Applications of Point Transformation on Third-Order Ordinary Differential Equations

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Abstract—**We have developed a new procedure for converting nonlinear third-order ordinary differential equations into linear forms using point transformation. These linear equations are more general and easier to solve. By applying the point transformation again, we can obtain the general solutions for the original nonlinear equations. The key feature of this work is the use of examples to demonstrate the application of the derived linearization criteria to various existing problems, including third-order ordinary differential equations, second-order ordinary differential equations under the Riccati transformation and third-order partial differential equations under travelling wave solutions.**

Index Terms—**linearization problem, point transformation, fiber preserving, nonlinear ODE.**

I. Introduction

DIFFERENTIAL equations are a powerful mathematical tool for modeling a wide range of phenomena in various fields such as science, engineering, economics, mathematics and physics. However, these equations are often nonlinear, making them difficult to solve. Numerical methods are a popular technique for solving such equations, but the solutions obtained are only approximations. Exact solutions are considered more significant because they allow for a deeper analysis of the properties of the equations being studied. One way to achieve the exact solution for a nonlinear differential equation is to transform it into a standard form, such as a linear equation, through a process called linearization. This method is an important discipline in the field of differential equations and is currently the focus of ongoing research by mathematicians. Modern research in this area has led to many benefits in scientific and technological developments.

To linearize a differential equation, a transformation is used to change the form of the equation. Different types of transformations can be used for linearization, such as tangent, point and contact transformation. The tangent transformation consists of derivatives, point transformation only relies on independent and dependent variables and contact transformation is a tangent transformation that also changes the independent variable, dependent variable and first-order derivative. Other transformations like generalized Sundman transformation and generalized linearizing transformation also contain nonlocal terms. In this work, the point transformation is chosen due to its versatility and wide range of applicability.

Nonlinear third-order ordinary differential equations are of interest to many researchers. Previous studies have focused on Laguerre forms and the simplest forms, but linearization through point transformations has yet to be explored. The linear form is important and can be applied in various cases, so we aim to develop a new algorithm for linearization via point transformations. Additionally, we will identify nonlinear equations found in the literature that can be linearized using our algorithm.

A. Historical Review

Linearization of nonlinear ordinary differential equations is a technique for converting a nonlinear equation into a linear one, making it easier to solve. The history of linearization starts with the work of Lie [1], who discovered a pattern of second-order ordinary differential equations that can be reduced to linear structures with point transformation. He also demonstrated that every second-order ordinary differential equation can be unconditionally transformed into a linear equation through contact transformation. Liouville [2] and Tresse [3] employed the concept of relative invariants of equivalence groups in order to address equivalence problems that arise in solving second-order ordinary differential equations. Recently, Suksern and Sawatdithep [4] reduced second-order ordinary differential equations to linear equations in general form and applied their results to various nonlinear equations found in nature. Other methods for linearizing second-order ordinary differential equations include Cartan's method [5], the generalized Sundman transformation [6], [7] and the linearizing transformation [8]-[11].

The linearization of third-order ordinary differential equations has been studied by multiple authors using various methods such as Cartan's method, point transformation and contact transformation. Chern [12] used Cartan's method to identify the linearizing criteria using contact transformation but did not provide the transformation. Bocharov, Sokolov and Svinolupov [13] later studied the same problem using point transformation. Grebot [14] also studied linearization using point transformation but only for specific cases. Ibragimov and Meleshko [15] studied the linearization in Laguerre form and provided linearizing criteria for both point and contact transformations. Furthermore, the linearization via the contact transformation was studied by many authors [see for examples, [15]-[18]]. The solution of the linearization was shown in [15] and [19]. The transformation algorithm is not shown in [19], but the linearizing criteria and the process of transformation are clearly explained in [15]. Additionally, the linearization via Sundman transformation has been investigated in [20]-[22].

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The complexity of the equation can make linearization difficult. Many researchers focus on simpler forms, such as the Laguerre form, but a general pattern that covers all formats is needed for applications. One approach to linearization is the use of point transformations, which can transform a third-order ordinary differential equation into a general linear form. One of the benefits of this approach is that it has a broad range of applicability to various nonlinear equations found in the natural world, which makes it a valuable and practical new method. This method's versatility allows for its use in a wide variety of contexts, increasing its utility as a solution to many different problems.

II. Formulation of the Linearization Theorems

A. Obtaining Necessary Condition of Linearization

The purpose of this research is to linearize the third-order ordinary differential equations

$$y''' = f(x, y, y', y'') \tag{1}$$

by using the point transformation

$$t = \varphi(x, y), \quad u = \psi(x, y). \tag{2}$$

The study starts with the necessary conditions for linearization. We obtained the general form of equation (1) that can be reduced to a linear equation via point transformation (2). The general linear third-order ordinary differential equation is written in the form

$$u''' + \gamma(t)u'' + \beta(t)u' + \alpha(t)u + \omega(t) = 0. \tag{3}$$

At the end, we attain two classes of equations candidating for linearization.

Let t and u be new independent and dependent variables, respectively. We get the following transformation of the derivatives

$$\begin{aligned} u'(t) &= \frac{D_x\psi}{D_x\varphi} = \frac{\psi_x + y'\psi_y}{\varphi_x + y'\varphi_y} = P(x, y, y'), \\ u''(t) &= \frac{D_xP}{D_x\varphi} = \frac{P_x + y'P_y + y''P_{y'}}{\varphi_x + y'\varphi_y} \\ &= \frac{\Delta}{(\varphi_x + y'\varphi_y)^3}[y'' + \frac{1}{\Delta}(\varphi_y\psi_{yy} - \varphi_{yy}\psi_y)y'^3 \\ &\quad +...] \\ &= Q(x, y, y', y''), \\ u'''(t) &= \frac{D_xQ}{D_x\varphi} = \frac{Q_x + y'Q_y + y''Q_{y'} + y'''Q_{y''}}{\varphi_x + y'\varphi_y} \\ &= \frac{\Delta}{(\varphi_x + y'\varphi_y)^5}[(\varphi_x + y'\varphi_y)y''' - 3\varphi_y y''^2 \\ &\quad +...], \end{aligned} \tag{4}$$

where

$$D_x = \frac{\partial}{\partial_x} + y'\frac{\partial}{\partial_y} + y''\frac{\partial}{\partial_{y'}} + y'''\frac{\partial}{\partial_{y''}}$$

is the total derivative. Here $\Delta = \varphi_x\psi_y - \varphi_y\psi_x \neq 0$ is Jacobian of the change of variables (2). From equation (4), we can see that the transformation (2) with the conditions $\varphi_y = 0$ and $\varphi_y \neq 0$ give two distinctly different candidates for linearization.

For $\varphi_y = 0$, we replace all results in equation (3) and derive the following equation:

$$\begin{aligned} y''' &+ (A_1y' + A_0)y'' + B_3y'^3 + B_2y'^2 \\ &+ B_1y' + B_0 = 0, \end{aligned} \tag{5}$$

where

$$A_1 = (3\psi_{yy})/\psi_y, \tag{6}$$
$$A_0 = (-3\varphi_{xx}\psi_y + \varphi_x^2\psi_y\gamma + 3\varphi_x\psi_{xy})/(\varphi_x\psi_y), \tag{7}$$
$$B_3 = \psi_{yyy}/\psi_y, \tag{8}$$
$$B_2 = (-3\varphi_{xx}\psi_{yy} + \varphi_x^2\psi_{yy}\gamma + 3\varphi_x\psi_{xyy})/(\varphi_x\psi_y), \tag{9}$$
$$\begin{aligned} B_1 = &(-\psi_{xxx}\varphi_x\psi_y + 3\varphi_{xx}^2\psi_y - \varphi_{xx}\varphi_x^2\psi_y\gamma \\ &- 6\varphi_{xx}\varphi_x\psi_{xy} + \varphi_x^4\psi_y\beta + 2\varphi_x^3\psi_{xy}\gamma \\ &+ 3\varphi_x^2\psi_{xxy})/(\varphi_x^2\psi_y), \end{aligned} \tag{10}$$
$$\begin{aligned} B_0 = &(-\psi_{xxx}\varphi_x\psi_x + 3\varphi_{xx}^2\psi_x - \varphi_{xx}\varphi_x^2\psi_x\gamma \\ &- 3\varphi_{xx}\varphi_x\psi_{xx} + \varphi_x^5\alpha\psi + \varphi_x^5\omega + \varphi_x^4\psi_x\beta \\ &+ \varphi_x^3\psi_{xx}\gamma + \varphi_x^2\psi_{xxx})/(\varphi_x^2\psi_y). \end{aligned} \tag{11}$$

