Adjacent Vertex Reducible Total Labeling of Graphs

Li Wang, Jingwen Li, Lijing Zhang

Abstract—**The Adjacent Vertex Reducible Total Labeling (AVRTL) of a graph $G(V,E)$ is a bijection from $V(G) \cup E(G)$ to the set of consecutive integers $\{1,2,\cdots,|V(G)|+|E(G)|\}$, and the sum of the labels is the same for any two adjacent vertices in the graph of the same degree, being the label of that vertex plus the labels of all associated edges of that vertex. Combining with real-life problems, a novel algorithm is designed by drawing on the ideas of traditional intelligent algorithms such as genetic algorithms and bee colony algorithms. The algorithm labels the vertices and edges of random graphs with the help of preprocessing function, adjustment function, backward function, etc., in a circular, iterative merit-seeking way and discriminates whether there is an Adjacent Vertex Reducible Total Labeling for all non-isomorphic graphs within finite vertices. By analyzing and summarizing the experimental results, the labeling rules of road graphs, circle graphs, star graphs, wheel graphs, fan graphs, friendship graphs and several joint graphs were obtained, the relevant theorems were summarized, and the corresponding proofs were given.**

Index Terms—**joint graphs, reducible total labeling, labeling algorithm, graph labeling**

I. INTRODUCTION

GRAPH theory [1] originated from the Seven Bridges of Konigsberg problem proposed in the 18th century, a branch of mathematics that uses graphs as objects for related research. Since 1736, when the Swiss mathematician Euler solved the Seven Bridges of Konigsberg problem and created a new branch of mathematics, graph theory has been studied by many scholars and developed continuously. Graph labeling is one of the important branches in graph theory, starting from a conjecture proposed by Ringel [2] in 1963: any tree is graceful. In the decades following the Graceful Tree Conjecture, researchers have launched on the study and research of graph labeling problems. Later, some scholars proposed the concepts of Graceful Labeling, (p,1)–Total Labeling and Magic Labeling, and the literature [3] introduced the concept of Vertex Magic Total Labeling. Through the study of special graphs and some more easily inscribed graphs, numerous researchers have published many research results on Vertex Magic Total Labeling.

In 2002, the idea of Adjacent Strong Edge Coloring of graphs was put up by Professor Zhongfu Zhang et al. [4], which satisfies that the color sets of any adjacent vertices are different on the basis of proper edge coloring. And later, Zhu et al. further studied and explored Distinguishable Coloring [5]–[7]. Based on the theory of Distinguishable Coloring, Prof. Zhongfu Zhang and colleagues proposed the idea of Reducible Coloring in 2009 [8], and whether graphs have reducible coloring has become one of the research hotspots of great interest. However, the research results in this area are still relatively few [9]–[10].

Suppose the weights of nodes and edges in a transportation network stand for a vehicle site's vehicle capacity and the amount of transit between sites, respectively. Then the Adjacent Vertex Reducible Total Labeling model may be used to depict a real-life situation where neighboring stations of the same degree are required to maintain about equal capacity, and the diversity of connecting roadways between stations reaches an extreme value. This paper proposes the concept of Adjacent Vertex Reducible Total Labeling based on the research of previous scholars, designs an AVRTL algorithm to solve the Adjacent Vertex Reducible Total Labeling of random graphs within finite points, and obtains some related theorems and gives proofs by analyzing and summarizing the experimental results.

II. BASIC KNOWLEDGE

In this paper, $G(p,q)$ represents a simple connected graph with p vertices and q edges. The central node of the fan graph F_m is v_0 and contains m fan vertices. W_n is a wheel graph with a total of $n+1$ vertices, and the central node u_0 is adjacent to the remaining n vertices.

