Signless Laplacian Energy of Partial Complement of a Graph

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Abstract—**The energy $E(G)$ of a graph G, defined as the sum of the absolute values of its eigenvalues, belongs to the most popular graph invariants in chemical graph theory. It originates from the $\pi-$electron energy in the Huckel molecular orbital model, but has also gained purely mathematical interest. Let $q_1, q_2, \ldots, q_n$ be the signless Laplacian eigenvalues of G. The signless Laplacian energy of G has recently been defined as $QE(G) = \sum_{i=1}^{n} |q_i - \frac{2m}{n}|$. In this paper, we define signless Laplacian energy of partial complements of a graph. Signless Laplacian spectrum of partial complements of the few classes of graphs are established. Some bounds and properties of signless Laplacian energy are obtained.**

Index Terms—**partial complement signless Laplacian energy, partial complement signless Laplacian spectrum, partial complement signless Laplacian eigenvalue.**

I. Introduction

LET G be a graph with n vertices $\{v_1, v_2, \ldots, v_n\}$ and m edges. Let $A = (a_{ij})$ be the adjacency matrix of graph G. $|A - \lambda I| = 0$ is characteristic equation of graph G. The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A, assumed in non increasing order, are eigenvalues of graph G. As A is real symmetric, the eigenvalues of G are real with sum equal to zero. The energy [4] $E(G)$ of G is defined to be sum of absolute values of eigenvalues of G. i.e., $E(G) = \sum_{i=1}^{n} |\lambda_i|$. The concept of energy of a graph was introduced by Ivan Gutman in the year 1978. More on Energy of the graphs we refer [1], [2], [3], [5], [6], [8], [11], [12].

For $v \in V$, degree of v, written by $d(v)$, is number of edges incident on v. Let $D(G)$ be diagonal matrix of vertex degrees. Then Laplacian matrix of G is $L(G) = D(G) - A(G)$. In 2006, Ivan Gutman and Bo Zhou defined Laplacian energy of graph [9] as $LE(G) = \sum_{i=1}^{n} |\mu_i - \frac{2m}{n}|$, where set $\{\mu_1, \mu_2, \ldots, \mu_n\}$ is the Laplacian spectrum of graph G.

The square matrix $Q(G) = D(G) + A(G)$ is referred as Signless Laplacian matrix of G and signless Laplacian energy is defined as $QE(G) = \sum_{i=1}^{n} |q_i - \frac{2m}{n}|$, where set $\{q_1, q_2, \ldots, q_n\}$ forms the signless Laplacian spectrum of graph G.

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Fedor V. Fomin et.al [7] introduced partial complements of graph. In this paper, we study signless Laplacian spectra/energy of partial complements of a graph.

This paper is organized as follows. In section 2, we give necessary definitions and some known results. In section 3, we give properties of signless Laplacian energy of partial complements of a graph. In section 4, we derive few upper and lower bounds for signless Laplacian energy of partial complements of a graph and in section 5, we obtain signless Laplacian spectrum of partial complements of families of a graph.

II. Preliminaries

Proposition 1. *Let $A = \begin{bmatrix} A_0 & A_1 \\ A_1 & A_0 \end{bmatrix}$ be a symmetric block matrix of order 2. Then spectrum of A is the union of spectra of $A_0 + A_1$ and $A_0 - A_1$.*

Proposition 2. *Let M, N, P, Q be square matrices, and let M be invertible. Let $S = \begin{bmatrix} M & N \\ P & Q \end{bmatrix}$ Then $\det S = \det M. \det[Q - PM^{-1}N]$. If M and P commute, then $\det S = \det[MQ - PN]$.*

Definition 3. *[7] Let $G = (V, E)$ be a graph and $S \subseteq V$. The partial complement of a graph G with respect to S, denoted by $G \oplus S$, is a graph (V, E_S), where for any two vertices $u, v \in V$, $uv \in E_S$ if and only if one of the following conditions holds good:*

1) *$u \notin S$ or $v \notin S$ and $uv \in E$.*
2) *$u, v \in S$ and $uv \notin E$.*
   Alternatively, we can also define partial complement of graph G with respect to a set S as graph obtained from G by removing edges of $\langle S \rangle$ and adding the edges which are not in $\langle S \rangle$.