For $\varphi_y \neq 0$, we have done in similar way. Setting $r(x, y) = \frac{\varphi_x}{\varphi_y}$, we derive the following equation:

$$\begin{aligned} y''' &+ \frac{1}{y' + r}[-3(y'')^2 + (C_2y'^2 + C_1y' + C_0)y'' \\ &+ D_5y'^5 + D_4y'^4 + D_3y'^3 + D_2y'^2 + D_1y' \\ &+ D_0] = 0, \end{aligned} \tag{12}$$

where

$$C_2 = -((6\varphi_{yy} - \varphi_y^2\gamma)\Delta - 3\varphi_y\Delta_y)/(\varphi_y\Delta), \tag{13}$$
$$\begin{aligned} C_1 = &(3(\Delta_x + \Delta_y r - 4r_y\Delta)\varphi_y \\ &- 2(6\varphi_{yy} - \varphi_y^2\gamma)r\Delta)/(\varphi_y\Delta), \end{aligned} \tag{14}$$
$$\begin{aligned} C_0 = &-(6\varphi_{yy}r^2\Delta - \varphi_y^2\gamma r^2\Delta + 6\varphi_y r_x\Delta \\ &+ 6\varphi_y r_y r\Delta - 3\varphi_y\Delta_x r)/(\varphi_y\Delta), \end{aligned} \tag{15}$$
$$\begin{aligned} D_5 = &-(\varphi_{yyy}\varphi_y\psi_y - 3\varphi_{yy}^2\psi_y + \varphi_{yy}\varphi_y^2\psi_y\gamma \\ &+ 3\varphi_{yy}\varphi_y\psi_{yy} - \varphi_y^5\alpha\psi - \varphi_y^5\omega \\ &- \varphi_y^4\psi_y\beta - \varphi_y^3\psi_{yy}\gamma - \varphi_y^2\psi_{yyy})/(\varphi_y\Delta), \end{aligned} \tag{16}$$
$$\begin{aligned} D_4 = &(((5\psi_{yyy}r - 2\Delta_y\gamma)\varphi_y - 3\Delta_{yy})\varphi_y^2 \\ &+ 15(\varphi_y\psi_y r - \Delta)\varphi_{yy}^2 - (5\varphi_y\psi_y r \\ &- 4\Delta)\varphi_{yyy}\varphi_y + ((5((\alpha\psi + \omega)\varphi_y + \psi_y\beta)\varphi_y r \\ &+ 5\psi_{yy}\gamma r - \beta\Delta)\varphi_y^3 - (5\varphi_y^2\psi_y\gamma r - 12\Delta_y \\ &+ 3(5\psi_{yy}r - \gamma\Delta)\varphi_y)\varphi_{yy})\varphi_y)/(\varphi_y^2\Delta), \end{aligned} \tag{17}$$
$$\begin{aligned} D_3 = &-(((3(\Delta_{xy} + 3\Delta_{yy}r - 4r_y\Delta_y - 2r_{yy}\Delta) \\ &+ ((\Delta_x + 7\Delta_y r - 3r_y\Delta)\gamma - 10\psi_{yyy}r^2)\varphi_y)\varphi_y \\ &+ 2((5(\varphi_y\psi_y\gamma + 3\psi_{yy})r - 6\gamma\Delta)\varphi_y r \\ &+ 3(3r_y\Delta - \Delta_x - 7\Delta_y r))\varphi_{yy})\varphi_y \\ &- 2((5((\alpha\psi + \omega)\varphi_y + \psi_y\beta)\varphi_y r + 5\psi_{yy}\gamma r \\ &- 2\beta\Delta)\varphi_y^4 - ((5\varphi_y\psi_y r - 8\Delta)\varphi_{yyy}\varphi_y \\ &- 15(\varphi_y\psi_y r - 2\Delta)\varphi_{yy}^2))r)/(\varphi_y^2\Delta), \end{aligned} \tag{18}$$

$$
\begin{aligned}
D_2 =&((((r_x\Delta - 3\Delta_x r - 9\Delta_y r^2)\gamma + 2(5\psi_{yyy}r^2 \\
&+ 4r_y\gamma\Delta)r)\varphi_y^2 - 2(3(r_x\Delta - 9\Delta_y r^2 \\
&+ (8r_y\Delta - 3\Delta_x)r) + (5\varphi_y\psi_y\gamma r + 15\psi_{yy}r \\
&- 9\gamma\Delta)\varphi_y r^2)\varphi_{yy})\varphi_y + 2((5((\alpha psi + \omega)\varphi_y \\
&+ \psi_y\beta)\varphi_y r + 5\psi_{yy}\gamma r - 3\beta\Delta)\varphi_y^4 - ((5\varphi_y\psi_y r \\
&- 12\Delta)\varphi_{yyy}\varphi_y - 15(\varphi_y\psi_y r - 3\Delta)\varphi_{yy}^2))r^2 \\
&+ (4r_{xy}\Delta - 7\Delta_{xy}r - 16r_y^2\Delta + 4r_x\Delta_y \\
&- (\Delta_{xx} + 10\Delta_{yy}r^2 - 14r_{yy}r\Delta \\
&- 8(\Delta_x + 3\Delta_y r)r_y))\varphi_y^2)/(\varphi_y^2\Delta), \qquad (19)
\end{aligned}
$$

$$
\begin{aligned}
D_1 =& -(((2\Delta_{xx} + 5\Delta_{yy}r^2 + 5\Delta_{xy}r + 19r_y^2\Delta \\
&- 11r_{yy}r\Delta)r - r_{xx}\Delta - 6r_{xy}r\Delta - (13\Delta_x \\
&+ 15\Delta_y r)r_y r - (3\Delta_x + 5\Delta_y r - 13r_y\Delta)r_x)\varphi_y^2 \\
&- (((5((\alpha\psi + \omega)\varphi_y + \psi_y\beta)\varphi_y r + 5\psi_{yy}\gamma r \\
&- 4\beta\Delta)\varphi_y^4 + 15(\varphi_y\psi_y r - 4\Delta)\varphi_{yy}^2 - (5\varphi_y\psi_y r \\
&- 16\Delta)\varphi_{yyy}\varphi_y)r^2 + (6((3\Delta_x + 5\Delta_y r \\
&- 7r_y\Delta)r - 2r_x\Delta) - (5\varphi_y\psi_y\gamma r + 15\psi_{yy}r \\
&- 12\gamma\Delta)\varphi_y r^2)\varphi_{yy}\varphi_y - (((3\Delta_x + 5\Delta_y r \\
&- 7r_y\Delta)r - 2r_x\Delta)\gamma \\
&- 5\psi_{yyy}r^3)\varphi_y^3)r)/(\varphi_y^2\Delta), \qquad (20)
\end{aligned}
$$

$$
\begin{aligned}
D_0 =& -(\varphi_{yyy}\varphi_y^2\psi_y r^5 - 4\varphi_{yyy}\varphi_y r^4\Delta \\
&- 3\varphi_{yy}^2\varphi_y\psi_y r^5 + 15\varphi_{yy}^2 r^4\Delta + \varphi_{yy}\varphi_y^3\psi_y\gamma r^5 \\
&+ 3\varphi_{yy}\varphi_y^2\psi_{yy}r^5 - 3\varphi_{yy}\varphi_y^2\gamma r^4\Delta \\
&+ 6\varphi_{yy}\varphi_y r_x r^2\Delta + 12\varphi_{yy}\varphi_y r_y r^3\Delta \\
&- 6\varphi_{yy}\varphi_y\Delta_x r^3 - 6\varphi_{yy}\varphi_y\Delta_y r^4 - \varphi_y^6\alpha\psi r^5 \\
&- \varphi_y^6\omega r^5 - \varphi_y^5\psi_y\beta r^5 - \varphi_y^4\psi_{yy}\gamma r^5 + \varphi_y^4\beta r^4\Delta \\
&- \varphi_y^3\psi_{yyy}r^5 - \varphi_y^3 r_x\gamma r^2\Delta - 2\varphi_y^3 r_y\gamma r^3\Delta \\
&+ \varphi_y^3\Delta_x\gamma r^3 + \varphi_y^3\Delta_y\gamma r^4 - 2\varphi_y^2 r_{xy}r^2\Delta \\
&- \varphi_y^2 r_{xx}r\Delta + 3\varphi_y^2 r_x^2\Delta + 7\varphi_y^2 r_x r_y r\Delta \\
&- 3\varphi_y^2 r_x\Delta_x r - \varphi_y^2 r_x\Delta_y r^2 - 3\varphi_y^2 r_{yy}r^3\Delta \\
&+ 6\varphi_y^2 r_y^2 r^2\Delta - 5\varphi_y^2 r_y\Delta_x r^2 - 3\varphi_y^2 r_y\Delta_y r^3 \\
&+ \varphi_y^2\Delta_{xy}r^3 + \varphi_y^2\Delta_{xx}r^2 + \varphi_y^2\Delta_{yy}r^4)/(\varphi_y^2\Delta). \qquad (21)
\end{aligned}
$$

Theorem 2.1: Any third-order ordinary differential equation linearizable by a point transformation has to be one of the forms either equation (5) or (12).

