Output-Feedback-Based Adaptive Control for H_infinity Anti-Synchronization of Uncertain Delayed Neural Networks

Xi Wang, Xinling Li, Xianghui Wu, Weipeng Tai, and Jianping Zhou

***Abstract*—This paper considers $\mathcal{H}_\infty$ anti-synchronization (HAS) for uncertain neural networks subject to time-varying delay. An output-feedback-based adaptive control scheme is used to ensure the HAS of the considered networks. An existence condition and update laws of the unknown parameters for the desired controller are developed by resorting to the Lyapunov-Krasovskii functional theory, Bessel-Legendre inequality, as well as reciprocal convex combination. Then, a linear-matrix-inequalities-based design method is developed for the output-feedback gain by decoupling nonlinear terms. Finally, a numerical example is given to illustrate the lower conservatism and effectiveness of the present HAS control method.**

***Index Terms*—Anti-synchronization, output-feedback control, $\mathcal{H}_\infty$ control, delayed neural networks.**

I. INTRODUCTION

NEURAL networks (NNs) are computing systems inspired by the human brain, consisting of interconnected nodes (called neurons). Such systems, which have many important characteristics including self-organization, self-learning, parallel processing, and high fault tolerance, have found widespread applications in various engineering areas from image encryption [1], consensus tracking [2], safety assessment [3], and video reconstruction [4], to nonlinear control [5] and satellite data prediction [6]. In practice, parameter uncertainty and time delay occur unavoidably, leading to complex dynamic behaviors such as Hopf bifurcations, oscillations, and strange chaos. Correspondingly, a great deal of research in the analysis and control of uncertain delayed NNs (UDNNs) has been carried out over the past few decades (see, e.g., [7–11]).

Anti-synchronization is a special synchronous behavior in which the states of coupled systems have the same amplitude but totally different signs. As early as 1665, Huygens observed such a phenomenon in his famous experiment of resonance clocks. In 1990, Pecora and Carroll proposed a drive-response synchronization framework [12]. Based on the framework, many researchers have investigated the anti-synchronization of different NN models. For UDNNs, Ahn studied the $\mathcal{H}_\infty$ anti-synchronization and gave an adaptive control method based on linear matrix inequalities (LMIs) [13]; In [14], Yan et al. further examined the mixed $\mathcal{H}_\infty$ anti-synchronization and presented a new adaptive controller design scheme with fewer LMI decision variables.

Note that in most literature on UDNNs, the time delay is assumed to be time-invariant, which may be overly restrictive. In reality, the time delay in biological and artificial NNs may dynamically change within a certain range. Therefore, the anti-synchronization of UDNNs with time-varying delays deserves more attention than that of UDNNs with constant delays. In addition, it is found that the adaptive controller mechanisms in [13, 14] are based on full-state feedback. However, in a practical dynamic system, comprehensive measurement of state information is often difficult and costly [15, 16].

In this paper, we revisit the $\mathcal{H}_\infty$ anti-synchronization (HAS) of UDNNs. Unlike [15, 16], we aim to determine an output-feedback-based adaptive control scheme for the time-varying delay case. Compared with the state-feedback mechanism, the considered output-feedback mechanism, which does not need all the system states to be measurable, is easier to implement. By resorting to the Lyapunov-Krasovskii functional (LKF) theory, Bessel-Legendre inequality (BLI) [17], as well as reciprocal convex combination (RCC) [18], we propose an existence condition and update laws of the unknown parameters for the desired output-feedback-based adaptive controller. Then, we develop an LMIs-based design method for the output-feedback gain by decoupling nonlinear terms. Lastly, we provide a numerical example to show the lower conservatism and validity of the present output-feedback-based adaptive HAS control method.

II. PRELIMINARIES

Throughout, $col\{\cdot\}$ represents a column vector and $diag\{\cdot\}$ denotes a block-diagonal matrix. For any matrix X, superscripts T and -1 stand for its transpose and inverse, respectively, and $He(X)$ denotes $X + X^T$.

Consider UDNN

$$\begin{aligned}\dot{\alpha}(t) =& A\alpha(t)+ \bar{A}\alpha(t-\upsilon(t))+Bf(\alpha(t))+ \bar{B}g(\alpha(t-\upsilon(t)))\\ &+\sum_{k=1}^{N_1}\Phi_k(\alpha(t))\theta_k+\sum_{l=1}^{N_2}\Psi_l(\alpha(t-\upsilon(t)))\phi_l \qquad \text{(1a)}\\ \beta(t) =& C\alpha(t) \qquad \text{(1b)}\end{aligned}$$

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Xi Wang is a postgraduate student at the School of Computer Science & Technology, Anhui University of Technology, Ma'anshan 243032, China (e-mail: xwang1010@126.com).

Xinling Li is a postgraduate student at the Research Institute of Information Technology, Anhui University of Technology, Ma'anshan 243000, China (e-mail: xinlingli0915@163.com).

Xianghui Wu is a postgraduate student at the Research Institute of Information Technology, Anhui University of Technology, Ma'anshan 243000, China (e-mail: 291374540@qq.com).

Weipeng Tai is an associate professor at the Research Institute of Information Technology, Anhui University of Technology, Ma'anshan 243000, China (corresponding author, e-mail: taiweipeng@ahut.edu.cn).

Jianping Zhou is a full professor at the School of Computer Science & Technology, Anhui University of Technology, Ma'anshan 243032, China (e-mail: jpzhou0@gmail.com).

