Cubic B-spline Technique for Numerical Solution of Singularly Perturbed Convection-Diffusion Equations with Discontinuous Source Term

Shilpkala Mane, Ram Kishun Lodhi

***Abstract*—In this paper, we have developed a cubic B-spline technique to obtain numerical solutions for second-order linear singularly perturbed boundary value problems of convection-diffusion type equations with discontinuous source terms. The solution to such types of problems contains boundaries and an interior layer. The error analysis of the proposed method has been studied and also proved the convergence of the method. The cubic B-spline technique has been implemented on two numerical examples which shows the efficiency and accuracy of the method. Obtained numerical results of the proposed method have compared with other existing methods and found that it gives better numerical solutions at the same number of mesh points.**

***Index Terms*—Singularly perturbed boundary value problems, convection-diffusion equation, interior layer, discontinuous source term, cubic B-spline method, convergence.**

I. INTRODUCTION

SINGULARLY perturbed problems (SPPs) with discontinuous source terms are differential equations (DE) where the highest derivative term of differential equations is multiplied by a small positive parameter and contains a discontinuity term. These types of problems occur in various engineering and applied science branches, such as fluid mechanics, modeling of semiconductor devices, mechanical and electrical frameworks, quantum mechanics, Navier–Stokes equation at high Reynolds number,etc. [1], [2]. The numerical solution of SPPs with discontinuous source terms present some major computational difficulties due to the presence of perturbation parameter ϵ, interior layer and discontinuous term. Farrell *et al.* [3], [4] have discussed the numerical solution of convection–diffusion problems with discontinuous source term.

We propose a cubic B-spline method (CBSM) to find the numerical solution of second order linear singularly perturbed convection-diffusion equations with discontinuous source term. It is appropriate to set up the notation $\Psi^- = (0, b)$, $\Psi^+ = (b, 1)$ and $\Psi = (0, 1)$ where, b is the point of discontinuity and $b \in \Psi$. We indicate the jump at the point of discontinuity b in any function with $z(b) = z(b, +) - z(b, -)$. The two point singularly perturbed boundary value problem (SPBVP) is defined as follows:

$$\mathrm{L}v(y) \equiv \varepsilon v''(y) + a(y)v'(y) = g(y), \forall y \in \Psi^- \cup \Psi^+ \quad (1)$$

$$v(0) = v_0 \, , \; v(1) = v_1 \quad (2)$$

where, $a, g \in C^2(\Psi^- \cup \Psi^+)$, $v \in C^1(\Psi) \cap C^2(\Psi^- \cup \Psi^+)$ and $\varepsilon > 0$ is a small parameter. We assume a single discontinuity at a point $b \in \Psi$. There is a boundary layer at $y = 0$ and interior layers with different widths at $y = b$ in the solution of the problem because of discontinuity at the point b, the solution v of equations (1)-(2) does not necessarily have a continuous second derivative at the point b, i.e.,$v \notin C^2(\Psi)$. This discontinuity produces a weak interior layer and boundary layer at the endpoint of the interval.

In literature, numerical methods like finite difference method, fitted mesh method and finite element method are presented in [5]–[8] to obtain the numerical solution of SPBVP with discontinuous term. Chandru and Shanti [9] have obtained a solution of second-order SPBVPs with Robin-type boundary conditions by using the fitted mesh method for non-smooth data. Debela and Duresssa [10] have used non-polynomial spline technique on uniform mesh to find the numerical solution of second-order SPBVPs. Nowadays, researchers are taking a much attention to develop the higher order convergence methods to find the numerical solution of singularly perturbed convection-diffusion problems with discontinuous source terms [11]–[13]. Spline methods are widely used to find the numerical solution of second and higher order singularly perturbed boundary value problems [14]–[22]. In [23], a class of polynomial spline curve with two free parameters is presented to achieve C^3 continuity which interpolated the data points. In [24], a new type of C^2 piecewise bivariate rational interpolation scheme with two parameters have been constructed. In [25], a class of singularly perturbed parabolic convection-diffusion problems with boundary turning point and retarded arguments have been solved by backward Euler finite difference method on a Shishkin mesh.

The main aim of this study is to build up a stable, convergent and efficient CBSM to solve SPBVPs with discontinuous source terms. The remaining part of the paper is organized as follows: In section II, we explained the proposed CBSM on a uniform mesh. The stability of the method is discussed in section III. In section IV, uniform convergence of the proposed method has been described. Section V presents numerical results, which demonstrate the applicability of the technique. In the last section VI, we discuss the conclusions.

II. CUBIC B-SPLINE METHOD

In this section, the CBSM is described to determine the approximate solution of equation (1)-(2). We divide the domain $\Psi = (0, 1)$ into two subdomains Ψ^- and Ψ^+.