B. Obtaining Sufficient Conditions of Linearization, Linearizing Transformation and Coefficients of Linear Equation

B.1 The First Class of Linearizable Equations

In the case $\varphi_y = 0$, the transformation (2) becomes a fiber preserving transformation, that is

$$t = \varphi(x), \quad u = \psi(x, y). \qquad (22)$$

For obtaining sufficient conditions, one has to solve the compatibility problem. We will consider the representations of the coefficients A_i and B_i through the unknown functions φ and ψ. At first, we rewrite the expression (6) for A_1 in the following form

$$\psi_{yy} = (\psi_y A_1)/3. \qquad (23)$$

From equation (8), one gets the condition

$$A_{1y} = (-A_1^2 + 9B_3)/3. \qquad (24)$$

One can determine γ from equation (7) as the following

$$\gamma = (3\varphi_{xx}\psi_y - 3\varphi_x\psi_{xy} + \varphi_x\psi_y A_0)/(\varphi_x^2\psi_y). \qquad (25)$$

Since $\varphi = \varphi(x)$, we have $\gamma_y = 0$ yields

$$A_{0y} = A_{1x}. \qquad (26)$$

Equation (9) provides the condition

$$A_{1x} = (-A_0A_1 + 3B_2)/3. \qquad (27)$$

One can determine β from equation (10) as the following

$$
\begin{aligned}
\beta =&(\varphi_{xxx}\psi_y^2 - 3\varphi_{xx}\psi_{xy}\psi_y + \varphi_{xx}\psi_y^2 A_0 \\
&+ 6\varphi_x\psi_{xy}^2 - 2\varphi_x\psi_{xy}\psi_y A_0 - 3\varphi_x\psi_{xxy}\psi_y \\
&+ \varphi_x\psi_y^2 B_1)/(\varphi_x^3\psi_y^2). \qquad (28)
\end{aligned}
$$

Since $\varphi = \varphi(x)$, we have $\beta_y = 0$ yields

$$B_{1y} = (-3A_{0x}A_1 + 9B_{2x} - A_0^2A_1 + 3A_0B_2)/9. \qquad (29)$$

One can determine α from equation (11) as the following

$$
\begin{aligned}
\alpha =&(-\varphi_x^3\psi_y^2\omega - 6\psi_{xy}^2\psi_x + 3\psi_{xy}\psi_{xx}\psi_y \\
&+ 2\psi_{xy}\psi_x\psi_y A_0 - \psi_{xxx}\psi_y^2 + 3\psi_{xxy}\psi_x\psi_y \\
&- \psi_{xx}\psi_y^2 A_0 - \psi_x\psi_y^2 B_1 + \psi_y^3 B_0)/(\varphi_x^3\psi_y^2\psi). \qquad (30)
\end{aligned}
$$

Since $\varphi = \varphi(x)$, we have $\alpha_y = 0$ yields

$$
\begin{aligned}
\omega =&(-3B_{oy}\psi_y^3\psi + 18\psi_{xy}^3\psi - 18\psi_{xy}^2\psi_x\psi_y \\
&- 6\psi_{xy}^2\psi_y A_0\psi - 18\psi_{xy}\psi_{xxy}\psi_y\psi + 9\psi_{xy}\psi_{xx}\psi_y^2 \\
&+ 6\psi_{xy}\psi_x\psi_y^2 A_0 + 3\psi_{xy}\psi_y^2 B_1\psi + 3\psi_{xxxy}\psi_y^2\psi \\
&- 3\psi_{xxx}\psi_y^3 + 9\psi_{xxy}\psi_x\psi_y^2 + 3\psi_{xxy}\psi_y^2 A_0\psi \\
&- 3\psi_{xx}\psi_y^3 A_0 - 3\psi_x\psi_y^3 B_1 + 3\psi_y^4 B_0 \\
&- \psi_y^3 A_1 B_0\psi)/(3\varphi_x^3\psi_y^3). \qquad (31)
\end{aligned}
$$

Since $\varphi = \varphi(x)$, we have $\omega_y = 0$ yields

$$
\begin{aligned}
B_{2xx} =&(9A_{0xx}A_1 + 18A_{0x}B_2 + 81B_{0yy} + 27B_{0y}A_1 \\
&- 18B_{2x}A_0 - 2A_0^3A_1 + 6A_0^2B_2 + 9A_0A_1B_1 \\
&- 9A_1^2B_0 + 81B_0B_3 - 27B_1B_2)/27. \qquad (32)
\end{aligned}
$$

All obtained results can be summarized in the following theorems.

Theorem 2.2: Sufficient conditions for equation (5) to be linearizable via the fiber- preserving transformation (22) are equations (24), (26), (27), (29) and (32).

Corollary 2.3: Provided that the sufficient conditions in Theorem 2.2 are satisfied, the transformation (22) mapping equation (5) to a linear equation (3) is obtained by solving the compatible system of equations $\varphi_y = 0$ and (23) for functions $\varphi(x)$ and $\psi(x, y)$. Finally, the coefficients γ, β, α and ω of the resulting linear equation (3) are given by equations (25), (28), (30) and (31).

B.2 The Second Class of Linearizable Equations

In this case, the problem is formulated as follows. At first, we give the coefficients C_i and D_i of equation (12).Then we find the necessary and sufficient conditions for integrability of the overdetermined system of equations (13)-(21) for the unknown functions $\varphi(x,y)$ and $\psi(x,y)$.

Recall that according to our notations, the following equations hold

$$\varphi_x = r\varphi_y, \quad (33)$$
$$\psi_x = (\varphi_y\psi_y r - \Delta)/\varphi_y, \quad (34)$$

and

$$\alpha_x = (\varphi_x\alpha_y)/\varphi_y, \quad (35)$$
$$\beta_x = (\varphi_x\beta_y)/\varphi_y, \quad (36)$$
$$\gamma_x = (\varphi_x\gamma_y)/\varphi_y, \quad (37)$$
$$\omega_x = (\varphi_x\omega_y)/\varphi_y. \quad (38)$$

Since $\varphi_y \neq 0$, equation (13) yields

$$\gamma = (6\varphi_{yy}\Delta - 3\varphi_y\Delta_y + \varphi_y C_2\Delta)/(\varphi_y^2\Delta). \quad (39)$$

The equation (37) leads to

$$\begin{aligned}\Delta_{xy} =&(3(\Delta_{yy}\Delta - \Delta_y^2)r + \Delta_x\Delta_y) + (3\Delta_y \\ &- C_2\Delta)r_y\Delta + 6r_{yy}\Delta^2 - C_{2y}r\Delta^2 \\ &+ C_{2x}\Delta^2)/(3\Delta). \quad (40)\end{aligned}$$

Equation (14) provides the derivative

$$\Delta_x = (12r_y\Delta + 3\Delta_y r + C_1\Delta - 2C_2 r\Delta)/3. \quad (41)$$

Substituting equation (41) into equation (40), one obtains the condition

$$r_{yy} = -(C_{1y} - C_{2x} - C_{2y}r - r_y C_2)/6. \quad (42)$$

From equation (15), one obtains

$$r_x = (6r_y r - C_0 + C_1 r - C_2 r^2)/6. \quad (43)$$

Since $\varphi_y \neq 0$, equation (16) yields

$$\begin{aligned}\alpha =&(\varphi_{yyy}\varphi_y\psi_y\Delta + 3\varphi_{yy}^2\psi_y\Delta - 3\varphi_{yy}\varphi_y\psi_{yy}\Delta \\ &- 3\varphi_{yy}\varphi_y\psi_y\Delta_y + \varphi_{yy}\varphi_y\psi_y C_2\Delta - \varphi_y^5\omega\Delta \\ &- \varphi_y^4\psi_y\beta\Delta - \varphi_y^2\psi_{yyy}\Delta + 3\varphi_y^2\psi_{yy}\Delta_y \\ &- \varphi_y^2\psi_{yy}C_2\Delta + \varphi_y D_5\Delta^2)/(\varphi_y^5\psi\Delta). \quad (44)\end{aligned}$$