In the UDNN considered, $\alpha(t)$ and $\beta(t) \in \mathbb{R}^n$ are the neuron state and output; A, $\bar{A}$, B and $\bar{B}$ are self-feedback and connection matrices, and $\upsilon(t)$ denotes a time-varying delay, which is continuous and fulfills $0 \le \upsilon_1 \le \upsilon(t) \le \upsilon_2$ as in [19], where υ_1 and υ_2 are constants corresponding to the infimum and supremum of $\upsilon(t)$, respectively. Furthermore $\Phi_k(\alpha(t))(k = 1,...,N_1) : \mathbb{R}^n \to \mathbb{R}^{n\times r}$ and $\Psi_l(l = 1,...,N_2) : \mathbb{R}^n \to \mathbb{R}^{n\times s}$ are nonlinear function matrices, $\theta_k \in \mathbb{R}^r(k = 1,...,N_1)$ and $\phi_l \in \mathbb{R}^s(l = 1,...,N_2)$ represent the unknown parameters, and $f(\alpha(\cdot)) : \mathbb{R}^n \to \mathbb{R}^n$ and $g(\alpha(\cdot)) : \mathbb{R}^n \to \mathbb{R}^n$ stand for activation functions satisfying [20, 21]:

$$||f(\alpha(\cdot)) + f(\beta(\cdot))|| \le L_f\, ||\alpha(\cdot) + \beta(\cdot)|| \tag{2a}$$
$$||g(\alpha(\cdot)) + g(\beta(\cdot))|| \le L_g\, ||\alpha(\cdot) + \beta(\cdot)|| \tag{2b}$$

for two given positive constants L_f and L_g and any $\alpha(\cdot)$, $\beta(\cdot) \in \mathbb{R}^n$. UDNN (1) is considered as a drive system. The response UDNN is given as

$$\begin{aligned}\dot{\hat{\alpha}}(t) =& A\hat{\alpha}(t) + \bar{A}\hat{\alpha}(t-\upsilon(t)) + Bf(\hat{\alpha}(t)) + \bar{B}g(\hat{\alpha}(t-\upsilon(t))) \\ &+ u(t) + Gd(t) \end{aligned}\tag{3a}$$
$$\hat{\beta}(t) = C\hat{\alpha}(t) \tag{3b}$$

where $\hat{\alpha}(t)$, $\hat{\beta}(t)$, $u(t) \in \mathbb{R}^n$ are the state, output, control input, respectively, $d(t)$ stands for a bounded disturbance belonging to $\mathcal{L}_2[0,\infty)$ [22], and $G \in \mathbb{R}^{n\times n}$ is a constant coefficient matrix. Defining $\kappa(t) = \hat{\alpha}(t)+\alpha(t)$, we can write the anti-synchronization error system (ASES) as

$$\begin{aligned}\dot{\kappa}(t) =& A\kappa(t) + \bar{A}\kappa(t-\upsilon(t)) + B(f(\hat{\alpha}(t)) + f(\alpha(t))) + u(t) \\ &+ \bar{B}(g(\hat{\alpha}(t-\upsilon(t))) + g(\alpha(t-\upsilon(t)))) + Gd(t) \\ &+ \sum_{k=1}^{N_1}\Phi_k(\alpha(t))\theta_k + \sum_{l=1}^{N_2}\Psi_l(\alpha(t-\upsilon(t)))\phi_l. \end{aligned}\tag{4}$$

Unlike [15, 16], the following output-feedback-based adaptive controller

$$\begin{aligned}u(t) =& K(\hat{\beta}(t) + \beta(t)) - \sum_{k=1}^{N_1}\Phi_k(\alpha(t))\bar{\theta}_k \\ &- \sum_{l=1}^{N_2}\Psi_l(\alpha(t-\upsilon(t)))\bar{\phi}_l \end{aligned}\tag{5}$$

will be used in this paper, where K is the control gain, and $\bar{\theta}_k(k = 1,...,N_1)$ and $\bar{\phi}_l(l = 1,...,N_2)$ are the estimates of θ_k and ϕ_l, respectively.

Definition 1. *Given a level $\gamma > 0$, UDNNs (1) and (3) are called $\mathcal{H}_\infty$ anti-synchronized if*

$$\int_0^\infty \kappa^T(\mu)S\kappa(\mu)d\mu < \gamma^2 \int_0^\infty d^T(\mu)d(\mu)d\mu \tag{6}$$

within the zero initial condition, where S is a positive symmetric matrix. The parameter γ is known as the $\mathcal{H}_\infty$ disturbance-attenuation performance level (DAPL) [23, 24].

Definition 2. *UDNNs (1) and (3) are called asymptotically anti-synchronized if $\lim_{t\to\infty}\kappa(t) = 0$ when $d(t) = 0$.*

Lemma 1. *[17] (BLI) Given a matrix $W \in \mathbb{S}^m_+$ and a differentiable function $\kappa(t)$: $[a, b] \to \mathbb{R}^m$,*

$$\int_a^b \dot{\kappa}^T(\mu)W\dot{\kappa}(\mu)d\mu \ge \frac{1}{b-a}\Omega^T diag(W, 3W, 5W)\Omega$$

holds, where $\sigma_{a,b}(\mu) = 2\frac{\mu-a}{b-a} - 1$ and

$$\Omega = \begin{bmatrix} \kappa(b) - \kappa(a) \\ \kappa(b) + \kappa(a) - \frac{2}{b-a}\int_a^b \kappa(\mu)d\mu \\ \kappa(b) - \kappa(a) - \frac{6}{b-a}\int_a^b \sigma_{a,b}(\mu)\kappa(\mu)d\mu \end{bmatrix}.$$

