_{N-1} \\
0 & 0 & \cdots & 0 & C_{N-1} \\
-6a_0 h & 0 & \cdots & 0 & R_N
\end{bmatrix}
\]
where \( R_0 = (-36ε + 12a_0 h_1) \) and \( R_N = (-36ε - 12a_N h_2) \). In the domain \( Ψ^- \cup Ψ^+ \), using the recurrence relationship of spline approximation given by equations (11) and (12), the difference relation is given by equation (17).

\[
α_i = H_i α_{i+1} + K_i,
\]

(17)

where,

\[
H_i = \frac{C_i}{B_i - A_i H_{i-1}}
\]

and

\[
K_i = \frac{A_i K_{i-1} - D_i}{B_i - A_i H_{i-1}}
\]

for \( (1 ≤ i ≤ N) \). To begin the iteration method of \( H_i \) and \( K_i \), we find \( H_0 \) and \( K_0 \) from equations (11) and (13) by using initial values of \( H_0 \) and \( K_0 \), we find further values of \( H_i \) and \( K_i \) using forward process and find \( α_N = K_N \).

### III. STABILITY OF THE CBSM METHOD

In this section, we prove that the proposed method is stable in the domain \( Ψ^- \cup Ψ^+ \). Stability means that the error obtained at one phase of calculations is not propagated into enormous errors in a further phase of calculations. Here local errors are not increased in successive calculations. Consider the recurrence relation given by equation (11). Let \( ε_i \) be a small error that has occurred in the computation of \( H_i \) then the exact value \( H_i \) is given as:

\[
H_i = H_i + ε_i
\]

(20)

and we are calculating

\[
\hat{H}_i = \frac{C_i}{(B_i - A_i H_{i-1})}.
\]

(21)

From equations (20) and (21), we have

\[
e_i = \frac{C_i}{B_i - A_i (H_{i-1} + ε_{i-1})} - \frac{C_i}{(B_i - A_i H_{i-1})}
\]

(22)

\[
e_i = \frac{A_i C_i ε_{i-1}}{(B_i - A_i (H_{i-1} + ε_{i-1}) (B_i - A_i H_{i-1})}
\]

\[
e_i = H_i^2 A_i \frac{C_i}{C_i} ε_{i-1}.
\]

Under the assumption that initially, the error is small. Then from the definition of \( A_i \), \( B_i \) and \( C_i \) with the belief that \( a(y) > 0 \), we can show that \( |B_i| ≥ |A_i + C_i| \), for all \( (1 ≤ i ≤ \frac{N}{2} - 1) \) \( \cup (\frac{N}{2} + 1 ≤ i ≤ N) \). From the initial condition of \( H_0 \), we can say that \( |H_0| < 1 \). Also, we have \( H_1 = \frac{C_1}{B_1 - A_1 H_0} \) and hence \( |H_1| < 1 \), since \( |H_0| < 1 \). Consequently, it follows that \( |H_i| < 1 \), for \( (1 ≤ i ≤ \frac{N}{2} - 1) \) \( \cup (\frac{N}{2} + 1 ≤ i ≤ N) \). Then from equation (22), we have,

\[
|e_i| = |H_i|^2 \frac{|A_i|}{|C_i|} |e_{i-1}|.
\]

(23)

Hence,

\[
|e_i| < |e_{i-1}|.
\]

Let a small error \( d_i \) has been made while calculating \( K_i \), then the exact value \( K_i \) is given as:

\[
K_i = K_i + d_i.
\]

(24)

Similar above arguments gives

\[
d_i = H_i \frac{A_i}{C_i} d_{i-1}.
\]

(25)

Hence, using the condition \( |H_i| < 1 \), for \( (1 ≤ i ≤ \frac{N}{2} - 1) \) \( \cup (\frac{N}{2} + 1 ≤ i ≤ N) \).

It follows that

\[
|d_i| = |H_i| \frac{|A_i|}{|C_i|} |d_{i-1}|.
\]

(26)

\[
< |d_{i-1}|
\]

(27)

Therefore, the recurrence relations \( H_i, K_i \) are stable.

### IV. UNIFORM CONVERGENCE OF CUBIC B-SPLINE METHOD

In this section, the convergence of CBSM is discussed by estimating the nodal errors. The following lemma is useful for the convergence proof of cubic B-spline method.

**Lemma 4.1.** The basis of cubic B-splines set \( χ = \{ ϖ_{−1}, ϖ_0, ϖ_1, \ldots, ϖ_{N+1} \} \) defined in equation (3) satisfies the inequality

\[
\sum_{i=−1}^{N+1} |ϖ_i(y)| ≤ 10, \quad 0 ≤ y ≤ 1.
\]

**Proof.** For the proof of the lemma, readers can refer [20].

**Theorem 4.1.** The collocation approximation \( S(y) \) from the space \( ξ_i (π) \) to the solution \( v(y) \) of the BVP (1)-(2) exists, further if \( g(y) ∈ Ψ^- \cup Ψ^+ \), then parameter uniform error estimate is given by

\[
∥ v(y) − S(y) ∥_∞ ≤ Rh^2
\]

(28)

for \( h \) sufficiently small and \( R \) is a positive constant (independent of \( N \) and \( ε \)).