Equation (17) provides

$$\begin{aligned}\beta =&(4\varphi_{yyy}\varphi_y\Delta^2 + 3\varphi_{yy}^2\Delta^2 - 9\varphi_{yy}\varphi_y\Delta_y\Delta \\ &+ 3\varphi_{yy}\varphi_y C_2\Delta^2 - 3\varphi_y^2\Delta_{yy}\Delta + 6\varphi_y^2\Delta_y^2 \\ &- 2\varphi_y^2\Delta_y C_2\Delta - \varphi_y^2 D_4\Delta^2 \\ &+ 5\varphi_y^2 D_5 r\Delta^2)/(\varphi_y^4\Delta^2). \quad (45)\end{aligned}$$

The equation (36) leads to

$$\begin{aligned}C_{1yy} =&(-C_{1y}C_2 + 24C_{2xy} + 13C_{2x}C_2 - 12C_{2yy}r \\ &- 24C_{2y}r_y - 11C_{2y}C_2 r + 18D_{4x} - 18D_{4y}r \\ &- 90D_{5x}r + 90D_{5y}r^2 - 11r_y C_2^2 - 36r_y D_4 \\ &+ 180r_y D_5 r + 15C_0 D_5 - 15C_1 D_5 r \\ &+ 15C_2 D_5 r^2)/6. \quad (46)\end{aligned}$$

From equations (18), (19), (20) and (21), one arrives at the following conditions

$$\begin{aligned}C_{2x} =&(3C_{2y}r - C_1C_2 + 2C_2^2 r - 3D_3 + 12D_4 r \\ &- 30D_5 r^2)/3, \quad (47)\\ C_{1x} =&(6C_{1y}r - 5C_0C_2 - 2C_1^2 + 9C_1C_2 r - 5C_2^2 r^2 \\ &- 18D_2 + 42D_3 r - 60D_4 r^2 + 60D_5 r^3)/6, \quad (48)\\ C_{0x} =&(6C_{0y}r - 7C_0C_1 + 9C_0C_2 r + 5C_1^2 r \\ &- 10C_1C_2 r^2 + 5C_2^2 r^3 - 36D_1 + 54D_2 r \\ &- 60D_3 r^2 + 60D_4 r^3 - 60D_5 r^4)/6, \quad (49)\\ D_0 =&(-C_0^2 + 2C_0C_1 r - 2C_0C_2 r^2 - C_1^2 r^2 \\ &+ 2C_1C_2 r^3 - C_2^2 r^4 + 12D_1 r - 12D_2 r^2 \\ &+ 12D_3 r^3 - 12D_4 r^4 + 12D_5 r^5)/12. \quad (50)\end{aligned}$$

Since $\varphi_y \neq 0$, equation (35) yields

$$\begin{aligned}\omega =&(3D_{5x}\varphi_y\psi\Delta^3 - 3D_{5y}\varphi_y\psi r\Delta^3 - 3\varphi_{yyyy}\psi\Delta^3 \\ &- 9\varphi_{yyy}\psi_y\Delta^3 + 9\varphi_{yyy}\Delta_y\psi\Delta^2 - 3\varphi_{yyy}C_2\psi\Delta^3 \\ &- 9\varphi_{yy}\psi_{yy}\Delta^3 + 18\varphi_{yy}\psi_y\Delta_y\Delta^2 - 6\varphi_{yy}\psi_y C_2\Delta^3 \\ &+ 9\varphi_{yy}\Delta_{yy}\psi\Delta^2 - 18\varphi_{yy}\Delta_y^2\psi\Delta + 6\varphi_{yy}\Delta_y C_2\psi\Delta^2 \\ &+ 3\varphi_{yy}D_4\psi\Delta^3 - 15\varphi_{yy}D_5\psi r\Delta^3 - 3\varphi_y\psi_{yyy}\Delta^3 \\ &+ 9\varphi_y\psi_{yy}\Delta_y\Delta^2 - 3\varphi_y\psi_{yy}C_2\Delta^3 + 9\varphi_y\psi_y\Delta_{yy}\Delta^2 \\ &- 18\varphi_y\psi_y\Delta_y^2\Delta + 6\varphi_y\psi_y\Delta_y C_2\Delta^2 + 3\varphi_y\psi_y D_4\Delta^3 \\ &- 15\varphi_y\psi_y D_5 r\Delta^3 + 3\varphi_y\Delta_{yyy}\psi\Delta^2 - 18\varphi_y\Delta_{yy}\Delta_y\psi\Delta \\ &+ 3\varphi_y\Delta_{yy}C_2\psi\Delta^2 + 18\varphi_y\Delta_y^3\psi - 6\varphi_y\Delta_y^2 C_2\psi\Delta \\ &- 3\varphi_y\Delta_y D_4\psi\Delta^2 + 15\varphi_y\Delta_y D_5\psi r\Delta^2 + \varphi_y C_1 D_5\psi\Delta^3 \\ &- 2\varphi_y C_2 D_5\psi r\Delta^3 + 3D_5\Delta^4)/(3\varphi_y^4\Delta^3). \quad (51)\end{aligned}$$

The equation (38) leads to

$$\begin{aligned}D_{3yy} =&(135C_{0y}D_5 - 45C_{1y}C_{2y} - 15C_{1y}C_2^2 \\ &- 36C_{1y}D_4 + 45C_{1y}D_5 r - 54C_{2yy}r_y \\ &- 18C_{2yy}C_1 + 36C_{2yy}C_2 r + 90C_{2y}^2 r \\ &- 63C_{2y}r_y C_2 - 51C_{2y}C_1C_2 + 132C_{2y}C_2^2 r \\ &- 117C_{2y}D_3 + 540C_{2y}D_4 r - 1395C_{2y}D_5 r^2 \\ &- 36D_{3y}C_2 + 162D_{4xy} + 54D_{4x}C_2 \\ &+ 54D_{4yy}r - 54D_{4y}r_y + 90D_{4y}C_2 r \\ &- 162D_{5xy}r - 324D_{5xx} + 162D_{5x}r_y \\ &- 108D_{5x}C_1 - 54D_{5x}C_2 r - 54D_{5yy}r^2 \\ &+ 108D_{5y}r_y r + 81D_{5y}C_0 + 27D_{5y}C_1 r \\ &- 225D_{5y}C_2 r^2 + 540r_y^2 D_5 + 189r_y C_1 D_5 \\ &- 9r_y C_2^3 - 36r_y C_2 D_4 - 198r_y C_2 D_5 r \\ &+ 99C_0C_2D_5 + 36C_1^2 D_5 - 13C_1C_2^3 \\ &- 48C_1C_2D_4 - 3C_1C_2D_5 r + 26C_2^4 r \\ &- 39C_2^2 D_3 + 252C_2^2 D_4 r - 627C_2^2 D_5 r^2 \\ &+ 324D_2D_5 - 144D_3D_4 - 252D_3D_5 r \\ &+ 576D_4^2 r - 2376D_4D_5 r^2 \\ &+ 3960D_5^2 r^3)/54. \quad (52)\end{aligned}$$

Theorem 2.4: Sufficient conditions for equation (12) to be linearizable via the point transformation (2) are equations (42), (43), (46), (47), (48), (49), (50) and (52).

Corollary 2.5: Provided that the sufficient conditions in Theorem 2.4 are satisfied, the transformation (2) mapping equation (12) to a linear equation (3) is obtained by solving the compatible system of equations (33), (34) and (41) for the functions $\varphi(x,y)$ and $\psi(x,y)$. Finally, the coefficients γ, α, β and ω of the resulting linear equation (3) are given by equations (39), (44), (45) and (51).

III. Some Applications

In this section we focus on finding some applications which satisfy the obtained theorems in section II. The obtained results are as follows.

A. Linearization for Some Interesting Third-Order Ordinary Differential Equations

Example 3.1: One equation in KAMKE's collection

- **The significance of the problem**

In [23], Jovan D. Kečkić considered the equation

$$y^2y''' + ayy'y'' + by'^3 = 0, \tag{53}$$

where a and b are arbitrary contants. By using KAMKA'S treatise, he found that equation (53) is equivalent to the equation $u''' = 0$ if and only if the condition $9b = a^2 - 3a$ holds. Since the general solution of the later is given by

$$u = C_1x^2 + C_2x + C_3,$$

he concluded that the general solution of

$$9y^2y''' + 9ayy'y'' + (a^2 - 3a)y'^3 = 0, \quad (a \neq -3)$$

is

$$y = (C_1x^2 + C_2x + C_3)^{\frac{3}{a+3}}, \tag{54}$$

where $y = F(u) = (\frac{a+3}{3}u)^{\frac{3}{a+3}}$.