Lemma 2. *[18] (RCC) Let $y_1(t)$, $y_2(t)$ be two functions that possess positive values in an open set $\digamma \in \mathbb{R}^n$. Then, for any $\epsilon \in (0,1)$ and symmetric function $o(t)$ defined on $\mathbb{R}^n$ with $\begin{bmatrix} y_1(t) & * \\ o(t) & y_2(t)\end{bmatrix} \ge 0$, the RCC of $y_1(t)$ and $y_2(t)$ over $\digamma$ satisfies*

$$\frac{1}{\epsilon}y_1(t) + \frac{1}{1-\epsilon}y_2(t) \ge y_1(t) + y_2(t) + 2o(t).$$

The following notations will be frequently employed in the subsequent section:

$$\begin{aligned}
\xi_i &= \begin{bmatrix}0_{n\times(i-1)n} & I_n & 0_{n\times(16-i)n}\end{bmatrix}, i = 1,...,16,\\
\mathcal{H}_2 &= \begin{bmatrix}\xi_1^T - \xi_2^T & \xi_1^T + \xi_2^T - 2\xi_5^T & \xi_1^T - \xi_2^T - 6\xi_6^T\end{bmatrix}^T\\
\mathcal{H}_3 &= \begin{bmatrix}\xi_2^T - \xi_3^T & \xi_2^T + \xi_3^T - 2\xi_7^T & \xi_2^T - \xi_3^T - 6\xi_8^T\end{bmatrix}^T\\
\mathcal{H}_4 &= \begin{bmatrix}\xi_3^T - \xi_4^T & \xi_3^T + \xi_4^T - 2\xi_9^T & \xi_3^T - \xi_4^T - 6\xi_{10}^T\end{bmatrix}^T\\
\Gamma &= \begin{bmatrix}\mathcal{H}_3^T & \mathcal{H}_4^T\end{bmatrix}^T, \kappa_t(s) = \kappa(t+s), \upsilon_{12} = \upsilon_2 - \upsilon_1\\
\eta_0(t) &= \begin{bmatrix}\kappa^T(t) & \kappa^T(t-\upsilon_1) & \kappa^T(t-\upsilon(t)) & \kappa^T(t-\upsilon_2)\end{bmatrix}^T\\
\eta_1(t) &= \frac{1}{\upsilon_1}\begin{bmatrix}\int_{-\upsilon_1}^0 \kappa_t^T(s)ds & \int_{-\upsilon_1}^0 \sigma_1(s)\kappa_t^T(s)ds\end{bmatrix}^T\\
\eta_2(t) &= \frac{1}{\upsilon(t)-\upsilon_1}\begin{bmatrix}\int_{-\upsilon(t)}^{-\upsilon_1} \kappa_t^T(s)ds & \int_{-\upsilon(t)}^{-\upsilon_1} \sigma_2(s)\kappa_t^T(s)ds\end{bmatrix}^T\\
\eta_3(t) &= \frac{1}{\upsilon_2-\upsilon(t)}\begin{bmatrix}\int_{-\upsilon_2}^{-\upsilon(t)} \kappa_t^T(s)ds & \int_{-\upsilon_2}^{-\upsilon(t)} \sigma_3(s)\kappa_t^T(s)ds\end{bmatrix}^T\\
\eta_4(t) &= (\upsilon(t)-\upsilon_1)\eta_2(t), \eta_5(t) = (\upsilon_2 - \upsilon(t))\eta_3(t),\\
\eta_6(t) &= \begin{bmatrix}\int_{-\upsilon_2}^{-\upsilon_1} \kappa_t^T(s)ds & \upsilon_{12}\int_{-\upsilon_2}^{-\upsilon_1} \sigma_4(s)\kappa_t^T(s)ds\end{bmatrix}^T\\
\sigma_1(s) &= 2\frac{s+\upsilon_1}{\upsilon_1} - 1, \sigma_2(s) = 2\frac{s+\upsilon(t)}{\upsilon(t)-\upsilon_1} - 1\\
\sigma_3(s) &= 2\frac{s+\upsilon_2}{\upsilon_2-\upsilon(t)} - 1, \sigma_4(s) = 2\frac{s+\upsilon_2}{\upsilon_{12}} - 1.
\end{aligned}$$

III. Main Results

Theorem 1. *Given a scalar $\gamma > 0$ and a matrix $S > 0$, suppose there exist constant $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ and positive matrices $P \in \mathbb{S}^{5n}_+$, S_1, S_2, W_1, $W_2 \in \mathbb{S}^n_+$, constant matrices N_1, $N_2 \in \mathbb{R}^{16n\times 2n}$, U_1, $U_2 \in \mathbb{R}^{n\times n}$, and $M \in \mathbb{R}^{3n\times 3n}$, such that , for any $\upsilon \in \{\upsilon_1, \upsilon_2\}$,*

$$\Pi_0 = \begin{bmatrix}\tilde{W}_2 & * \\ M^T & \tilde{W}_2\end{bmatrix} \ge 0 \tag{7}$$

$$\begin{bmatrix}\Delta(\upsilon) & (\xi_1^T U_1 + \xi_{15}^T U_2)B & (\xi_1^T U_1 + \xi_{15}^T U_2)\bar{B} \\ * & -\varepsilon_1 I & 0 \\ * & * & -\varepsilon_2 I\end{bmatrix} < 0 \tag{8}$$

hold, where

$$\begin{aligned}
\Delta(\upsilon) =& He(\hat{\mathcal{H}}_0^T(\upsilon)P\mathcal{H}_1 + N_1 g_1(\upsilon) + N_2 g_2(\upsilon) + \hat{S}\\
&+ \xi_{15}^T(\upsilon_1^2 W_1 + \upsilon_{12}^2 W_2)\xi_{15} - \mathcal{H}_2^T\tilde{W}_1\mathcal{H}_2 + \Pi_{ij}\\
&+ \xi_1^T S\xi_1 - \gamma^2\xi_{16}^T\xi_{16} - \Gamma^T\Pi_0\Gamma\\
\mathcal{H}_0(\upsilon) =& (\upsilon_2 - \upsilon)(\xi_{11} + \xi_{14}) + (\upsilon - \upsilon_1)(\xi_{12} - \xi_{13})\\
\hat{\mathcal{H}}_0(\upsilon) =& col\{\xi_1, \upsilon_1\xi_5, \upsilon_1\xi_6, \xi_{11} + \xi_{13}, \mathcal{H}_0(\upsilon)\}\\
\mathcal{H}_1 =& \upsilon_{12}(\xi_2 + \xi_4) - 2(\xi_{11} + \xi_{13})
\end{aligned}$$

