Linearization Criteria for a System of Two Second-Order Ordinary Differential Equations Via Generalized Linearizing Transformation

Supakit Tammakun, and Supaporn Suksern

***Abstract*—The generalized linearizing transformation is commonly used in the literature to study the linearization problem of single second-order ordinary differential equations. Necessary and sufficient conditions have been discovered for the linearization of such equations. However, the use of the generalized linearizing transformation to solve the linearization problem of a system of two second-order ordinary differential equations has not been explored. This paper aims to address this gap by applying the generalized linearizing transformation to a system of two second-order ordinary differential equations and solving the linearization problem.**

***Index Terms*—linearization problem, linearizing transformation, Sundman transformation, system of nonlinear ODEs.**

I. Introduction

COLLABORATING with scientists, engineers, and researchers from different fields, mathematicians create mathematical models to explain and forecast a wide range of real-world problems and physical phenomena, such as radioactive decay, Newton's law of cooling, chemical reactions, germ propagation, weather forecasting, electrical circuits, and economic issues. They use differential equations, including nonlinear ordinary differential equations, to model such phenomena. Nonetheless, solving these equations accurately is difficult. Therefore, mathematicians are continuously conducting research to overcome this challenge.

Linearization is an important research area for solving nonlinear ordinary differential equations. This applies to second, third, or higher-order equations. The key strategy involves converting the nonlinear equation into a linear one through an invertible transformation that enables any solution of one equation to be transformed into a solution of the other. Furthermore, several methods are commonly used to solve linearization problems, such as point transformation, fiber-preserving transformation, tangent transformation, contact transformation, generalized Sundman transformation and generalized linearizing transformation. Once the equation is linearized, the solution can be found using standard techniques for linear differential equations.

Historically, mathematicians have commonly used transformations to solve linearization problems for single second-order ordinary differential equations, but they are not frequently used for systems of two second-order ordinary differential equations. Sakka and Meleshko [1] have discovered a criterion for linearizing such systems under fiber-preserving transformations, and Moyo and Meleshko [2] have explored the use of generalized Sundman transformations for this problem. However, there is currently no research extending the use of generalized linearizing transformations to a system of ordinary differential equations, specifically for a system of two second-order ordinary differential equations. Therefore, the goal of this research is to use generalized linearizing transformations to determine the criteria for linearizing such a system.

A. Historical Review

For more than a century, mathematicians have been interested in solving the problem of linearizing ordinary differential equations. Esteemed mathematicians such as S. Lie and E. Cartan have dedicated their efforts to this problem. Lie [3] was the first to investigate the linearization problem of a single second-order ordinary differential equation by using the point transformation. Later, Liouville [4] and Tress [5] studied the similar problem by using relative invariants. Lie also pointed out that all second-order ordinary differential equations can be transformed into a linear equation through contact transformations without any restrictions.

To solve the linearization problem of second-order ordinary differential equations, there are multiple techniques available. One such method is using differential geometry, which is employed by Cartan [6]. Another transformation that can be used is the generalized Sundman transformation, which is an interesting approach that has not yet been mentioned. Duarte, Moreira, and Santos [7] use the Laguerre form in conjunction with the generalized Sundman transformation to solve the equivalence problem. However, Nakpim and Meleshko [8] have demonstrated through examples that only Laguerre form is not adequate for the linearization problem via generalized Sundman transformations. They suggest using the general form of a linear equation instead of the Laguerre form as the standard linear equation for the linearization problem. The generalized linearizing transformation is the other transformation which is used to solve the linearization problem in [9]-[10].

Although the above focuses on single second-order differential equations, the next part of the discussion will describe research on solving the linearization problem for a system of second-order ordinary differential equations.

In the field of differential equations, various researchers have developed tools to linearize systems of differential equations. For the system of two second-order differential equations, Wafo and Mahomed [11] employed a four-dimensional

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Lie algebra to linearize such system. Aminova and Aminov [12] discovered linearizing criteria by examining equivalent differential systems. Mahomed and Qadir [13] developed a linearization criterion that is specifically applicable to a particular class of systems: second-order quadratically semi-linear ordinary differential equations. Sookmee [14] made important contributions to the field of differential equations by discovering first- and second-order relative invariants via point transformations. Building on this work, Sookmee and Meleshko [15] developed a novel approach to linearizing systems of ordinary differential equations known as sequential linearization. This technique was successfully applied to such system, and the authors provide examples of systems that cannot be linearized using point transformations, but can be linearized using the new method. Neut, Petitot, and Dridi [16] used Cartan's method to establish the conditions for a system to be reduced to the simplest form. Bagderina [17] defined the criteria for mapping a system to a linear system of general form, providing the conditions for linearization by point transformation in terms of system coefficients. Sookmee and Meleshko [1] contributed to this field by identifying the required form of a linearizable system with point transformations. In addition to this, they established the criteria for a system to be transformed into a linear system with constant coefficients of general form via fiber-preserving transformations. Moyo and Meleshko [2] made significant contributions to the linearization of such systems by extending the application of the generalized Sundman transformation. Their work has helped to expand our understanding of how the generalized Sundman transformation can be applied in the linearization of differential equations.

This paper aims to apply the generalized linearizing transformation to solve the linearization of a system of two second-order ordinary differential equations, which has not been done before.

II. Formulation of the Linearization Theorems

A. *Obtaining Necessary Condition of Linearization*

In order to linearize a system of two second-order ordinary differential equations, it is necessary to first express the system in its general form

$$x'' = f(t,x,y,x',y'), \quad y'' = g(t,x,y,x',y'), \tag{1}$$

which can then be transformed into a linear system

$$u'' = 0, \quad v'' = 0, \tag{2}$$

using a generalized linearizing transformation

$$\begin{aligned} u = F_1(t,x,y), \quad v = F_2(t,x,y), \\ dT = [G_1(t,x,y)\,y' + G_2(t,x,y)]\,dt, \end{aligned} \tag{3}$$

with $G_1 \neq 0$. This process leads to a theorem that outlines the necessary conditions for linearizing the system.

Theorem 2.1: Any system of two second-order ordinary differential equations (1) obtained from a linear system (2) by a generalized linearizing transformation (3) has to be the form

$$\begin{aligned} x'' =& \lambda_1 x'^3 + \lambda_2 x'^2 y' + \lambda_3 y'^2 x' + \lambda_4 x'y' + \lambda_5 x'^2 \\ &+ \lambda_6 y'^2 + \lambda_7 x' + \lambda_8 y' + \lambda_9, \\ y'' =& \beta_1 y'^3 + \beta_2 x'^2 y' + \beta_3 y'^2 x' + \beta_4 x'y' + \beta_5 x'^2 \\ &+ \beta_6 y'^2 + \beta_7 x' + \beta_8 y' + \beta_9, \end{aligned} \tag{4}$$

where

$$\lambda_1 = (\Delta^{-1})(G_1F_{1xx}F_{2x} - G_1F_{1x}F_{2xx}), \tag{5}$$

$$\begin{aligned}\lambda_2 =& (\Delta^{-1})(2F_{1xy}F_{2x}G_1 - 2F_{1x}F_{2xy}G_1 \\ &+ F_{1x}F_{2y}G_{1x} - F_{1y}F_{2x}G_{1x}),\end{aligned} \tag{6}$$

$$\begin{aligned}\lambda_3 =& (\Delta^{-1})(-F_{1x}F_{2yy}G_1 + F_{1x}F_{2y}G_{1y} \\ &+ F_{1yy}F_{2x}G_1 - F_{1y}F_{2x}G_{1y}),\end{aligned} \tag{7}$$

$$\begin{aligned}\lambda_4 =& (\Delta^{-1})(2F_{1ty}F_{2x}G_1 - 2F_{1t}F_{2xy}G_1 + F_{1t}F_{2y}G_{1x} \\ &+ 2F_{1xy}F_{2t}G_1 - 2F_{1xy}F_{2y}G_2 - 2F_{1x}F_{2ty}G_1 \\ &+ F_{1x}F_{2y}G_{1t} + F_{1x}F_{2y}G_{2y} - F_{1y}F_{2t}G_{1x} \\ &+ 2F_{1y}F_{2xy}G_2 - F_{1y}F_{2x}G_{1t} - F_{1y}F_{2x}G_{2y}),\end{aligned} \tag{8}$$

$$\begin{aligned}\lambda_5 =& (\Delta^{-1})(2F_{1tx}F_{2x}G_1 - F_{1t}F_{2xx}G_1 + F_{1xx}F_{2t}G_1 \\ &- F_{1xx}F_{2y}G_2 - 2F_{1x}F_{2tx}G_1 + F_{1x}F_{2y}G_{2x} \\ &+ F_{1y}F_{2xx}G_2 - F_{1y}F_{2x}G_{2x}),\end{aligned} \tag{9}$$

$$\begin{aligned}\lambda_6 =& (\Delta^{-1})(-F_{1t}F_{2yy}G_1 + F_{1t}F_{2y}G_{1y} + F_{1yy}F_{2t}G_1 \\ &- F_{1yy}F_{2y}G_2 - F_{1y}F_{2t}G_{1y} + F_{1y}F_{2yy}G_2),\end{aligned} \tag{10}$$

$$\begin{aligned}\lambda_7 =& (\Delta^{-1})(2F_{1tx}F_{2t}G_1 - 2F_{1tx}F_{2y}G_2 + F_{1tt}F_{2x}G_1 \\ &- 2F_{1t}F_{2tx}G_1 + F_{1t}F_{2y}G_{2x} - F_{1x}F_{2tt}G_1 \\ &+ F_{1x}F_{2y}G_{2t} + 2F_{1y}F_{2tx}G_2 - F_{1y}F_{2t}G_{2x} \\ &- F_{1y}F_{2x}G_{2t}),\end{aligned} \tag{11}$$

$$\begin{aligned}\lambda_8 =& (\Delta^{-1})(2F_{1ty}F_{2t}G_1 - 2F_{1ty}F_{2y}G_2 \\ &- 2F_{1t}F_{2ty}G_1 + F_{1t}F_{2y}G_{1t} + F_{1t}F_{2y}G_{2y} \\ &+ 2F_{1y}F_{2ty}G_2 - F_{1y}F_{2t}G_{1t} - F_{1y}F_{2t}G_{2y}),\end{aligned} \tag{12}$$

$$\begin{aligned}\lambda_9 =& (\Delta^{-1})(F_{1tt}F_{2t}G_1 - F_{1tt}F_{2y}G_2 - F_{1t}F_{2tt}G_1 \\ &+ F_{1t}F_{2y}G_{2t} + F_{1y}F_{2tt}G_2 - F_{1y}F_{2t}G_{2t}),\end{aligned} \tag{13}$$

$$\begin{aligned}\beta_1 =& (\Delta^{-1})(-F_{1x}F_{2yy}G_1 + F_{1x}F_{2y}G_{1y} \\ &+ F_{1yy}F_{2x}G_1 - F_{1y}F_{2x}G_{1y}),\end{aligned} \tag{14}$$

$$\beta_2 = (\Delta^{-1})(G_1F_{1xx}F_{2x} - G_1F_{1x}F_{2xx}), \tag{15}$$

$$\begin{aligned}\beta_3 =& (\Delta^{-1})(2F_{1xy}F_{2x}G_1 - 2F_{1x}F_{2xy}G_1 \\ &+ F_{1x}F_{2y}G_{1x} - F_{1y}F_{2x}G_{1x}),\end{aligned} \tag{16}$$

$$\begin{aligned}\beta_4 =& (\Delta^{-1})(2F_{1tx}F_{2x}G_1 - F_{1t}F_{2x}G_{1x} \\ &+ 2F_{1xy}F_{2x}G_2 - 2F_{1x}F_{2tx}G_1 + F_{1x}F_{2t}G_{1x} \\ &- 2F_{1x}F_{2xy}G_2 + F_{1x}F_{2y}G_{2x} - F_{1y}F_{2x}G_{2x}),\end{aligned} \tag{17}$$

$$\beta_5 = (\Delta^{-1})(G_2F_{1xx}F_{2x} - G_2F_{1x}F_{2xx}), \tag{18}$$

$$\begin{aligned}\beta_6 =& (\Delta^{-1})(2F_{1ty}F_{2x}G_1 - F_{1t}F_{2x}G_{1y} \\ &- 2F_{1x}F_{2ty}G_1 + F_{1x}F_{2t}G_{1y} - F_{1x}F_{2yy}G_2 \\ &+ F_{1x}F_{2y}G_{1t} + F_{1x}F_{2y}G_{2y} + F_{1yy}F_{2x}G_2 \\ &- F_{1y}F_{2x}G_{1t} - F_{1y}F_{2x}G_{2y}),\end{aligned} \tag{19}$$

$$\begin{aligned}\beta_7 =& (\Delta^{-1})(2F_{1tx}F_{2x}G_2 - F_{1t}F_{2x}G_{2x} \\ &- 2F_{1x}F_{2tx}G_2 + F_{1x}F_{2t}G_{2x}),\end{aligned} \tag{20}$$

$$\begin{aligned}\beta_8 =& (\Delta^{-1})(2F_{1ty}F_{2x}G_2 + F_{1tt}F_{2x}G_1 - F_{1t}F_{2x}G_{1t} \\ &- F_{1t}F_{2x}G_{2y} - 2F_{1x}F_{2ty}G_2 - F_{1x}F_{2tt}G_1 \\ &+ F_{1x}F_{2t}G_{1t} + F_{1x}F_{2t}G_{2y} + F_{1x}F_{2y}G_{2t} \\ &- F_{1y}F_{2x}G_{2t}),\end{aligned} \tag{21}$$

$$\begin{aligned}\beta_9 =& (\Delta^{-1})(F_{1tt}F_{2x}G_2 - F_{1t}F_{2x}G_{2t} - F_{1x}F_{2tt}G_2 \\ &+ F_{1x}F_{2t}G_{2t}),\end{aligned} \tag{22}$$

where

$$\Delta = F_{1t}F_{2x}G_1 - F_{1x}F_{2t}G_1 + F_{1x}F_{2y}G_2 - F_{1y}F_{2x}G_2 \neq 0.$$

Proof. By utilizing a generalized linearizing transformation (3), the initial first-order derivatives undergo a transformation as follows. Differentiating (3) with respect to t, we find

$$\begin{aligned}
u' =& \frac{D_t F_1}{D_t \int [G_1 y' + G_2] dt} \\
=& \frac{F_{1t} + x' F_{1x} + y' F_{1y}}{G_1 y' + G_2} \\
=& H_1(t, x, y, x', y'), \\
v' =& \frac{D_t F_2}{D_t \int [G_1 y' + G_2] dt} \\
=& \frac{F_{2t} + x' F_{2x} + y' F_{2y}}{G_1 y' + G_2} \\
=& H_2(t, x, y, x', y').
\end{aligned}$$