- **Applying the obtained theorems to the problem**

Considering the nonlinear ordinary differential equation (53)

$$y^2y''' + ayy'y'' + by'^3 = 0,$$

it is an equation of the form (5) in Theorem 2.1 with the coefficients

$$A_1 = \frac{a}{y},\ A_0 = 0,\ B_3 = \frac{b}{y^2},\ B_2 = B_1 = B_0 = 0.$$

One can check that the equations (26), (27), (29), (32) in Theorem 2.2 are satisfied. Now, the equation (24) is satisfied when the following condition holds, that is,

$$b = \frac{a^2 - 3a}{9}.$$

Applying Corollary 2.3, the linearizing transformation is found by solving the following equations

$$\varphi_y = 0,\ \psi_{yy} = \frac{\psi_y a}{3y}. \tag{55}$$

One can find the particular solution for equations in (55) as

$$\varphi = x,\ \psi = y^{\frac{a+3}{3}}.$$

So that, one obtains the linearizing transformation

$$t = x, \quad u = y^{\frac{a+3}{3}}. \tag{56}$$

From Corollary 2.3, the coefficients γ, β, α and ω of the resulting linear equation (1) are

$$\gamma = 0,\ \beta = 0,\ \alpha = 0,\ \omega = 0.$$

Hence, the nonlinear equation (53) can be mapped into the linear equation

$$u''' = 0.$$

So that,

$$u = C_1t^2 + C_2t + C_3, \tag{57}$$

where C_1, C_2 and C_3 are arbitrary constants. Substituting equation (56) into equation (57), we get the solution

$$y^{\frac{a+3}{3}} = C_1x^2 + C_2x + C_3.$$

This implies that

$$y = (C_1x^2 + C_2x + C_3)^{\frac{3}{a+3}}.$$

Notice that the solution by our method is the same as the solution in equation (54).

Remark : We also found some interesting results in [24].

Example 3.2: Equation in the article [24]

In [24], Aeeman Fatima considered the equation

$$yy''' + 3y'y'' = 0. \tag{58}$$

He studied about the generator that is a symmetry of the derived ODE.

By using our method, equation (58) can be reduced to the linear equation $u''' = 0$ with the transformation $t = x$, $u = y^2$.

Example 3.3: Equation in the article [15]

- **The significance of the problem**

In [15], Ibragimov and Meleshko gave the relatively trivial example of the third-order Lie-linearizable scalar ODE of the second type

$$y''' - (3y''^2 + xy'^5)/y' = 0. \tag{59}$$

By using their method, they obtained that equation (59) can be reduced to the linear equation in the Laguerre's form

$$u''' + u = 0$$

by the point transformation $t = y, \quad u = x$.

- **Applying the obtained theorems to the problem**

Considering the nonlinear ordinary differential equation (59)

$$y''' - (3y''^2 + xy'^5)/y' = 0,$$

it is an equation of the form (12) in Theorem 2.1 with the coefficients

$$r = 0,\ C_0 = C_1 = C_2 = 0,$$
$$D_0 = D_1 = D_2 = D_3 = D_4 = 0,\ D_5 = -x.$$

One can check that these coefficients obey the conditions in Theorem 2.4. Hence, an equation (59) is linearizable via a point transformation. Applying Corollary 2.5, the linearizing transformation is found by solving the following equations

$$\varphi_x = 0,\ \psi_x = -\frac{\Delta}{\varphi_y},\ \Delta_x = 0. \tag{60}$$

One can find the particular solution for equations in (60) as

$$\varphi = y,\ \Delta = -1,\ \psi = x.$$

So that, one obtains the linearizing transformation

$$t = y, \quad u = x. \tag{61}$$

From Corollary 2.5, the coefficients γ, β, α and ω of the resulting linear equation (3) are

$$\gamma = 0,\, \beta = 0,\, \alpha = 1,\, \omega = 0.$$

Hence, the nonlinear equation (59) can be mapped by transformation (61) into the linear equation

$$u''' + u = 0. \tag{62}$$

The solution of equation (62) is

$$u\,(t) = (C_1 + C_2t + C_3t^2)e^{-t}, \tag{63}$$

where $C_1,\ C_2$ and C_3 are arbitrary constants. Substituting equation (61) into equation (63), we get

$$x = (C_1 + C_2y + C_3y^2)e^{-y}.$$

B. Linearization for Some Interesting Second-Order Ordinary Differential Equations Under the Riccati Transformation

Example 3.4: The modified Painlevé-Ince equation

- **The significance of the problem**

Abraham-Shrauner [25] examined the process of linearizing the modified Painlev-Ince equation, given by

$$y'' + \sigma yy' + \beta y^3 = 0, \tag{64}$$

where σ and β are constants, as well as the underlying symmetries that are not immediately apparent. By applying a nonlocal transformation to the linearized form of the equation, a damped or growing harmonic oscillator equation can be derived for values of β greater than zero.

We can find equation (64), with or without specific values of the parameters σ and β, in many distinct circumstances. A lot of research topics emerges from this equation [26]-[30].

- **Applying the obtained theorems to the problem**

Let us consider the nonlinear second-order ordinary differential equation (64)

$$y'' + \sigma yy' + \beta y^3 = 0.$$

Under the Riccati transformation $y = \frac{a\omega'}{\omega}$, equation (64) becomes

$$\begin{aligned}&\omega'''\omega^2 + \omega''\omega' a\omega\sigma - 3\omega''\omega'\omega + {\omega'}^3 a^2\beta\\ &-{\omega'}^3 a\sigma + 2{\omega'}^3 = 0.\end{aligned} \tag{65}$$

It is an equation of the form (5) in Theorem 2.1 with the coefficients

$$\begin{aligned}&A_1 = \frac{a\sigma - 3}{\omega},\, A_0 = 0,\, B_3 = \frac{a^2\beta - a\sigma + 2}{\omega^2},\\ &B_2 = B_1 = B_0 = 0.\end{aligned}$$

One can check that the equations (26), (27), (29), (32) in Theorem 2.2 are satisfied. Now, the equation (24) is satisfied when the following condition holds, that is,

$$\beta = \frac{\sigma^2}{9}.$$

Applying Corollary 2.3, the linearizing transformation is found by solving the equations

$$\varphi_\omega = 0,\ \psi_{\omega\omega} = \frac{\psi_\omega(a\sigma - 3)}{3\omega}. \tag{66}$$

One can find the particular solution for equations in (66) as

$$\varphi = x,\ \psi = \omega^{\frac{a\sigma}{3}}.$$

So that, one obtains the linearizing transformation

$$t = x, \quad u = \omega^{\frac{a\sigma}{3}}. \tag{67}$$

From Corollary 2.3, the coefficients $\tilde{\gamma}$, $\tilde{\beta}$, $\tilde{\alpha}$ and $\tilde{\omega}$ of the resulting linear equation (3) are

$$\tilde{\gamma} = 0,\, \tilde{\beta} = 0,\, \tilde{\alpha} = 0,\, \tilde{\omega} = 0.$$

Hence, the nonlinear equation (65) can be mapped by transformation (67) into the linear equation

$$u''' = 0.$$

So that,

$$u = C_1t^2 + C_2t + C_3, \tag{68}$$

where C_1, C_2 and C_3 are arbitrary constants. Substituting equation (67) into equation (68), we get

$$\omega^{\frac{a\sigma}{3}} = C_1x^2 + C_2x + C_3.$$

So that,

$$\omega = (C_1x^2 + C_2x + C_3)^{\frac{3}{a\sigma}}.$$

Hence, the original solution is

$$y = \frac{3}{\sigma}\frac{(2C_1x + C_2)}{(C_1x^2 + C_2x + C_3)}.$$

Remark : We also found some interesting results in [20], [25], [26], [28]-[31].

Example 3.5: Equation in the article [25], [26], [28]-[31]

The Painlevé-Ince equation

$$y'' + 3yy' + y^3 = 0, \tag{69}$$

has been the focus of several studies in recent decades, including [25], [26], [28]-[31]. If the solution of the equation is characterized by a movable singularity, it can be expressed as a power law function of the form $y(x) \simeq (xx_0)^p$ where p is a negative number and x_0 represents the location of the singularity.

By using our method, equation (69) can be reduced to the linear equation $u''' = 0$ with the transformation $t = x$, $u = \omega^a$.

Example 3.6: Equation in the article [20]

In [20], the equation

$$y'' + yy' + \kappa y^3 = 0, \tag{70}$$

can be linearized by our method if and only if

$$\kappa = \frac{1}{9}.$$

The linearized equation is $u''' = 0$ and the linearizing transformation is $t = x$, $u = \omega^{\frac{a}{3}}$. Notice that for $\kappa = \frac{1}{9}$, in [20] showed that equation (70) admits Lie isomorphic to sl$(3, R)$.