Definition 1: Assume that $G(V,E)$ is a simple undirected connected graph, if there exists a bijection f: $V(G) \cup E(G) \to \{1,2,\cdots,|V|+|E|\}$, for any two adjacent vertices $uv \in E(G)$, if $d(u)=d(v)$, we get $S(u)=S(v)$, where $S(u)=f(u)+\sum_{uw\in E(G)} f(uw)$, $d(u)$ denotes the degree of vertex u, then f is said to be the Adjacent Vertex Reducible Total Labeling of the graph G, or AVRTL for short, and graph G is an AVRTL graph. If a graph does not have an AVRTL, it is said to be a NAVRTL graph.

Definition 2: The graph $T_{(m,n)}(n \geq 1, m \geq 3)$ denotes the graph formed by joining n C_m circle graphs with common vertex u_0 as shown in Figure 1(a). The joint graph formed by

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bonding the common vertex u_0 of $T_{(m,n)}$ to the circle graph C_r is $T_{(m,n)} \uparrow C_r$.

Definition 3: Suppose the vertex set of the graph G_n is $\{u_1, u_2, \cdots, u_n\}$, the vertex set of the graph S_m is $\{v_1, v_2, \cdots, v_m\}$, and the central vertex of graph S_m is denoted by v_0. $S_m^{G_n}$ denotes the graph obtained by connecting all the m vertices in the star graph except the star center with a G_n, as illustrated in Figure 1(b).

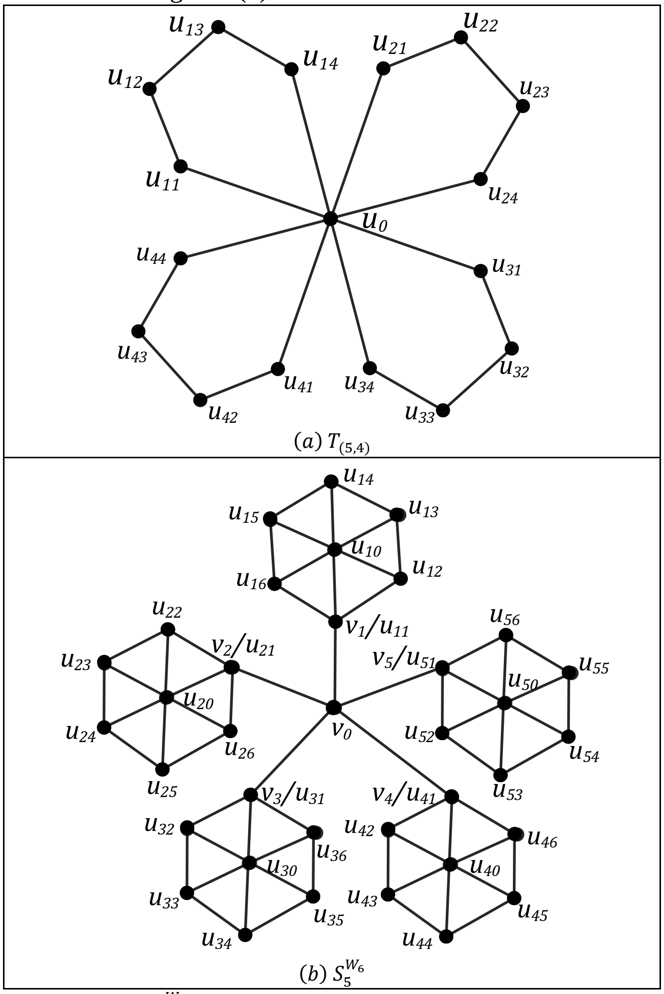


Fig 1. $T_{(5,4)}$ *and* $S_5^{W_6}$

Definition 4: The joint graph $F_n \uparrow C_l \uparrow S_m$ is the graph obtained by connecting the central vertex v_0 of the fan graph F_n with any vertex of C_l, and then connecting the connection vertex to the central vertex of the star graph S_m, $|V(F_n \uparrow C_l \uparrow S_m)| = |V(F_n)| + |V(C_l)| + |V(S_m)| - 2$, $|E(F_n \uparrow C_l \uparrow S_m)| = |E(F_n)| + |E(C_l)| + |E(S_m)|$.