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Each subdomains are divided into $N/2$ equal points and piecewise uniform mesh points are generated by $y_i = ih_1$ in the subdomain Ψ^-, where $h_1 = 2b/N$, for $i = 0, 1, 2, \ldots, N/2$ and $y_i = ih_2$ in the subdomain Ψ^+, where $h_2 = 2(1-b)/N$, for $i = N/2+1, N/2+2, \ldots, N$. The set of partition of subdomains Ψ^- and Ψ^+ are defined as: $\pi_1 = \{0 = y_0, y_1, \cdots, y_{N/2}\}$, $\pi_2 = \left\{y_{\frac{N}{2}+1}, y_{\frac{N}{2}+2}, \ldots, y_N = 1\right\}$. We define $L_2[0,1]$ as a vector space of all square integrable functions on $\overline{\Psi}$ and X be the linear subspace of $L_2\ [0,1]$. We introduce two additional mesh points on each side of the partition $\pi = \pi_1 \cup \pi_2$ as $y_{-2} < y_{-1} < y_0$ and $y_N < y_{N+1} < y_{N+2}$. Now the cubic B-spline basis functions are defined [26], [27] for $i = -1, 0, \ldots, N+1$, as :

$$\varpi_i(y) = \frac{1}{h^3}\begin{cases} (y-y_{i-2})^3, & \text{if } y \in [y_{i-2}, y_{i-1}] \\ h^3 + 3h^2(y - y_{i-1}) + 3h(y - y_{i-1})^2 & \\ -3(y - y_{i-1})^3, & \text{if } y \in [y_{i-1}, y_i] \\ h^3 + 3h^2(y_{i+1}-y) + 3h(y_{i+1}-y)^2 & \\ -3(y_{i+1}-y)^3, & \text{if } y \in [y_i, y_{i+1}] \\ (y_{i+2}-y)^3, & \text{if } y \in [y_{i+1}, y_{i+2}] \\ 0, \quad \text{otherwise} & \end{cases} \tag{3}$$

The functions $\varpi_i(y)$ given by equation (3) is a piecewise cubic polynomial with mesh points y_i for $0 \le i \le N$ and $\varpi_i(y) \in X$, also, each $\varpi_i(y)$ is twice continuously differentiable functions on the whole interval. The values of basis functions $\varpi_i(y)$, $\varpi_i'(y)$, $\varpi_i''(y)$ at the mesh points y_i are given in Table I.

TABLE I: BASIS FUNCTIONS AND THEIR DERIVATIVES VALUES AT MESH POINTS

	y_{i-1}	y_i	y_{i+1}
$\varpi_i(y)$	1	4	4
$\varpi_i'(y)$	$-3/h$	0	$-3/h$
$\varpi_i''(y)$	$6/h^2$	$-12/h^2$	$6/h^2$

Let $\chi = \{\varpi_{-1}, \varpi_0, \varpi_1 \ldots, \varpi_{N+1}\}$ and $\xi_3(\pi)$ = span χ. The function χ is linearly independent on $\overline{\Psi}$ and hence $\xi_3(\pi)$ is N+3 dimensional. It is observed that $\xi_3(\pi)$ is a linear subspace of X. Let L is a linear operator whose domain and range are X. Let $S(y)$ be an approximate solution of $v(y)$ defined as $S(y) = \sum_{i=-1}^{N+1} \alpha_i \varpi_i(y)$,where α_i are unknowns real coefficients. Hence

$$L_c^N(S(y_i)) = g(y_i),\ y_i \in \Psi^- \cup \Psi^+ \tag{4}$$

$$S(y_0) = v_0\ ,\ S(y_N) = v_N \tag{5}$$

At the point of discontinuity $y_{\frac{N}{2}} = b$, we shall use the hybrid difference operator:

$$L_t^N y_{\frac{N}{2}} \equiv \frac{-y_{\frac{N}{2}+2} + 4y_{\frac{N}{2}+1} - 3y_{\frac{N}{2}}}{2h} - \frac{y_{\frac{N}{2}-2} - 4y_{\frac{N}{2}-1} + 3y_{\frac{N}{2}}}{2h} = 0$$

To find the approximate solution $S(y)$ by using CBSM, we calculate the spline functions at the mesh points $y = y_i$. From Table I and assumed cubic B-spline solution $S(y)$,we can obtain the following relationships in the domain $\Psi^- \cup \Psi^+$:

$$S(y_i) = \alpha_{i-1} + 4\alpha_i + \alpha_{i+1}. \tag{6}$$

$$S'(y_i) = \frac{3}{h}(-\alpha_{i-1} + \alpha_{i+1}). \tag{7}$$

$$S''(y_i) = \frac{6}{h^2}(\alpha_{i-1} - 2\alpha_i + \alpha_{i+1}). \tag{8}$$

Hence, we get

$$\varepsilon S''(y) + a(y) S'(y) = g(y)\ , \qquad y \in \Psi^- \cup \Psi^+ \tag{9}$$

Discretizing equation (9) at the mesh points y_i for $i = 0, 1, \ldots, \frac{N}{2}-1, \frac{N}{2}+1, \ldots, N$, we obtain

$$\varepsilon S'' + a(y_i) S'(y_i) = g(y_i) \tag{10}$$

Using equations (7) and (8) in equation (10) and simplifying, we get

$$A_i\alpha_{i-1} + B_i\alpha_i + C_i\alpha_{i+1} = D_i\ , \tag{11}$$

where, $A_i = (6\varepsilon - 3a_ih)$, $B_i = -12\varepsilon$, $C_i = (6\varepsilon + 3a_ih)$,

$$D_i = g_ih^2, for\ i = 0, \ldots, \frac{N}{2}-1, \frac{N}{2}+1, \ldots, N \tag{12}$$

with boundary conditions

$$\alpha_{-1} + 4\alpha_0 + \alpha_1 = v_0 \tag{13}$$

and

$$\alpha_{N-1} + 4\alpha_N + \alpha_{N+1} = v_N \tag{14}$$

where $h = \begin{cases} h_1, 0 \le i \le \frac{N}{2} \\ h_2, \frac{N}{2}+1 \le i \le N \end{cases}$.