Definition 4. *[10] Let $G \oplus S$ be a partial complement of a graph G and S be an induced subset of a vertex set of graph G. Then partial complement adjacency matrix of graph $G \oplus S$ is a $n \times n$ matrix denoted as $A_p(G \oplus S) = (a_{ij})$., where*

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent with } i \neq j \\ 1, & \text{if } i = j \text{ and } v_i \in S \\ 0, & \text{otherwise.} \end{cases}$$

Characteristic polynomial of partial complement of a graph G is denoted by $|\lambda I - A_p(G \oplus S)|$ and partial complement energy of G is denoted by $E_p(G \oplus S)$ and is defined as $E_p(G \oplus S) = \sum_{i=1}^{n} |\lambda_i|$.

Let $G \oplus S = (V, E_S)$ be partial complement of a graph G on n vertices and m_S edges. Let $D(G \oplus S)$ be diagonal matrix of vertex degrees. Then partial complement signless

Laplacian matrix of $G \oplus S$ is

$$Q_p(G \oplus S) = D(G \oplus S) + A_p(G \oplus S).$$

Partial complement signless Laplacian energy of graph G is defined as $QE_p(G \oplus S) = \sum_{i=1}^{n} \left| q_i - \frac{2m_S}{n} \right|$.

The set $\{q_1, q_2, \ldots, q_n\}$ is partial complement signless Laplacian spectrum of the graph G.

III. Properties of signless Laplacian energy of partial complements of a graph

Theorem 5. *If $q_1, q_2, \ldots, q_n$ represent eigenvalues of a signless Laplacian matrix of partial complement of a graph G with n vertices and m_S edges, then*

1) $\sum_{i=1}^{n} q_i = 2m_S + |S|$.
2) $\sum_{i=1}^{n} q_i^2 = 2m_S + \sum_{i=1}^{n} (d_i + s_i)^2$, *where*

$$s_i = \begin{cases} 1 & \text{if } v_i \in \langle S \rangle \\ 0 & \text{if } v_i \notin \langle S \rangle \end{cases}$$

Proof:

1) *Sum of principal diagonal elements of*

$$Q_p(G \oplus S) = \sum_{i=1}^{n} d_i + |S| = 2m_S + |S|.$$

Also sum of eigenvalues of

$$Q_p(G \oplus S) = trace\ of\ Q_p(G \oplus S).$$

Thus $\sum_{i=1}^{n} q_i = 2m_S + |S|$.

2) *We know that sum of squares of eigenvalues of*

$$Q_p(G \oplus S) = trace\ of\ Q_p^2(G \oplus S).$$

$$\begin{aligned} i.e., \sum_{i=1}^{n} q_i^2 &= \sum_{i=1}^{n} \sum_{j=1}^{n} l_{ij} l_{ji} \\ &= \sum_{i=1}^{n} l_{ii}^2 + \sum_{i \neq j} l_{ij} l_{ji} \\ &= \sum_{i=1}^{n} (d_i + s_i)^2 + 2 \sum_{i<j} l_{ij}^2 \\ &= 2m_S + \sum_{i=1}^{n} (d_i + s_i)^2, \end{aligned}$$

$$where\ s_i = \begin{cases} 1 & \text{if } v_i \in \langle S \rangle \\ 0 & \text{if } v_i \notin \langle S \rangle \end{cases}$$

So $\sum_{i=1}^{n} q_i^2 = 2R$, *where* $R = m_S + \sum_{i=1}^{n} (d_i + s_i)^2$. ∎

IV. Bounds for signless Laplacian energy of partial complements of a graph

Theorem 6. *Let $G \oplus S$ be partial complement of a graph G with n vertices and m_S edges. Then* $\sqrt{2R + n(n-1)D^{2/n}} - 2m_S \leq QE_p(G \oplus S) \leq \sqrt{2Rn} + 2m_S$, *where* $D = |Q_p(G \oplus S)$.

Proof: By taking $a_i = 1$ and $b_i = |q_i|$ in Cauchy-Schwarz inequality, we get

$$\left(\sum_{i=1}^{n} |q_i| \right)^2 \leq n \sum_{i=1}^{n} |q_i|^2$$

$$\left(\sum_{i=1}^{n} |q_i| \right)^2 \leq n2R$$

$$\sum_{i=1}^{n} |q_i| \leq \sqrt{2Rn}.$$

By triangle inequality,

$$\begin{aligned} \left| q_i - \frac{2m_S}{n} \right| &\leq |q_i| + \left| \frac{2m_S}{n} \right|, \forall i = 1, 2, \ldots, n \\ &\leq |q_i| + \frac{2m_S}{n}, \forall i = 1, 2, \ldots, n \end{aligned}$$

Hence, $\left| q_i - \frac{2m_S}{n} \right| \leq \sqrt{2Rn} + 2m_S$.