The second-order derivatives after the transformation are expressed as

$$\begin{aligned}
u'' =& \frac{D_t H_1}{D_t \int [G_1 y' + G_2] dt} \\
=& [x'' F_{1x}(G_1 y' + G_2) + y''(-F_{1t} G_1 - F_{1x} G_1 x' \\
& + F_{1y} G_2) + 2F_{1tx} G_1 x' y' + 2F_{1tx} G_2 x' \\
& + 2F_{1ty} G_1 {y'}^2 + 2F_{1ty} G_2 y' + F_{1tt} G_1 y' \\
& + F_{1tt} G_2 - F_{1t} G_{1t} y' - F_{1t} G_{1x} x' y' \\
& - F_{1t} G_{1y} {y'}^2 - F_{1t} G_{2t} - F_{1t} G_{2x} x' \\
& - F_{1t} G_{2y} y' + 2F_{1xy} G_1 x' {y'}^2 + 2F_{1xy} G_2 x' y' \\
& + F_{1xx} G_1 {x'}^2 y' + F_{1xx} G_2 {x'}^2 - F_{1x} G_{1t} x' y' \\
& - F_{1x} G_{1x} {x'}^2 y' - F_{1x} G_{1y} x' {y'}^2 - F_{1x} G_{2t} x' \\
& - F_{1x} G_{2x} {x'}^2 - F_{1x} G_{2y} x' y' + F_{1yy} G_1 {y'}^3 \\
& + F_{1yy} G_2 {y'}^2 - F_{1y} G_{1t} {y'}^2 - F_{1y} G_{1x} x' {y'}^2 \\
& - F_{1y} G_{1y} {y'}^3 - F_{1y} G_{2t} y' - F_{1y} G_{2x} x' y' \\
& - F_{1y} G_{2y} {y'}^2] / (G_1^3 {y'}^3 + 3G_1^2 G_2 {y'}^2 \\
& + 3G_1 G_2^2 y' + G_2^3),
\end{aligned}$$

$$\begin{aligned}
v'' =& \frac{D_t H_2}{D_t \int [G_1 y' + G_2] dt} \\
=& [x'' F_{2x}(G_1 y' + G_2) + y''(-F_{2t} G_1 - F_{2x} G_1 x' \\
& + F_{2y} G_2) + 2F_{2tx} G_1 x' y' + 2F_{2tx} G_2 x' \\
& + 2F_{2ty} G_1 {y'}^2 + 2F_{2ty} G_2 y' + F_{2tt} G_1 y' \\
& + F_{2tt} G_2 - F_{2t} G_{1t} y' - F_{2t} G_{1x} x' y' \\
& - F_{2t} G_{1y} {y'}^2 - F_{2t} G_{2t} - F_{2t} G_{2x} x' \\
& - F_{2t} G_{2y} y' + 2F_{2xy} G_1 x' {y'}^2 + 2F_{2xy} G_2 x' y' \\
& + F_{2xx} G_1 {x'}^2 y' + F_{2xx} G_2 {x'}^2 - F_{2x} G_{1t} x' y' \\
& - F_{2x} G_{1x} {x'}^2 y' - F_{2x} G_{1y} x' {y'}^2 - F_{2x} G_{2t} x' \\
& - F_{2x} G_{2x} {x'}^2 - F_{2x} G_{2y} x' y' + F_{2yy} G_1 {y'}^3 \\
& + F_{2yy} G_2 {y'}^2 - F_{2y} G_{1t} {y'}^2 - F_{2y} G_{1x} x' {y'}^2 \\
& - F_{2y} G_{1y} {y'}^3 - F_{2y} G_{2t} y' - F_{2y} G_{2x} x' y' \\
& - F_{2y} G_{2y} {y'}^2] / (G_1^3 {y'}^3 + 3G_1^2 G_2 {y'}^2 \\
& + 3G_1 G_2^2 y' + G_2^3), \qquad (23)
\end{aligned}$$

where $D_t = \frac{\partial}{\partial t} + x' \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + x'' \frac{\partial}{\partial x'} + y'' \frac{\partial}{\partial y'}$ is a total derivative. After replacing the obtained expression into the linear system (2), one obtains the following system

$$\begin{aligned}
& x'' F_{1x}(G_1 y' + G_2) + y''(-F_{1t} G_1 \\
& - F_{1x} G_1 x' + F_{1y} G_2) + 2F_{1tx} G_1 x' y' \\
& + 2F_{1tx} G_2 x' + 2F_{1ty} G_1 {y'}^2 + 2F_{1ty} G_2 y' \\
& + F_{1tt} G_1 y' + F_{1tt} G_2 - F_{1t} G_{1t} y' \\
& - F_{1t} G_{1x} x' y' - F_{1t} G_{1y} {y'}^2 - F_{1t} G_{2t} \\
& - F_{1t} G_{2x} x' - F_{1t} G_{2y} y' + 2F_{1xy} G_1 x' {y'}^2 \\
& + 2F_{1xy} G_2 x' y' + F_{1xx} G_1 {x'}^2 y' + F_{1xx} G_2 {x'}^2 \\
& - F_{1x} G_{1t} x' y' - F_{1x} G_{1x} {x'}^2 y' - F_{1x} G_{1y} x' {y'}^2 \\
& - F_{1x} G_{2t} x' - F_{1x} G_{2x} {x'}^2 - F_{1x} G_{2y} x' y' \\
& + F_{1yy} G_1 {y'}^3 + F_{1yy} G_2 {y'}^2 - F_{1y} G_{1t} {y'}^2 \\
& - F_{1y} G_{1x} x' {y'}^2 - F_{1y} G_{1y} {y'}^3 - F_{1y} G_{2t} y' \\
& - F_{1y} G_{2x} x' y' - F_{1y} G_{2y} {y'}^2 = 0, \qquad (24)
\end{aligned}$$

$$\begin{aligned}
& x'' F_{2x}(G_1 y' + G_2) + y''(-F_{2t} G_1 \\
& - F_{2x} G_1 x' + F_{2y} G_2) + 2F_{2tx} G_1 x' y' \\
& + 2F_{2tx} G_2 x' + 2F_{2ty} G_1 {y'}^2 + 2F_{2ty} G_2 y' \\
& + F_{2tt} G_1 y' + F_{2tt} G_2 - F_{2t} G_{1t} y' \\
& - F_{2t} G_{1x} x' y' - F_{2t} G_{1y} {y'}^2 - F_{2t} G_{2t} \\
& - F_{2t} G_{2x} x' - F_{2t} G_{2y} y' + 2F_{2xy} G_1 x' {y'}^2 \\
& + 2F_{2xy} G_2 x' y' + F_{2xx} G_1 {x'}^2 y' + F_{2xx} G_2 {x'}^2 \\
& - F_{2x} G_{1t} x' y' - F_{2x} G_{1x} {x'}^2 y' - F_{2x} G_{1y} x' {y'}^2 \\
& - F_{2x} G_{2t} x' - F_{2x} G_{2x} {x'}^2 - F_{2x} G_{2y} x' y' \\
& + F_{2yy} G_1 {y'}^3 + F_{2yy} G_2 {y'}^2 - F_{2y} G_{1t} {y'}^2 \\
& - F_{2y} G_{1x} x' {y'}^2 - F_{2y} G_{1y} {y'}^3 - F_{2y} G_{2t} y' \\
& - F_{2y} G_{2x} x' y' - F_{2y} G_{2y} {y'}^2 = 0. \qquad (25)
\end{aligned}$$

From equation (24), one has $F_{1x}(G_1 y' + G_2) \neq 0$, then equation (24) can be rewritten as

$$\begin{aligned}
x'' =& [y'' x' F_{1x} G_1 + y''(F_{1t} G_1 - F_{1y} G_2) \\
& + {x'}^2 y'(-F_{1xx} G_1 + F_{1x} G_{1x}) \\
& + {x'}^2(-F_{1xx} G_2 + F_{1x} G_{2x}) \\
& + x' {y'}^2(-2F_{1xy} G_1 + F_{1x} G_{1y} + F_{1y} G_{1x}) \\
& + x' y'(-2F_{1tx} G_1 + F_{1t} G_{1x} - 2F_{1xy} G_2 \\
& + F_{1x} G_{1t} + F_{1x} G_{2y} + F_{1y} G_{2x}) \\
& + x'(-2F_{1tx} G_2 + F_{1t} G_{2x} + F_{1x} G_{2t}) \\
& + {y'}^3(-F_{1yy} G_1 + F_{1y} G_{1y}) \\
& + {y'}^2(-2F_{1ty} G_1 + F_{1t} G_{1y} - F_{1yy} G_2 \\
& + F_{1y} G_{1t} + F_{1y} G_{2y}) + y'(-2F_{1ty} G_2 \\
& - F_{1tt} G_1 + F_{1t} G_{1t} + F_{1t} G_{2y} \\
& + F_{1y} G_{2t}) - F_{1tt} G_2 + F_{1t} G_{2t}] \\
& / [F_{1x}(G_1 y' + G_2)]. \qquad (26)
\end{aligned}$$

Substituting equation (26) into equation (25), one gets the equation

$$\begin{aligned}
&y''(F_{1t}F_{2x}G_1 - F_{1x}F_{2t}G_1 + F_{1x}F_{2y}G_2\\
&- F_{1y}F_{2x}G_2) + x'^2y'G_1(-F_{1xx}F_{2x} + F_{1x}F_{2xx})\\
&+ x'^2G_2(-F_{1xx}F_{2x} + F_{1x}F_{2xx})\\
&+ x'y'^2(-2F_{1xy}F_{2x}G_1 + 2F_{1x}F_{2xy}G_1\\
&- F_{1x}F_{2y}G_{1x} + F_{1y}F_{2x}G_{1x}) + x'y'(-2F_{1tx}F_{2x}G_1\\
&+ F_{1t}F_{2x}G_{1x} - 2F_{1xy}F_{2x}G_2 + 2F_{1x}F_{2tx}G_1\\
&- F_{1x}F_{2t}G_{1x} + 2F_{1x}F_{2xy}G_2 - F_{1x}F_{2y}G_{2x}\\
&+ F_{1y}F_{2x}G_{2x}) + x'(-2F_{1tx}F_{2x}G_2 + F_{1t}F_{2x}G_{2x}\\
&+ 2F_{1x}F_{2tx}G_2 - F_{1x}F_{2t}G_{2x}) + y'^3(F_{1x}F_{2yy}G_1\\
&- F_{1x}F_{2y}G_{1y} - F_{1yy}F_{2x}G_1 + F_{1y}F_{2x}G_{1y})\\
&+ y'^2(-2F_{1ty}F_{2x}G_1 + F_{1t}F_{2x}G_{1y} + 2F_{1x}F_{2ty}G_1\\
&- F_{1x}F_{2t}G_{1y} + F_{1x}F_{2yy}G_2 - F_{1x}F_{2y}G_{1t}\\
&- F_{1x}F_{2y}G_{2y} - F_{1yy}F_{2x}G_2 + F_{1y}F_{2x}G_{1t}\\
&+ F_{1y}F_{2x}G_{2y}) + y'(-2F_{1ty}F_{2x}G_2 - F_{1tt}F_{2x}G_1\\
&+ F_{1t}F_{2x}G_{1t} + F_{1t}F_{2x}G_{2y} + 2F_{1x}F_{2ty}G_2\\
&+ F_{1x}F_{2tt}G_1 - F_{1x}F_{2t}G_{1t} - F_{1x}F_{2t}G_{2y}\\
&- F_{1x}F_{2y}G_{2t} + F_{1y}F_{2x}G_{2t}) - F_{1tt}F_{2x}G_2\\
&+ F_{1t}F_{2x}G_{2t} + F_{1x}F_{2tt}G_2 - F_{1x}F_{2t}G_{2t} = 0. \qquad (27)
\end{aligned}$$

From equation (27), one has $F_{1t}F_{2x}G_1 - F_{1x}F_{2t}G_1 + F_{1x}F_{2y}G_2 - F_{1y}F_{2x}G_2 \neq 0$, then equation (27), can be rewritten as

$$\begin{aligned}
y'' =&[x'^2y'G_1(F_{1xx}F_{2x} - F_{1x}F_{2xx})\\
&+ x'^2G_2(F_{1xx}F_{2x} - F_{1x}F_{2xx})\\
&+ x'y'^2(2F_{1xy}F_{2x}G_1 - 2F_{1x}F_{2xy}G_1\\
&+ F_{1x}F_{2y}G_{1x} - F_{1y}F_{2x}G_{1x})\\
&+ x'y'(2F_{1tx}F_{2x}G_1 - F_{1t}F_{2x}G_{1x}\\
&+ 2F_{1xy}F_{2x}G_2 - 2F_{1x}F_{2tx}G_1 + F_{1x}F_{2t}G_{1x}\\
&- 2F_{1x}F_{2xy}G_2 + F_{1x}F_{2y}G_{2x} - F_{1y}F_{2x}G_{2x})\\
&+ x'(2F_{1tx}F_{2x}G_2 - F_{1t}F_{2x}G_{2x} - 2F_{1x}F_{2tx}G_2\\
&+ F_{1x}F_{2t}G_{2x}) + y'^3(-F_{1x}F_{2yy}G_1\\
&+ F_{1x}F_{2y}G_{1y} + F_{1yy}F_{2x}G_1 - F_{1y}F_{2x}G_{1y})\\
&+ y'^2(2F_{1ty}F_{2x}G_1 - F_{1t}F_{2x}G_{1y}\\
&- 2F_{1x}F_{2ty}G_1 + F_{1x}F_{2t}G_{1y} - F_{1x}F_{2yy}G_2\\
&+ F_{1x}F_{2y}G_{1t} + F_{1x}F_{2y}G_{2y} + F_{1yy}F_{2x}G_2\\
&- F_{1y}F_{2x}G_{1t} - F_{1y}F_{2x}G_{2y}) + y'(2F_{1ty}F_{2x}G_2\\
&+ F_{1tt}F_{2x}G_1 - F_{1t}F_{2x}G_{1t} - F_{1t}F_{2x}G_{2y}\\
&- 2F_{1x}F_{2ty}G_2 - F_{1x}F_{2tt}G_1 + F_{1x}F_{2t}G_{1t}\\
&+ F_{1x}F_{2t}G_{2y} + F_{1x}F_{2y}G_{2t} - F_{1y}F_{2x}G_{2t})\\
&+ F_{1tt}F_{2x}G_2 - F_{1t}F_{2x}G_{2t} - F_{1x}F_{2tt}G_2\\
&+ F_{1x}F_{2t}G_{2t}]/(F_{1t}F_{2x}G_1 - F_{1x}F_{2t}G_1\\
&+ F_{1x}F_{2y}G_2 - F_{1y}F_{2x}G_2). \qquad (28)
\end{aligned}$$

