Stability of Riemann Solutions for the Hyperbolic System

Yujin Liu and Wenhua Sun

***Abstract*—In the present paper, we discuss mainly the elementary wave interactions for the simplified hyperbolic equations which shows the rich internal mechanisms. By virtue of the characteristic method, we construct the unique perturbed solution. Moreover, we find the Riemann solutions are stable.**

***Index Terms*—Wave interaction, Hyperbolic conservation laws, Delta shock, Riemann problem.**

I. INTRODUCTION

IN this paper we study mainly the following hyperbolic equations

$$\begin{cases} u_t + (\psi u)_x = 0, \\ v_t + (\psi v)_x = 0, \end{cases} \tag{1}$$

where $\psi = \psi(\nu)$ is the given smooth function of $\nu = au+bv$ which satisfies the condition $a^2 + b^2 \neq 0$, a and b are the constants. For the general case $\psi = \psi(u, v)$, the authors [1] investigated the existence of the global solutions.

In [2], the authors studied the Riemann problem for (1) with

$$(u,v)(x,t)|_{t=0} = \begin{cases} (u_+, v_+), & as\ x > 0, \\ (u_-, v_-), & as\ x < 0, \end{cases} \tag{2}$$

and they obtained the stability of the Riemann solutions and the delta shock waves appeared.

In [3], by using the viscous vanishing method the authors investigated

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x = 0, \end{cases} \tag{3}$$

where $\rho(x,t) \geq 0$ and u is respectively the density and the velocity.

In [4], the author studied the following coupled systems

$$\begin{cases} v_t + (vg(u))_x = 0, \\ (vu)_t + (vug(u))_x = 0, \end{cases} \tag{4}$$

which included (3). In [5], we obtained the global explicit solutions for the Cauchy problem of (4).

In [6], the author discussed the following Riemann problem

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p)_x = 0, \end{cases} \tag{5}$$

and $p = -\frac{1}{\rho}$. In [7], the authors investigated the general solutions for system (5) and gave some conjectures on the solutions structures. Many works about the initial value problem are recommended to [8], [9], [10] and the references cited therein.

In this study, we make the investigations of the elementary wave interactions for (1). We consider the wave interactions containing $\delta-$shock in another paper, and here we just study the cases containing no $\delta-$shock. Many conclusions about $\delta-$shock can be referred to [11], [12], [13].

We consider the following initial value question for (1)

$$(u,v)(x,t)|_{t=0} = \begin{cases} (u_-, v_-), & as\ x < -\eta, \\ (u_m, v_m), & as\ -\eta < x < \eta, \\ (u_+, v_+), & as\ x > \eta, \end{cases} \tag{6}$$

where the perturbation parameter $\eta > 0$ is small enough. We will research the above problem by studying detailedly the elementary wave interactions. We will investigate the two Riemann problems and analyze the wave interactions. Moreover, by letting $\eta \to 0$, we conclude that the perturbed solution of (1) and (6) has convergence, which reveals the stability of the Riemann solutions of (1) and (2).

The present paper is continued as follows. we list the studies for (1) and (2) in Section II. In Section III, we investigate the elementary waves interactions and obtain that the Riemann solutions are globally stable. In Section IV we get the main result.

II. PRELIMINARIES

In what follows, we give briefly the Riemann problem of (1) and (2) [2].

The characteristic roots of (1) are $\mu_1 = \psi$, $\mu_2 = \psi + \nu\psi_r$, and the right characteristic vectors of μ_i $(i = 1, 2)$ are respectively

$$\overrightarrow{\chi}_1 = (b, -a)^T, \qquad \overrightarrow{\chi}_2 = (u, v)^T. \tag{7}$$

It is easily known that when $\nu\psi_\nu = 0$, (1) is non-strictly hyperbolic. From

$$\nabla\mu_1 \cdot \overrightarrow{\chi}_1 \equiv 0, \qquad \nabla\mu_2 \cdot \overrightarrow{\chi}_2 = \nu(\nu\psi)_{\nu\nu}, \tag{8}$$