By Arithmetic mean and Geometric mean inequality,

$$\begin{aligned} \frac{1}{n(n-1)} \sum_{i \neq j} |q_i||q_j| &\geq \left[\prod_{i \neq j} |q_i||q_j| \right]^{\frac{1}{n(n-1)}} \\ &\geq [\det Q_p(G \oplus S)]^{2/n} \\ \sum_{i \neq j} |q_i||q_j| &\geq n(n-1) [\det Q_p(G \oplus S)]^{2/n}. \\ &\geq n(n-1)D^{2/n} \end{aligned}$$

Consider,

$$\left(\sum_{i=1}^{n} |q_i| \right)^2 = \sum_{i=1}^{n} |q_i|^2 + \sum_{i \neq j} |q_i||q_j|$$

$$\left(\sum_{i=1}^{n} |q_i| \right)^2 \geq 2R + n(n-1)D^{2/n}$$

We know that.

$$|q_i| - \left| \frac{2m_S}{n} \right| \leq \left| q_i - \frac{2m_S}{n} \right|, \forall i,$$

$$\sum_{i=1}^{n} |q_i| - \sum_{i=1}^{n} \frac{2m_S}{n} \leq \sum_{i=1}^{n} \left| q_i - \frac{2m_S}{n} \right|, \forall i,$$

$$\sum_{i=1}^{n} |q_i| - 2m_S \leq QE_p(G \oplus S),$$

$$QE_p(G \oplus S) \geq \sqrt{2R + n(n-1)D^{2/n}}.$$

∎

Theorem 7. *Let $G \oplus S$ be partial complement of a graph G with induced set S of vertex set of G. Then*

$$QE_p(G \oplus S) \leq \sqrt{2Rn - 4m_S(|S| + m_S)}.$$

Proof: By applying Cauchy-Schwarz inequality for $a_i = 1$ and $b_i = q_i - \frac{2m_S}{n}$, we obtain

$$\left(\sum_{i=1}^{n} \left| q_i - \frac{2m_S}{n} \right| \right)^2 \leq \left(\sum_{i=1}^{n} 1 \right) \left(\sum_{i=1}^{n} \left| q_i - \frac{2m_S}{n} \right|^2 \right)$$

$$\begin{aligned} [QE_p(G \oplus S)]^2 &\leq n \left[\sum_{i=1}^{n} q_i^2 + \sum_{i=1}^{n} \frac{4m_S^2}{n^2} - 4\frac{4m_S}{n}(2m_S + |S|) \right] \\ &\leq 2Rn - 4m_S(|S| + m_S) \\ QE_p(G \oplus S) &\leq \sqrt{2Rn - 4m_S(|S| + m_S)}. \end{aligned}$$

■

Theorem 8. *If $q_1 \geq q_2 \geq q_3 \geq \ldots \geq q_n$ are signless Laplacian eigenvalues of G, then* $QE_p(G \oplus S) \leq \left(q_1 - \frac{2m_S}{n}\right) + \sqrt{(n-1)\left[2R - q_1^2 + (n-1)\frac{4m_S^2}{n^2} - \frac{4m_S}{n}(2m_S + |S| - q_1)\right]}$.

Proof: We have $\left(\sum\limits_{i=2}^{n} a_i b_i\right)^2 \leq \left(\sum\limits_{i=2}^{n} a_i^2\right)\left(\sum\limits_{i=2}^{n} b_i^2\right)$, by Cauchy-Schwarz inequality for $(n-1)$ terms.

$$\left(\sum_{i=2}^{n}\left|q_i - \frac{2m_S}{n}\right|\right)^2 \leq \left(\sum_{i=2}^{n} 1\right)\left(\sum_{i=2}^{n}\left|q_i - \frac{2m_S}{n}\right|^2\right)$$

$$\left[QE_p(G \oplus S) - \left|q_1 - \frac{2m_S}{n}\right|\right]^2 \leq (n-1)$$
$$\left[\sum_{i=2}^{n} q_i^2 + \sum_{i=2}^{n}\frac{4m_S^2}{n^2} - 4\frac{4m_S}{n}(2m_S + |S|)\right]$$

$$QE_p(G \oplus S) \leq \left(1 - \frac{2m_S}{n}\right) + \sqrt{(n-1)\left[2R - q_1^2 + (n-1)\frac{4m_S^2}{n^2} - 4\frac{4m_S}{n}(2m_S + |S| - q_1)\right]}.$$

■

V. Partial complement signless Laplacian spectrum of some classes of graphs

Now we compute partial complement signless Laplacian spectrum for various classes of graphs. We adopt approach of eigenvector to prove Theorems 9, 12 and 15 (Case 1). In this approach, the result is proved by showing $Q_pW = qW$ for certain vector W and by making use of fact that geometric multiplicity and algebraic multiplicity of each eigenvalue q is same, as $Q_p(G \oplus S)$ is real and symmetric.