Hence in the interval Ψ there are $N+3$ equations and $N+3$ unknowns, including discontinuity condition at point b and boundary conditions. Now eliminating α_{-1} from the first equation of equation (11) and(13) and α_{N+1} from the last equation of equation (11) and (14),we have

$$(-36\varepsilon+12a_0h)\alpha_0+(6a_0h)\alpha_1 = g_0h^2 - v_0(6\varepsilon-3a_0h) \tag{15}$$

and

$$(-6a_Nh)\alpha_{N-1}+(-36\varepsilon-12a_Nh)\alpha_N = g_Nh^2 - v_1(6\varepsilon+3a_Nh) \tag{16}$$

Coupling equation (15) and (16) with the second $(N-2)$ equations of (11) including discontinuity condition at point b, this lead to $N+1$ linear equations in $N+1$ unknowns. These system of equations can be expressed in matrix form $Ax_n = t_n$, where a matrix of unknown $x_n = (\alpha_0, \ldots, \alpha_N)'$, right-hand column matrix $t_n$is given by $t_n = \left(g_0h^2 - v_0(6\varepsilon - 3a_0h), h_1^2 g(y_1), \ldots, h_1^2 g(y_{N-1}), g_Nh^2 - v_1(6\varepsilon + 3a_Nh)\right)'$.

Here matrix A is nonsingular, we can solve the system $Ax_n = t_n$ for $\alpha_0, \ldots, \alpha_N$ and substitute in the boundary conditions (13) and (14) to obtain α_{-1} and α_{N+1}.

$$A = \begin{bmatrix} R_0 & 6a_0h_1 & 0 & \cdots & 0 & 0 \\ A_1 & B_1 & C_1 & \cdots & 0 & 0 \\ 0 & A_2 & B_2 & \cdots & 0 & 0 \\ 0 & 0 & A_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & A_{N-1} & B_{N-1} & C_{N-1} \\ 0 & 0 & \cdots & 0 & -6a_Nh_2 & R_N \end{bmatrix}.$$

where $R_0 = (-36\varepsilon + 12a_0h_1)$ and $R_N = (-36\varepsilon - 12a_Nh_2)$. In the domain $\Psi^- \cup \Psi^+$, using the recurrence relationship of spline approximation given by equations (11) and (12), the difference relation is given by equation (17).

$$\alpha_i = H_i\alpha_{i+1} + K_i, \tag{17}$$

where,

$$H_i = \frac{C_i}{B_i - A_iH_{i-1}} \tag{18}$$

and

$$K_i = \frac{A_iK_{i-1} - D_i}{B_i - A_iH_{i-1}} \tag{19}$$

for $\left(1 \le i \le \frac{N}{2} - 1\right) \cup \left(\frac{N}{2} + 1 \le i \le N\right)$. To begin the iteration method of H_i and K_i, we find H_0 and K_0 from equations (11) and (13) by using initial values of H_0 and K_0, we find further values of H_i and K_i using forward process and find α_i in the backward process from equation (17) by taking $\alpha_N = K_N$.

III. STABILITY OF THE CBSM METHOD

In this section, we prove that the proposed method is stable in the domain $\Psi^- \cup \Psi^+$. Stability means that the error obtained at one phase of calculations is not propagated into more enormous errors in a further phase of calculations. Here local errors are not increased in successive calculations. Consider the recurrence relation given by equation (11). Let e_i be a small error that has occurred in the computation of H_i then the exact value $\overline{H_i}$ is given as:

$$\bar{H}_i = H_i + e_i \tag{20}$$

and we are calculating

$$\bar{H}_i = \frac{C_i}{\left(B_i - A_i\bar{H}_{i-1}\right)}. \tag{21}$$

From equations (20) and (21), we have

$$e_i = \frac{C_i}{B_i - A_i\left(H_{i-1} + e_{i-1}\right)} - \frac{C_i}{\left(B_i - A_iH_{i-1}\right)}$$

$$e_i = \frac{A_iC_ie_{i-1}}{B_i - A_i\left(H_{i-1} + e_{i-1}\right)\left(B_i - A_iH_{i-1}\right)}$$