In our paper, let $\psi_\nu > 0$, $(\nu\psi)_{\nu\nu} > 0$, and $\psi(0) = 0$. Let $(u, \rho)(x,t) = (u, \rho)(\xi)$, $\xi = \frac{x}{t}$, and (1) and (2) become the following problem

$$\begin{cases} -\zeta u_\zeta + (\psi u)_\zeta = 0, \\ -\zeta v_\zeta + (\psi v)_\zeta = 0, \end{cases} \tag{9}$$

and $(u, v)(\pm\infty) = (u_\pm, v_\pm)$. For the smooth solutions, (9) becomes

$$A(\varsigma)\varsigma_\zeta = 0, \tag{10}$$

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where $\varsigma = (u,v)^T$, and

$$A(\varsigma) = \begin{pmatrix} -\zeta+\psi+au\psi_\nu & bu\psi_\nu \\ av\psi_\nu & -\zeta+\psi+bv\psi_\nu \end{pmatrix}.$$

Besides $(u,v) = constant$, (10) has the singular solution

$$\begin{cases} \zeta = \phi, \\ au+bv = au_- + bv_-, \end{cases} \tag{11}$$

and the rarefaction wave solution

$$\begin{cases} \zeta = \psi + \nu\psi_\nu, \\ \frac{u}{v} = \frac{u_-}{v_-}, \quad \nu_- < \nu. \end{cases} \tag{12}$$

Denote $\overleftarrow{R}$ when $\nu_- < \nu < 0$, and $\overrightarrow{R}$ when $\nu > \nu_- > 0$.

At $\zeta = \omega$, it holds the Rankine-Hugoniot equations

$$\begin{cases} -\omega[u] + [\psi u] = 0, \\ -\omega[v] + [\psi v] = 0, \end{cases} \tag{13}$$

where $[u] = u_r - u_l$ denotes the jump of u, $u_l = u(\omega - 0)$, $u_r = u(\omega + 0)$, etc.

From (13) we obtain the contact discontinuity

$$\begin{cases} \zeta = \psi(\nu_-) = \psi(\nu_+), \quad \nu_- = \nu_+, \\ au+bv = au_- + bv_-, \quad \nu = \nu_-, \end{cases} \tag{14}$$

and the shock wave

$$\begin{cases} \zeta = \omega = \frac{\nu_+\psi(\nu_+) - \nu_-\psi(\nu_-)}{\nu_+ - \nu_-}, \\ \frac{u}{v} = \frac{u_-}{v_-}, \quad \nu < \nu_-. \end{cases} \tag{15}$$

Denote $\overleftarrow{S}$ when $\nu < \nu_- < 0$, and $\overrightarrow{S}$ when $\nu_- > \nu > 0$.

Note that R coincide with S, we know that (1) is the Temple class [14], [15].

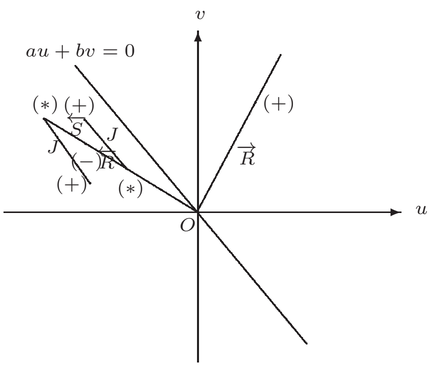

Fig. 1. Wave curves when $\nu_- < 0$.

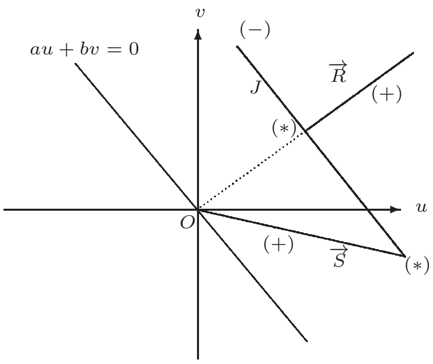

Fig. 2. Wave curves when $\nu_- > 0$, $\nu_+ > 0$.