Theorem 9. *Let $K_{1,n-1} \oplus S$ be partial complement of star graph with $|S| = k$ vertices including central vertex. Then*

$$Q_p spec(K_{1,n-1} \oplus S) = \begin{Bmatrix} 1 & k-2 & 2k-3 & q_1 & q_2 \\ n-k-1 & k-2 & 1 & 1 & 1 \end{Bmatrix}.$$

Proof: Partial complement signless Laplacian matrix of $K_{1,n-1}$ is given by $Q_p(K_{1,n-1} \oplus S)$

$$= \begin{bmatrix} (n-k+1)J_1 & 0_{1\times(k-1)} & J_{1\times(n-k)} \\ 0_{(k-1)\times 1} & [(k-2)I + J]_{k-1} & 0_{(k-1)\times(n-k)} \\ J_{(n-k)\times 1} & 0_{(n-k)\times(k-1)} & I_{n-k} \end{bmatrix}_n$$

Let $W = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$ be an eigenvector of order n partitioned conformally with $Q_p(K_{1,n-1} \oplus S)$.

Consider

$$[Q_p(K_{1,n-1} \oplus S) - qI]\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{bmatrix} (n-k+1-q)X + JZ \\ (J + (k-2-q)I)Y \\ JX - qIZ \end{bmatrix}_n \tag{10}$$

Case 1: Let $X = 0, Y = 1_{k-1}$ and $Z = 0_{n-k}$.

From equation (10), we have

$$[(q-k+2)I - J]_{k-1}.1_{n-k} = (q-2k+3)1_{n-k}.$$

This implies that $q = 2k-3$ is an eigenvalue with multiplicity of at least one.

Case 2: Let $X = q-1$, where q be any root of the equation

$$q^2 - q(n-k+2) + 1 = 0 \tag{11}$$

Let $Y = 0_{k-1}$ and $Z = 1_{n-k}$.

From equation (10),

$[q-n+k-1)]I.(q-1) - J.1 = q^2 - q(n-k+2) + 1.$

From equation (11), $q_1 = \dfrac{(n-k+2) + \sqrt{(n-k+2)^2 - 4}}{2}$ and $q_2 = \dfrac{(n-k+2) - \sqrt{(n-k+2)^2 - 4}}{2}$ are eigenvalues both with multiplicity of at least one.

Case 3: Let $X = 0, Y = Y_j$ and $Z = 0_{n-k}$, where Y_j denote the vector with 1^{st} element 1, j^{th} element -1, where $j = 2, 3, \ldots, k-1$ and remaining $0's$.

From equation (10),

$$\begin{aligned} [(q-k+2)I - J]Y_j &= (q-k+2)Y_j + 0 \\ &= (q-k+2)Y_j. \end{aligned}$$

Hence, $q = k-2$ is an eigenvalue with multiplicity of at least $k-2$, as there are $k-2$ independent eigenvectors of the form Y_j.

Case 4: Let $X = 0, Y = 0_{k-1}$ and $Z = Z_j$, Z_j denote the vector with 1^{st} element 1, j^{th} element -1, where $j = 2, 3, \ldots, n-k$ and remaining $0's$.

From equation (10), $-J.0 + 0.0_{k-1} + (q-1)IZ_j = (q-1)Z_j$.

So $q = 1$ is an eigenvalue with multiplicity of at least $n-k-1$, as there are $n-k-1$ independent eigenvectors of the form Z_j.

Since the order of star graph is n, partial complement signless Laplacian spectrum of $K_{1,n-1} \oplus S$ is

$$\begin{Bmatrix} 1 & k-2 & 2k-3 & q_1 & q_2 \\ n-k-1 & k-2 & 1 & 1 & 1 \end{Bmatrix}.$$

■

Theorem 12. *Partial complement signless Laplacian spectrum of complete graph with $|S| = k$ vertices is*

$$Q_p spec(K_n \oplus S) = \begin{Bmatrix} n-k+1 & n-2 & q_1 & q_2 \\ k-1 & n-k-1 & 1 & 1 \end{Bmatrix}.$$

Proof: Partial complement signless Laplacian matrix of K_n is

$$Q_p(K_n \oplus S) = \left[\begin{array}{c|c} [(n-k+1)I]_k & J_{k\times n-k} \\ \hline J_{n-k\times 2} & [(n-2)I + J]_{n-k} \end{array}\right]_n$$

Let $W = \begin{bmatrix} X \\ Y \end{bmatrix}$ be an eigenvector of order n partitioned conformally with $Q_p(K_n \oplus S)$.