Substituting equation (28) into equation (26), one gets the equation

$$\begin{aligned}
x'' =&[x'^3G_1(F_{1xx}F_{2x} - F_{1x}F_{2xx})\\
&+ x'^2y'(2F_{1xy}F_{2x}G_1 - 2F_{1x}F_{2xy}G_1\\
&+ F_{1x}F_{2y}G_{1x} - F_{1y}F_{2x}G_{1x})\\
&+ x'^2(2F_{1tx}F_{2x}G_1 - F_{1t}F_{2xx}G_1 + F_{1xx}F_{2t}G_1\\
&- F_{1xx}F_{2y}G_2 - 2F_{1x}F_{2tx}G_1 + F_{1x}F_{2y}G_{2x}\\
&+ F_{1y}F_{2xx}G_2 - F_{1y}F_{2x}G_{2x})\\
&+ x'y'^2(-F_{1x}F_{2yy}G_1 + F_{1x}F_{2y}G_{1y}\\
&+ F_{1yy}F_{2x}G_1 - F_{1y}F_{2x}G_{1y})\\
&+ x'y'(2F_{1ty}F_{2x}G_1 - 2F_{1t}F_{2xy}G_1\\
&+ F_{1t}F_{2y}G_{1x} + 2F_{1xy}F_{2t}G_1 - 2F_{1xy}F_{2y}G_2\\
&- 2F_{1x}F_{2ty}G_1 + F_{1x}F_{2y}G_{1t} + F_{1x}F_{2y}G_{2y}\\
&- F_{1y}F_{2t}G_{1x} + 2F_{1y}F_{2xy}G_2 - F_{1y}F_{2x}G_{1t}\\
&- F_{1y}F_{2x}G_{2y}) + x'(2F_{1tx}F_{2t}G_1\\
&- 2F_{1tx}F_{2y}G_2 + F_{1tt}F_{2x}G_1 - 2F_{1t}F_{2tx}G_1\\
&+ F_{1t}F_{2y}G_{2x} - F_{1x}F_{2tt}G_1 + F_{1x}F_{2y}G_{2t}\\
&+ 2F_{1y}F_{2tx}G_2 - F_{1y}F_{2t}G_{2x} - F_{1y}F_{2x}G_{2t})\\
&+ y'^2(-F_{1t}F_{2yy}G_1 + F_{1t}F_{2y}G_{1y} + F_{1yy}F_{2t}G_1\\
&- F_{1yy}F_{2y}G_2 - F_{1y}F_{2t}G_{1y} + F_{1y}F_{2yy}G_2)\\
&+ y'(2F_{1ty}F_{2t}G_1 - 2F_{1ty}F_{2y}G_2 - 2F_{1t}F_{2ty}G_1\\
&+ F_{1t}F_{2y}G_{1t} + F_{1t}F_{2y}G_{2y} + 2F_{1y}F_{2ty}G_2\\
&- F_{1y}F_{2t}G_{1t} - F_{1y}F_{2t}G_{2y}) + F_{1tt}F_{2t}G_1\\
&- F_{1tt}F_{2y}G_2 - F_{1t}F_{2tt}G_1 + F_{1t}F_{2y}G_{2t}\\
&+ F_{1y}F_{2tt}G_2 - F_{1y}F_{2t}G_{2t}]/(F_{1t}F_{2x}G_1\\
&- F_{1x}F_{2t}G_1 + F_{1x}F_{2y}G_2 - F_{1y}F_{2x}G_2). \qquad (29)
\end{aligned}$$

Denoting λ_i and β_i, $(i = 1, 2, ..., 9)$ as equations (5)-(22) for equations (28) and (29), so we obtain the necessary form (4).

B. *Obtaining Sufficient Conditions of Linearization and Linearizing Transformation*

To determine the sufficient conditions for linearizing system (4), it is necessary to address the compatibility problem of the system of equations (5)-(22). This system of equations should be considered as an overdetermined system of partial differential equations for the functions F_1, F_2, G_1 and G_2, taking into account the given coefficients λ_i and β_i of system (4). To solve this problem, a comprehensive solution will be derived specifically for the scenario where $F_{1y} = 0$ and $F_{2x} = 0$.

Setting the notation $K = F_{2y}G_2 - F_{2t}G_1$, we define the derivative F_{2t} as

$$F_{2t} = (F_{2y}G_2 - K)/G_1. \qquad (30)$$

From equations $(5), (15)$ and (18), one gets the conditions

$$\lambda_1 = 0, \quad \beta_2 = 0, \quad \beta_5 = 0. \qquad (31)$$

From equation (7), one gets the derivative

$$F_{2yy} = (F_{2y}G_{1y} - K\lambda_3)/G_1. \qquad (32)$$

Equation (14) provides the condition

$$\beta_1 = \lambda_3. \qquad (33)$$

From equation (8), one gets the derivative

$$K_y=(-F_{1t}F_{2y}G_{1x}G_1-F_{1x}F_{2y}G_{1t}G_1+F_{1x}F_{2y}G_{2y}G_1+2F_{1x}G_{1y}K+F_{1x}G_1K\lambda_4-2F_{1x}G_2K\lambda_3)/(2F_{1x}G_1). \quad (34)$$

Comparing the mixed derivative $(F_{2t})_x=0$, one gets the derivative

$$K_x=(-F_{2y}G_{1x}G_2+F_{2y}G_{2x}G_1+G_{1x}K)/G_1. \quad (35)$$

From equations (9), (11), (13), (19), (21) and (22), one can find the derivatives

$$F_{1xx}=(F_{1x}F_{2y}G_{2x}-F_{1x}K\lambda_5)/K, \quad (36)$$

$$F_{1tx}=(-F_{1t}F_{2y}G_{1x}G_2+2F_{1t}F_{2y}G_{2x}G_1+F_{1x}F_{2y}G_{1t}G_2-F_{1x}F_{2y}G_{2y}G_2-2F_{1x}G_{1t}K-2F_{1x}G_1K\lambda_7+2F_{1x}G_1K_t+F_{1x}G_2K\lambda_4)/(4G_1K), \quad (37)$$

$$F_{1tt}=(-F_{1t}^2F_{2y}G_{1x}G_2+F_{1t}F_{1x}F_{2y}G_{1t}G_2-F_{1t}F_{1x}F_{2y}G_{2y}G_2-2F_{1t}F_{1x}G_{1t}K+2F_{1t}F_{1x}G_1K_t+F_{1t}F_{1x}G_2K\lambda_4-2F_{1x}^2G_1K\lambda_9)/(2F_{1x}G_1K), \quad (38)$$

$$G_{1y}=(-F_{1t}F_{2y}G_{1x}G_1+F_{1x}G_1K(-\beta_6+\lambda_4)+F_{1x}G_2K\lambda_3)/(F_{1x}K), \quad (39)$$

$$K_t=(3F_{1t}F_{2y}G_{1x}G_2-F_{1x}F_{2y}G_{1t}G_2+F_{1x}F_{2y}G_{2y}G_2+4F_{1x}G_{1t}K+2F_{1x}G_{2y}K+2F_{1x}G_1K\beta_8-3F_{1x}G_2K\lambda_4)/(2F_{1x}G_1), \quad (40)$$

$$G_{2t}=(F_{1t}F_{2y}G_{1x}G_2^2+F_{1x}G_{1t}G_2K+F_{1x}G_{2y}G_2K-F_{1x}G_1^2K\beta_9+F_{1x}G_1G_2K\beta_8-F_{1x}G_2^2K\lambda_4)/(F_{1x}G_1K). \quad (41)$$

Equation (6) becomes

$$F_{1x}K\lambda_2-F_{1x}F_{2y}G_{1x}=0. \quad (42)$$

The compatibility analysis depend on the value of F_{2y} in equation (42) : it is separated into two cases; that is, $F_{2y}\neq 0$ and $F_{2y}=0$.

B.1. Case $F_{2y}=0$

As $F_{2y}=0$, we can replace it in the expression for F_{2yy} given in equation (32), which yields the following condition

$$\lambda_3=0, \quad (43)$$

and this satisfied equations $(F_{2y})_x=0$ and $(F_{2yy})_x=0$. By comparing the mixed derivative $(F_{2y})_t$ with $(F_{2t})_y$, we obtain the following condition

$$\lambda_4=0, \quad (44)$$

and this satisfied equations $(F_{2yy})_t=(F_{2t})_{yy}$ and $(K_x)_y=(K_y)_x$. The equations $(6), (42), (12)$ and (16) leads to the conditions

$$\lambda_2=0, \quad \lambda_6=0, \quad \lambda_8=0, \quad \beta_3=0. \quad (45)$$

Comparing the mixed derivatives $(F_{1xx})_y=0$, $(F_{1tx})_y=0$, $(F_{1tt})_y=0$ and $(F_{1tx})_x=(F_{1xx})_t$, one obtains the conditions

$$\lambda_{5y}=0, \quad \lambda_{7y}=0, \quad \lambda_{9y}=0, \quad \lambda_{5t}=\lambda_{7x}/2. \quad (46)$$

From equation (17), one gets the derivative

$$G_{1x}=-G_1\beta_4. \quad (47)$$

Comparing the mixed derivative $(G_{1x})_y=(G_{1y})_x$, one obtains the condition

$$\beta_{4y}=\beta_{6x}. \quad (48)$$

From equation (20), one gets the derivative

$$G_{2x}=-G_1\beta_7. \quad (49)$$

Comparing the mixed derivatives $(K_t)_y=(K_y)_t$ and $(F_{1tt})_x=(F_{1tx})_t$, one can find the derivatives

$$G_{2yy}=-G_{2y}\beta_6+G_1(\beta_{6t}-\beta_{8y}), \quad (50)$$

$$G_{1tt}=(3G_{1t}^2+2G_{1t}G_{2y}+2G_{1t}G_1\beta_8-G_{2y}^2+G_1^2(-2\beta_{8t}+2\beta_{9y}+2\lambda_{7t}-4\lambda_{9x}-2\beta_6\beta_9+\beta_8^2+4\lambda_5\lambda_9-\lambda_7^2))/(2G_1). \quad (51)$$

By comparing the mixed derivative $(K_t)_x$ with $(K_x)_t$, we obtain the following equation

$$G_{2y}K\beta_4+G_1K(-\beta_{4t}-\beta_{7y}+\beta_{8x}+\beta_6\beta_7)=0. \quad (52)$$

The compatibility analysis depend on the value of β_4 in equation (52) : it is separated into two cases; that is, $\beta_4\neq 0$ and $\beta_4=0$.

B.1.1. Case $\beta_4\neq 0$

Equation (52) provides the derivative

$$G_{2y}=G_1(\beta_{4t}+\beta_{7y}-\beta_{8x}-\beta_6\beta_7)/\beta_4. \quad (53)$$

By comparing the mixed derivative $(G_{2y})_x$ with $(G_{2x})_y$, we obtain the following condition

$$\beta_{4tx}=(\beta_{4t}\beta_{4x}+\beta_{4t}\beta_4^2+\beta_{4x}\beta_{7y}-\beta_{4x}\beta_{8x}-\beta_{4x}\beta_6\beta_7+\beta_{6x}\beta_4\beta_7-\beta_{7xy}\beta_4+\beta_{7x}\beta_4\beta_6+\beta_{8xx}\beta_4-\beta_{8x}\beta_4^2)/\beta_4. \quad (54)$$

Substituting G_{2y} from equation (53) into G_{2yy} in equation (50), one gets the condition

$$\beta_{6tx}=(\beta_{4t}\beta_{6x}+\beta_{6t}\beta_4^2+\beta_{6x}\beta_{7y}-\beta_{6x}\beta_{8x}-\beta_{6x}\beta_6\beta_7+\beta_{6y}\beta_4\beta_7-\beta_{7yy}\beta_4+\beta_{7y}\beta_4\beta_6+\beta_{8xy}\beta_4-\beta_{8y}\beta_4^2)/\beta_4, \quad (55)$$

and this satisfied equation $(G_{2yy})_x=(G_{2x})_{yy}$. Comparing the mixed derivatives $(G_{2y})_t=(G_{2t})_y$, $(G_{2x})_t=(G_{2t})_x$, $(G_{1tt})_y=(G_{1y})_{tt}$ and $(G_{1tt})_x=(G_{1x})_{tt}$, one obtains the conditions

$$\beta_{4tt}=(2\beta_{4t}^2+3\beta_{4t}\beta_{7y}-3\beta_{4t}\beta_{8x}+\beta_{4t}\beta_4\beta_8-3\beta_{4t}\beta_6\beta_7+\beta_{6t}\beta_4\beta_7-\beta_{7ty}\beta_4+\beta_{7t}\beta_4\beta_6+\beta_{7y}^2-2\beta_{7y}\beta_{8x}+\beta_{7y}\beta_4\beta_8-2\beta_{7y}\beta_6\beta_7+\beta_{8tx}\beta_4+\beta_{8x}^2-\beta_{8x}\beta_4\beta_8+2\beta_{8x}\beta_6\beta_7-\beta_{9y}\beta_4^2+\beta_4^2\beta_6\beta_9-\beta_4\beta_6\beta_7\beta_8+\beta_6^2\beta_7^2)/\beta_4, \quad (56)$$

$$\beta_{7t}=(\beta_{4t}\beta_7+\beta_{7y}\beta_7-\beta_{8x}\beta_7+\beta_{9x}\beta_4-\beta_4^2\beta_9+\beta_4\beta_7\beta_8-\beta_6\beta_7^2)/\beta_4, \quad (57)$$

$$\beta_{6tt}=(2\beta_{4t}\beta_{6t}-\beta_{4t}\beta_{8y}+2\beta_{6t}\beta_{7y}-2\beta_{6t}\beta_{8x}+\beta_{6t}\beta_4\beta_8-2\beta_{6t}\beta_6\beta_7+\beta_{6y}\beta_4\beta_9-\beta_{7y}\beta_{8y}+\beta_{8ty}\beta_4+\beta_{8x}\beta_{8y}-\beta_{8y}\beta_4\beta_8+\beta_{8y}\beta_6\beta_7-\beta_{9yy}\beta_4+\beta_{9y}\beta_4\beta_6)/\beta_4, \quad (58)$$

$$\lambda_{7tx}=-2\lambda_{5x}\lambda_9+\lambda_{7x}\lambda_7+2\lambda_{9xx}-2\lambda_{9x}\lambda_5, \quad (59)$$

and these satisfied equation $(G_{2yy})_t = (G_{2t})_{yy}$.

B.1.2. Case $\beta_4 = 0$

As $\beta_4 = 0$, we can replace it in the expression for β_{4y} given in equation (48), which yields the following condition

$$\beta_{6x} = 0. \tag{60}$$

Equation (52) provides the condition

$$\beta_{7y} = \beta_{8x} + \beta_6\beta_7, \tag{61}$$

and this satisfied equation $(G_{2yy})_x = (G_{2x})_{yy}$. By comparing the mixed derivative $(G_{2x})_t$ with $(G_{2t})_x$, we obtain the following equation

$$G_{2y}\beta_7 + G_1(-\beta_{7t} + \beta_{9x} + \beta_7\beta_8) = 0. \tag{62}$$

The compatibility analysis depend on the value of β_7 in equation (62) : it is separated into two cases; that is, $\beta_7 \neq 0$ and $\beta_7 = 0$.