When $\nu_- < 0$ (Fig. 1.), there are three possibilities:

if $\nu_+ < \nu_- < 0$, the unique solution is $\overleftarrow{S} + J$, if $\nu_- < \nu_+ < 0$, the unique solution is $\overleftarrow{R} + J$, if $\nu_+ < \nu_- < 0$, the unique solution is $\overleftarrow{R} + \overleftarrow{R}$;

When $\nu_- > 0$ and $\nu_+ > 0$ (Fig. 2.), there are two possibilities:

if $\nu_+ > \nu_- > 0$, the unique solution is $J + \overrightarrow{R}$, if $\nu_- > \nu_+ > 0$, the unique solution is $J + \overrightarrow{S}$.

When $\nu_- \geq 0 \geq \nu_+$, we obtain $\delta-$shock solution which satisfies

$$\begin{cases} \omega = \frac{r_+\psi(\nu_+) - \nu_-\psi(\nu_-)}{\nu_+ - \nu_-}, \\ x = \omega t, \\ \varphi(t) = \frac{1}{\sqrt{1+\omega^2}} \frac{\psi(\nu_+) - \psi(\nu_-)}{\nu_+ - \nu_-}(u_+v_- - u_-v_+)t, \\ \psi(\nu)|_{x=\omega t} = \omega. \end{cases} \tag{16}$$

The δ-entropy condition is

$$\mu_2(\nu_+) \leq \mu_1(\nu_+) \leq \omega \leq \mu_1(\nu_-) \leq \mu_2(\nu_-). \tag{17}$$

Theorem 2.1 We construct the unique Riemann solution of (1) and (2).

III. Solutions of the Perturbed Initial Value Problem (1) and (6)

Now we consider the elementary wave interactions for (1) and (6). There are four different cases from $(-\eta, 0)$ and $(\eta, 0)$:

$J + \overrightarrow{S}$ and $J + \overrightarrow{S}$, $J + \overrightarrow{R}$ and $J + \overrightarrow{R}$, $J + \overrightarrow{S}$ and $J + \overrightarrow{R}$, $\overleftarrow{S} + J$ and $\overleftarrow{R} + J$.

Case 1. When $J + \overrightarrow{S}$ overtakes $J + \overrightarrow{S}$.

Since $\nu_- > \nu_m > \nu_+ > 0$ (Fig. 3.), we get (Fig. 4.)

$$\begin{aligned} &(u_-, v_-) + J_1 + (u_1, v_1) + \overrightarrow{S}_1 + (u_m, v_m) + J_2 + \\ &(u_2, v_2) + \overrightarrow{S}_2 + (u_+, v_+). \end{aligned} \tag{18}$$

The propagating speed of $\overrightarrow{S}_1$ is $\omega_1 = \frac{\nu_1\psi(\nu_1) - \nu_m\psi(\nu_m)}{\nu_1 - \nu_m}$ and that of J_2 is $\mu_2 = \psi(\nu_m) = \psi(\nu_2)$. Thus, $\overrightarrow{S}_1$ will overtake J_2 in the finite time. The intersection point (x_1, t_1) satisfies

$$\begin{cases} x_1 + \eta = \omega_1 t_1, \\ x_1 - \eta = \mu_2 t_1, \end{cases} \tag{19}$$

which shows that $(x_1, t_1) = \left(\frac{\omega_1 + \mu_2}{\omega_1 - \mu_2}\eta, \frac{2\eta}{\omega_1 - \mu_2}\right)$.

After (x_1, t_1), it generates a new $\overrightarrow{S}_3$ and a new J_3. Notice $\omega_3 = \omega_1$, and J_3 is parallel to J_1 due to $\mu_3 = \mu_1$.