Consider

$$(qI - Q_p(K_n \oplus S))\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{bmatrix} (q-n+k-1)IX - JY \\ JX + [(q-n+2)I - J]Y \end{bmatrix} \tag{13}$$

Case 1: Let $X = X_j = e_1 - e_j, j = 2, 3, \ldots, k$ and $Y = 0_{n-k}$.

From equation (13), $(q-n+k-1)IX_j + J0_{n-k} = (q-n+k-1)X_j$.

Then, $q = n-k+1$ is an eigenvalue with multiplicity of at least $k-1$ since there are $k-1$ independent vectors of the form X_j.

Case 2: Let $X = O_{k-1}$ and $Y = Y_j = e_1 - e_j$, where $j = 2, 3, \ldots, n-k$.

From equation (13), $J0_{k-1} + [(q-n+2)I - J]Y_j = (q-n+2)Y_j$.

So $q = n-2$ is the eigenvalue with multiplicity of at least $n-k-1$ since there are $n-k-1$ independent vectors of form Y_j.

Case 3: Let $Y = 1_{n-k}$ and $X = \left(\dfrac{n-k}{q-n+k-1}\right)1_k$, where q be any root of the equation

$$q^2 - (3n-2k-1)q + \{2(k-n)^2 + k - 2\} = 0 \tag{14}$$

From equation (13),

$$-J\left(\frac{n-k}{q-n+k-1}\right)1_k+[(q-n+2)I-J]1_{n-k}$$
$$=\frac{-k(n-k)}{q-n+k+1}1_{n-k}+(q-2n+k+2)1_{n-k}$$
$$=\frac{q^2-(3n-2k-1)q+\{2(k-n)^2+k-2\}}{q-n+k+1}1_{n-k}.$$

From equation (14),
$q_1=\frac{3n-2k-1}{2}+\frac{\sqrt{(3n-2k-1)^2-4\{2(n-k)^2+(k-2)\}}}{2}$
and $q_2=\frac{3n-2k-1}{2}-\frac{\sqrt{(3n-2k-1)^2-4\{2(n-k)^2+(k-2)\}}}{2}$ are eigenvalues with multiplicity of at least one.

Hence, partial complement signless Laplacian spectrum of complete graph K_n with $|S|=k$ vertices is
$\begin{Bmatrix} n-k+1 & n-2 & q_1 & q_2 \\ k-1 & n-k-1 & 1 & 1 \end{Bmatrix}$. ■

Theorem 15. *Let $S_n^0\oplus S$ be partial complement of a crown graph with $|S|=k$.*

Case 1: If $k=2n$,
$Q_p spec(S_n^0\oplus S)=\begin{Bmatrix} n-1 & n+1 & 2n+1 & 2n-1 \\ n-1 & n-1 & 1 & 1 \end{Bmatrix}$.

Case 2: If $k=n, Q_p spec(S_n^0\oplus S)=$
$\begin{Bmatrix} \frac{4n-3+p}{2} & \frac{4n-3-p}{2} & \frac{3n-3+R}{2} & \frac{3n-3-R}{2} \\ 1 & 1 & n-1 & n-1 \end{Bmatrix}$,
where $P=\sqrt{8n^2-12n+5}$ and $R=\sqrt{n^2-2n+5}$.

Proof: Case 1: Partial complement signless Laplacian matrix of $S_n^0\oplus S$ is

$$Q_p(S_n^0\oplus S)=\left[\begin{array}{c|c} (nI+J)_n & I_n \\ \hline I_n & (nI+J)_n \end{array}\right]_{2n}$$

Let $W=\begin{bmatrix} X \\ Y \end{bmatrix}$ be an eigenvector of order $2n$ partitioned conformally with $Q_p(S_n^0\oplus S)$.

Consider

$$[qI-Q_p(S_n^0\oplus S)]\begin{bmatrix} X \\ Y \end{bmatrix}=\begin{bmatrix} [(q-n)I-J]_nX-I_nY \\ -I_nX+[(q-n)I-J]_nY \end{bmatrix} \tag{16}$$

Subcase 1: Let $X=X_j=e_1-e_j, j=2,3,\ldots,n$ and $Y=(q-n)X_j$, where q be any root of

$$q^2-2nq+(n^2-1)=0. \tag{17}$$

From equation (16),
$-IX_j+[(q-n)I+J](q-n)X_j=-X_j+(q-n)^2X_j$.