$$e_i = {H_i}^2\frac{A_i}{C_i}e_{i-1}. \tag{22}$$

Under the assumption that initially, the error is small. Then from the definition of A_i, B_i and C_i with the belief that $a(y) > 0$, we can show that $|B_i| \ge |A_i + C_i|$, for all $\left(1 \le i \le \frac{N}{2} - 1\right) \cup \left(\frac{N}{2} + 1 \le i \le N\right)$. From the initial condition of H_0, we can say that $|H_0| < 1$. Also, we have $H_1 = \frac{C_1}{B_1 - A_1H_0}$ and hence $|H_1| < 1$, since $|H_0| < 1$. Consecutively, it follows that $|H_i| < 1$, for $\left(1 \le i \le \frac{N}{2} - 1\right) \cup \left(\frac{N}{2} + 1 \le i \le N\right)$. Then from equation (22), we have,

$$|e_i| = |H_i|^2\frac{|A_i|}{|C_i|}|e_{i-1}|.$$

Hence,

$$|e_i| < |e_{i-1}|. \tag{23}$$

Let a small error d_i has been made while calculating K_i, then the exact value $\overline{K_i}$ is given as:

$$\overline{K_i} = K_i + d_i\ . \tag{24}$$

Similar above arguments gives

$$d_i = H_i\frac{A_i}{C_i}d_{i-1.} \tag{25}$$

Hence, using the condition
$|H_i| < 1$, for $\left(1 \le i \le \frac{N}{2} - 1)\right) \cup \left(\frac{N}{2} + 1 \le i \le N\right)$.
It follows that

$$|d_i| = |H_i|\frac{|A_i|}{|C_i|}|d_{i-1}|. \tag{26}$$

$$< |\, d_{i-1}\,| \tag{27}$$

Therefore, the recurrence relations H_i,K_i are stable.

IV. UNIFORM CONVERGENCE OF CUBIC B-SPLINE METHOD

In this section, the convergence of CBSM is discussed by estimating the nodal errors. The following lemma is useful for the convergence proof of cubic B-spline method.

Lemma 4.1: The basis of cubic B-splines set $\chi = \{\varpi_{-1}, \varpi_0,\ \varpi_1, \ldots, \varpi_{N+1}\}$ defined in equation (3) satisfies the inequality

$$\sum_{i=-1}^{N+1}|\varpi_i(y)| \le 10, \qquad 0 \le y \le 1.$$

Proof. For the proof of the lemma, readers can refer [20].

Theorem 4.1. The collocation approximation $S(y)$ from the space $\xi_3(\pi)$ to the solution $v(y)$ of the BVP (1)- (2) exists, further if $g(y) \in \Psi^- \cup \Psi^+$, then parameter uniform error estimate is given by

$$\|\, v(y) - S(y)\,\|_\infty \le Rh^2 \tag{28}$$

for h sufficiently small and R is a positive constant (independent of N and ε).

Proof: We determine the error $\|\, v(y) - S(y)\,\|_\infty$ in the domain $y \in \Psi$. Let Z_n be the unique spline interpolate from $\xi_3(\pi)$ to the solution $v(y)$ of BVP (1)-(2). If g $(y) \in \Psi^- \cup \Psi^+$ and $v(y) \in C^4([0,\, b) \cup (b\,,\, 1])$ then we have

$$\left\|D^j\left(v\left(y\right) - S\left(y\right)\right)\right\|_\infty \le k_j\, {h_c}^{4-j}, \quad j = 0, 1, 2, \tag{29}$$

where, $h_c = \max\{h_1, h_2\}$ and k_j are independent of h and N. Let

$$Z_n(y) = \sum_{i=-1}^{N+1,}\mu_i\varpi_i(y) \tag{30}$$

and $S(y) = \sum_{i=-1}^{N+1}\alpha_i\varpi_i(y)$ is collocation solution.
From the equation (29), we have

$$|LS(y_i) - LZ_n(y_i)| \le ph_c^2, \tag{31}$$

where $p = \varepsilon k_2 + ||a(y)||_\infty k_1h_c$. Now collocating conditions are $LS(y_i) = Lv(y_i) = g(y_i)$. Let $LZ_n(y_i) = g_n^1(y_i)$, for all $i = -1, 0, \ldots, N + 1$ and $g_n^1 = \left(g_n^1(y_{-1}), g_n^1(y_0), \ldots, g_n^1(y_{N+1})\right)$. Using the system $Ax_n = t_n$ and equation (31), the

i^{th} coordinate $|A(x_n - z^n)|_i$ of $A(x_n - z^n)$, where $z^n = (\mu_0, \mu_1, \ldots \mu_{N+1})^{'}$ satisfy the inequality

$$|[A(x_n - z^n)]_i| \leq ph_c^4. \quad (32)$$

Since$(Ax_n)_i = h_c^2 g(y_i)$ and $(Az^n)_i = h_c^2 g_1^n(y_i)$, for $0 \leq i \leq N$. The i^{th} coordinate of $A(x_n - z^n)$ is the i^{th} equation given as:

$$(6\varepsilon - 3a_i h_c)\tau_{i-1} + (-12\varepsilon)\tau_i \; + (6\varepsilon + 3a_i h_c)\tau_{i+1} = \rho_i \quad (33)$$

where $\tau_i = \alpha_i - \mu_i$, for $-1 \leq i \leq N+1$ and $\rho_i = h_c^2\left(g(y_i) - g_n^1(y_i)\right)$, for $1 \leq i \leq N-1$.
Clearly, $|\rho_i| \leq ph_c^4$. Let $\rho = \max|\rho_i|$, for $1 \leq i \leq N-1$.
Consider $\tau = (\tau_{-1}, \tau_0, \ldots, \tau_{N+1})$, $e_i^{'} = |\tau_i| = |\alpha_i - \mu_i|$ and $\widetilde{e} = \max\left|e_i^{'}\right|$, for $1 \leq i \leq N-1$. Now, equation (33) gives

$$-12\varepsilon\tau_i = \rho_i - 6\varepsilon(\tau_{i-1} + \tau_{i+1}) + 3a_i h_c(\tau_{i-1} - \tau_{i+1}) \quad (34)$$