B.1.2.1. Case $\beta_7 \neq 0$

Since $\beta_7 \neq 0$, equation (62) provides the derivative

$$G_{2y} = (G_1(\beta_{7t} - \beta_{9x} - \beta_7\beta_8))/\beta_7. \tag{63}$$

By comparing the mixed derivative $(G_{2y})_x$ with $(G_{2x})_y$, we obtain the following condition

$$\beta_{7tx} = (\beta_{7t}\beta_{7x} - \beta_{7x}\beta_{9x} + \beta_{9xx}\beta_7)/\beta_7. \tag{64}$$

Substituting G_{2y} from equation (63) into G_{2yy} in equation (50), one gets the condition

$$\beta_{8tx} = (\beta_{7t}\beta_{8x} - \beta_{8x}\beta_{9x} + \beta_{9xy}\beta_7 - \beta_{9x}\beta_6\beta_7)/\beta_7. \tag{65}$$

Comparing the mixed derivatives $(G_{2y})_t = (G_{2t})_y$, $(G_{1tt})_y = (G_{1y})_{tt}$ and $(G_{1tt})_x = (G_{1x})_{tt}$, one obtains the conditions

$$\begin{aligned}\beta_{7tt} =&(2\beta_{7t}^2 - 3\beta_{7t}\beta_{9x} - \beta_{7t}\beta_7\beta_8 + \beta_{8t}\beta_7^2 + \beta_{9tx}\beta_7 \\ &+ \beta_{9x}^2 + \beta_{9x}\beta_7\beta_8 - \beta_{9y}\beta_7^2 + \beta_6\beta_7^2\beta_9)/\beta_7, \end{aligned} \tag{66}$$

$$\begin{aligned}\beta_{6tt} =&(2\beta_{6t}\beta_{7t} - 2\beta_{6t}\beta_{9x} - \beta_{6t}\beta_7\beta_8 + \beta_{6y}\beta_7\beta_9 \\ &- \beta_{7t}\beta_{8y} + \beta_{8ty}\beta_7 + \beta_{8y}\beta_{9x} - \beta_{9yy}\beta_7 \\ &+ \beta_{9y}\beta_6\beta_7)/\beta_7, \end{aligned} \tag{67}$$

$$\lambda_{7tx} = -2\lambda_{5x}\lambda_9 + \lambda_{7x}\lambda_7 + 2\lambda_{9xx} - 2\lambda_{9x}\lambda_5, \tag{68}$$

and these satisfied equation $(G_{2yy})_t = (G_{2t})_{yy}$.

B.1.2.2. Case $\beta_7 = 0$

As $\beta_7 = 0$, we can replace it in the expression for β_{7y} given in equation (61), which yields the following condition

$$\beta_{8x} = 0. \tag{69}$$

From equation (62), one gets the condition

$$\beta_{9x} = 0. \tag{70}$$

By comparing the mixed derivative $(G_{1tt})_x$ with $(G_{1x})_{tt}$, we obtain the following condition

$$\lambda_{7tx} = -2\lambda_{5x}\lambda_9 + \lambda_{7x}\lambda_7 + 2\lambda_{9xx} - 2\lambda_{9x}\lambda_5. \tag{71}$$

By comparing the mixed derivative $(G_{1tt})_y$ with $(G_{1y})_{tt}$, we obtain the following equation

$$G_{2y}\gamma_2 + G_1\gamma_1 = 0, \tag{72}$$

where

$$\begin{aligned}\gamma_1 =&\beta_{6tt} - \beta_{6t}\beta_8 - \beta_{6y}\beta_9 - \beta_{8ty} \\ &+ \beta_{8y}\beta_8 + \beta_{9yy} - \beta_{9y}\beta_6, \\ \gamma_2 =& -2\beta_{6t} + \beta_{8y}. \end{aligned} \tag{73}$$

The relation $(\beta_{6t})_t = \beta_{6tt}$ provides the condition

$$\begin{aligned}\gamma_{2t} =& -2\beta_{6y}\beta_9 - \beta_{8ty} + \beta_{8y}\beta_8 + 2\beta_{9yy} \\ &- 2\beta_{9y}\beta_6 + \beta_8\gamma_2 - 2\gamma_1. \end{aligned} \tag{74}$$

The compatibility analysis depend on the value of γ_2 in equation (72) : it is separated into two cases; that is, $\gamma_2 \neq 0$ and $\gamma_2 = 0$.

B.1.2.2.1. Case $\gamma_2 \neq 0$

Since $\gamma_2 \neq 0$, equation (72) provides the derivative

$$G_{2y} = (-G_1\gamma_1)/\gamma_2, \tag{75}$$

and this satisfied equation $(G_{2yy})_t = (G_{2t})_{yy}$. By comparing the mixed derivative $(G_{2y})_x$ with $(G_{2x})_y$, we obtain the following condition

$$\gamma_{1x} = (\gamma_{2x}\gamma_1)/\gamma_2. \tag{76}$$

Substituting G_{2y} from equation (75) into G_{2yy} in equation (50), one gets the condition

$$\gamma_{1y} = (\beta_{8y}\gamma_2^2 + 2\gamma_{2y}\gamma_1 + \gamma_2^3)/(2\gamma_2). \tag{77}$$

Comparing the mixed derivative $(G_{2y})_t = (G_{2t})_y$, one obtains the condition

$$\begin{aligned}\gamma_{1t} =&(-2\beta_{6y}\beta_9\gamma_1 - \beta_{8ty}\gamma_1 + \beta_{8y}\beta_8\gamma_1 + 2\beta_{9yy}\gamma_1 \\ &- 2\beta_{9y}\beta_6\gamma_1 + \beta_{9y}\gamma_2^2 - \beta_6\beta_9\gamma_2^2 + 2\beta_8\gamma_1\gamma_2 \\ &- 3\gamma_1^2)/\gamma_2. \end{aligned} \tag{78}$$

B.1.2.2.2. Case $\gamma_2 = 0$

As $\gamma_2 = 0$, we can replace it in the expression for γ_{2t} given in equation (74), which yields the following condition

$$\beta_{8ty} = -2\beta_{6y}\beta_9 + \beta_{8y}\beta_8 + 2\beta_{9yy} - 2\beta_{9y}\beta_6 - 2\gamma_1. \tag{79}$$

From equation (72), one gets the condition

$$\gamma_1 = 0, \tag{80}$$

and this satisfied equation $(G_{2yy})_t = (G_{2t})_{yy}$.

Theorem 2.2: Sufficient conditions for equation (4) to be equivalent to a linear system (2) via generalized linearizing transformation (3) with the functions $F_1(t,x)$, $F_2(t)$, $G_1(t,x,y)$ and $G_2(t,x,y)$ are the equations (31), (33), (43), (44), (45), (46) and the additional conditions are as follows.

I. If $\beta_4 \neq 0$, then the conditions are equations (48), (54), (55), (56), (57), (58) and (59).

II. If $\beta_4 = 0$ and $\beta_7 \neq 0$, then the conditions are equations (60), (61), (64), (65), (66), (67) and (68).

III. If $\beta_4 = 0$, $\beta_7 = 0$ and $\gamma_2 \neq 0$, then the conditions are equations (60), (69), (70), (71), (74), (76), (77) and (78).

IV. If $\beta_4 = 0$, $\beta_7 = 0$ and $\gamma_2 = 0$, then the conditions are equations (60), (69), (70), (71), (79) and (80).

Corollary 2.3: Provided that the sufficient conditions in Theorem 2.2 are satisfied, the transformation (3) mapping equation (4) to a linear system (2) is obtained by solving the following compatible system of equations for the functions $F_1(t,x)$, $F_2(t)$, $G_1(t,x,y)$ and $G_2(t,x,y)$:

I. (30), (34), (35), (36), (37), (38), (39), (40), (41), (47), (49), (51) and (53).

II. (30), (34), (35), (36), (37), (38), (39), (40), (41), (47), (49), (51) and (63).

III. (30), (34), (35), (36), (37), (38), (39), (40), (41), (47), (49), (51) and (75).

IV. (30), (34), (35), (36), (37), (38), (39), (40), (41), (47), (49), (50) and (51).

B.2 Case $F_{2y} \neq 0$

On this case we focus on the special case $G_2 = 0$.

As $G_2 = 0$, we can replace it in the expression for G_{2t} given in equation (41), which yields the following condition

$$\beta_9 = 0, \tag{81}$$

and this satisfied equations $(G_2)_x = 0$ and $(G_2)_y = 0$. From equation (20), one gets the condition

$$\beta_7 = 0. \tag{82}$$

The relations $(F_{1xx})_y = 0$ and $(F_{1tx})_x = (F_{1xx})_t$ provide the conditions

$$\lambda_{5y} = 0, \quad \lambda_{5t} = \lambda_{7x}/2. \tag{83}$$

Equation (6) provides the derivative

$$G_{1x} = (K\lambda_2)/F_{2y}. \tag{84}$$

From equation (20), one gets the condition

$$\beta_3 = \lambda_2. \tag{85}$$

Equation (17) becomes

$$F_{1x}F_{2y}G_1K\beta_4 + F_{1x}K^2\lambda_2 = 0. \tag{86}$$

The compatibility analysis depend on the value of β_4 in equation (86) : it is separated into two cases; that is, $\beta_4 \neq 0$ and $\beta_4 = 0$.

B.2.1. Case $\beta_4 \neq 0$

Equation (86) provides the derivative

$$F_{2y} = (-K\lambda_2)/(G_1\beta_4). \tag{87}$$

Since $F_{2y} \neq 0$, then $\lambda_2 \neq 0$. The relations $(F_{2y})_x = 0$, $(F_{2yy})_x = 0$ and $(K_t)_x = (K_x)_t$ provide the conditions

$$\lambda_{2x} = (\beta_{4x}\lambda_2)/\beta_4, \lambda_{3x} = (\beta_{4y}\lambda_2)/\beta_4, \beta_{4t} = \beta_{8x}. \tag{88}$$

Substituting F_{2y} from equation (87) into F_{2yy} in equation (32), one gets the derivative

$$\begin{aligned} G_{1t} =& (-F_{1t}G_1\beta_4\lambda_2^2 + F_{1x}G_1(2\beta_{4y}\lambda_2 - 2\lambda_{2y}\beta_4 \\ &+ 2\beta_4^2\lambda_3 - 2\beta_4\beta_6\lambda_2 + \beta_4\lambda_2\lambda_4))/(F_{1x}\lambda_2^2). \end{aligned} \tag{89}$$

Comparing the mixed derivatives $(F_{2y})_t = (F_{2t})_y$ and $(F_{2yy})_t = (F_{2t})_{yy}$, one obtains the conditions

$$\begin{aligned} \lambda_{2y} =& (\beta_{4y}\beta_4\lambda_2 - \beta_{8x}\lambda_2^2 + \lambda_{2t}\beta_4\lambda_2 + \beta_4^3\lambda_3 \\ &- \beta_4^2\beta_6\lambda_2 + \beta_4\beta_8\lambda_2^2)/\beta_4^2, \end{aligned} \tag{90}$$

$$\begin{aligned} \lambda_{2tt} =& (\beta_{6t}\beta_4^3 + \beta_{8tx}\beta_4\lambda_2 - \beta_{8t}\beta_4^2\lambda_2 - 2\beta_{8x}^2\lambda_2 \\ &+ 2\beta_{8x}\lambda_{2t}\beta_4 + \beta_{8x}\beta_4\beta_8\lambda_2 - \beta_{8y}\beta_4^3 \\ &- \lambda_{2t}\beta_4^2\beta_8)/\beta_4^2. \end{aligned} \tag{91}$$

Equation (12) becomes

$$F_{1t}^2\lambda_2 - F_{1t}F_{1x}\lambda_4 + F_{1x}^2\lambda_8 = 0. \tag{92}$$

Equation (92) provides the derivative

$$F_{1t} = (F_{1x}(\lambda_4 \pm \sqrt{\nu}))/(2\lambda_2), \tag{93}$$

where $\nu = \lambda_4^2 - 4\lambda_2\lambda_8$. The compatibility analysis depend on the value of F_{1t} in equation (93) : it is separated into two cases; that is, $F_{1t} = (F_{1x}(\lambda_4 + \sqrt{\nu}))/(2\lambda_2)$ and $F_{1t} = (F_{1x}(\lambda_4 - \sqrt{\nu}))/(2\lambda_2)$.

B.2.1.1. Case $F_{1t} = (F_{1x}(\lambda_4 + \sqrt{\nu}))/(2\lambda_2)$

As $F_{1t} = (F_{1x}(\lambda_4 + \sqrt{\nu}))/(2\lambda_2)$, we can replace it in the expression for F_{1tx} given in equation (37), which yields the following condition

$$\begin{aligned} \nu_x =& (2\sqrt{\nu}\beta_{4x}\lambda_4 + 2\beta_{4x}\nu + 4\sqrt{\nu}\beta_{8x}\lambda_2 - 4\sqrt{\nu}\lambda_{2t}\beta_4 \\ &- 2\sqrt{\nu}\lambda_{4x}\beta_4 + \sqrt{\nu}\beta_4^2\lambda_4 - 2\sqrt{\nu}\beta_4\beta_8\lambda_2 \\ &- 2\sqrt{\nu}\beta_4\lambda_2\lambda_7 + 2\sqrt{\nu}\beta_4\lambda_4\lambda_5 \\ &- \beta_4^2\nu + 2\beta_4\lambda_5\nu)/\beta_4. \end{aligned} \tag{94}$$

The relation $(F_{1t})_y = 0$ provides the condition

$$\begin{aligned} \nu_y =& (2(\sqrt{\nu}\beta_{4y}\beta_4\lambda_2\lambda_4 + \beta_{4y}\beta_4\lambda_2\nu - \sqrt{\nu}\beta_{8x}\lambda_2^2\lambda_4 \\ &- \beta_{8x}\lambda_2^2\nu + \sqrt{\nu}\lambda_{2t}\beta_4\lambda_2\lambda_4 + \lambda_{2t}\beta_4\lambda_2\nu \\ &- \sqrt{\nu}\lambda_{4y}\beta_4^2\lambda_2 + \sqrt{\nu}\beta_4^3\lambda_3\lambda_4 - \sqrt{\nu}\beta_4^2\beta_6\lambda_2\lambda_4 \\ &+ \sqrt{\nu}\beta_4\beta_8\lambda_2^2\lambda_4 + \beta_4^3\lambda_3\nu - \beta_4^2\beta_6\lambda_2\nu \\ &+ \beta_4\beta_8\lambda_2^2\nu))/(\beta_4^2\lambda_2). \end{aligned} \tag{95}$$

Substituting F_{1t} into F_{1tt} in equation (38), one gets the condition

$$\begin{aligned} \nu_t =& (4\sqrt{\nu}\beta_{8x}\lambda_2\lambda_4 + 4\beta_{8x}\lambda_2\nu - 4\sqrt{\nu}\lambda_{4t}\beta_4\lambda_2 \\ &+ \sqrt{\nu}\beta_4^2\lambda_4^2 - \sqrt{\nu}\beta_4^2\nu - 2\sqrt{\nu}\beta_4\beta_8\lambda_2\lambda_4 \\ &- 8\sqrt{\nu}\beta_4\lambda_2^2\lambda_9 + 2\sqrt{\nu}\beta_4\lambda_2\lambda_4\lambda_7 - 2\beta_4\beta_8\lambda_2\nu \\ &+ 2\beta_4\lambda_2\lambda_7\nu)/(2\beta_4\lambda_2). \end{aligned} \tag{96}$$