Then by equation (17), $q=n+1$ and $q=n-1$ are eigenvalues with multiplicity of at least $n-1$ since there are $n-1$ independent vectors of the form X_j.

Subcase 2: Let $X=1_n$ and $Y=(q-2n)1_n$.
$-I1_n+[(q-n)I-J](q-2n)1_n=$
$-1_n+[(q-2n)(q-n)-n(q-2n)]1_n$.

So, $q=2n+1$ and $q=2n-1$ are eigenvalues with multiplicity of at least one.

Hence, partial complement signless Laplacian spectrum is
$\begin{Bmatrix} n-1 & n+1 & 2n+1 & 2n-1 \\ n-1 & n-1 & 1 & 1 \end{Bmatrix}$.

Case 2 : Partial complement signless Laplacian matrix of $S_n^0\oplus S=\left[\begin{array}{c|c} [(2n-2)I+J]_n & \mathcal{B}_n \\ \hline \mathcal{B}_n & (n-1)I_n \end{array}\right]_{2n}$,

where J is matrix of all $1's$, I is an identity matrix and $\mathcal{B}$ is the adjacency matrix of complete subgraph.

Consider $|qI-Q_p(S_n^0\oplus S)|$.

Step 1: On replacing R_i by $R_i\longrightarrow R_i-R_{i+1}$, where $i=1,2,\ldots,n-1,n+1,\ldots,2n-1$ in above determinant, we obtain $\det(A)$.

Step 2: Applying column operations $C_i\longrightarrow C_i+C_{i-1}+\ldots+C_1, i=n,n-1,\ldots,2$ and $C_j\longrightarrow C_j+C_{j-1}+\ldots+C_{n+1}, i=2n,2n-1,\ldots,n+1$ on $\det(A)$, we get $\det(B)$.

Step 3: In $\det(B)$, replacing R_i by $(q-n+1)R_i-R_{n+i}$, where $i=1,2,\ldots,n-1$, we get $\det(C)$.

Step 4: On expanding the $\det(C)$ along the row $R_i, i\neq n,2n$, we obtain
$|qI-Q_p(S_n^0\oplus S)|=[(q-2n+2)(q-n+1)-1]^{n-1}\begin{vmatrix} q-3n+2 & 1-n \\ 1-n & q-n+1 \end{vmatrix}$

Step 5: On simplification we get, $|qI-Q_p(S_n^0\oplus S)|=[q^2-(4n-3)q+2n^2-3n+1][q^2-3(n-1)q+(2n^2-4n+1)]^{n-1}$.

Hence, partial complement signless Laplacian spectrum is $\begin{Bmatrix} \frac{4n-3+p}{2} & \frac{4n-3-p}{2} & \frac{3n-3+R}{2} & \frac{3n-3-R}{2} \\ 1 & 1 & n-1 & n-1 \end{Bmatrix}$,
where $P=\sqrt{8n^2-12n+5}$ and $R=\sqrt{n^2-2n+5}$. ■

Theorem 18. *Let $K_{l,m}\oplus S$ be partial complement of complete bipartite graph with partitions V_1 and V_2 of l and m vertices respectively and S be an induced set of order k which consists of p vertices of V_1 and $k-p$ vertices of V_2. Then partial complement signless Laplacian characteristic polynomial of $K_{l,m}\oplus S$ is $(q-m+k-2p+1)^{p-1}(q-m)^{l-p-1}(q-l+2p-k+1)^{k-p-1}(q-l)^{m-k+p-1}[q^4+(2-2l-2m-k)q^3+(-2k^2+3km+10kp-k+l^2+3lm+3lp-3l+m^2-3mp-3m-10p^2+1)q^2+(2k^2m+kl^2-klm-5klp+2kl-2km^2-9kmp+km-l^2m-3l^2p+l^2-lm^2+2lm+7lp^2-lp-l+3m^2p+m^2+7mp^2+mp-m)q+kl-kp-lp+mp-k^2l-2k^3l-kp^2+k^2p+16kp^3+2k^3p+8lp^3-8mp^3+p^2-8p^4-10k^2p^2+2k^2lm-16klp^2+10k^2lp+8kmp^2-2k^2mp+4lmp^2-klm+klp-kmp-4klmp]$.*