Due to the fact that $\omega_3 = \omega_1 = \frac{\nu_1\psi(\nu_1) - \nu_m\psi(\nu_m)}{\nu_1 - \nu_m}$, $\omega_2 = \frac{\nu_2\psi(\nu_2) - \nu_+\psi(\nu_+)}{\nu_2 - \nu_+}$, we know that $\omega_3 > \omega_2$ and $\overrightarrow{S}_3$ will overtake $\overrightarrow{S}_2$ at (x_2, t_2) and generate a new $\overrightarrow{S}_4$ where the point (x_2, t_2) satisfies

$$\begin{cases} x_2 - x_1 = \omega_3(t_2 - t_1), \\ x_2 - \eta = \omega_2 t_2. \end{cases} \tag{20}$$

It follows that

$$(x_2, t_2) = \left(\frac{\omega_3 + \omega_2}{\omega_3 - \omega_2}\eta, \frac{2\eta}{\omega_3 - \omega_2}\right). \tag{21}$$

When the time is large enough, the solution is given as follows $(u_-, v_-) + J_1 + (u_1, v_1) + J_3 + (u_3, v_3) + \overrightarrow{S}_4 + (u_+, v_+)$.

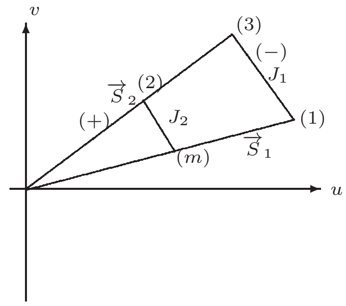

Fig. 3. Wave curves when $\nu_- > \nu_m > \nu_+ > 0$.

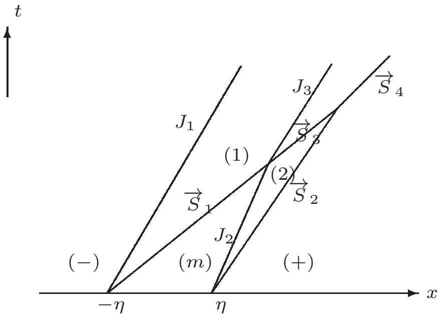

Fig. 4. $J + \overrightarrow{S}$ and $J + \overrightarrow{S}$.

Theorem 3.1 When $J+\overrightarrow{S}$ overtakes $J+\overrightarrow{S}$, we observe that the perturbed solution is $J + \overrightarrow{S}$ which remains unchanged.

Case 2. When $J + \overrightarrow{R}$ overtakes $J + \overrightarrow{R}$.

Since $\nu_m > \nu_- > 0$ and $\nu_+ > \nu_m > 0$, i.e., $\nu_+ > \nu_m > \nu_- > 0$ (Fig. 5.). The propagation speed of $\overrightarrow{R}_1$ is $\zeta_1 = \psi(\nu_m) + \nu_m\psi_\nu(\nu_m)$ and that of J_2 is $\mu_2 = \psi(\nu_m) = \psi(\nu_2)$ (Fig. 6.). It is clear that $\overrightarrow{R}_1$ will overtake J_2 at (x_1, t_1) which is determined by

$$\begin{cases} x_1 + \eta = \zeta_1 t_1, \\ x_1 - \eta = \mu_2 t_1. \end{cases} \tag{22}$$

It yields that

$$(x_1, t_1) = (\frac{\zeta_1 + \nu_2}{\zeta_1 - \mu_2}, \frac{2\eta}{\zeta_1 - \mu_2}). \tag{23}$$

Then J_2 crosses the whole of $\overrightarrow{R}_1$, which satisfies

$$\begin{cases} \frac{dx}{dt} = \psi(\nu), \\ x + \varepsilon = [\psi(\nu) + \nu\psi_\nu(\nu)]t, \\ \frac{u}{v} = \frac{u_1}{v_1} = \frac{u_m}{v_m}, \\ x(t_1) = x_1. \end{cases} \tag{24}$$