Comparing the mixed derivatives $(G_{1t})_x = (G_{1x})_t$ and $(G_{1x})_y = (G_{1y})_x$, one obtains the conditions

$$\begin{aligned} \beta_{6x} =& (-2\sqrt{\nu}\beta_{4x} - 2\beta_{4x}\lambda_4 + 8\beta_{4y}\beta_4 + 4\lambda_{2t}\beta_4 \\ &+ 4\lambda_{4x}\beta_4 + \sqrt{\nu}\beta_4^2 - 2\sqrt{\nu}\beta_4\lambda_5 - \beta_4^2\lambda_4 \\ &+ 2\beta_4\beta_8\lambda_2 + 2\beta_4\lambda_2\lambda_7 - 2\beta_4\lambda_4\lambda_5)/(8\beta_4), \end{aligned} \tag{97}$$

$$\begin{aligned} \lambda_{4x} =& (2\sqrt{\nu}\beta_{4x} + 2\beta_{4x}\lambda_4 + 8\beta_{8x}\lambda_2 - 4\lambda_{2t}\beta_4 \\ &- \sqrt{\nu}\beta_4^2 + 2\sqrt{\nu}\beta_4\lambda_5 + \beta_4^2\lambda_4 - 2\beta_4\beta_8\lambda_2 \\ &- 2\beta_4\lambda_2\lambda_7 + 2\beta_4\lambda_4\lambda_5)/(4\beta_4), \end{aligned} \tag{98}$$

and these satisfied equations $(K_x)_y = (K_y)_x$ and $(K_t)_y = (K_y)_t$. By comparing the mixed derivative $(G_{1t})_y$ with

$(G_{1y})_t$, we obtain the following condition

$$\lambda_{3t} =(-4\sqrt{\nu}\beta_{4y}\beta_4\lambda_2 - 4\beta_{4y}\beta_4\lambda_2\lambda_4 + 8\beta_{6t}\beta_4\lambda_2^2 + 4\sqrt{\nu}\beta_{8x}\lambda_2^2 - 16\beta_{8x}\beta_4\lambda_2\lambda_3 + 12\beta_{8x}\lambda_2^2\lambda_4 + 16\lambda_{2t}\beta_4^2\lambda_3 - 8\lambda_{2t}\beta_4\lambda_2\lambda_4 - 8\lambda_{4t}\beta_4\lambda_2^2 + 8\lambda_{4y}\beta_4^2\lambda_2 - 2\sqrt{\nu}\beta_4\beta_8\lambda_2^2 + 2\sqrt{\nu}\beta_4\lambda_2^2\lambda_7 - 8\beta_4^3\lambda_3\lambda_4 + 8\beta_4^2\beta_6\lambda_2\lambda_4 + \beta_4^2\lambda_2\lambda_4^2 - \beta_4^2\lambda_2\nu - 10\beta_4\beta_8\lambda_2^2\lambda_4 - 8\beta_4\lambda_2^3\lambda_9 + 2\beta_4\lambda_2^2\lambda_4\lambda_7)/(16\beta_4^2\lambda_2). \tag{99}$$

From equation (42), one gets the condition

$$\lambda_6 = (\lambda_3(\sqrt{\nu} + \lambda_4))/(2\lambda_2). \tag{100}$$

Comparing the mixed derivatives $(F_{1tx})_y = 0$, $(F_{1tt})_y = 0$ and $(F_{1tt})_x = (F_{1tx})_t$, one obtains the conditions

$$- 8\sqrt{\nu}\beta_{6t}\beta_4\lambda_2 - 4\sqrt{\nu}\beta_{8x}\lambda_2\lambda_4 - 4\beta_{8x}\lambda_2\nu + 8\sqrt{\nu}\beta_{8y}\beta_4\lambda_2 + 8\sqrt{\nu}\lambda_{4t}\beta_4\lambda_2 - 8\sqrt{\nu}\lambda_{7y}\beta_4\lambda_2 - \sqrt{\nu}\beta_4^2\lambda_4^2 + \sqrt{\nu}\beta_4^2\nu + 2\sqrt{\nu}\beta_4\beta_8\lambda_2\lambda_4 + 8\sqrt{\nu}\beta_4\lambda_2^2\lambda_9 - 2\sqrt{\nu}\beta_4\lambda_2\lambda_4\lambda_7 + 2\beta_4\beta_8\lambda_2\nu - 2\beta_4\lambda_2\lambda_7\nu = 0, \tag{101}$$

$$\sqrt{\nu}\lambda_{7y}\lambda_4 + \lambda_{7y}\nu - 2\sqrt{\nu}\lambda_{9y}\lambda_2 = 0, \tag{102}$$

$$- 4\sqrt{\nu}\beta_{8t}\lambda_2^2 - 4\beta_{8x}\lambda_2\nu - 2\sqrt{\nu}\lambda_{2t}\beta_4\lambda_4 + 2\lambda_{2t}\beta_4\nu + 4\sqrt{\nu}\lambda_{4t}\beta_4\lambda_2 + 4\sqrt{\nu}\lambda_{7t}\lambda_2^2 - 8\sqrt{\nu}\lambda_{7y}\beta_4\lambda_2 - 8\sqrt{\nu}\lambda_{9x}\lambda_2^2 + \sqrt{\nu}\beta_4^2\nu - \sqrt{\nu}\beta_4\beta_8\lambda_2\lambda_4 + 4\sqrt{\nu}\beta_4\lambda_2^2\lambda_9 - \sqrt{\nu}\beta_4\lambda_2\lambda_4\lambda_7 + 2\sqrt{\nu}\beta_8^2\lambda_2^2 + 8\sqrt{\nu}\lambda_2^2\lambda_5\lambda_9 - 2\sqrt{\nu}\lambda_2^2\lambda_7^2 - \beta_4^2\lambda_4\nu + 3\beta_4\beta_8\lambda_2\nu - \beta_4\lambda_2\lambda_7\nu = 0. \tag{103}$$

B.2.1.2. Case $F_{1t} = (F_{1x}(\lambda_4 - \sqrt{\nu}))/(2\lambda_2)$
As $F_{1t} = (F_{1x}(\lambda_4 - \sqrt{\nu}))/(2\lambda_2)$, we can replace it in the expression for F_{1tx} given in equation (37), which yields the following condition

$$\nu_x =(-2\sqrt{\nu}\beta_{4x}\lambda_4 + 2\beta_{4x}\nu - 4\sqrt{\nu}\beta_{8x}\lambda_2 + 4\sqrt{\nu}\lambda_{2t}\beta_4 + 2\sqrt{\nu}\lambda_{4x}\beta_4 - \sqrt{\nu}\beta_4^2\lambda_4 + 2\sqrt{\nu}\beta_4\beta_8\lambda_2 + 2\sqrt{\nu}\beta_4\lambda_2\lambda_7 - 2\sqrt{\nu}\beta_4\lambda_4\lambda_5 - \beta_4^2\nu + 2\beta_4\lambda_5\nu)/\beta_4 \tag{104}$$

The relation $(F_{1t})_y = 0$ provides the condition

$$\nu_y =(2(-\sqrt{\nu}\beta_{4y}\beta_4\lambda_2\lambda_4 + \beta_{4y}\beta_4\lambda_2\nu + \sqrt{\nu}\beta_{8x}\lambda_2^2\lambda_4 - \beta_{8x}\lambda_2^2\nu - \sqrt{\nu}\lambda_{2t}\beta_4\lambda_2\lambda_4 + \lambda_{2t}\beta_4\lambda_2\nu + \sqrt{\nu}\lambda_{4y}\beta_4^2\lambda_2 - \sqrt{\nu}\beta_4^3\lambda_3\lambda_4 + \sqrt{\nu}\beta_4^2\beta_6\lambda_2\lambda_4 - \sqrt{\nu}\beta_4\beta_8\lambda_2^2\lambda_4 + \beta_4^3\lambda_3\nu - \beta_4^2\beta_6\lambda_2\nu + \beta_4\beta_8\lambda_2^2\nu))/(\beta_4^2\lambda_2). \tag{105}$$

Substituting F_{1t} into F_{1tt} in equation (38), one gets the condition

$$\nu_t =(-4\sqrt{\nu}\beta_{8x}\lambda_2\lambda_4 + 4\beta_{8x}\lambda_2\nu + 4\sqrt{\nu}\lambda_{4t}\beta_4\lambda_2 - \sqrt{\nu}\beta_4^2\lambda_4^2 + \sqrt{\nu}\beta_4^2\nu + 2\sqrt{\nu}\beta_4\beta_8\lambda_2\lambda_4 + 8\sqrt{\nu}\beta_4\lambda_2^2\lambda_9 - 2\sqrt{\nu}\beta_4\lambda_2\lambda_4\lambda_7 - 2\beta_4\beta_8\lambda_2\nu + 2\beta_4\lambda_2\lambda_7\nu)/(2\beta_4\lambda_2). \tag{106}$$

Comparing the mixed derivatives $(G_{1t})_x = (G_{1x})_t$ and $(G_{1x})_y = (G_{1y})_x$, one obtains the conditions

$$\beta_{6x} =(2\sqrt{\nu}\beta_{4x} - 2\beta_{4x}\lambda_4 + 8\beta_{4y}\beta_4 + 4\lambda_{2t}\beta_4 + 4\lambda_{4x}\beta_4 - \sqrt{\nu}\beta_4^2 + 2\sqrt{\nu}\beta_4\lambda_5 - \beta_4^2\lambda_4 + 2\beta_4\beta_8\lambda_2 + 2\beta_4\lambda_2\lambda_7 - 2\beta_4\lambda_4\lambda_5)/(8\beta_4), \tag{107}$$

$$\lambda_{4x} =(-2\sqrt{\nu}\beta_{4x} + 2\beta_{4x}\lambda_4 + 8\beta_{8x}\lambda_2 - 4\lambda_{2t}\beta_4 + \sqrt{\nu}\beta_4^2 - 2\sqrt{\nu}\beta_4\lambda_5 + \beta_4^2\lambda_4 - 2\beta_4\beta_8\lambda_2 - 2\beta_4\lambda_2\lambda_7 + 2\beta_4\lambda_4\lambda_5)/(4\beta_4), \tag{108}$$

and these satisfied equations $(K_x)_y = (K_y)_x$ and $(K_t)_y = (K_y)_t$. By comparing the mixed derivative $(G_{1t})_y$ with $(G_{1y})_t$, we obtain the following condition

$$\lambda_{3t} =(4\sqrt{\nu}\beta_{4y}\beta_4\lambda_2 - 4\beta_{4y}\beta_4\lambda_2\lambda_4 + 8\beta_{6t}\beta_4\lambda_2^2 - 4\sqrt{\nu}\beta_{8x}\lambda_2^2 - 16\beta_{8x}\beta_4\lambda_2\lambda_3 + 12\beta_{8x}\lambda_2^2\lambda_4 + 16\lambda_{2t}\beta_4^2\lambda_3 - 8\lambda_{2t}\beta_4\lambda_2\lambda_4 - 8\lambda_{4t}\beta_4\lambda_2^2 + 8\lambda_{4y}\beta_4^2\lambda_2 + 2\sqrt{\nu}\beta_4\beta_8\lambda_2^2 - 2\sqrt{\nu}\beta_4\lambda_2^2\lambda_7 - 8\beta_4^3\lambda_3\lambda_4 + 8\beta_4^2\beta_6\lambda_2\lambda_4 + \beta_4^2\lambda_2\lambda_4^2 - \beta_4^2\lambda_2\nu - 10\beta_4\beta_8\lambda_2^2\lambda_4 - 8\beta_4\lambda_2^3\lambda_9 + 2\beta_4\lambda_2^2\lambda_4\lambda_7)/(16\beta_4^2\lambda_2). \tag{109}$$

From equation (42), one gets the condition

$$\lambda_6 = (\lambda_3(-\sqrt{\nu} + \lambda_4))/(2\lambda_2). \tag{110}$$

Comparing the mixed derivatives $(F_{1tx})_y = 0$, $(F_{1tt})_y = 0$ and $(F_{1tt})_x = (F_{1tx})_t$, one obtains the conditions

$$- 8\sqrt{\nu}\beta_{6t}\beta_4\lambda_2 - 4\sqrt{\nu}\beta_{8x}\lambda_2\lambda_4 + 4\beta_{8x}\lambda_2\nu + 8\sqrt{\nu}\beta_{8y}\beta_4\lambda_2 + 8\sqrt{\nu}\lambda_{4t}\beta_4\lambda_2 - 8\sqrt{\nu}\lambda_{7y}\beta_4\lambda_2 - \sqrt{\nu}\beta_4^2\lambda_4^2 + \sqrt{\nu}\beta_4^2\nu + 2\sqrt{\nu}\beta_4\beta_8\lambda_2\lambda_4 + 8\sqrt{\nu}\beta_4\lambda_2^2\lambda_9 - 2\sqrt{\nu}\beta_4\lambda_2\lambda_4\lambda_7 - 2\beta_4\beta_8\lambda_2\nu + 2\beta_4\lambda_2\lambda_7\nu = 0, \tag{111}$$

$$\sqrt{\nu}\lambda_{7y}\lambda_4 - \lambda_{7y}\nu - 2\sqrt{\nu}\lambda_{9y}\lambda_2 = 0, \tag{112}$$

$$- 4\sqrt{\nu}\beta_{8t}\lambda_2^2 + 4\beta_{8x}\lambda_2\nu - 2\sqrt{\nu}\lambda_{2t}\beta_4\lambda_4 - 2\lambda_{2t}\beta_4\nu + 4\sqrt{\nu}\lambda_{4t}\beta_4\lambda_2 + 4\sqrt{\nu}\lambda_{7t}\lambda_2^2 - 8\sqrt{\nu}\lambda_{7y}\beta_4\lambda_2 - 8\sqrt{\nu}\lambda_{9x}\lambda_2^2 + \sqrt{\nu}\beta_4^2\nu - \sqrt{\nu}\beta_4\beta_8\lambda_2\lambda_4 + 4\sqrt{\nu}\beta_4\lambda_2^2\lambda_9 - \sqrt{\nu}\beta_4\lambda_2\lambda_4\lambda_7 + 2\sqrt{\nu}\beta_8^2\lambda_2^2 + 8\sqrt{\nu}\lambda_2^2\lambda_5\lambda_9 - 2\sqrt{\nu}\lambda_2^2\lambda_7^2 + \beta_4^2\lambda_4\nu - 3\beta_4\beta_8\lambda_2\nu + \beta_4\lambda_2\lambda_7\nu = 0. \tag{113}$$

B.2.2. Case $\beta_4 = 0$
Since $\beta_4 = 0$, then the equations (86) and $(F_{2yy})_x = 0$, one gets the conditions

$$\lambda_2 = 0, \tag{114}$$

$$\lambda_{3x} = 0. \tag{115}$$

Comparing the mixed derivatives $(K_x)_y = (K_y)_x$, $(K_t)_x = (K_x)_t$ and $(G_{1x})_y = (G_{1y})_x$, one obtains the conditions

$$\lambda_{4x} = 0, \tag{116}$$

$$\beta_{8x} = 0, \quad \beta_{6x} = 0. \tag{117}$$

By comparing the mixed derivative $(F_{1tt})_x$ with $(F_{1tx})_t$, we obtain the following derivative

$$G_{1tt} =(3G_{1t}^2 + 2G_{1t}G_1\beta_8 + G_1^2(-2\beta_{8t} + 2\lambda_{7t} - 4\lambda_{9x} + \beta_8^2 + 4\lambda_5\lambda_9 - \lambda_7^2))/(2G_1). \quad (118)$$

By comparing the mixed derivative $(G_{1tt})_x$ with $(G_{1x})_{tt}$, we obtain the following condition

$$\lambda_{7tx} = -2\lambda_{5x}\lambda_9 + \lambda_{7x}\lambda_7 + 2\lambda_{9xx} - 2\lambda_{9x}\lambda_5. \quad (119)$$

Equation (12) becomes

$$F_{1x}K\lambda_8 - F_{1t}K\lambda_4 = 0. \quad (120)$$

The compatibility analysis depend on the value of λ_4 in equation (120) : it is separated into two cases; that is, $\lambda_4 \neq 0$ and $\lambda_4 = 0$.