Case 1: If $p=l$, $Q_p spec(K_{l,m}\oplus S)=$
$\begin{pmatrix} m-k+2l-1 & k-l-1 & l & 2k-2l-1 & q_1 & q_2 \\ l-1 & k-l-1 & m-k+l-1 & 1 & 1 & 1 \end{pmatrix}$,
where

$$q_1=(m-k+4l-1)+\frac{\sqrt{(m-k+4l-1)^2-4l(2l-1)}}{2}$$

and

$$q_2=(m-k+4l-1)-\frac{\sqrt{(m-k+4l-1)^2-4l(2l-1)}}{2}.$$

Case 2: If $k=l$,
$Q_p spec(K_{l,m}\oplus S)=\begin{pmatrix} m+l-1 & l & q_1 & q_2 \\ l-1 & m-1 & 1 & 1 \end{pmatrix}$,
where $q_1=(m+3l-1)+\frac{\sqrt{(m+3l-1)^2-4(2l^2-l)}}{2}$
and $q_2=(m+3l-1)-\frac{\sqrt{(m+3l-1)^2-4(2l^2-l)}}{2}$.

Proof: Partial complement signless Laplacian matrix of $K_{l,m}\oplus S$ is given by $Q_p(K_{l,m}\oplus S)=$

$$\left[\begin{array}{c|c|c|c} C_p & 0_{p\times l-p} & 0_{p\times k-p} & J_{p\times M} \\ \hline 0_{l-p\times p} & mI_{l-p} & J_{l-p\times k-p} & J_{l-p\times M} \\ \hline 0_{k-p\times l-p} & J_{k-p\times l-p} & D_{k-p} & J_{k-p\times M} \\ \hline J_{M\times p} & J_{M\times l-p} & 0_{M\times k-p} & lI_M \end{array}\right]_n,$$

where $C = (M+p-1)I+J, D = (l-2p+k-1)I+J, M = m-k+p$ *and* 0 *is the matrix of all* $0's$.

Consider $|qI - Q_p(K_{l,m} \oplus S)|$.

Step 1: Applying row operation $R_i \longrightarrow R_i - R_{i+1}$, *where* $i = 1, 2, \ldots, p-1, p+1, \ldots, l-p-1, l-p+1, \ldots, k-p-1, k-p+1, \ldots, M-1$ *and simplifying, determinant reduces to* $(q-m+k-2p+1)^{p-1}(q-m)^{l-p-1}(q-l+2p-k+1)^{k-p-1}(q-l)^{m-k+p-1}\det(B)$.

Step 2: By applying column operations $C_i \longrightarrow C_i + C_{i-1} + \ldots + C_1, i = p, p-1, \ldots, 2, C_j \longrightarrow C_j + C_{j-1} + \ldots + C_{p+1}, i = l-p, l-p-1, \ldots, p+2, Cr \longrightarrow C_r + C_{r-1} + \ldots + C_{l-p+1}, i = k-p, k-p-1, \ldots, l-p+2$ *and* $C_s \longrightarrow C_s + C_{s-1} + \ldots + C_{k-p+1}, i = M, M-1, \ldots, k-p+2$ *on* $\det(B)$, *we obtain* $\det(C)$. *We observe that in each row of* $\det(C)$, *except* $p^{th}, (l-p)^{th}, (k-p)^{th}$ *and* M^{th} *rows, there will be only one non zero integer* -1.

Step 3: On expanding $\det(C)$ *using elementary row and column operations we obtain determinant of order* 4.

Thus $|qI - Q_p(K_{l,m} \oplus S)| = (q-m+k-2p+1)^{p-1}(q-m)^{l-p-1}(q-l+2p-k)^{k-p-1}(q-l)^{m-k+p-1}$

$$\begin{vmatrix} q-m+k-3p+1 & 0 & 0 & -M \\ 0 & q-m & p-k & -M \\ 0 & p-l & q-l+3p-2k+1 & 0 \\ -p & p-l & 0 & q-l \end{vmatrix}$$

On expansion, we get

$(q-m+k-2p+1)^{p-1}(q-m)^{l-p-1}(q-l+2p-k+1)^{k-p-1}(q-l)^{m-k+p-1}[q^4+(2-2l-2m-k)q^3+(-2k^2+3km+10kp-k+l^2+3lm+3lp-3l+m^2-3mp-3m-10p^2+1)q^2+(2k^2m+kl^2-klm-5klp+2kl-2km^2-9kmp+km-l^2m-3l^2p+l^2-lm^2+2lm+7lp^2-lp-l+3m^2p+m^2+7mp^2+mp-m)q+kl-kp-lp+mp-k^2l-2k^3l-kp^2+k^2p+16kp^3+2k^3p+8lp^3-8mp^3+p^2-8p^4-10k^2p^2+2k^2lm-16klp^2+10k^2lp+8kmp^2-2k^2mp+4lmp^2-klm+klp-kmp-4klmp]$. ■

Definition 19. *Friendship Graph* F_n *consists of* n *triangles with a common vertex.*

Theorem 20. *For a friendship graph* F_n *with* $\langle S \rangle = K_3$, $Q_p spec(F_n \oplus S) =$

$$\left\{\begin{matrix} 1 & 3 & n+1+L & n+1-L \\ n+1 & n-2 & 1 & 1 \end{matrix}\right\},$$

where $L = \sqrt{n^2-2n+2}$.