$$\begin{aligned}
\hat{\mathcal{H}}_1 =& col\{\xi_{15}, \xi_1 - \xi_2, \xi_1 + \xi_2 - 2\xi_5, \xi_2 - \xi_4, \mathcal{H}_1\} \\
\hat{S} =& diag\{S_1, -S_1 + S_2, 0_{n\times n}, -S_2, 0_{12n\times 12n}\} \\
\tilde{W}_i =& diag\{W_i, 3W_i, 5W_i\}, i = 1, 2 \\
\Pi_{ij} =& He(\xi_1^T(U_1A+U_1KC)\xi_1+\xi_{15}^T(U_2A+U_2KC)\xi_1 \\
&+ \xi_1^TU_1\bar{A}\xi_3 + \xi_{15}^TU_2\bar{A}\xi_3 + \xi_1^TU_1G\xi_{16} \\
&+ \xi_{15}^TU_2G\xi_{16} - \xi_1^TU_1\xi_{15} - \xi_{15}^TU_2\xi_{15}) \\
&+ \varepsilon_1L_f^2\xi_1^T\xi_1 + \varepsilon_2L_g^2\xi_3^T\xi_3 \\
g_1(\upsilon) =& (\upsilon - \upsilon_1)\left[\, \xi_7^T \;\; \xi_8^T \,\right]^T - \left[\, \xi_{11}^T \;\; \xi_{12}^T \,\right]^T \\
g_2(\upsilon) =& (\upsilon_2 - \upsilon)\left[\, \xi_9^T \;\; \xi_{10}^T \,\right]^T - \left[\, \xi_{13}^T \;\; \xi_{14}^T \,\right]^T.
\end{aligned}$$

Then, under adaptive laws

$$\dot{\breve{\theta}}_k = \Lambda\Phi_k^T(\alpha(t))(U_1^T\kappa(t) + U_2^T\dot{\kappa}(t)) - \sigma_k\breve{\theta} \quad \text{(9a)}$$

$$\dot{\breve{\phi}}_l = \Upsilon\Psi_l^T(\alpha(t-\upsilon(t)))(U_1^T\kappa(t) + U_2^T\dot{\kappa}(t)) - \sigma_l\breve{\phi} \quad \text{(9b)}$$

the HAS for UDNNs (1) and (3) is achieved under controller (5).

Proof: Using schur's complement to (8), one has

$$\Delta(\upsilon) + (\xi_1^TU_1 + \xi_{15}^TU_2)\frac{1}{\varepsilon_1}BB^T(U_1^T\xi_1 + U_2^T\xi_{15}) + (\xi_1^TU_1 + \xi_{15}^TU_2)\frac{1}{\varepsilon_2}\bar{B}\bar{B}^T(U_1^T\xi_1 + U_2^T\xi_{15}) < 0$$

which can be rewritten as

$$\Phi_0(\upsilon) - \Gamma^T\Pi_0\Gamma < 0$$

where

$$\begin{aligned}
\Phi_0(\upsilon) =& He(\hat{\mathcal{H}}_0^T(\upsilon)P\hat{\mathcal{H}}_1 + N_1g_1(\upsilon) + N_2g_2(\upsilon)) + \hat{S} \\
&+ \xi_{15}^T(\upsilon_1^2W_1 + \upsilon_{12}^2W_2)\xi_{15} - \mathcal{H}_2^T\tilde{W}_1\mathcal{H}_2 + \Pi_{ij} \\
&+ (\xi_1^TU_1 + \xi_{15}^TU_2)\frac{1}{\varepsilon_1}BB^T(U_1^T\xi_1 + U_2^T\xi_{15}) \\
&+ (e_1^TU_1 + \xi_{15}^TU_2)\frac{1}{\varepsilon_2}\bar{B}\bar{B}^T(U_1^T\xi_1 + U_2^T\xi_{15}) \\
&+ \xi_1^TS\xi_1 - \gamma^2\xi_{16}^T\xi_{16}.
\end{aligned}$$

Substituting (5) into (4) gives

$$\begin{aligned}
\dot{\kappa}(t) =& (A+KC)\kappa(t) + \bar{A}\kappa(t-\upsilon(t)) + B(f(\hat{\alpha}(t)) + f(\alpha(t))) \\
&+ \bar{B}(g(\hat{\alpha}(t-\upsilon(t))) + g(\alpha(t-\upsilon(t)))) + Gd(t) \\
&- \sum_{k=1}^{N_1}\Phi_k(\alpha(t))\breve{\theta}_k - \sum_{l=1}^{N_2}\Psi_l(\alpha(t-\upsilon(t)))\breve{\phi}_l
\end{aligned}$$

where $\breve{\theta}_k = \bar{\theta}_k - \theta_k$ and $\breve{\phi}_l = \bar{\phi}_l - \phi_l$.