Since

$$\begin{aligned} \frac{dx}{dt} &= \psi(\nu) + \nu\psi_\nu(\nu) + t[2\psi_\nu(\nu) + \nu\psi_{\nu\nu}(\nu)](a\frac{du}{dt} + b\frac{dv}{dt}), \\ &= \psi(\nu), \end{aligned} \tag{25}$$

we have $a\frac{du}{dt} + b\frac{dv}{dt} = \frac{\nu\psi_\nu(\nu)}{t[2\psi_\nu(\nu)+\nu\psi_{\nu\nu}(\nu)]}$. Considering that the condition $(r\phi)_{rr} > 0$, we have

$$(\nu\psi(\nu))''_{\nu\nu} = [(\nu\psi(\nu))_\nu]' = [\psi+\nu\psi_\nu]' = 2\psi'(\nu)+\nu\psi''(\nu) > 0.$$

And together with $\psi_\nu > 0$, it follows that

$$\frac{d^2x}{dt^2} = \psi'(\nu)[a\frac{du}{dt} + b\frac{dv}{dt}] > 0. \tag{26}$$

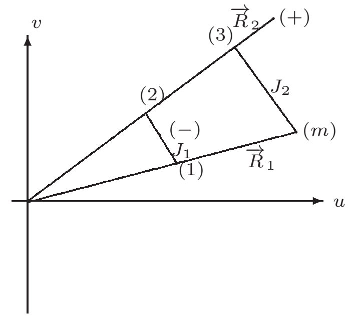

Fig. 5. Wave curves when $\nu_+ > \nu_m > \nu_- > 0$.

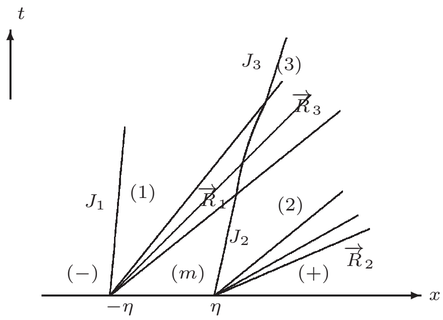

Fig. 6. $J + \overrightarrow{R}$ and $J + \overrightarrow{R}$.

From (24), we know that J_2 will pass through $\overrightarrow{R}_1$ completely in the finite time. After the completion of the penetration, we denote the contact discontinuity with J_3 and $\mu_3 = \psi(\nu_-) = \psi(\nu_1)$, which shows that J_3 is parallel to J_1. Since $\psi(\nu_1) = \psi\nu_3)$ and $\psi(\nu_m) = \psi(\nu_2)$, we know that the propagation rate of $\overrightarrow{R}_1$ keeps unchanged.

The solution is $(u_-, v_-) + J_1 + (u_1, v_1) + J_3 + (u_3, v_3) + \overrightarrow{R}_3 + (u_2, v_2) + \overrightarrow{R}_2 + (u_+, v_+)$. As $\eta \to 0$, J_1 and J_3 will coincide with each other and the two rarefaction waves $\overrightarrow{R}_3$ and $\overrightarrow{R}_2$ will coalesce into one.

Theorem 3.2 When $J+\overrightarrow{R}$ intersects with $J+\overrightarrow{R}$, we observe that the perturbed solution is still $J + \overrightarrow{R}$.

Case 3. When $J + \overrightarrow{S}$ intersects with $J + \overrightarrow{R}$.

Since $\nu_- > \nu_m > 0$ and $\nu_+ > \nu_m > 0$ (Fig. 7. and Fig. 9.). Similar discussions as Case 1, the interaction of $\overrightarrow{S}$ and J_2 results in a new $\overrightarrow{S}_1$ and a new J_3.

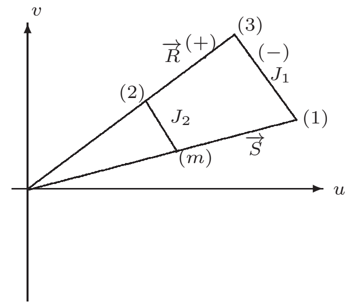

Fig. 7. Wave curves in (u, v) for Subcase 3.1.