Taking absolute value with a sufficiently small value of h_c, we obtain

$$12\varepsilon e_i^{'} \leq \rho + 2\widetilde{e}(-6\varepsilon + 3a_i h_c) \; . \quad (35)$$

Since, $0 < a^* \leq a(y)$, we get

$$12\varepsilon e_i^{'} \leq \rho + 2\,\widetilde{e}(6\varepsilon + 3a^* h_c) \; .$$

In particular,

$$12\varepsilon\widetilde{e} \leq \rho + 2\widetilde{e}(6\varepsilon + 3a^* h_c)$$

Solving for $\widetilde{e}$, we get

$$-6a^* h_c\widetilde{e} \leq \rho \leq ph_c^4$$

Hence,

$$\widetilde{e} \leq \frac{-ph_c^3}{6a^*}, 1 \leq i \leq N-1. \quad (36)$$

Now to determine $e_{-1}^{'}$, $e_0^{'}$, e_N, $e_{N+1}^{'}$, using 1^{st} and last equation of the system

$$A[(x_n - z^n)] = h_c^2\left(g_n - g_n^1\right), \quad (37)$$

where $g_n = (g_0, g_1 \ldots, g_N)^{'}$
From 1^{st} equation of the above system, we have $(-36\varepsilon + 12a_0 h_c)\tau_0 + (6a_0 h_c)\tau_1 = h_c^2\left(g_0 - g_0^1\right)$.
After simplification, we get

$$e_0^{'} \leq \frac{p\,h_c^4}{36\,\varepsilon - 12a^*\,h_c} \quad (38)$$

Now $e_{-1}^{'}$ can be obtained using boundary conditions, we have

$$e_{-1}^{'} \leq \frac{-4p\,h_c^4}{(-36\varepsilon + 12a^* h_c)} + \frac{ph_c^3}{6a^*} \quad (39)$$

Similarily,

$$e_N^{'} \leq \frac{p\,h_c^4}{-36\,\varepsilon - 12a^* h_c} \quad (40)$$

$$e_{N+1}^{'} \leq \frac{-4p\,h_c^4}{(-36\varepsilon - 12a^* h_c)} + \frac{ph_c^3}{6a^*} \quad (41)$$

Using the value of $p = \varepsilon k_2 + ||a(y)||_\infty k_1 h_c$ and since there exist a constant $\widetilde{C}$ such that

$$e = \max\{\,e_i^{'}\,\} \leq \widetilde{C}h_c^2 \quad (42)$$

The above equation (42) helps us to determine $\|S(y) - Z_n(y)\|_\infty$ and therefore $\|v(y) - S(y)\|_\infty$
In particular,

$$S(y) - Z_n(y) = \sum_{i=-1}^{N+2}(\alpha_i - \mu_i)\varpi_i(y) \quad (43)$$

$$|S(y) - Z_n(y)| \leq \max|\alpha_i - \mu_i|\sum_{i=-1}^{N+2}|\varpi_i(y)| \quad (44)$$

Combining equations (43), (44) and using Lemma 1, we get

$$\|S(y) - Z_n(y)\|_\infty \leq 10\,\widetilde{C}\,h_c^2$$

Since $\|v(y) - Z_n(y)\|_\infty \leq k_0 h_c^4$

$$\begin{aligned}\|v(y) - S(y)\|_\infty &\leq \|v(y) - Z_n(y)\|_\infty + \|Z_n(y) - S(y)\|_\infty \\ &\leq k_0 h_c^4 + 10\widetilde{C}h_c^2 \\ &= Rh_c^2 \end{aligned} \quad (45)$$

where , $R = 10\widetilde{C} + k_0 h_c^2$.

Hence, $||v(y) - S(y)||_\infty \leq Rh^2$.

V. NUMERICAL ILLUSTRATION

In this section, we have implemented the CBSM on two test problems which shows the accuracy and order of convergence of the method and all the computation work are performed by MATLAB software.