Denote $\zeta(t) = \text{col}\{\eta_0(t), \cdots, \eta_5(t), \dot{\kappa}(t), d(t)\}$. Then, for slack variables N_1 and N_2, one can establish that

$$0 = \zeta^T(t)\left(He(N_1g_1(\upsilon(t)) + N_2g_2(\upsilon(t)))\right)\zeta(t).$$

For slack variables U_1 and U_2, from the Lipschitz conditions (2), one can obtain

$$\begin{aligned}
0 =& 2(\kappa^T(t)U_1 + \dot{\kappa}^T(t)U_2)(-\dot{\kappa}(t) + (A+KC)\kappa(t) \\
&+ \bar{A}\kappa(t-\upsilon(t)) + B(f(\hat{\alpha}(t)) + f(\alpha(t))) \\
&+ \bar{B}(g(\hat{\alpha}(t-\upsilon(t))) + g(\alpha(t-\upsilon(t)))) + Gd(t)) \\
&- 2(\kappa^T(t)U_1 + \dot{\kappa}^T(t)U_2)(\sum_{k=1}^{N_1}\Phi_k(\alpha(t))\breve{\theta}_k \\
&+ \sum_{l=1}^{N_2}\Psi_l(\alpha(t-\upsilon(t)))\breve{\phi}_l) \\
\leq& \zeta^T(t)\Pi_{ij}\zeta(t) - 2(\kappa^T(t)U_1 + \dot{\kappa}^T(t)U_2)(\sum_{k=1}^{N_1}\Phi_k(\alpha(t))\breve{\theta}_k \\
&+ \sum_{l=1}^{N_2}\Psi_l(\alpha(t-\upsilon(t)))\breve{\phi}_l) \\
&+ \frac{1}{\varepsilon_1}(\kappa^T(t)U_1 + \dot{\kappa}^T(t)U_2)BB^T(U_1^T\kappa(t) + U_2^T\dot{\kappa}(t)) \\
&+ \frac{1}{\varepsilon_2}(\kappa^T(t)U_1 + \dot{\kappa}^T(t)U_2)\bar{B}\bar{B}^T(U_1^T\kappa(t) + U_2^T\dot{\kappa}(t)).
\end{aligned}$$

Consider LKF

$$V(\kappa_t, \dot{\kappa}_t) = V_1(\kappa_t) + V_2(\kappa_t) + V_3(\kappa_t, \dot{\kappa}_t) + V_4(\kappa_t) \quad (10)$$

where

$$\begin{aligned}
V_1(\kappa_t) =& \tilde{\kappa}^T(t)P\tilde{\kappa}(t) \\
V_2(\kappa_t) =& \int_{t-\upsilon_1}^{t}\kappa^T(s)S_1\kappa(s)ds + \int_{t-\upsilon_2}^{t-\upsilon_1}\kappa^T(s)S_2\kappa(s)ds \\
V_3(\kappa_t, \dot{\kappa}_t) =& \upsilon_1\int_{-\upsilon_1}^{0}\int_{t+\lambda}^{t}\dot{\kappa}^T(s)W_1\dot{\kappa}(s)dsd\lambda \\
&+ \upsilon_{12}\int_{-\upsilon_2}^{-\upsilon_1}\int_{t+\lambda}^{t}\dot{\kappa}^T(s)W_2\dot{\kappa}(s)dsd\lambda \\
V_4(\kappa_t) =& \sum_{k=1}^{N_1}\breve{\theta}_k^T\Lambda^{-1}\breve{\theta}_k + \sum_{l=1}^{N_2}\breve{\phi}_l^T\Upsilon^{-1}\breve{\phi}_l
\end{aligned}$$

in which $\tilde{\kappa}(t) = \text{col}\{\kappa(t), \upsilon_1\eta_1(t), \eta_6(t)\}$. One calculates that

$$\begin{aligned}
\dot{V}_1(\kappa_t) =& \zeta^T(t)He(\hat{\mathcal{H}}_0^T(\upsilon(t))P\hat{\mathcal{H}}_1 + N_1g_1(\upsilon(t)) \\
&+ N_2g_2(\upsilon(t)))\zeta(t) \\
\dot{V}_2(\kappa_t) =& \zeta^T(t)\hat{S}\zeta(t) \\
\dot{V}_3(\kappa_t, \dot{\kappa}_t) =& \zeta^T(t)\xi_{15}^T(\upsilon_1^2W_1 + \upsilon_{12}^2W_2)\xi_{15}\zeta(t) \\
&- \upsilon_1\int_{t-\upsilon_1}^{t}\dot{\kappa}^T(s)W_1\dot{\kappa}(s)ds \\
&- \upsilon_{12}\int_{t-\upsilon_2}^{t-\upsilon_1}\dot{\kappa}^T(s)W_2\dot{\kappa}(s)ds \qquad (11) \\
\dot{V}_4(\kappa_t) =& 2\sum_{k=1}^{N_1}\breve{\theta}_k^T\Lambda^{-1}\dot{\breve{\theta}}_k + 2\sum_{l=1}^{N_2}\breve{\phi}_l^T\Upsilon^{-1}\dot{\breve{\phi}}_l.
\end{aligned}$$