Example 3.7: The Liénard type ODEs

- **The significance of the problem**

The Liénard type ordinary differential equations are expressed as

$$y'' + (b + 3ky)y' + k^2y^3 + bky^2 + \lambda y = 0, \quad (71)$$

where b, k and λ are arbitrary constants. These equations are utilized as models of various physical and other phenomena, such as the stability of gaseous spheres and nonlinear oscillations. Johnpillai and Mahomed [32] investigated the linearization of a group of Linard type nonlinear second-order ordinary differential equations using the generalized Sundman transformation. They created the linearizing generalized Sundman transformation for this group and utilized it to transform the underlying equations into a linear second-order ordinary differential equations, which is not in the Laguerre form. By integrating the linearized equation and then applying the generalized Sundman transformation, they derived the general solution for this group of equations.

- **Applying the obtained theorems to the problem**

Let us consider the nonlinear second-order ordinary differential equation (71)

$$y'' + (b + 3ky)y' + k^2y^3 + bky^2 + \lambda y = 0.$$

Under the Riccati transformation $y = \frac{a\omega'}{\omega}$, equation (71) becomes

$$\begin{aligned}&\omega'''\omega^2 + 3\omega''\omega' ak\omega - 3\omega''\omega'\omega + \omega'' b\omega^2\\&+\omega'^3 a^2k^2 - 3\omega'^3 ak + 2\omega'^3 + \omega'^2 abk\omega\\&-\omega'^2 b\omega + \omega'\lambda\omega^2 = 0. \quad (72)\end{aligned}$$

It is an equation of the form (5) in Theorem 2.1 with the coefficients

$$A_1 = \frac{3\,(ak-1)}{\omega},\ A_0 = b,\ B_3 = \frac{a^2k^2 - 3ak + 2}{\omega^2},$$
$$B_2 = \frac{b(ak-1)}{\omega},\ B_1 = \lambda,\ B_0 = 0.$$

One can check that these coefficients obey the conditions in Theorem 2.2. Hence, an equation (72) is linearizable via a point transformations. Applying Corollary 2.3, the linearizing transformation is found by solving the equation

$$\varphi_\omega = 0,\ \psi_{\omega\omega} = \frac{\psi_\omega(ak-1)}{\omega}. \quad (73)$$

One can find the particular solution for equations in (73) as

$$\varphi = x,\ \psi = \omega^{ak}.$$

So that, one obtains the linearizing transformation

$$t = x, \quad u = \omega^{ak}. \quad (74)$$

From Corollary 2.3, the coefficients $\tilde{\gamma}$, $\tilde{\beta}$, $\tilde{\alpha}$ and $\tilde{\omega}$ of the resulting linear equation (3) are

$$\tilde{\gamma} = b,\ \tilde{\beta} = \lambda,\ \tilde{\alpha} = 0,\ \tilde{\omega} = 0.$$

Hence, the nonlinear equation (72) can be mapped by transformation (74) into the linear equation

$$u''' + bu'' + \lambda u' = 0. \quad (75)$$

- Case $b^2 - 4\lambda = 0$, the solution of equation (75) is

$$u(t) = C_1 + (C_2 t + C_3)e^{-\frac{b}{2}t}, \quad (76)$$

where C_1, C_2 and C_3 are arbitrary constants. Substituting equation (74) into equation (76), we get

$$\omega^{ak} = C_1 + (C_2 x + C_3)e^{-\frac{b}{2}x}.$$

So that,

$$\omega = [C_1 + (C_2 x + C_3)e^{-\frac{b}{2}x}]^{\frac{1}{ak}}.$$

Hence, the original solution is

$$y = \frac{1}{k}[\frac{-\frac{b}{2}(C_2x + C_3)e^{-\frac{b}{2}x} + C_3 e^{-\frac{b}{2}x}}{C_1 + (C_2x + C_3)e^{-\frac{b}{2}x}}].$$

- Case $b^2 - 4\lambda > 0$, the solution of equation (75) is

$$u(t) = C_1 + C_2 e^{\frac{-b+\sqrt{b^2-4\lambda}}{2}t} + C_3 e^{\frac{-b-\sqrt{b^2-4\lambda}}{2}t}, \quad (77)$$

where C_1, C_2 and C_3 are arbitrary constants. Substituting equation (74) into equation (77), we get

$$\omega^{ak} = C_1 + C_2 e^{\frac{-b+\sqrt{b^2-4\lambda}}{2}x} + C_3 e^{\frac{-b-\sqrt{b^2-4\lambda}}{2}x}.$$

So that,

$$\omega = [C_1 + C_2 e^{(\frac{-b+\sqrt{b^2-4\lambda}}{2})x} + C_3 e^{(\frac{-b-\sqrt{b^2-4\lambda}}{2})x}]^{\frac{1}{ak}}.$$

Hence, the original solution is

$$\begin{aligned}y = \quad &((\frac{-b+\sqrt{b^2-4\lambda}}{2})C_2 e^{(\frac{-b+\sqrt{b^2-4\lambda}}{2})x}\\&-(\frac{b+\sqrt{b^2-4\lambda}}{2})C_3 e^{(\frac{-b-\sqrt{b^2-4\lambda}}{2})x})/\\&(k(C_1 + C_2 e^{(\frac{-b+\sqrt{b^2-4\lambda}}{2})x}\\&+C_3 e^{(\frac{-b-\sqrt{b^2-4\lambda}}{2})x})).\end{aligned}$$

- Case $b^2 - 4\lambda < 0$, the solution of equation (75) is

$$\begin{aligned}u(t) &= C_1 + e^{-bt}(C_2\cos(\frac{\sqrt{b^2-4\lambda}}{2})t\\&\quad +C_3\sin(\frac{\sqrt{b^2-4\lambda}}{2})t), \quad (78)\end{aligned}$$

where C_1, C_2 and C_3 are arbitrary constants. Substituting equation (74) into equation (78), we get

$$\begin{aligned}\omega^{ak} &= C_1 + e^{-bx}(C_2\cos(\frac{\sqrt{b^2-4\lambda}}{2})x\\&\quad +C_3\sin(\frac{\sqrt{b^2-4\lambda}}{2})x).\end{aligned}$$

So that,

$$\begin{aligned}\omega &= [C_1 + e^{-bx}(C_2\cos(\frac{\sqrt{b^2-4\lambda}}{2})x\\&\quad +C_3\sin(\frac{\sqrt{b^2-4\lambda}}{2})x)]^{\frac{1}{ak}}.\end{aligned}$$

Hence, the original solution is

$$\begin{aligned} y = \quad & e^{-bx}[-(\frac{\sqrt{b^2-4\lambda}}{2})C_2\sin(\frac{\sqrt{b^2-4\lambda}}{2})x \\ & -bC_2\cos(\frac{\sqrt{b^2-4\lambda}}{2})x \\ & +(\frac{\sqrt{b^2-4\lambda}}{2})C_3\cos(\frac{\sqrt{b^2-4\lambda}}{2})x \\ & -bC_3\sin\frac{\sqrt{b^2-4\lambda}}{2})x]/[k(C_1 \\ & +e^{-bx}(C_2\cos\frac{\sqrt{b^2-4\lambda}}{2})x \\ & +C_3\sin\frac{\sqrt{b^2-4\lambda}}{2})x))]. \end{aligned}$$

C. Linearization for Some Interesting Third-Order Partial Differential Equations Under the Travelling Wave Solutions

Travelling waves are observed in many areas of science such as a result of a chemical reaction in combustion [33] and the impulses that are apparent in nerve fibres [34]. Travelling wave solutions are derived from solving the corresponding partial differential equations. These solutions are in the form

$$u(x,t) = H(z) \ \text{,where } z = x - Dt.$$

Here, the spatial and time domains are represented as x and t, with the velocity of the wave given as D.

Example 3.8: Equation in the article [35]

- **The significance of the problem**

The symmetry reductions of a class of nonlinear third-order partial differential equation given by

$$u_t - \varepsilon u_{xxt} + 2\kappa u_x = uu_{xxx} + \alpha uu_x + \beta u_x u_{xx}, \qquad (79)$$

where α, β, κ, and ε are arbitrary constants, were studied by Clarkson, Mansfield and Priestley [35]. There are three special cases of (79) as following:
(i) the Fornberg-Whitham equation [36]-[38], for the parameters $\varepsilon = 1,\ \alpha = -1,\ \beta = 3$ and $\kappa = 1/2$,
(ii) the Rosenau-Hyman equation [39], for the parameters $\varepsilon = 0,\ \alpha = 1,\ \beta = 3$ and $\kappa = 0$,
(iii) the Fuchssteiner-Fokas-Camassa-Holm equation [40]-[43], for the parameters $\epsilon = 1,\ \alpha = -1$ and $\beta = 2$.
Whitham [37] looked at qualitative behaviours of wave-breaking by considering the Fornberg-Whitham (FW) equation

$$u_t - u_{xxt} + u_x = uu_{xxx} - uu_x + 3u_x u_{xx}.$$

Rosenau and Hyman [39] used the Rosenau-Hyman (RH) equation

$$u_t = uu_{xxx} + uu_x + 3u_x u_{xx},$$

to model the effect of nonlinear dispersion in the formation of patterns in liquid drops.