Example 1. Consider the following two points singularly perturbed BVP with discontinuous source term:

$$\varepsilon v''(y) + v'(y) = g(y), \quad y \in \Psi^- \bigcup \Psi^+$$
$$v(0) = -1,\, v(1) = 0$$
$$\text{where } g(y) = \begin{cases} -9,\; y \leq \frac{1}{3} \\ 9(y-1)^2,\; y > \frac{1}{3} \end{cases} .$$

TABLE II: MAE AND RATE OF CONVERGENCE OF EXAMPLE 1 FOR DIFFERENT VALUES OF ε AND N

ε	$N = 2^6$	$N = 2^7$	$N = 2^8$	$N = 2^9$	$N = 2^{10}$
2^{-1}	8.6558E-2	4.3949E-2	2.1604E-2	1.0162E-2	4.3727E-3
2^{-2}	7.6482E-2	4.1314E-2	2.0932E-2	9.9944E-3	4.3324E-3
2^{-3}	5.7357E-2	3.6037E-2	1.9546E-2	9.6426E-3	4.2476E-3
2^{-4}	5.9834E-2	2.7425E-2	1.7148E-2	9.0149E-3	4.0937E-3
2^{-5}	7.7543E-2	2.9694E-2	1.3090E-2	7.8923E-3	3.8103E-3
2^{-6}	1.3084E-1	4.1458E-2	1.4723E-2	5.9010E-3	3.2805E-3
2^{-7}	3.2191E-1	8.5405E-2	2.4877E-2	7.2445E-3	2.3405E-3
2^{-8}	8.6559E-1	2.5938E-1	6.3524E-2	1.7166E-2	4.4448E-3
E^N	8.6559E-1	2.5938E-1	6.3523E-2	1.7166E-2	4.4448E-3
P_N	1.738	2.0297	1.8878	1.9494	

Example 2. Consider the following two points singularly perturbed BVP with discontinuous source term:

$$\varepsilon v''(y) + v'(y) = g(y), \quad y \in \Psi^- \bigcup \Psi^+$$
$$v(0) = 1,\, v(1) = 1,$$

TABLE III: MAE AND RATE OF CONVERGENCE OF EXAMPLE 2 FOR DIFFERENT VALUES OF ε AND N

ε	$N=2^6$	$N=2^7$	$N=2^8$	$N=2^9$	$N=2^{10}$
2^{-1}	1.4629E-2	7.3794E-3	3.6148E-3	1.6973E-3	7.2964E-4
2^{-2}	1.3066E-2	6.9808E-3	3.5144E-3	1.6723E-3	7.2368E-4
2^{-3}	1.0505E-2	6.2906E-3	3.3354E-3	1.6272E-3	7.1283E-4
2^{-4}	9.2531E-3	5.2464E-3	3.0496E-3	1.5531E-3	6.9476E-4
2^{-5}	2.4389E-2	6.4515E-3	2.5097E-3	1.4077E-3	6.5855E-4
2^{-6}	8.3806E-2	2.0056E-2	5.0851E-3	1.2847E-3	5.8925E-4
2^{-7}	2.9990E-1	7.6389E-2	1.7847E-2	4.4031E-3	1.0578E-3
2^{-8}	7.4754E-1	2.8511E-1	7.2514E-2	1.6577E-2	3.9047E-3
E^N	7.4754E-1	2.8511E-1	7.2514E-2	1.6577E-2	3.9047E-3
P_N	1.3906	1.9752	2.1291	2.0859	

TABLE IV: MAE AND RATE OF CONVERGENCE OF EXAMPLE 1 BY FARRELL *ET AL.* [3] FOR DIFFERENT VALUES OF ε AND N.

ε	$N=2^6$	$N=2^7$	$N=2^8$	$N=2^9$	$N=2^{10}$
2^{-1}	1.3098E-2	6.6301E-3	3.2538E-3	1.5292E-3	6.5771E-4
2^{-2}	2.7701E-2	1.4069E-2	6.9155E-3	3.2527E-3	1.3995E−3
2^{-3}	4.3567E-2	2.1967E-2	1.0756E-2	5.0489E-3	2.1701E−3
2^{-4}	3.2122E-2	2.2100E-2	1.2656E-2	5.9267E-3	2.5444E−3
2^{-5}	4.4781E-2	2.0447E-2	2.0447E-2	3.9835E-3	3.9835E−3
2^{-6}	6.1320E-2	2.9076E-2	1.3544E-2	6.0891E-3	2.5202E−3
2^{-7}	7.3582E-2	3.4644E-2	1.6305E-2	7.4720E-3	3.1487E−3
2^{-8}	8.4455E-2	4.0685E-2	1.9279E-2	8.8565E-3	3.7539E−3
E^N	8.4455E-2	4.0685E-2	1.9279E-2	8.8565E-3	3.7539E-3
P_N	1.0537	1.0775	1.1222	1.2383	

$$\text{where } g(y) = \begin{cases} -1\,, & y \le \frac{1}{2} \\ 1\,, & y > \frac{1}{2} \end{cases}.$$

The maximum absolute error (MAE) and rate of convergence of examples 1 and 2 are shown in Tables II and III by using CBSM. Tables IV presents MAE with existing method [3]. We calculate MAE as the difference between the numerical solutions for various values of N and the numerical solution for N= 4096, which is given by

$E_{\varepsilon}^{N} = \max\limits_{y_i \in \Psi^- \cup \Psi^+} \left| V_{\varepsilon}^{N} - \overline{V}_{\varepsilon}^{4096} \right|$ and $E^N = \max\limits_{\varepsilon} E_{\varepsilon}^{N}$

Rate of convergence is calculated with the given below formula [28]:

$$P_N = \log_2 \frac{E^N}{E^{2N}}.$$

Obtained order of convergence results using CBSM have been compared with the existing method [3] which are presented in Tables IV and it is concluded that present method gives better order of convergence result than [3]. Figs 1, 2, 3 and 4 provide the graphical behavior of error and cubic B-spline solution of examples 1 and 2 for different values of N and ε.