By applying Lemma 1 and Lemma 2, one has

$$
\begin{aligned}
&-\upsilon_1\int_{t-\upsilon_1}^{t}\dot{\kappa}^T(s)W_1\dot{\kappa}(s)ds-\upsilon_{12}\int_{t-\upsilon_2}^{t-\upsilon(t)}\dot{\kappa}^T(s)W_2\dot{\kappa}(s)ds\\
&-\upsilon_{12}\int_{t-\upsilon(t)}^{t-\upsilon_1}\dot{\kappa}^T(s)W_2\dot{\kappa}(s)ds\\
&\le-\zeta^T(t)\mathcal{H}_2^T\tilde{W}_1\mathcal{H}_2\zeta(t)-\zeta^T(t)\begin{pmatrix}\mathcal{H}_3^T & \mathcal{H}_4^T\end{pmatrix}\\
&\quad\times\begin{pmatrix}\frac{\upsilon_{12}}{\upsilon(t)-\upsilon_1}\tilde{W}_2 & \\ & \frac{\upsilon_{12}}{\upsilon_2-\upsilon(t)}\tilde{W}_2\end{pmatrix}\begin{pmatrix}\mathcal{H}_3\\ \mathcal{H}_4\end{pmatrix}\zeta(t)\\
&\le-\zeta^T(t)\mathcal{H}_2^T\tilde{W}_1\mathcal{H}_2\zeta(t)-\zeta^T(t)\Gamma^T\begin{pmatrix}\tilde{W}_2 & *\\ M^T & \tilde{W}_2\end{pmatrix}\Gamma\zeta(t)\\
&=-\zeta^T(t)\mathcal{H}_2^T\tilde{W}_1\mathcal{H}_2\zeta(t)-\zeta^T(t)\Gamma^T\Pi_0\Gamma\zeta(t)
\end{aligned}
$$

which together with (11), gives

$$
\begin{aligned}
\dot{V}_3(\kappa_t,\dot{\kappa}_t)\le&\zeta^T(t)\xi_{15}^T(\upsilon_1^2W_1+\upsilon_{12}^2W_2)\xi_{15}\zeta(t)\\
&-\zeta^T(t)\mathcal{H}_2^T\tilde{R}_1\mathcal{H}_2\zeta(t)-\zeta^T(t)\Gamma^T\Pi_0\Gamma\zeta(t).
\end{aligned}
$$

Combining the previous expressions, one is able to write

$$
\begin{aligned}
&\dot{V}(\kappa_t,\dot{\kappa}_t)\\
&\le\zeta^T(t)He(\hat{\mathcal{H}}_0^T(\upsilon(t))P\hat{\mathcal{H}}_1+N_1g_1(\upsilon(t))\\
&\quad+N_2g_2(\upsilon(t)))\zeta(t)+\zeta^T(t)\hat{S}\zeta(t)+\zeta^T(t)\xi_1^TS\xi_1\zeta(t)\\
&\quad-\gamma^2\zeta^T(t)\xi_{16}^T\xi_{16}\zeta(t)+\zeta^T(t)\xi_{15}^T(\upsilon_1^2W_1+\upsilon_{12}^2W_2)\xi_{15}\zeta(t)\\
&\quad-\zeta^T(t)\mathcal{H}_2^T\tilde{W}_1\mathcal{H}_2\zeta(t)-\zeta^T(t)\Gamma^T\Pi_0\Gamma\zeta(t)\\
&\quad+\zeta^T(t)\Pi_{ij}\zeta(t)+2\sum_{k=1}^{N_1}\breve{\theta}_k^T\Lambda^{-1}\dot{\theta}_k+2\sum_{l=1}^{N_2}\breve{\phi}_l^T\Upsilon^{-1}\dot{\phi}_l\\
&\quad+\zeta^T(t)((\xi_1^TU_1+\xi_{15}^TU_2)\frac{1}{\varepsilon_1}BB^T(U_1^T\xi_1+U_2^T\xi_{15})\\
&\quad+(e_1^TU_1+\xi_{15}^TU_2)\frac{1}{\varepsilon_2}\bar{B}\bar{B}^T(U_1^T\xi_1+U_2^T\xi_{15}))\zeta(t)\\
&\quad-2(\kappa^T(t)U_1+\dot{\kappa}^T(t)U_2)(\sum_{k=1}^{N_1}\Phi_k(\alpha(t))\breve{\theta}_k\\
&\quad+\sum_{l=1}^{N_2}\Psi_l(\alpha(t-\upsilon(t)))\breve{\phi}_l)-\kappa^T(t)S\kappa(t)+\gamma^2d^T(t)d(t)\\
&=\zeta^T(t)(\Phi_0(\upsilon)-\Gamma^T\Pi_0\Gamma)\zeta(t)+2\sum_{k=1}^{N_1}\breve{\theta}_k^T\Lambda^{-1}(\dot{\theta}_k\\
&\quad-\Lambda\Phi_k^T(\alpha(t))(U_1^T\kappa(t)+U_2^T\dot{\kappa}(t)))+2\sum_{l=1}^{N_2}\breve{\phi}_l^T\Upsilon^{-1}\\
&\quad\times(\dot{\phi}_l-\Upsilon\Psi_l^T(\alpha(t-\upsilon(t)))(U_1^T\kappa(t)+U_2^T\dot{\kappa}(t)))\\
&\quad-\kappa^T(t)S\kappa(t)+\gamma^2d^T(t)d(t).
\end{aligned}
$$

Using further the adaptive laws (9), one gets

$$
\dot{V}(\kappa_t,\dot{\kappa}_t)\le-\kappa^T(t)S\kappa(t)+\gamma^2d^T(t)d(t). \tag{12}
$$

From (12) one has

$$
V(\infty)-V(0)\le-\int_0^{\infty}\kappa^T(t)S\kappa(t)dt+\gamma^2\int_0^{\infty}d^T(t)d(t)dt.
$$

Noticing that $V(\infty)\ge 0$ and $V(0)=0$, one can obtain inequality (6), which means that ASES (4) possesses the $\mathcal{H}_\infty$ DAPL γ. Thus, the proof is finished. ■

Theorem 1 provides an existence condition and the update laws of the unknown parameters for the desired output-feedback-based adaptive controller. Following the proof steps in [25], it is easy to demonstrate that the inequality conditions in this theorem can also ensure the asymptotical anti-synchronization of UDNNs (1) and (3). However, the inequality conditions are not linear due to the existence of nonlinear coupling terms consisting of K and decision variables U_1 and U_2. To fix the issue, we introduce a new decision variable L and set $L=U_1K$, $U_2=\iota U_1$, where ι is a pre-set constant. Then, we arrive at the following theorem:

TABLE I
MINIMUM ALLOWABLE VALUE γ FOR GIVEN K

γ	k				
	-10	-20	-30	-40	-50
Theorem 1 in [13]	0.3475	0.0933	0.0589	0.0436	0.0347
Theorem 1 in [14]	0.3138	0.1825	0.1415	0.1195	0.1055
Theorem 1	0.2283	0.0696	0.0411	0.0291	0.0226

Theorem 2. *Given scalars $\gamma>0$ and $\iota>0$, and a matrix $S\in\mathbb{S}_+^n$, suppose there exist constant $\varepsilon_1>0$ and $\varepsilon_2>0$, and matrices $P\in\mathbb{S}_+^{5n}$, S_1, S_2, W_1, $W_2\in\mathbb{S}_+^n$, N_1, $N_2\in\mathbb{R}^{15n\times 2n}$, $U_1\in\mathbb{R}^{n\times n}$, L, and $M\in\mathbb{R}^{3n\times 3n}$, for any $\upsilon\in\{\upsilon_1,\upsilon_2\}$, such that LMIs (7) and*

$$
\begin{bmatrix}\hat{\Delta}(\upsilon) & (\xi_1^T+\iota\xi_{15}^T)U_1B & (\xi_1^T+\iota\xi_{15}^T)U_1\bar{B}\\ * & -\varepsilon_1I & 0\\ * & * & -\varepsilon_2I\end{bmatrix}<0
$$

hold, where

$$
\begin{aligned}
\hat{\Delta}(\upsilon)=&He(\hat{\mathcal{H}}_0^T(\upsilon)P\hat{\mathcal{H}}_1+N_1g_1(\upsilon)+N_2g_2(\upsilon))+\hat{S}\\
&+\xi_{15}^T(\upsilon_1^2W_1+\upsilon_{12}^2W_2)\xi_{15}-\mathcal{H}_2^T\tilde{W}_1\mathcal{H}_2\\
&+\hat{\Pi}_{ij}+\xi_1^TS\xi_1-\gamma^2\xi_{16}^T\xi_{16}-\Gamma^T\Pi_0\Gamma\\
\hat{\Pi}_{ij}=&He(\xi_1^T(U_1A+LC)\xi_1+\xi_{15}^T(\iota U_1A+\iota LC)\xi_1\\
&+\xi_1^TU_1\bar{A}\xi_3+\xi_{15}^T\iota U_1\bar{A}\xi_3+\xi_1^TU_1G\xi_{16}\\
&+\xi_{15}^T\iota U_1G\xi_{16}-\xi_1^TU_1\xi_{15}-\xi_{15}^T\iota U_1\xi_{15})\\
&+\varepsilon_1L_f^2\xi_1^T\xi_1+\varepsilon_2L_g^2\xi_3^T\xi_3.
\end{aligned}
$$

Then, under controller (5) with control gain $K=U_1^{-1}L$ and adaptive laws (9), the HAS for UDNNs (1) and (3) can be achieved.

IV. NUMERICAL EXAMPLE

Example 1. *Consider UDNN (1) with*

$$
\left[\begin{array}{c|c}A & f(\cdot)\\ \hline \bar{A} & g(\alpha(\cdot))\\ \hline B & \Phi_1(\cdot)\\ \hline \bar{B} & \Psi_1(\alpha(\cdot))\end{array}\right]=\left[\begin{array}{cc|c}-1 & 0 & tanh(\alpha_1(\cdot))\\ 0 & -1 & tanh(\alpha_2(\cdot))\\ \hline 0 & 0 & tanh(\alpha_1(\cdot))\\ 0 & 0 & tanh(\alpha_2(\cdot))\\ \hline 2 & 0 & -tanh(\alpha_2(\cdot))\\ -5 & 1.5 & 0\\ \hline -1.5 & -0.1 & 0\\ 0 & -1 & -tanh(\alpha_1(\cdot))\end{array}\right].
$$

For numerical calculation and simulation, we set $\theta_1=0.1$, $\phi_2=0.2$, $L_f=L_g=1$, $G=S=diag\{1,1\}$, $\Lambda=50$, $\Upsilon=200$, $\sigma_k=0.8$, $\sigma_l=0.8$, and $d(t)=1.8e^{(-0.5t)}$.

Consider the following two cases:

Case 1: $C=diag\{1,1\}$ and $\upsilon(t)=1$. In this case, the output feedback reduces to the state feedback. For this situation, Theorem 1 of this paper, Theorem 1 in [13] and Theorem 1 in [14] are capable of checking adaptive HAS of