Proof: Partial complement signless Laplacian matrix of $F_n \oplus S$ *is given by* $Q_p(F_n \oplus S) =$

$$\begin{bmatrix} (2n-1)J_1 & 0_{1\times 2} & J_{1\times 2} & \ldots & J_{1\times 2} \\ 0_{2\times 1} & I_2 & 0_2 & \ldots & 0_2 \\ J_{2\times 1} & 0_2 & (J+I)_2 & \ldots & 0_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ J_{2\times 1} & 0_2 & 0_2 & \ldots & (J+I)_2 \end{bmatrix}_{2n+1},$$

Consider $|qI - Q_p(F_n \oplus S)|$.

Step 1: Applying row operation $R_i \longrightarrow R_i - R_{i+1}$, *where* $i = 3, 4, \ldots, n-1$ *and column operations* $C_i \longrightarrow C_i + C_{i-1} + \ldots + C_2 + C_1, i = n, n-1, \ldots, 2$ *on* $Q_p(F_n \oplus S)$, *we obtain* $(q-1)^2((q-2)^2-1)^{n-2}\det(B)$.

Step 2: On expanding $\det(B)$, *we get* $(q-1)[q^2-(2n+2)q+4n-1]$.

Hence, we obtain the spectrum $Q_p spec(F_n \oplus S) =$

$$\left\{\begin{matrix} 1 & 3 & n+1+L & n+1-L \\ n+1 & n-2 & 1 & 1 \end{matrix}\right\},$$

where $L = \sqrt{n^2-2n+2}$. ■

VI. Conclusion

Partial complement of graph depends on the induced subset of vertex set of a graph. In this paper, we have defined signless Laplacian energy of partial complement of a graph. We have studied properties of signless Laplacian eigenvalues and established several bounds for signless Laplacian energy of partial complements of a graph. Signless Laplacian spectrum of partial complements of families of graphs are derived.

References

[1] C. S. Adiga, A. Bayed, I. Gutman and A. S. Shrikanth, "The minimum covering energy of a graph," *Kragujevac J. Science*, vol. 34, pp. 39-56, 2012.

[2] R. B. Bapat, *G*raphs and Matrices, first edition, Hindustan Book Agency, New Delhi, 2010.

[3] R. B. Bapat, *L*inear Algebra and Linear Models, Third edition, Hindustan Book Agency, New Delhi, 2011.

[4] R. Balakrishnan, "The energy of a graph," *linear algebra Appl.*, vol. 387, pp. 287-295, 2004.

[5] P. G. Bhat and S. D' Souza, "Color signless Laplacian energy of graphs," *AKCE International Journal of Graphs and Combinatorics*, vol. 14, pp. 140-148, 2017.

[6] D. M. Cvetkovic, M. Doob and H. Sachs, *S*pectra of Graphs, Academic Press, New York, 1980.

[7] F. V. Fomin, P. A. Golovach, T. JF. StrÃ¸mme and D. M. Thilikos, "Partial complementation of graphs," rXiv: 1804.10920 [cs. CC], 2018.

[8] I. Gutman, "The energy of a graph," *Ber. Math-Statist. Sekt. Forschungsz. Graz*, vol. 103, pp. 1-22, 1978.

[9] I. Gutman and B. Zhou, "Laplacian energy of a graph," *Lin. Algebra Appl.*, vol. 414, pp. 29-37, 2006.

[10] S. Nayak, S. D' Souza, H. J. Gowtham and P. G. Bhat, "Energy of partial complements of a graph," *Proceedings of Jangjeon Mathematical Society*, vol. 22, no. 3, pp. 369–379, 2019.

[11] B. Zhou, "More on energy and laplacian energy," *MATCH Commun. Math. Comput. Chem.*, vol. 64 , pp. 75-84, 2010.

[12] S. Nayak, S. D' Souza, H. J. Gowtham and P. G. Bhat, "Color Energy of Generalised Complements of a Graph," *IAENG International Journal of Applied Mathematics*, vol. 50, no. 4, pp. 720–728, 2020.

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