B.2.2.1. Case $\lambda_4 \neq 0$

Since $\lambda_4 \neq 0$, equation (120), one gets the derivative

$$F_{1t} = (F_{1x}\lambda_8)/\lambda_4. \quad (121)$$

Equations $(F_{1t})_y = 0$ and equation (42), provide the conditions

$$\lambda_{8y} =(\lambda_{4y}\lambda_8)/\lambda_4, \quad (122)$$
$$\lambda_6 =(\lambda_3\lambda_8)/\lambda_4. \quad (123)$$

Substituting F_{1t} from equation (121) into F_{1tx} in equation (37), one gets the derivative

$$G_{1t} = G_1(2\lambda_{8x} - \beta_8\lambda_4 + \lambda_4\lambda_7 - 2\lambda_5\lambda_8)/\lambda_4. \quad (124)$$

Comparing the mixed derivatives $(G_{1tt})_y = (G_{1y})_{tt}$ and $(G_{1t})_x = (G_{1x})_t$, one obtains the conditions

$$\beta_{6tt} =(2\beta_{6t}\lambda_{8x} + \beta_{6t}\lambda_4\lambda_7 - 2\beta_{6t}\lambda_5\lambda_8 + \beta_{8ty}\lambda_4 - 2\beta_{8y}\lambda_{8x} - \beta_{8y}\lambda_4\lambda_7 + 2\beta_{8y}\lambda_5\lambda_8 + \lambda_{4tt}\lambda_4 - 2\lambda_{4t}\lambda_{8x} - \lambda_{4t}\lambda_4\lambda_7 + 2\lambda_{4t}\lambda_5\lambda_8 - \lambda_{7ty}\lambda_4 + \lambda_{7y}\lambda_4\lambda_7 + 2\lambda_{9xy}\lambda_4 - 2\lambda_{9y}\lambda_4\lambda_5)/\lambda_4, \quad (125)$$
$$\lambda_{8xx} =(2\lambda_{5x}\lambda_8 - \lambda_{7x}\lambda_4 + 2\lambda_{8x}\lambda_5)/2. \quad (126)$$

Substituting G_{1t} from equation (124) into G_{1tt} in equation (118), one gets the condition

$$\lambda_{8tx} =(2\lambda_{4t}\lambda_{8x} - 2\lambda_{4t}\lambda_5\lambda_8 + \lambda_{7x}\lambda_4\lambda_8 + 2\lambda_{8t}\lambda_4\lambda_5 + 2\lambda_{8x}^2 + 2\lambda_{8x}\lambda_4\lambda_7 - 4\lambda_{8x}\lambda_5\lambda_8 - 2\lambda_{9x}\lambda_4^2 + 2\lambda_4^2\lambda_5\lambda_9 - 2\lambda_4\lambda_5\lambda_7\lambda_8 + 2\lambda_5^2\lambda_8^2)/(2\lambda_4). \quad (127)$$

Substituting F_{1t} from equation (121) into F_{1tt} in equation (38), one gets the condition

$$\lambda_{8t} =(\lambda_{4t}\lambda_8 + \lambda_{8x}\lambda_8 - \lambda_4^2\lambda_9 + \lambda_4\lambda_7\lambda_8 - \lambda_5\lambda_8^2)/\lambda_4, \quad (128)$$

and this satisfied equation $(\lambda_{8t})_x = \lambda_{8tx}$. By comparing the mixed derivative $(G_{1t})_y$ with $(G_{1y})_t$, we obtain the following condition

$$\beta_{6t} = \beta_{8y} + \lambda_{4t} - \lambda_{7y}. \quad (129)$$

The relation $(\beta_{6t})_t = \beta_{6tt}$ provides the condition

$$\lambda_{9xy} = (\lambda_{7y}\lambda_{8x} - \lambda_{7y}\lambda_5\lambda_8 + \lambda_{9y}\lambda_4\lambda_5)/\lambda_4. \quad (130)$$

By comparing the mixed derivative $(K_t)_y$ with $(K_y)_t$, we obtain the following equation

$$F_{2y}G_1\mu_1 + K\mu_2 = 0, \quad (131)$$

where

$$\mu_1 =- 2\beta_{8t} + 2\lambda_{7t} - 4\lambda_{9x} + \beta_8^2 + 4\lambda_5\lambda_9 - \lambda_7^2,$$
$$\mu_2 =- 2\lambda_{4t} + 4\lambda_{7y} - 2\lambda_{8x} + \beta_8\lambda_4 - \lambda_4\lambda_7 + 2\lambda_5\lambda_8.$$

The relations $(\lambda_{7t})_x = \lambda_{7tx}$, $(\lambda_{8x})_x = \lambda_{8xx}$ and $(\lambda_{8x})_t = \lambda_{8tx}$ provide the conditions

$$\mu_{1x} =0, \qquad \mu_{2x} = 4\lambda_{7xy}, \quad (132)$$
$$\mu_{1y} =(-8\beta_{8ty}\lambda_4 + 8\beta_{8y}\beta_8\lambda_4 + 4\lambda_{4tt}\lambda_4 + 8\lambda_{4t}\lambda_{7y} - 4\lambda_{4t}\beta_8\lambda_4 + 2\lambda_{4t}\mu_2 - 16\lambda_{7y}^2 + 2\mu_{2t}\lambda_4 - 2\beta_8\lambda_4\mu_2 + \lambda_4^2\mu_1 + \mu_2^2)/(4\lambda_4). \quad (133)$$

The compatibility analysis depend on the value of μ_1 in equation (131) : it is separated into two cases; that is, $\mu_1 \neq 0$ and $\mu_1 = 0$.

B.2.2.1.1. Case $\mu_1 \neq 0$

Since $\mu_1 \neq 0$, then the equation (131), one gets the derivative

$$F_{2y} = (-K\mu_2)/(G_1\mu_1), \quad (134)$$

and this satisfied equation $(F_{1tx})_y = 0$. Because of $F_{2y} \neq 0$, then $\mu_2 \neq 0$. The equations $(F_{1tt})_y = 0$ and $(F_{2y})_x = 0$ provide the conditions

$$\lambda_{9y} = (\lambda_{7y}\lambda_8)/\lambda_4, \quad \lambda_{7xy} = 0, \quad (135)$$

and these satisfied equation $(\lambda_{9y})_x = \lambda_{9xy}$. Substituting F_{2y} from equation (134) into F_{2yy} in equation (32), one gets the condition

$$\beta_{8ty} =(8\beta_{8y}\beta_8\lambda_4\mu_2 + 4\lambda_{4tt}\lambda_4\mu_2 + 8\lambda_{4t}\lambda_{7y}\mu_2 - 4\lambda_{4t}\beta_8\lambda_4\mu_2 + 6\lambda_{4t}\mu_2^2 - 16\lambda_{7y}^2\mu_2 - 8\lambda_{7y}\mu_2^2 + 2\mu_{2t}\lambda_4\mu_2 - 4\mu_{2y}\lambda_4\mu_1 - 4\beta_6\lambda_4\mu_1\mu_2 - 2\beta_8\lambda_4\mu_2^2 + 4\lambda_3\lambda_4\mu_1^2 + 3\lambda_4^2\mu_1\mu_2 + 3\mu_2^3)/(8\lambda_4\mu_2). \quad (136)$$

Comparing the mixed derivatives $(F_{2yy})_t = (F_{2t})_{yy}$ and $(F_{2y})_t = (F_{2t})_y$, one obtains the conditions

$$\lambda_{7yy} =(-4\lambda_{3t}\lambda_4^2\mu_1 + 4\lambda_{4ty}\lambda_4\mu_2 - 4\lambda_{4t}\lambda_{4y}\mu_2 + 4\lambda_{4t}\lambda_3\lambda_4\mu_1 - 2\lambda_{4t}\lambda_4^2\mu_2 + 8\lambda_{4y}\lambda_{7y}\mu_2 + 2\lambda_{4y}\lambda_4^2\mu_1 - 2\lambda_{4y}\mu_2^2 - 8\lambda_{7y}\lambda_3\lambda_4\mu_1 + 4\lambda_{7y}\lambda_4^2\mu_2 + 2\mu_{2y}\lambda_4\mu_2 + 2\beta_6\lambda_4^3\mu_1 - 4\beta_8\lambda_3\lambda_4^2\mu_1 + 2\lambda_3\lambda_4\mu_1\mu_2 - \lambda_4^4\mu_1 - \lambda_4^2\mu_2^2)/(8\lambda_4\mu_2), \quad (137)$$
$$\mu_{2t} =(2\lambda_{4t}\mu_1\mu_2 - 4\lambda_{7y}\mu_1\mu_2 + 2\mu_{1t}\lambda_4\mu_2 - 2\beta_8\lambda_4\mu_1\mu_2 + \lambda_4^2\mu_1^2 + \mu_1\mu_2^2)/(2\lambda_4\mu_1). \quad (138)$$

B.2.2.1.2. Case $\mu_1 = 0$

As $\mu_1 = 0$, we can replace it in the expression for μ_{1y} given in equation (133), which yields the following condition

$$\beta_{8ty} =(8\beta_{8y}\beta_8\lambda_4 + 4\lambda_{4tt}\lambda_4 + 8\lambda_{4t}\lambda_{7y} - 4\lambda_{4t}\beta_8\lambda_4 + 2\lambda_{4t}\mu_2 - 16\lambda_{7y}^2 + 2\mu_{2t}\lambda_4 - 2\beta_8\lambda_4\mu_2 + \mu_2^2)/(8\lambda_4), \quad (139)$$

and this satisfied equation $(\mu_1)_x = \mu_{1x}$. From equation (131), one gets the condition

$$\mu_2 = 0. \quad (140)$$

The relation $(\mu_2)_x = \mu_{2x}$ provides the condition

$$\lambda_{7xy} = 0, \quad (141)$$

and this satisfied equation $(F_{1tx})_y = 0$. From equation $(F_{1tt})_y = 0$, one obtains the condition

$$\lambda_{9y} = (\lambda_{7y}\lambda_8)/\lambda_4, \quad (142)$$

and this satisfied equation $(\lambda_{9y})_x = \lambda_{9xy}$. By comparing the mixed derivative $(F_{2yy})_t$ with $(F_{2t})_{yy}$, we obtain the following equation

$$2F_{2y}G_1\mu_3 + K\lambda_4\mu_4 = 0, \quad (143)$$

where

$$\mu_3 = - 2\lambda_{4ty}\lambda_4 + 2\lambda_{4t}\lambda_{4y} + \lambda_{4t}\lambda_4^2 - 4\lambda_{4y}\lambda_{7y} + 4\lambda_{7yy}\lambda_4 - 2\lambda_{7y}\lambda_4^2,$$
$$\mu_4 = - 4\lambda_{3t}\lambda_4 + 4\lambda_{4t}\lambda_3 + 2\lambda_{4y}\lambda_4 - 8\lambda_{7y}\lambda_3 + 2\beta_6\lambda_4^2 - 4\beta_8\lambda_3\lambda_4 + 2\lambda_3\mu_2 - \lambda_4^3.$$

The compatibility analysis depend on the value of μ_3 in equation (143) : it is separated into two cases; that is, $\mu_3 \neq 0$ and $\mu_3 = 0$.

B.2.2.1.2.1. Case $\mu_3 \neq 0$

Since $\mu_3 \neq 0$, then the equation (143), one gets the derivative

$$F_{2y} = (-K\lambda_4\mu_4)/(2G_1\mu_3). \quad (144)$$

Because of $F_{2y} \neq 0$, then $\mu_4 \neq 0$. From equation $(F_{2y})_x = 0$, one obtains the condition

$$\mu_{4x} = (\mu_{3x}\mu_4)/\mu_3. \quad (145)$$

Substituting F_{2y} from equation (144) into F_{2yy} in equation (32), one gets the condition

$$\mu_{4y} =(\lambda_{4t}\lambda_4\mu_4^2 - 2\lambda_{4y}\mu_3\mu_4 - 2\lambda_{7y}\lambda_4\mu_4^2 + 2\mu_{3y}\lambda_4\mu_4 - 2\beta_6\lambda_4\mu_3\mu_4 + 4\lambda_3\mu_3^2 + \lambda_4^2\mu_3\mu_4)/(2\lambda_4\mu_3). \quad (146)$$

Comparing the mixed derivative $(F_{2y})_t = (F_{2t})_y$, one obtains the condition

$$\mu_{4t} =(-2\lambda_{7y}\mu_3\mu_4 + \mu_{3t}\lambda_4\mu_4 - \beta_8\lambda_4\mu_3\mu_4 + \lambda_4\mu_3^2)/(\lambda_4\mu_3). \quad (147)$$

B.2.2.1.2.2. Case $\mu_3 = 0$

Since $\mu_3 = 0$, then the equation (143), one gets the condition

$$\mu_4 = 0. \quad (148)$$

B.2.2.2. Case $\lambda_4 = 0$

Since $\lambda_4 = 0$, then the equation (120), one gets the condition

$$\lambda_8 = 0, \quad (149)$$

and this satisfied equation $(\lambda_4)_x = \lambda_{4x}$. Equation (42) becomes

$$F_{1x}K\lambda_6 - F_{1t}K\lambda_3 = 0. \quad (150)$$

The compatibility analysis depend on the value of λ_3 in equation (150) : it is separated into two cases; that is, $\lambda_3 \neq 0$ and $\lambda_3 = 0$.