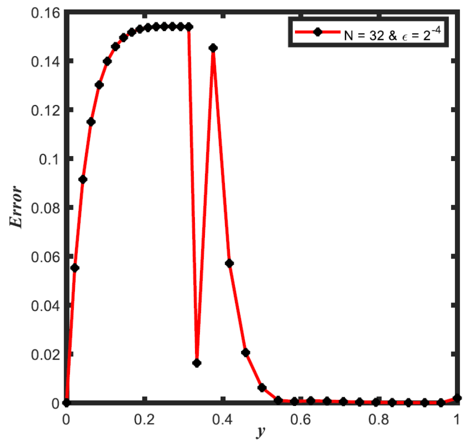

Fig. 1: Error Graph of example 1 for $N = 32$ and $\varepsilon = 2^{-4}$

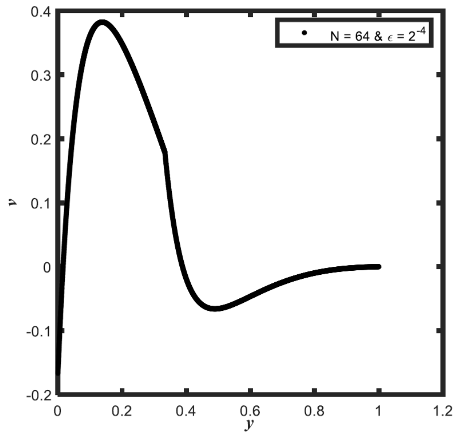

Fig. 2: Numerical solution of example 1 for N = 64 and $\varepsilon = 2^{-4}$.

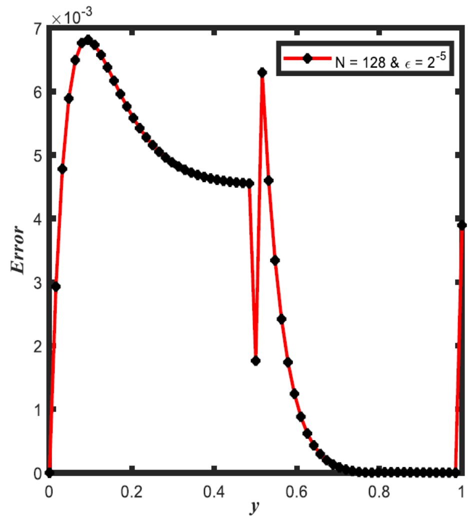

Fig. 3: Error graph of example 2 for N = 128 and ε = 2^{-5}

VI. CONCLUSION

In this paper, we have described a cubic B-spline technique for the numerical treatment of two-point SPBVPs with discontinuous source term. We proved the stability and the convergence of the proposed method, where the order of convergence comes to be approximately two. The CBSM has been applied on two numerical examples to show the accuracy of the method. Also, spline function gives the solution at any point in the interval, whereas the finite difference method only provides the solution at selected mesh points.

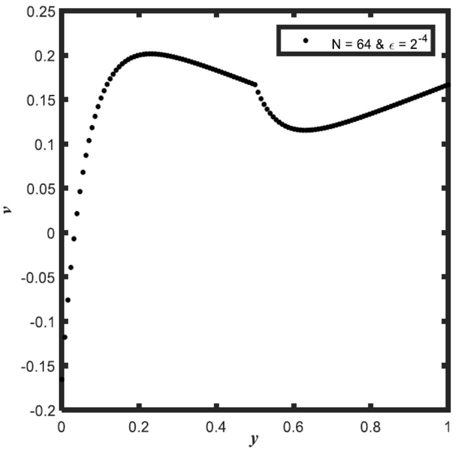

Fig. 4: Numerical solution of example 2 for $N = 64$ and $\varepsilon = 2^{-4}$

REFERENCES

[1] E. P. Doolan, J. J. H. Miller, and W. H. A. Schilders, *Uniform numerical methods for problems with initial and boundary layers*. Boole Press, 1980.

[2] H. G. Roos, M. Stynes, and L. Tobiska, *Numerical methods for singularly perturbed differential equations*. Springer–Verlag, 2008.

[3] P. A. Farrell, A. F. Hegarty, J. J. H. Miller, E. O'Riordan, and G. I. Shishkin, "Singularly perturbed convection–diffusion problems with boundary and weak interior layers," *Journal of Computational and Applied Mathematics*, vol. 166, pp. 133–151, 2004.

[4] ——, "Global maximum norm parameter-uniform numerical method for a singularly perturbed convection-diffusion problem with discontinuous convection coefficient," *Mathematical and Computer Modelling*, vol. 40, pp. 1375–1392, 2004.