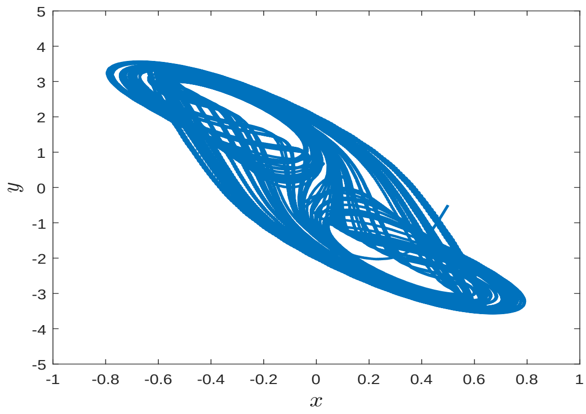

Fig. 1. Chaotic drive system

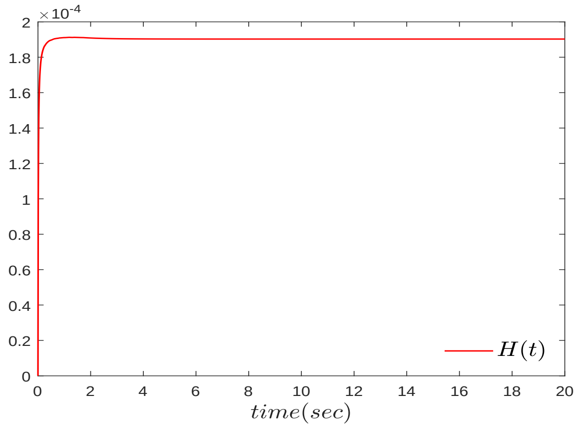

Fig. 2. The trajectory of $H(t)$

the NN with the state feedback. However, as listed in TABLE I, the results of [13] and [14] are much conservative than present Theorem 1.

Case 2: $C \neq diag\{1, 1\}$. In this case, the theorems in [13] and [14] are not applicable, while the present method may be resorted to. We take $C = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$ as an example, use scalar parameter $\iota = 1$ to analyze output-feedback adaptive HAS of the NN, and choose $\left[\alpha_1^T(0) \;\; \alpha_2^T(0)\right]^T = \left[0.5 \;\; -0.5\right]^T$, $\left[\hat{\alpha}_1^T(0) \;\; \hat{\alpha}_2^T(0)\right] = \left[-2.6 \;\; 3.3\right]^T$, $\bar{\theta}_1(0) = -0.5$, and $\bar{\phi}_1(0) = 0.3$ as initial conditions. The time-varying delay is given by $\upsilon(t) = 0.15 + 1.6\,|\sin(t)|$. Fig. 1 depicts the trajectory of the chaotic attractor. When γ is assigned a value of 0.5, the trajectory of $\mathcal{H}_\infty$ is plotted (6). Based on Theorem 2 we can obtain a solution, where $\varepsilon_1 = 1.0735$, $\varepsilon_2 = 0.0306$, and

$$\left[\begin{array}{c|c} S_1 & S_2 \\ \hline R_1 & R_2 \\ \hline U_1 & L \end{array}\right] = \left[\begin{array}{cc|cc} 0.7698 & -0.0505 & 0.3347 & -0.0238 \\ -0.0505 & 0.5883 & -0.0238 & 0.2498 \\ \hline 0.0023 & 0.0000 & 0.0046 & 0.0003 \\ 0.0000 & 0.0024 & 0.0003 & 0.0058 \\ \hline 0.0124 & 0.0011 & 0.0359 & 1.6506 \\ 0.0013 & 0.0164 & -1.5568 & 1.5210 \end{array}\right].$$

Therefore, we can get the control gain

$$K = \begin{bmatrix} 11.6019 & 125.3661 \\ -96.0699 & 83.3791 \end{bmatrix}.$$

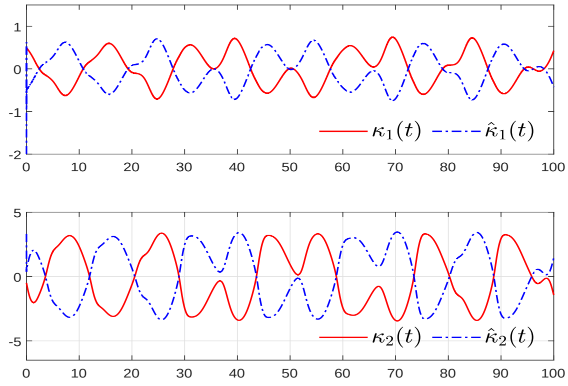

Fig. 3. State trajectories

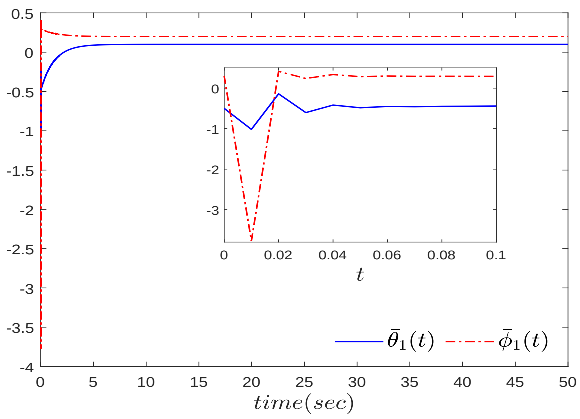

Fig. 4. Trajectories of $\bar{\theta}_1(t)$ and $\bar{\phi}_1(t)$

The curve $H(t)$ versus time is shown in Fig. 2. It can be found that the $\mathcal{H}_\infty$ norm from $\mathrm{d}(t)$ to anti-synchronization error $\kappa(t)$ is bounded and smaller than the given γ. Fig. 3 gives the curve of state trajectories, while Fig. 4 shows estimated values $\bar{\theta}_1(t)$ and $\bar{\phi}_1(t)$ for $\theta_1(t)$ and $\phi_1(t)$, respectively. As can be seen from these three figures, the anti-synchronization of UDNNs is well guaranteed, and the estimates $\bar{\theta}_1(t)$ and $\bar{\phi}_1(t)$ eventually converge to 0.1 and 0.2, respectively.

V. CONCLUSIONS

The issue HAS for UDNNs was considered. An output-feedback-based adaptive control scheme was used to ensure the HAS of the considered networks. An existence condition and update laws of the unknown parameters for the desired controller were developed by resorting to the BLI, LKF (10), as well as the RCC. Then, a LMIs-based design method for the output-feedback gain was developed by decoupling nonlinear terms. Finally, a numerical example was provided to explain the lower conservatism and effectiveness of the present HAS control method.

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