Camassa and Holm [40] used the Fuchssteiner-Fokas-Camassa-Holm (FFCH) equation

$$u_t - u_{xxt} + 2\kappa u_x = uu_{xxx} - 3uu_x + 2u_x u_{xx}$$

to model dispersive shallow water waves.

- **Applying the obtained theorems to the problem**

We will consider the nonlinear third-order partial differential equation (79) given by:

$$u_t - \varepsilon u_{xxt} + 2\kappa u_x u u_{xxx} + \alpha uu_x + \beta u_x u_{xx} = 0.$$

Of particular interest are the travelling wave solutions for equation (79) , which can be expressed as

$$u(x,t) = H(x - Dt),$$

where D is a constant phase velocity, and the argument $x - Dt$ represents the phase of the wave. By substituting this representation of a solution into equation (79) , we get

$$\begin{aligned} & -DH' + \varepsilon DH''' + 2\kappa H' - HH''' \\ & -\alpha HH' - \beta H'H'' = 0. \end{aligned} \qquad (80)$$

It is an equation of the form (5) in Theorem 2.1 with the coefficients

$$\begin{aligned} & A_1 = -\frac{\beta}{d\varepsilon - y},\ A_0 = 0,\ B_1 = -\frac{\alpha y - d + 2\kappa}{d\varepsilon - y}, \\ & B_3 = B_2 = B_0 = 0. \end{aligned}$$

One can check that the equations (26), (27) and (32) in Theorem 2.2 are satisfied. Now, the equations (24) and (29) are satisfied when the following conditions holds, that are,

$$\beta = 0 \ \text{ or } \ \beta = 3 \ \text{ and } \ \kappa = \frac{d\,(\alpha\varepsilon + 1)}{2}. \qquad (81)$$

- Case $\beta = 0$ and $\kappa = \frac{d(\alpha\varepsilon+1)}{2}$

Applying Corollary 2.3, the linearizing transformation is found by solving the equations

$$\varphi_H = 0,\ \psi_{HH} = 0. \qquad (82)$$

One can find the particular solution for equations in (82) as

$$\varphi = x - Dt,\ \psi = H.$$

So that, one obtains the linearizing transformation

$$\tilde{t} = x - Dt, \quad \tilde{u} = H. \qquad (83)$$

From Corollary 2.3, the coefficients $\tilde{\gamma}$, $\tilde{\beta}$, $\tilde{\alpha}$ and $\tilde{\omega}$ of the resulting linear equation (3) are

$$\tilde{\gamma} = 0,\ \tilde{\beta} = \alpha,\ \tilde{\alpha} = 0,\ \tilde{\omega} = 0.$$

Hence, the nonlinear equation (80) can be mapped by transformation (83) into the linear equation

$$\tilde{u}''' + \alpha\tilde{u}' = 0. \qquad (84)$$

The solution of equation (84) is

$$\tilde{u}(\tilde{t}) = C_1 + C_2\cos\sqrt{\alpha}\tilde{t} + C_3\sin\sqrt{\alpha}\tilde{t}, \qquad (85)$$

where C_1, C_2 and C_3 are arbitrary constants. Substituting equation (83) into equation (85), we get the solution of ordinary differential equation

$$H = C_1 + C_2\cos\left(\sqrt{\alpha}\,(x - Dt)\right) + C_3\sin\left(\sqrt{\alpha}\,(x - Dt)\right).$$

So that, the solution of partial differential equation is

$$u = C_1 + C_2\cos\left(\sqrt{\alpha}\,(x - Dt)\right) + C_3\sin\left(\sqrt{\alpha}\,(x - Dt)\right).$$

- Case $\beta = 3$ and $\kappa = \frac{d(\alpha\varepsilon+1)}{2}$

Applying Corollary 2.3, the linearizing transformation is found by solving the equations

$$\varphi_H = 0,\ \psi_{HH} = -\frac{\psi_H}{\varepsilon d - H}. \qquad (86)$$

One can find the particular solution for equations in (86) as

$$\varphi = x - Dt,\ \psi = H^2 - 2\varepsilon dH.$$

So that, one obtains the linearizing transformation

$$\tilde{t} = x - Dt, \quad \tilde{u} = H^2 - \varepsilon dH. \qquad (87)$$

From Corollary 2.3, the coefficients $\tilde{\gamma}$, $\tilde{\beta}$, $\tilde{\alpha}$ and $\tilde{\omega}$ of the resulting linear equation (3) are

$$\tilde{\gamma} = 0,\ \tilde{\beta} = \alpha,\ \tilde{\alpha} = 0,\ \tilde{\omega} = 0.$$

Hence, the nonlinear equation (80) can be mapped by transformation (87) into the linear equation

$$\tilde{u}''' + \alpha\tilde{u}' = 0. \qquad (88)$$

The solution of equation (88) is

$$\tilde{u}(\tilde{t}) = C_1 + C_2 \cos\sqrt{\alpha}\tilde{t} + C_3 \sin\sqrt{\alpha}\tilde{t}, \qquad (89)$$

where C_1, C_2 and C_3 are arbitrary constants. Substituting equation (87) into equation (89), we get the solution of ordinary differential equation

$$\begin{aligned} H^2 - 2\varepsilon dH &= C_1 + C_2 \cos\left(\sqrt{\alpha}\,(x - Dt)\right) \\ &\quad + C_3 \sin\left(\sqrt{\alpha}\,(x - Dt)\right). \end{aligned}$$

So that, the solution of partial differential equation is

$$u^2 - \varepsilon du = C_1 + C_2 \cos\left(\sqrt{\alpha}\,(x - Dt)\right) + C_3 \sin\left(\sqrt{\alpha}\,(x - Dt)\right).$$

Example 3.9: The regularized Burgers equation

- **The significance of the problem**

One of attracted interests is equations of the class of evolutionary PDEs

$$m_t + Au_x m + Bum_x + Cuu_x + Du_{xxt} = Ku_x,$$

where $m = u - \alpha^2 u_{xx}$ is the Helmholtz operator acting on the dependent variable u, function of the spatial variable x and time t and A, B, C, D, K are constants. The derivative of this equation is known as the Camassa-Holm equation.

If $A = 0$, $B = 1$, $C = 0$, $D = 0$ and $K = 0$, then the above equation becomes

$$u_t + uu_x = \alpha^2(u_{xxt} + uu_{xxx}), \qquad (90)$$

and is called a regularized Burgers equation.

Bhat and Fetecau [44] proposed equation (90) in the aspect of Burgers equation $u_t + uu_x = 0$. Solution properties of members of this class are investigated by Camassa et al. [45].

- **Applying the obtained theorems to the problem**

Let us consider the nonlinear third-order partial differential equation (90)

$$u_t + uu_x = \alpha^2(u_{xxt} + uu_{xxx}).$$

Of particular interest among solutions of equation (90) are travelling wave solutions:

$$u(x, t) = H(x - Dt),$$

where D is a constant phase velocity and the argument $x - Dt$ is a phase of the wave.
Substituting the representation of a solution into equation (90), one finds

$$-DH' + HH' - \alpha^2(-DH''' + HH''') = 0. \qquad (91)$$

It is an equation of the form (5) in Theorem 2.1 with the coefficients

$$A_1 = A_0 = 0,\ B_1 = -\frac{1}{\alpha^2},\ B_3 = B_2 = B_0 = 0.$$

One can check that these coefficients obey the conditions in Theorem 2.2. Hence, an equation (91) is linearizable via a point transformations. Applying Corollary 2.3, the linearizing transformation is found by solving the equations

$$\varphi_H = 0,\ \psi_{HH} = 0. \qquad (92)$$

One can find the particular solution for equations in (92) as

$$\varphi = x - Dt,\ \psi = H.$$

So that, one obtains the linearizing transformation

$$\tilde{t} = x - Dt, \quad \tilde{u} = H. \qquad (93)$$

From Corollary 2.3, the coefficients $\tilde{\gamma}$, $\tilde{\beta}$, $\tilde{\alpha}$ and $\tilde{\omega}$ of the resulting linear equation (3) are

$$\tilde{\gamma} = 0,\ \tilde{\beta} = -\frac{1}{\alpha^2},\ \tilde{\alpha} = 0,\ \tilde{\omega} = 0.$$