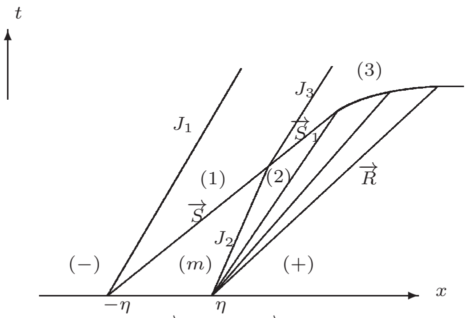

Fig. 8. $J+\overrightarrow{S}$ and $J+\overrightarrow{R}$, $\nu_->\nu_+>\nu_m>0$.

$\overrightarrow{S}_1$ must overtake $\overrightarrow{R}$ at the intersection point (x_2,t_2) which satisfies

$$\begin{cases} x_2-x_1=\omega_1(t_2-t_1), \\ x_2-\eta=[\psi(\nu_m)+\nu_m\psi_\nu(\nu_m)]t_2, \end{cases} \tag{27}$$

ω_1 is the velocity of $\overrightarrow{S}_1$, which yields

$$(x_2,t_2)=\left(\eta+\frac{2[\psi(\nu_m)+\nu_m\psi_\nu(\nu_m)]}{\omega_1-\mu_2}\eta,\frac{2\eta}{\omega_1-\mu_2}\right).$$

When $t>t_2$, $\overrightarrow{S}_1$ begins to penetrate $\overrightarrow{R}$ and

$$\begin{cases} \frac{dx}{dt}=\frac{r\psi(\nu)-\nu_-\psi(\nu_-)}{\nu-\nu_-}=\frac{\nu\psi(\nu)-\nu_3\psi(\nu_3)}{\nu-\nu_3}, \\ x-\eta=[\psi(\nu)+\nu\psi_\nu(\nu)]t, \\ \frac{u}{v}=\frac{u_2}{v_2}=\frac{u_+}{v_+}, \\ x(t_2)=x_2. \end{cases} \tag{28}$$

Since

$$\begin{aligned} \frac{dx}{dt}&=\psi(\nu)+\nu\psi_\nu(\nu)+[2\psi'(\nu)+\nu\psi''(\nu)]t(a\frac{du}{dt}+b\frac{dv}{dt}), \\ &=\frac{\nu\psi(\nu)-\nu_3\phi(\nu_3)}{\nu-\nu_3}, \end{aligned} \tag{29}$$

and $\frac{du}{dt}=\frac{u_2}{v_2}\frac{dv}{dt}$, we know

$$a\frac{du}{dt}+b\frac{dv}{dt}=\frac{1}{[2\psi'(\nu)+\nu\psi''(\nu)]t}[\frac{\psi(\nu)-\psi(\nu_3)}{\nu-\nu_3}\nu_3-\nu\psi_\nu(\nu)],$$

it follows that

$$\frac{du}{dt}=\frac{\frac{\psi(\nu)-\psi(\nu_3)}{\nu-\nu_3}\nu_3-\nu\psi_\nu(\nu)}{[2\psi'(\nu)+\nu\psi''(\nu)](a+b\frac{v_2}{u_2})t}.$$

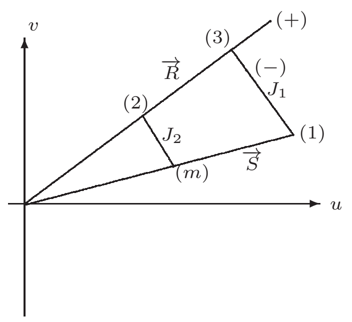

Fig. 9. Wave curves in (u,v) for Subcase 3.2.

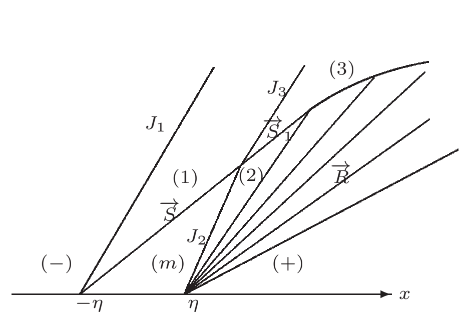

Fig. 10. $J+\overrightarrow{S}$ and $J+\overrightarrow{R}$, $\nu_+>\nu_->\nu_m>0$.