[5] A. R. Babu and N. Ramanujam, "The sdfem for singularly perturbed convection–diffusion problems with discontinuous source term arising in the chemical reactor theory," *International Journal of Computer Mathematics*, vol. 88, pp. 1664–1680, 2011.

[6] Z. Cen, "A hybrid difference scheme for a singularly perturbed convection-diffusion problem with discontinuous convection coefficient," *Applied Mathematics and Computation*, vol. 169, pp. 689–699, 2005.

[7] T. Linß, "Finite difference schemes for convection-diffusion problems with a concentrated source and a discontinuous convection field," *Computational Methods in Applied Mathematics*, vol. 2, pp. 41–49, 2001.

[8] H. G. Roos and H. Zarin, "The streamline-diffusion method for a convection–diffusion problem with a point source," *Journal of Computational and Applied Mathematics*, vol. 150, pp. 109–128, 2003.

[9] M. Chandru and V. Shanthi, "Fitted mesh method for singularly perturbed robin type boundary value problem with discontinuous source term," *International Journal of Applied and Computational Mathematics*, vol. 1, pp. 491–501, 2015.

[10] H. G. Debela and G. F. Duresssa, "Uniformly convergent nonpolynomial spline method for singularly perturbed robin-type boundary value problems with discontinuous source term," *Hindawi Abstract and Applied Analysis*, vol. 2021, 7569209, 2021.

[11] G. J. Jayalakshmi and A. Tamilselvan, "Hybrid difference scheme for singularly perturbed convection diffusion boundary turning point problems with discontinuous source term," *International Journal of Applied and Computational Mathematics*, vol. 5, pp. 1–14, 2019.

[12] M. Priyadharshini and N. Ramanujam, "High-order method for a singularly perturbed second-order ordinary differential equation with discontinuous source term subject to mixed type boundary conditions," *International Conference on Mathematical Modelling and Scientific Computation*, pp. 351–359, 2012.

[13] M. Stynes and H. G. Roos, "The midpoint upwind scheme," *Applied Numerical Mathematics*, vol. 23, pp. 361–374, 1997.

[14] R. K. Bawa and S. Natesan, "A computational method for self-adjoint singular perturbation problems using quintic spline," *Computers and Mathematics with Applications*, vol. 50, pp. 1371–1382, 2005.

[15] M. K. Kadalbajoo and V. K. Aggarwal, "Fitted mesh b-spline method for solving a class of singular singularly perturbed boundary value problems," *International Journal of Computer Mathematics*, vol. 82, pp. 67–76, 2005.

[16] M. K. Kadalbajoo and V. Gupta, "A parameter uniform b-spline collocation method for solving singularly perturbed turning point problem having twin boundary layers," *International Journal of Computer Mathematics*, vol. 87, pp. 3218–3235, 2010.

[17] M. K. Kadalbajoo and K. Patidar, "Variable mesh spline in compression for the numerical solution of singular perturbation problems," *International Journal of Computer Mathematics*, vol. 80, pp. 83–93, 2003.

[18] M. K. Kadalbajoo and V. Gupta, "Numerical solution of singularly perturbed convection–diffusion problem using parameter uniform b-spline collocation method," *Journal of Mathematical Analysis and Applications*, vol. 355, pp. 439–452, 2009.

[19] M. K. Kadalbajoo and A. S. Yadaw, "B-splines collocation method for a two parameter singularly perturbed convection–diffusion boundary value problem," *Applied Mathematics and Computation*, vol. 201, pp. 504–513, 2008.

[20] M. K. Kadalbajoo and P. Arora, "B-splines with artificial viscosity for solving singularly perturbed boundary value problems," *Mathematical and Computer Modelling*, vol. 52, pp. 654–666, 2010.

[21] ——, "B-spline collocation method for the singular-perturbation problem using artificial viscosity," *Computers and Mathematics with Applications*, vol. 57, pp. 650–663, 2009.

[22] R. K. Lodhi and H. K. Mishra, "Quintic b-spline method for solving second order linear and nonlinear singularly perturbed two-point boundary value problems," *Journal of Computational and Applied Mathematics*, vol. 319, pp. 170–187, 2017.

[23] J. Li and C. Liu, "A class of polynomial spline curve with free parameters that naturally interpolates the data points," *IAENG International Journal of Applied Mathematics*, vol. 50, no. 3, pp. 643–647, 2020.

[24] X. B. Qin and Q. Xu, "A new c2 piecewise bivariate rational interpolation scheme with bi-quadratic denominator," *IAENG International Journal of Applied Mathematics*, vol. 47, no. 4, p. 454–458, 2017.

[25] P. Rai and K. K. Sharma, "Singularly perturbed parabolic differential equations with turning point and retarded arguments," *IAENG International Journal of Applied Mathematics*, vol. 45, no.4, pp. 404–409, 2015.

[26] G. Micula, *Handbook of splines*. Kluwer Academic Publishers, 1999.

[27] P. M. Prenter, *Splines and variational methods*. Wiley Classics Library, John Wiley and Sons, 1989.

[28] P. A. Farrell, A. F. Hegarty, J. J. H. Miller, E. O'Riordan, and G. I. Shishkin, *Robust computational techniques for boundary layers*. Chapman and hall, 2000.

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