**Proof.** We determine the error \( ∥ v(y) − S(y) ∥_∞ \) in the domain \( ϖ ∈ Ψ \). Let \( Z_n \) be the unique spline interpolant from \( ξ_i (π) \) to the solution \( v(y) \) of BVP (1)-(2). If \( g(y) ∈ Ψ^- \cup Ψ^+ \) and \( v(y) ∈ C^4(0, b) \cup (a, 1) \) then we have

\[
∥ D^j (v(y) − S(y)) ∥_∞ ≤ k_j h_c^{4−j}, \quad j = 0, 1, 2
\]

(29)

where, \( h_c = \max \{h_1, h_2 \} \) and \( k_j \) are independent of \( h \) and \( N \). Let

\[
Z_n (y) = \sum_{i=−1}^{N+1} \mu_i ϖ_i (y)
\]

(30)

and \( S(y) = ∑_{i=−1}^{N+1} ϖ_i (y) \) is collocation solution. From the equation (29), we have

\[
|LS(ϕ_i) − LZ_n (ϕ_i)| ≤ ph^2,
\]

(31)

where \( p = εk_2 + ||a(y)||_∞ k_1 h_c \). Now collocating conditions are \( LS(ϕ_i) = L(ϕ_i) = g(ϕ_i) \). Let \( LZ_n (ϕ_i) = g_n^1 (ϕ_i) \) for all \( i = 1, 0, \ldots, N + 1 \) and \( g_n^1 (ϕ_i) = (g_n^1 (ϕ−1), g_n^1 (ϕ_0), \ldots, g_n^1 (ϕ_{N+1}) \). Using the system \( Ax_n = t_n \) and equation (31), the
After simplification, we get
\[ h = \frac{1}{\beta} \left( \int |f''(\tau)| \, d\tau \right)^{1/2} \]

Now to determine \( h \), we consider \( \tau = \int_{0}^{1} \frac{1}{g(\tau)} \, d\tau \), which is the equation of the system.

Combining equations (43), (44) and using Lemma 1, we get
\[ |v(y) - S(y)|_{\infty} \leq \frac{1}{\alpha} \left( \frac{1}{\beta} \right)^{1/2} \]

where \( \tau = \frac{1}{\alpha} \left( \frac{1}{\beta} \right)^{1/2} \).

Hence, \( |v(y) - S(y)|_{\infty} \leq \frac{1}{\alpha} \left( \frac{1}{\beta} \right)^{1/2} R^2 \).

V. NUMERICAL ILLUSTRATION

In this section, we have implemented the CBSM on two test problems which shows the accuracy and order of convergence of the method and all the computation work is performed by MATLAB software.

Example 1. Consider the following two points singularly perturbed BVP with discontinuous source term:
\[ \varepsilon v''(y) + v'(y) = g(y), \quad y \in \Psi - \bigcup_{i=1}^{N} \Psi_{i} \]

where \( g(y) = \begin{cases} -9, & y < \frac{1}{3} \\ \frac{1}{9}(y - 1)^2, & y > \frac{1}{3} \end{cases} \).

TABLE II: MAE AND RATE OF CONVERGENCE OF EXAMPLE 1 FOR DIFFERENT VALUES OF \( \varepsilon \) AND \( N \)

<table>
<thead>
<tr>
<th>( \varepsilon )</th>
<th>( N = 2^6 )</th>
<th>( N = 2^7 )</th>
<th>( N = 2^8 )</th>
<th>( N = 2^9 )</th>
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<td>4.4448E-3</td>
</tr>
</tbody>
</table>

Example 2. Consider the following two points singularly perturbed BVP with discontinuous source term:
\[ \varepsilon v''(y) + v'(y) = g(y), \quad y \in \Psi - \bigcup_{i=1}^{N} \Psi_{i} \]

where \( v(0) = -1, v(1) = 1 \).
TABLE III: MAE AND RATE OF CONVERGENCE OF EXAMPLE 2 FOR DIFFERENT VALUES OF $\varepsilon$ AND $N$

<table>
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</tr>
</tbody>
</table>

$P_N = \log_2 \frac{E_N}{E_2^{10}}$

The maximum absolute error (MAE) and rate of convergence of examples 1 and 2 are shown in Tables II and III by using CBSM. Tables IV presents MAE with existing method [3]. We calculate MAE as the difference between the numerical solutions for various values of $N$ and the numerical solution for $N=4096$, which is given by $E_N = \max_{y_i \in \Psi} |V_N - V_4096^{10}|$ and $E_N^\varepsilon = \max_{\varepsilon} E_N^\varepsilon$

Rate of convergence is calculated with the given below formula [28]:

$P_N = \log_2 \frac{E_N}{E_2^{10}}$

Obtained order of convergence results using CBSM have been compared with the existing method [3] which are presented in Tables IV and it is concluded that present method gives better order of convergence result than [3]. Figs 1, 2, 3 and 4 provide the graphical behavior of error and cubic B-spline solution of examples 1 and 2 for different values of $N$ and $\varepsilon$.

VI. CONCLUSION

In this paper, we have described a cubic B-spline technique for the numerical treatment of two-point SPBVPs with discontinuous source term. We proved the stability and the convergence of the proposed method, where the order of convergence comes to be approximately two. The CBSM has been applied on two numerical examples to show the accuracy of the method. Also, spline function gives the solution at any point in the interval, whereas the finite difference method only provides the solution at selected mesh points.
REFERENCES


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