Subcase 3.1. $\nu_->\nu_+$.

For this subcase, it holds $\nu_->\nu_+>\nu_m>0$ (Fig. 7.). When $t>t_3$, the solution is (Fig. 8.)

$$(u_-,v_-)+J_1+(u_1,v_1)+J_3+(u_3,v_3)+\overrightarrow{S}_1+(u_+,v_+).$$

Subcase 3.2. $\nu_-<\nu_+$.

From $\nu_+>\nu_->\nu_m>0$ (Fig. 9.) we know $\overrightarrow{S}_1$ cannot penetrate $\overrightarrow{R}$ completely. The solution is (Fig. 10.)

$$(u_-,v_-)+J_1+(u_1,v_1)+J_3+(u_3,v_3)+\overrightarrow{R}+(u_+,v_+).$$

Theorem 3.3 When $J+\overrightarrow{S}$ intersects with $J+\overrightarrow{R}$, we observe that as $\nu_->\nu_+$, the perturbed solution is $J+\overrightarrow{S}$ as the perturbation parameter tends to zero, as $\nu_-<\nu_+$, the perturbed solution is $J+\overrightarrow{S}$.

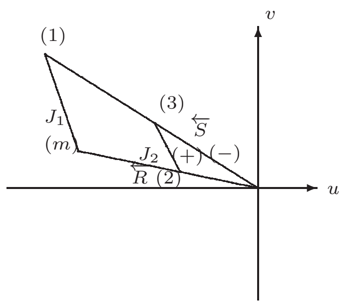

Fig. 11. Wave curves in (u,v) for Subcase 4.1.

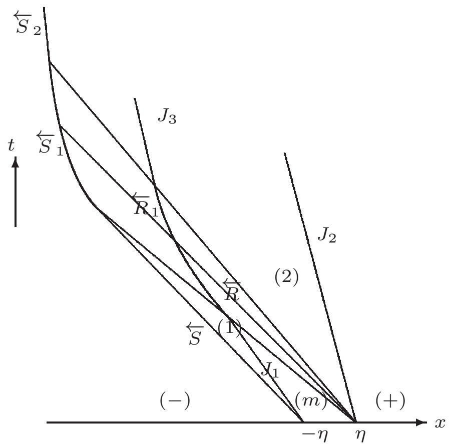

Fig. 12. $\overleftarrow{S}+J$ and $\overleftarrow{R}+J$, $\nu_m<\nu_+<\nu_-<0$.

Case 4. When $\overleftarrow{S}+J$ intersects with $\overleftarrow{R}+J$.

Since $\nu_m<\nu_-<0$ and $\nu_m<\nu_+<0$ is satisfied. J_1 emitting from $(-\eta,0)$ will catch up with $\overleftarrow{R}$ and begin to penetrate it. Like as the arguments in Case 2, J_1 will penetrate $\overleftarrow{R}$ completely. And J_3 is parallel to J_2 emitting from $(\eta,0)$.

Now we turn to study the intersection between $\overleftarrow{S}$ with $\overleftarrow{R}_1$. Denote their first intersection point is (x_1, t_1) which is determined by

$$\begin{cases} x_1 + \eta = \frac{\nu_m \psi(\nu_m) - \nu_- \psi(\nu_-)}{\nu_m - \nu_-} t_1, \\ x_1 - \eta = [\psi(\nu_m) + \nu_m \psi_\nu(\nu_m)] t_1. \end{cases}$$