B.2.2.2.1. Case $\lambda_3 \neq 0$

Since $\lambda_3 \neq 0$, then the equation (150), one gets the derivative

$$F_{1t} = (F_{1x}\lambda_6)/\lambda_3. \quad (151)$$

Substituting F_{1t} from equation (151) into F_{1tx} in equation (37), one gets the derivative

$$G_{1t} = G_1(2\lambda_{6x} - \beta_8\lambda_3 + \lambda_3\lambda_7 - 2\lambda_5\lambda_6)/\lambda_3. \quad (152)$$

Comparing the mixed derivatives $(G_{1t})_x = (G_{1x})_t$ and $(G_{1t})_y = (G_{1y})_t$, one obtains the conditions

$$\lambda_{6xx} =(2\lambda_{5x}\lambda_6 + 2\lambda_{6x}\lambda_5 - \lambda_{7x}\lambda_3)/2, \quad (153)$$
$$\lambda_{6xy} =(-\beta_{6t}\lambda_3^2 + \beta_{8y}\lambda_3^2 + 2\lambda_{3y}\lambda_{6x} - 2\lambda_{3y}\lambda_5\lambda_6 + 2\lambda_{6y}\lambda_3\lambda_5 - \lambda_{7y}\lambda_3^2)/(2\lambda_3). \quad (154)$$

Substituting G_{1t} from equation (152) into G_{1tt} in equation (118), one gets the condition

$$\lambda_{6tx} =(2\lambda_{3t}\lambda_{6x} - 2\lambda_{3t}\lambda_5\lambda_6 + 2\lambda_{6t}\lambda_3\lambda_5 + 2\lambda_{6x}^2 + 2\lambda_{6x}\lambda_3\lambda_7 - 4\lambda_{6x}\lambda_5\lambda_6 + \lambda_{7x}\lambda_3\lambda_6 - 2\lambda_{9x}\lambda_3^2 + 2\lambda_3^2\lambda_5\lambda_9 - 2\lambda_3\lambda_5\lambda_6\lambda_7 + 2\lambda_5^2\lambda_6^2)/(2\lambda_3). \quad (155)$$

By comparing the mixed derivative $(G_{1tt})_y$ with $(G_{1y})_{tt}$, we obtain the following condition

$$\beta_{6tt} =(2\beta_{6t}\lambda_{6x} + \beta_{6t}\lambda_3\lambda_7 - 2\beta_{6t}\lambda_5\lambda_6 + \beta_{8ty}\lambda_3 - 2\beta_{8y}\lambda_{6x} - \beta_{8y}\lambda_3\lambda_7 + 2\beta_{8y}\lambda_5\lambda_6 - \lambda_{7ty}\lambda_3 + \lambda_{7y}\lambda_3\lambda_7 + 2\lambda_{9xy}\lambda_3 - 2\lambda_{9y}\lambda_3\lambda_5)/\lambda_3. \quad (156)$$

From equation $(F_{1t})_y = 0$, one obtains the condition

$$\lambda_{6y} = (\lambda_{3y}\lambda_6)/\lambda_3. \quad (157)$$

The relation $(\lambda_{6y})_x = \lambda_{6xy}$ provides the condition

$$\beta_{6t} = \beta_{8y} - \lambda_{7y}. \quad (158)$$

The relation $(\beta_{6t})_t = \beta_{6tt}$ provides the condition

$$\lambda_{9xy} = (\lambda_{6x}\lambda_{7y} - \lambda_{7y}\lambda_5\lambda_6 + \lambda_{9y}\lambda_3\lambda_5)/\lambda_3. \quad (159)$$

Substituting F_{1t} from equation (151) into F_{1tt} in equation (38), one gets the condition

$$\lambda_{6t} = (\lambda_{3t}\lambda_6 + \lambda_{6x}\lambda_6 - \lambda_3^2\lambda_9 + \lambda_3\lambda_6\lambda_7 - \lambda_5\lambda_6^2)/\lambda_3, \quad (160)$$

and this satisfied equation $(\lambda_{6t})_x = \lambda_{6tx}$. By comparing the mixed derivative $(K_t)_y$ with $(K_y)_t$, we obtain the following equation

$$F_{2y}G_1\mu_5 + 4K\lambda_{7y} = 0, \quad (161)$$

where

$$\mu_5 = -2\beta_{8t} + 2\lambda_{7t} - 4\lambda_{9x} + \beta_8^2 + 4\lambda_5\lambda_9 - \lambda_7^2.$$

The compatibility analysis depend on the value of μ_5 in equation (161) : it is separated into two cases; that is, $\mu_5 \neq 0$ and $\mu_5 = 0$.

B.2.2.2.1.1. Case $\mu_5 \neq 0$
Since $\mu_5 \neq 0$, then the equation (161), one gets the derivative

$$F_{2y} = (-4K\lambda_{7y})/(G_1\mu_5), \tag{162}$$

and this satisfied equation $(F_{1tx})_y = 0$. Because of $F_{2y} \neq 0$, then $\lambda_{7y} \neq 0$.
The equation $(F_{2y})_x = 0$, one obtains the condition

$$\mu_{5x} = (\lambda_{7xy}\mu_5)/\lambda_{7y}. \tag{163}$$

Substituting F_{2y} from equation (162) into F_{2yy} in equation (32), one gets the condition

$$\begin{aligned}\mu_{5y} =&(16\lambda_{6x}\lambda_{7y}^2 + 4\lambda_{7yy}\lambda_3\mu_5 - 8\lambda_{7y}^2\beta_8\lambda_3 \\ &+ 8\lambda_{7y}^2\lambda_3\lambda_7 - 16\lambda_{7y}^2\lambda_5\lambda_6 + 4\lambda_{7y}\beta_6\lambda_3\mu_5 \\ &- \lambda_3^2\mu_5^2)/(4\lambda_{7y}\lambda_3).\end{aligned} \tag{164}$$

Comparing the mixed derivative $(F_{2y})_t = (F_{2t})_y$, one gets the condition

$$\begin{aligned}\mu_{5t} =&(\mu_5(2\lambda_{6x}\lambda_{7y} + 2\lambda_{7ty}\lambda_3 + \lambda_{7y}\beta_8\lambda_3 \\ &+ \lambda_{7y}\lambda_3\lambda_7 - 2\lambda_{7y}\lambda_5\lambda_6))/(2\lambda_{7y}\lambda_3).\end{aligned} \tag{165}$$

The equation $(F_{1tt})_y = 0$, one obtains the condition

$$\lambda_{9y} = (\lambda_{7y}\lambda_6)/\lambda_3. \tag{166}$$

Comparing the mixed derivative $(F_{2yy})_t = (F_{2t})_{yy}$, one gets the condition

$$\begin{aligned}\lambda_{6x} =&(4\beta_{8y}\lambda_{7y} - 2\lambda_{3t}\mu_5 - 4\lambda_{7y}^2 - \beta_8\lambda_3\mu_5 \\ &- \lambda_3\lambda_7\mu_5 + 2\lambda_5\lambda_6\mu_5)/(2\mu_5).\end{aligned} \tag{167}$$

The relations $(\lambda_{6x})_x = \lambda_{6xx}$, $(\lambda_{6x})_y = \lambda_{6xy}$ and $(\lambda_{6x})_t = \lambda_{6tx}$, provides the conditions

$$\lambda_{7xy} =0, \tag{168}$$

$$\begin{aligned}\lambda_{3ty} =&(2\beta_{8yy}\lambda_{7y}\lambda_3\mu_5^2 - 16\beta_{8y}^2\lambda_{7y}^3 \\ &+ 8\beta_{8y}\lambda_{3t}\lambda_{7y}^2\mu_5 - 2\beta_{8y}\lambda_{3y}\lambda_{7y}\mu_5^2 \\ &+ 32\beta_{8y}\lambda_{7y}^4 + 8\beta_{8y}\lambda_{7y}^2\beta_8\lambda_3\mu_5 \\ &- 2\beta_{8y}\lambda_{7y}\beta_6\lambda_3\mu_5^2 + \lambda_{3t}\lambda_{3y}\mu_5^3 \\ &- 8\lambda_{3t}\lambda_{7y}^3\mu_5 + 2\lambda_{3y}\lambda_{7y}^2\mu_5^2 \\ &- 2\lambda_{7yy}\lambda_{7y}\lambda_3\mu_5^2 - 16\lambda_{7y}^5 \\ &- 8\lambda_{7y}^3\beta_8\lambda_3\mu_5 + 2\lambda_{7y}^2\beta_6\lambda_3\mu_5^2 \\ &- \lambda_{7y}\lambda_3^2\mu_5^3)/(\lambda_3\mu_5^3),\end{aligned} \tag{169}$$

$$\begin{aligned}\lambda_{3tt} =&(-16\beta_{8y}^2\lambda_{7y}^2 + 8\beta_{8y}\lambda_{3t}\lambda_{7y}\mu_5 \\ &+ 8\beta_{8y}\lambda_{7y}\beta_8\lambda_3\mu_5 + 8\lambda_{3t}\lambda_{7y}^2\mu_5 \\ &- 2\lambda_{3t}\beta_8\lambda_3\mu_5^2 - 2\lambda_{7t}\lambda_3^2\mu_5^2 - 2\lambda_{7yy}\lambda_3\mu_5^2 \\ &+ 16\lambda_{7y}^4 + 8\lambda_{7y}^2\beta_8\lambda_3\mu_5 - 2\lambda_{7y}\beta_6\lambda_3\mu_5^2 \\ &+ 4\lambda_{9x}\lambda_3^2\mu_5^2 - \beta_8^2\lambda_3^2\mu_5^2 - 4\lambda_3^2\lambda_5\lambda_9\mu_5^2 \\ &+ \lambda_3^2\lambda_7^2\mu_5^2 + \lambda_3^2\mu_5^3)/(2\lambda_3\mu_5^2),\end{aligned} \tag{170}$$

and these satisfied equation $(\lambda_{9y})_x = \lambda_{9xy}$.

B.2.2.2.1.2. Case $\mu_5 = 0$
Since $\mu_5 = 0$, then the equation (161) provides the condition

$$\lambda_{7y} = 0, \tag{171}$$

and this satisfied equation $(F_{1tx})_y = 0$. From the equation $(F_{1tt})_y = 0$, one obtains the condition

$$\lambda_{9y} = 0, \tag{172}$$

and this satisfied equation $(\lambda_{9y})_x = \lambda_{9xy}$. Comparing the mixed derivative $(F_{2yy})_t = (F_{2t})_{yy}$, one gets the equation

$$-F_{2y}G_1\beta_{8y} + K\mu_6 = 0, \tag{173}$$

where

$$\mu_6 = -2\lambda_{3t} - 2\lambda_{6x} - \beta_8\lambda_3 - \lambda_3\lambda_7 + 2\lambda_5\lambda_6.$$

The relations $(\lambda_{6x})_x = \lambda_{6xx}$, $(\lambda_{6x})_y = \lambda_{6xy}$ and $(\lambda_{6x})_t = \lambda_{6tx}$, provides the conditions

$$\mu_{6x} =0, \tag{174}$$

$$\mu_{6y} =(-\beta_{8y}\lambda_3^2 - 2\lambda_{3ty}\lambda_3 + 2\lambda_{3t}\lambda_{3y} + \lambda_{3y}\mu_6)/\lambda_3, \tag{175}$$

$$\begin{aligned}\mu_{6t} =&(-4\lambda_{3tt}\lambda_3 - 4\lambda_{3t}\beta_8\lambda_3 - 2\lambda_{3t}\mu_6 - 4\lambda_{7t}\lambda_3^2 \\ &+ 8\lambda_{9x}\lambda_3^2 - 2\beta_8^2\lambda_3^2 - 2\beta_8\lambda_3\mu_6 - 8\lambda_3^2\lambda_5\lambda_9 \\ &+ 2\lambda_3^2\lambda_7^2 - \mu_6^2)/(2\lambda_3).\end{aligned} \tag{176}$$

The compatibility analysis depend on the value of β_{8y} in equation (173) : it is separated into two cases; that is, $\beta_{8y} \neq 0$ and $\beta_{8y} = 0$.

B.2.2.2.1.2.1. Case $\beta_{8y} \neq 0$
Since $\beta_{8y} \neq 0$, then the equation (173), one gets the derivative

$$F_{2y} = (K\mu_6)/(G_1\beta_{8y}). \tag{177}$$

Because of $F_{2y} \neq 0$, then $\mu_6 \neq 0$. Substituting F_{2y} from equation (177) into F_{2yy} in equation (32), one gets the condition

$$\begin{aligned}\lambda_{3ty} =&(-2\beta_{8yy}\lambda_3\mu_6 + 4\beta_{8y}\lambda_{3t}\lambda_{3y} + 2\beta_{8y}\lambda_{3y}\mu_6 \\ &+ 2\beta_{8y}\beta_6\lambda_3\mu_6 + 2\lambda_{3t}\mu_6^2 + 2\beta_8\lambda_3\mu_6^2 + \mu_6^3) \\ &/(4\beta_{8y}\lambda_3).\end{aligned} \tag{178}$$

Comparing the mixed derivative $(F_{2y})_t = (F_{2t})_y$, one gets the condition

$$\begin{aligned}\lambda_{3tt} =&(-2\lambda_{3t}\beta_8\lambda_3 - 2\lambda_{3t}\mu_6 - 2\lambda_{7t}\lambda_3^2 \\ &+ 4\lambda_{9x}\lambda_3^2 - \beta_8^2\lambda_3^2 - 2\beta_8\lambda_3\mu_6 \\ &- 4\lambda_3^2\lambda_5\lambda_9 + \lambda_3^2\lambda_7^2 - \mu_6^2)/(2\lambda_3),\end{aligned} \tag{179}$$

and this satisfied equation $(F_{2y})_x = 0$.

B.2.2.2.1.2.2. Case $\beta_{8y} = 0$
Since $\beta_{8y} = 0$, then the equation (173) provides the condition

$$\mu_6 = 0. \tag{180}$$

The relations $(\mu_6)_y = \mu_{6y}$ and $(\mu_6)_t = \mu_{6t}$, provides the conditions

$$\lambda_{3ty} =(\lambda_{3t}\lambda_{3y})/\lambda_3, \tag{181}$$

$$\begin{aligned}\lambda_{3tt} =&(-2\lambda_{3t}\beta_8 - 2\lambda_{7t}\lambda_3 + 4\lambda_{9x}\lambda_3 - \beta_8^2\lambda_3 \\ &- 4\lambda_3\lambda_5\lambda_9 + \lambda_3\lambda_7^2)/2,\end{aligned} \tag{182}$$

and this satisfied equation $(\mu_6)_x = \mu_{6x}$.

B.2.2.2.2. Case $\lambda_3 = 0$

Since $\lambda_3 = 0$, then the equation (150), one gets the condition

$$\lambda_6 = 0, \tag{183}$$

and this satisfied equation $(\lambda_3)_x = \lambda_{3x}$. Comparing the mixed derivative $(F_{2yy})_t = (F_{2t})_{yy}$, one gets the condition

$$\beta_{6t} = 0. \tag{184}$$

By comparing the mixed derivative $(K_t)_y$ with $(K_y)_t$, we obtain the following equation

$$F_{2y}G_1\mu_7 + 4K\beta_{8y} = 0, \tag{185}$$

where

$$\mu_7 = -2\beta_{8t} + 2\lambda_{7t} - 4\lambda_{9x} + \beta_8^2 + 4\lambda_5\lambda_9 - \lambda_7^2.$$

The relation $(\lambda_{7t})_x = \lambda_{7tx}$, provides the condition

$$\mu_{7x} = 0. \tag{186}$$

The compatibility analysis depend on the value of μ_7 in equation (185) : it is separated into two cases; that is, $\mu_7 \neq 0$ and $\mu_7 = 0$.