Hence, the nonlinear equation (91) can be mapped by transformation (93) into the linear equation

$$\tilde{u}''' - \frac{1}{\alpha^2}\tilde{u}' = 0. \qquad (94)$$

The solution of equation (94) is

$$\tilde{u}(\tilde{t}) = C_1 + C_2 e^{-\frac{1}{\alpha}\tilde{t}} + C_3 e^{\frac{1}{\alpha}\tilde{t}}, \qquad (95)$$

where C_1, C_2 and C_3 are arbitrary constants. Substituting equation (93) into equation (95), we get the solution of ordinary differential equation

$$H = C_1 + C_2 e^{-\frac{1}{\alpha}(x - Dt)} + C_3 e^{\frac{1}{\alpha}(x - Dt)}.$$

So that, the solution of partial differential equation is

$$u = C_1 + C_2 e^{-\frac{1}{\alpha}(x - Dt)} + C_3 e^{\frac{1}{\alpha}(x - Dt)}.$$

Example 3.10: Equation in the article [46]

- **The significance of the problem**

Basarab-Horwath and Güngör [46] presented criteria for testing equations for linearity for third-order evolution equations. Linearizing transformations are found by using symmetry structure and local conservation laws. They cosidered here an evolution given by

$$u_t = u_{xxx} + \frac{3\alpha}{u}u_x u_{xx} + \frac{\beta}{u^2}{u_x}^3 + a\left(\frac{\alpha}{u}u_x^2 + u_{xx}\right), \qquad (96)$$

where α, β and a are arbitrary constants.

- **Applying the obtained theorems to the problem**

Let us consider the nonlinear third-order partial differential equation (96)

$$u_t = u_{xxx} + \frac{3\alpha}{u}u_x u_{xx} + \frac{\beta}{u^2}{u_x}^3 + a(\frac{\alpha}{u}u_x^2 + u_{xx}).$$

Of particular interest among solutions of equation (96) are travelling wave solutions:

$$u(x,t) = H(x - Dt),$$

where D is a constant phase velocity and the argument $x-Dt$ is a phase of the wave.

Substituting the representation of a solution into equation (96), one finds

$$\begin{aligned} &-DH' - H''' - \frac{3\alpha}{H}H'H'' - \frac{\beta}{H^2}H'^3 \\ &-a(\frac{\alpha}{H}H'^2 + H'') = 0. \end{aligned} \quad (97)$$

It is an equation of the form (5) in Theorem 2.1 with the coefficients

$$A_1 = \frac{3\alpha}{y},\ A_0 = a,\ B_3 = \frac{\beta}{y^2},\ B_2 = \frac{a\alpha}{y},\ B_1 = d, B_0 = 0.$$

One can easily check that the equations (26), (27), (29) and (32) in Theorem 2.2 are satisfied. Now, the equation (24) is satisfied when the following condition holds, that is,

$$\beta = \alpha(\alpha - 1).$$

Applying Corollary 2.3, the linearizing transformation is found by solving the equations

$$\varphi_H = 0,\ \psi_{HH} = \frac{\psi_H \alpha}{H}. \quad (98)$$

One can find the particular solution for equations in (98) as

$$\varphi = x - Dt,\ \psi = H^{\alpha+1}.$$

So that, one obtains the linearizing transformation

$$\tilde{t} = x - Dt, \quad \tilde{u} = H^{\alpha+1}. \quad (99)$$

From Corollary 2.3, the coefficients $\tilde{\gamma}$, $\tilde{\beta}$, $\tilde{\alpha}$ and $\tilde{\omega}$ of the resulting linear equation (3) are

$$\tilde{\gamma} = a,\ \tilde{\beta} = D,\ \tilde{\alpha} = 0,\ \tilde{\omega} = 0.$$

Hence, the nonlinear equation (97) can be mapped by transformation (99) into the linear equation

$$\tilde{u}''' + a\tilde{u}'' + D\tilde{u}' = 0. \quad (100)$$

- Case $a^2 - 4D = 0$, the solution of equation (100) is

$$\tilde{u}(\tilde{t}) = C_1 + (C_2\tilde{t} + C_3)e^{-\frac{a}{2}\tilde{t}}, \quad (101)$$

where C_1, C_2 and C_3 are arbitrary constants. Substituting equation (99) into equation (101), we get the solution of ordinary differential equation

$$H^{\alpha+1} = C_1 + (C_2(x - Dt) + C_3)e^{-\frac{a}{2}(x-Dt)},$$

we have

$$H = [C_1 + (C_2(x - Dt) + C_3)e^{-\frac{a}{2}(x-Dt)}]^{\frac{1}{\alpha+1}}.$$

So that, the solution of partial differential equation is

$$u = [C_1 + (C_2(x - Dt) + C_3)e^{-\frac{a}{2}(x-Dt)}]^{\frac{1}{\alpha+1}}.$$

- Case $a^2 - 4D > 0$, the solution of equation (100) is

$$\tilde{u}(\tilde{t}) = C_1 + C_2 e^{(\frac{-a+\sqrt{a^2-4D}}{2})\tilde{t}} + C_3 e^{(\frac{-a-\sqrt{a^2-4D}}{2})\tilde{t}}, \quad (102)$$

where C_1, C_2 and C_3 are arbitrary constants. Substituting equation (99) into equation (102), we get the solution of ordinary differential equation

$$\begin{aligned} H^{\alpha+1} &= C_1 + C_2 e^{(\frac{-a+\sqrt{a^2-4D}}{2})(x-Dt)} \\ &\quad + C_3 e^{(\frac{-a-\sqrt{a^2-4D}}{2})(x-Dt)}, \end{aligned}$$

we have

$$\begin{aligned} H &= [C_1 + C_2 e^{(\frac{-a+\sqrt{a^2-4D}}{2})(x-Dt)} \\ &\quad + C_3 e^{(\frac{-a-\sqrt{a^2-4D}}{2})(x-Dt)}]^{\frac{1}{\alpha+1}}. \end{aligned}$$

So that, the solution of partial differential equation is

$$\begin{aligned} u &= [C_1 + C_2 e^{(\frac{-a+\sqrt{a^2-4D}}{2})(x-Dt)} \\ &\quad + C_3 e^{(\frac{-a-\sqrt{a^2-4D}}{2})(x-Dt)}]^{\frac{1}{\alpha+1}}. \end{aligned}$$

- Case $a^2 - 4D < 0$, the solution of equation (100) is

$$\begin{aligned} \tilde{u}(\tilde{t}) &= [C_1 + e^{-a\tilde{t}}(C_2\cos(\frac{\sqrt{a^2-4D}}{2})\tilde{t} \\ &\quad + C_3\sin(\frac{\sqrt{a^2-4D}}{2})\tilde{t})]^{\frac{1}{\alpha+1}}, \end{aligned} \quad (103)$$

where C_1, C_2 and C_3 are arbitrary constants. Substituting equation (99) into equation (103), we get solution of the ordinary differential equation

$$\begin{aligned} H^{\alpha+1} &= C_1 + e^{-a(x-Dt)}(C_2\cos(\frac{\sqrt{a^2-4D}}{2})(x - Dt) \\ &\quad + C_3\sin(\frac{\sqrt{a^2-4D}}{2})(x - Dt)), \end{aligned}$$

we have

$$\begin{aligned} H &= [C_1 + e^{-a(x-Dt)}(C_2\cos(\frac{\sqrt{a^2-4D}}{2})(x - Dt) \\ &\quad + C_3\sin(\frac{\sqrt{a^2-4D}}{2})(x - Dt))]^{\frac{1}{\alpha+1}}. \end{aligned}$$

So that, the solution of partial differential equation is

$$\begin{aligned} u &= [C_1 + e^{-a(x-Dt)}(C_2\cos(\frac{\sqrt{a^2-4D}}{2})(x - Dt) \\ &\quad + C_3\sin(\frac{\sqrt{a^2-4D}}{2})(x - Dt))]^{\frac{1}{\alpha+1}}. \end{aligned}$$

IV. Conclusion

In summary, if a third-order ordinary differential equation is not in one of the forms specified in Theorem 2.1, it definitely cannot be linearized by the point transformation. The form that satisfies corresponding conditions in either Theorem 2.2 or Theorem 2.4 is linearizable via the point transformation. The original solution can be attained by applying the transformations derived from Corollary 2.3 and Corollary 2.5. This method has been proven to be effective for various third-order ordinary differential equations in literature, as well as some second-order ordinary differential equations and third-order partial differential equations under specific conditions.

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