It yields that

$$\begin{cases} x_1 = \frac{2(\nu_m \psi(\nu_m) - \nu_- \psi(\nu_-))\eta}{\nu_m \psi(\nu_m) - \nu_- \psi(\nu_-) - [\psi(\nu_m) + \nu_m \psi_\nu(\nu_m)](\nu_m - \nu_-)} - \eta, \\ t_1 = \frac{2\eta}{\frac{\nu_m \psi(\nu_m) - \nu_- \psi(\nu_-)}{\nu_m - \nu_-} - [\psi(\nu_m) + \nu_m \psi_\nu(\nu_m)]}. \end{cases}$$

When $t > t_1$, $\overleftarrow{S}$ begins to penetrate $\overleftarrow{R}_1$ and

$$\begin{cases} \frac{dx}{dt} = \frac{\nu \psi(\nu) - \nu_- \psi(\nu_-)}{\nu - \nu_-}, \\ x - \eta = [\psi(\nu) + \nu \psi_\nu(\nu)] t, \\ \frac{u}{v} = \frac{u_1}{v_1} = \frac{u_3}{v_3}, \\ x(t_1) = x_1. \end{cases} \tag{30}$$

Similar discussions with the second part of Case 3, we proceed as follows.

Subcase 4.1. $\nu_- > \nu_+$.

For this subcase, we have $\nu_m < \nu_+ < \nu_- < 0$ (Fig. 11.). The shock wave $\overleftarrow{S}$ will cross the whole of $\overleftarrow{R}_1$ completely.

When t is large enough, the solution is (Fig. 12.)

$$(u_-, v_-) + \overleftarrow{S}_1 + (u_3, v_3) + J_3 + (u_2, v_2) + J_2 + (u_+, v_+).$$

Subcase 4.2. $\nu_- < \nu_+$.

Here we have $\nu_m < \nu_- < \nu_+ < 0$ (Fig. 13.), the solution is (Fig. 14.)

$$(u_-, v_-) + \overleftarrow{R}_1 + (u_3, v_3) + J_3 + (u_2, v_2) + J_2 + (u_+, v_+).$$

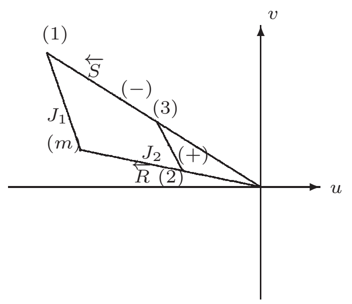

Fig. 13. Wave curves in (u, v) for Subcase 4.2.

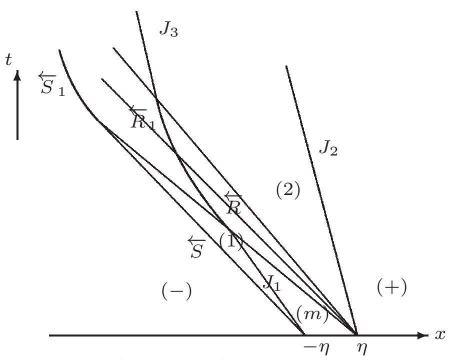

Fig. 14. $\overleftarrow{S} + J$ and $\overleftarrow{R} + J$, $\nu_m < \nu_- < \nu_+ < 0$.

Theorem 3.4 When $\overleftarrow{S} + J$ intersects with $\overleftarrow{R} + J$, it follows that as $\nu_- > \nu_+$, the perturbed solution is $\overleftarrow{S} + J$ as the perturbation parameter tends to zero, as $\nu_- < \nu_+$, the perturbed solution is $\overleftarrow{R} + J$.

IV. Conclusion

Now we have finished the discussions of the perturbed initial value problem (1) and (6). And we find that the structures of the perturbed solutions are much more simple due to the fact that (1) belongs to the Temple types. We summarize our main results in this paper.

Theorem 4.1 The perturbed Riemann solutions of (1) and (6) converge to the corresponding cases of (1) and (2) as $\eta \to 0$. Therefore, we observe that the Riemann solutions of (1) and (2) are stable.

It is important to study the elementary wave interactions of (1), since it is significant in the practical applications and enlightening for the general mathematical theory for the hyperbolic equations. In the future study, we will probe into the perturbed solutions which is added the perturbation source term.

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