B.2.2.2.2.1. Case $\mu_7 \neq 0$
Since $\mu_7 \neq 0$, then the equation (185), one gets the derivative

$$F_{2y} = (-4K\beta_{8y})/(G_1\mu_7), \tag{187}$$

and this satisfied equation $(F_{2y})_x = 0$. The equation $(F_{1tt})_y = 0$, one obtains the derivative

$$F_{1t} = (F_{1x}\lambda_{9y})/\beta_{8y}. \tag{188}$$

By comparing the mixed derivative $(G_{1tt})_y$ with $(G_{1y})_{tt}$, we obtain the following condition

$$G_{1t} = (-G_1\mu_{7y})/(2\beta_{8y}), \tag{189}$$

and this satisfied equation $(G_{1t})_x = (G_{1x})_t$. By comparing the mixed derivative $(G_{1t})_y$ with $(G_{1y})_t$, we obtain the following condition

$$\mu_{7yy} = (\beta_{8yy}\mu_{7y})/\beta_{8y}. \tag{190}$$

Substituting G_{1t} from equation (189) into G_{1tt} in equation (118), one gets the condition

$$\begin{aligned}\mu_{7ty} =&(4\beta_{8ty}\mu_{7y} - 4\beta_{8y}^2\mu_7 + 4\beta_{8y}\mu_{7y}\beta_8 \\ &- \mu_{7y}^2)/(4\beta_{8y}).\end{aligned} \tag{191}$$

Substituting F_{1t} from equation (188) into F_{1tx} in equation (37), one gets the condition

$$\lambda_{9xy} = (2\beta_{8y}\beta_8 - 2\beta_{8y}\lambda_7 + 4\lambda_{9y}\lambda_5 - \mu_{7y})/4. \tag{192}$$

The equation $(F_{1t})_y = 0$, one obtains the condition

$$\lambda_{9yy} = (\beta_{8yy}\lambda_{9y})/\beta_{8y}. \tag{193}$$

Substituting F_{1t} from equation (188) into F_{1tt} in equation (38), one gets the condition

$$\begin{aligned}\lambda_{9ty} =&(4\beta_{8ty}\lambda_{9y} - 4\beta_{8y}^2\lambda_9 + 2\beta_{8y}\lambda_{9y}\beta_8 \\ &+ 2\beta_{8y}\lambda_{9y}\lambda_7 - \lambda_{9y}\mu_{7y})/(4\beta_{8y}).\end{aligned} \tag{194}$$

The equation $(F_{1tx})_y = 0$, one obtains the condition

$$\lambda_{7y} = \beta_{8y}. \tag{195}$$

Substituting F_{2y} from equation (187) into F_{2yy} in equation (32), one gets the condition

$$\beta_{8yy} = (\beta_{8y}(2\mu_{7y} - \beta_6\mu_7))/\mu_7. \tag{196}$$

Comparing the mixed derivative $(F_{2y})_t = (F_{2t})_y$, one gets the condition

$$\beta_{8ty} = (4\beta_{8y}\mu_{7t} - 4\beta_{8y}\beta_8\mu_7 + \mu_{7y}\mu_7)/(4\mu_7). \tag{197}$$

B.2.2.2.2.2. Case $\mu_7 = 0$
Since $\mu_7 = 0$, then the equation (185), one gets the condition

$$\beta_{8y} = 0, \tag{198}$$

and this satisfied equations $(\mu_7)_x = \mu_{7x}$ and $(G_{1tt})_y = (G_{1y})_{tt}$.
The equations $(F_{1tx})_y = 0$ and $(F_{1tt})_y = 0$ provide the conditions

$$\lambda_{7y} = 0, \quad \lambda_{9y} = 0. \tag{199}$$

Theorem 2.4: Sufficient conditions for equation (4) to be equivalent to a linear system (2) via generalized linearizing transformation (3) with the functions $F_1(t,x)$, $F_2(t,y)$ and $G_1(t,x,y)$ are the equations (31), (33), (81), (82), (83), (85) and the additional conditions are as follows.

I. If $\beta_4 \neq 0$ and $F_{1t} = (F_{1x}(\lambda_4 + \sqrt{\nu}))/(2\lambda_2)$, then the conditions are equations (88), (90), (91), (94), (95), (96), (97), (98), (99), (100), (101), (102) and (103).

II. If $\beta_4 \neq 0$ and $F_{1t} = (F_{1x}(\lambda_4 - \sqrt{\nu}))/(2\lambda_2)$, then the conditions are equations (88), (90), (91), (104), (105), (106), (107), (108), (109), (110), (111), (112) and (113).

III. If $\beta_4 = 0$, $\lambda_4 \neq 0$ and $\mu_1 \neq 0$, then the conditions are equations (114), (115), (116), (117), (122), (123), (128), (129), (132), (133), (135), (136), (137) and (138).

IV. If $\beta_4 = 0$, $\lambda_4 \neq 0$, $\mu_1 = 0$ and $\mu_3 \neq 0$, then the conditions are equations (114), (115), (116), (117), (122), (123), (128), (129), (139), (140), (141), (142), (145), (146) and (147).

V. If $\beta_4 = 0$, $\lambda_4 \neq 0$, $\mu_1 = 0$ and $\mu_3 = 0$, then the conditions are equations (114), (115), (116), (117), (122), (123), (128), (129), (139), (140), (141), (142) and (148).

VI. If $\beta_4 = 0$, $\lambda_4 = 0$, $\lambda_3 \neq 0$ and $\mu_5 \neq 0$, then the conditions are equations (114), (115), (117), (119), (149), (157), (158), (160), (163), (164), (165), (166), (167), (168), (169) and (170).

VII. If $\beta_4 = 0$, $\lambda_4 = 0$, $\lambda_3 \neq 0$, $\mu_5 = 0$ and $\beta_{8y} \neq 0$, then the conditions are equations (114), (115), (117), (119), (149), (157), (158), (160), (171), (172), (174), (175), (176), (178) and (179).

VIII. If $\beta_4 = 0$, $\lambda_4 = 0$, $\lambda_3 \neq 0$, $\mu_5 = 0$ and $\beta_{8y} = 0$, then the conditions are equations (114), (115), (117), (119), (149), (157), (158), (160), (171), (172), (180), (181) and (182).

IX. If $\beta_4 = 0$, $\lambda_4 = 0$, $\lambda_3 = 0$ and $\mu_7 \neq 0$, then the conditions are equations (114), (117), (149), (183), (184), (186), (190), (191), (192), (193), (194), (195), (196) and (197).

X. If $\beta_4 = 0$, $\lambda_4 = 0$, $\lambda_3 = 0$ and $\mu_7 = 0$, then the conditions are equations (114), (117), (149), (183), (184), (198) and (199).

Corollary 2.5: Provided that the sufficient conditions in Theorem 2.4 are satisfied, the transformation (3) mapping equation (4) to a linear system (2) is obtained by solving the following compatible system of equations for the functions $F_1(t,x)$, $F_2(t,y)$ and $G_1(t,x,y)$:

I. (30), (34), (35), (36), (39), (40), (84), (87), (89) and $F_{1t} = (F_{1x}(\lambda_4 + \sqrt{\nu}))/(2\lambda_2)$.

II. (30), (34), (35), (36), (39), (40), (84), (87), (89) and $F_{1t} = (F_{1x}(\lambda_4 - \sqrt{\nu}))/(2\lambda_2)$.

III. (30), (34), (35), (36), (39), (40), (84), (121), (124) and (134).

IV. (30), (34), (35), (36), (39), (40), (84), (121), (124) and (144).

V. (30), (32), (34), (35), (36), (39), (40), (84), (121) and (124).

VI. (30), (34), (35), (36), (39), (40), (84), (151), (152) and (162).

VII. (30), (34), (35), (36), (39), (40), (84), (151), (152) and (177).

$VIII$. (30), (32), (34), (35), (36), (39), (40), (84), (151) and (152).

IX. (30), (34), (35), (36), (39), (40), (84), (187), (188) and (189).

X. (30), (32), (34), (35), (36), (37), (38), (39), (40), (84) and (118).

III. Examples

Example 3.1: Consider the system of two second-order ordinary differential equations

$$\begin{aligned} x''txy - x'^2y't^2 - 2x'y'tx + 2x'xy - y'x^2 = 0, \\ y''txy - x'y'^2t^2 - x'y'ty - y'^2tx + y'xy = 0. \end{aligned} \quad (200)$$

The system under consideration has coefficients that conform to the form of (4) in Theorem 2.1, which are

$$\begin{gathered} \lambda_1 = 0, \quad \lambda_2 = \tfrac{t}{xy}, \quad \lambda_3 = 0, \quad \lambda_4 = \tfrac{2}{y}, \quad \lambda_5 = 0, \\ \lambda_6 = 0, \quad \lambda_7 = -\tfrac{2}{t}, \quad \lambda_8 = \tfrac{x}{ty}, \quad \lambda_9 = 0, \quad \beta_1 = 0, \\ \beta_2 = 0, \quad \beta_3 = \tfrac{t}{xy}, \quad \beta_4 = \tfrac{1}{x}, \quad \beta_5 = 0, \quad \beta_6 = \tfrac{1}{y}, \\ \beta_7 = 0, \quad \beta_8 = -\tfrac{1}{t}, \quad \beta_9 = 0, \quad \nu = 0. \end{gathered}$$

It can be verified that these coefficients satisfy the conditions outlined in Theorem 2.4. Therefore, it is possible to linearize equation (200) by utilizing a generalized linearizing transformation. To determine the functions F_1, F_2, G_1 and G_2, we must solve the equations presented in Corollary 2.5, specifically in case (I). These equations are

$$\begin{gathered} F_{1t} = \tfrac{F_{1x}x}{t}, \quad F_{1xx} = 0, \quad F_{1y} = 0, \\ F_{2t} = -\tfrac{K}{G_1}, \quad F_{2x} = 0, \quad F_{2y} = -\tfrac{Kt}{G_1 y}, \\ G_{1t} = \tfrac{G_1}{t}, \quad G_{1x} = -\tfrac{G_1}{x}, \quad G_{1y} = 0, \\ K_t = \tfrac{K}{t}, \quad K_x = -\tfrac{K}{x}, \quad K_y = \tfrac{K}{y}, \quad G_2 = 0. \end{gathered} \quad (201)$$

The particular solution for the equations presented in (201) can be obtained as follows

$$F_1 = tx, \quad F_2 = ty, \quad G_1 = \tfrac{t}{x}, \quad G_2 = 0, \quad K = -\tfrac{ty}{x}.$$

Thus, we obtain the linearizing transformation as

$$u = tx, \quad v = ty, \quad dT = \frac{ty'}{x}dt. \quad (202)$$

As a result, the system (200) is transformed by the mappings presented in (202) into the following linear system

$$u'' = 0, \quad v'' = 0.$$

The solution for this linear system can be expressed as follows

$$u(T) = c_1T + c_2, \quad v(T) = c_3T + c_4, \quad (203)$$

where $c_i, (i = 1,2,3,4)$ are arbitrary constants. After applying the transformation (202) to equation (203), we can determine the general solution for equation (200) as follows

$$tx = c_1\phi(t) + c_2, \quad ty = c_3\phi(t) + c_4,$$

here, the function $T = \phi(t)$ represents a solution to the equation

$$\frac{dT}{dt} = \frac{ty'}{x}.$$

Example 3.2: Consider the system of two second-order ordinary differential equations

$$\begin{aligned} x''tx - 2x'^2t + 2x'x = 0, \\ y''xy + x'y'y + x't + y'^2x + x = 0. \end{aligned} \quad (204)$$

The system under consideration has coefficients that conform to the form of (4) in Theorem 2.1, which are

$$\begin{gathered} \lambda_1 = 0, \quad \lambda_2 = 0, \quad \lambda_3 = 0, \quad \lambda_4 = 0, \quad \lambda_5 = \tfrac{2}{x}, \\ \lambda_6 = 0, \quad \lambda_7 = -\tfrac{2}{t}, \quad \lambda_8 = 0, \quad \lambda_9 = 0, \quad \beta_1 = 0, \\ \beta_2 = 0, \quad \beta_3 = 0, \quad \beta_4 = -\tfrac{1}{x}, \quad \beta_5 = 0, \quad \beta_6 = -\tfrac{1}{y}, \\ \beta_7 = -\tfrac{t}{xy}, \quad \beta_8 = 0, \quad \beta_9 = -\tfrac{1}{y}. \end{gathered}$$

It can be verified that these coefficients satisfy the conditions outlined in Theorem 2.2. Therefore, it is possible to linearize equation (204) by utilizing a generalized linearizing transformation. To determine the functions F_1, F_2, G_1 and G_2, we must solve the equations presented in Corollary 2.3, specifically in case (I). These equations are

$$\begin{gathered} F_{1tt} = \tfrac{F_{1t}G_{1t}}{G_1}, \quad F_{1tx} = \tfrac{F_{1x}(G_{1t}t + 2G_1)}{2G_1 t}, \\ F_{1xx} = -\tfrac{2F_{1x}}{x}, \quad F_{1y} = 0, \quad F_{2t} = -\tfrac{K}{G_1}, \\ F_{2x} = 0, \quad F_{2y} = 0, \quad G_{1tt} = \tfrac{3G_{1t}^2}{2G_1}, \\ G_{1x} = \tfrac{G_1}{x}, \quad G_{1y} = \tfrac{G_1}{y}, \quad G_{2t} = \tfrac{G_{1t}G_{2}y + G_1{}^2}{G_1 y}, \\ G_{2x} = \tfrac{G_1 t}{xy}, \quad G_{2y} = 0, \quad K_t = \tfrac{2G_{1t}K}{G_1}, \\ K_x = \tfrac{K}{x}, \quad K_y = \tfrac{K}{y}. \end{gathered} \quad (205)$$

The particular solution for the equations presented in (205) can be obtained as follows

$$F_1 = \tfrac{t}{x}, \quad F_2 = t, \quad G_1 = xy, \quad G_2 = tx, \quad K = -xy.$$

Thus, we obtain the linearizing transformation as

$$u = \frac{t}{x}, \quad v = t, \quad dT = (xyy' + tx)dt. \quad (206)$$

As a result, the system (204) is transformed by the mappings presented in (206) into the following linear system

$$u'' = 0, \quad v'' = 0.$$

The solution for this linear system can be expressed as follows

$$u(T) = c_1T + c_2, \quad v(T) = c_3T + c_4, \quad (207)$$

where $c_i, (i = 1,2,3,4)$ are arbitrary constants. After applying the transformation (206) to equation (207), we can determine the general solution for equation (204) as follows

$$\tfrac{t}{x} = c_1\phi(t) + c_2, \quad t = c_3\phi(t) + c_4,$$

here, the function $T = \phi(t)$ represents a solution to the equation

$$\frac{dT}{dt} = xyy' + tx.$$

IV. CONCLUSION

This paper presents a generalized linearizing transformation to convert a system of two second-order ordinary differential equations into a linear system. It outlines the necessary conditions for the system to be linearizable in Theorem 2.1 and provides sufficient conditions in Theorems 2.2 and 2.4, which are identified by the linearizing transformation. The paper provides illustrative examples to demonstrate the presented theorems.

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