Definition 5: The joint graph $S_m \uparrow F_n \downarrow P_t$ is obtained by merging the center vertex of the star graph S_m with the center vertex of the fan graph F_n, setting the merged vertex as u_0, and then merging any vertex other than u_0 with any vertex of the road graph P_t.

Definition 6: The graph $I_r(P_m)$ denotes the r-crown graph of the road graph P_m and is the graph obtained by bonding r hanging edges at each vertex of the road graph P_m.

Definition 7: $G_{m,n}$ denotes the graph formed by joining the central vertices of graph G_m and G_n with the common vertex u_0, as shown in Figure 2(a). Then the joint graph obtained by bonding u_0 of graph $G_{m,n}$ to a road graph P_t is called $G_{m,n} \uparrow P_t$.

Definition 8: $F_n^{(2)}$ denotes the graph formed by bonding the central vertices of two fan graphs F_n, and the common vertex of this joint graph is u_0.

Definition 9: The tiller graph H_n is the graph obtained by adding a hanging edge to each vertex u_i of W_n except for the central vertex u_0, as shown in Figure 2(b).

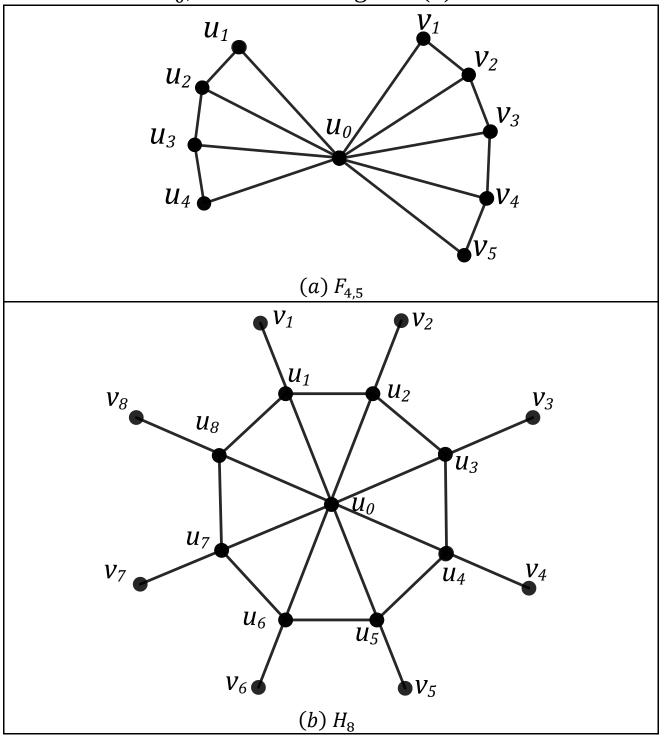


Fig 2. $F_{4,5}$ *and* H_8

III. Algorithm

A. *Fundamentals of the algorithm*

The restrictions are built in accordance with the definition of the Adjacent Vertex Reducible Total Labeling, drawing on the design principles of conventional intelligent algorithms. The vertices and edges of graph $G(p, q)$ are labeled using an iterative loop once all vertices have been classified by degree and whether or not they are adjacent. Finally, the labeling matrix satisfying the requirements is obtained to finish the labeling process of the graph.

Step 1: The preprocessing function reads the graph set file to get the graph's adjacency matrix, as well as the necessary attribute values, like the number of vertices and edges and the set of nearby vertices with the same degree.

Step 2: The construction of the adjustment function, according to the adjustment criteria, determines the current matrix of all positions in the adjustment order, as well as the adjustment span of the label value (generally set to 1).

Step 3: Determine whether the difference between the sum of labels of adjacent points of the same degree is greater than 2 after adjustment or whether the attribute values in the set of labels are discontinuous. The matrix that meets the requirement is recorded as the middle matrix $MidAdjust$.

Step 4: If the label's value in the matrix reaches the maximum or the intermediate matrix does not change, the final intermediate matrix is recorded as the final matrix, and the cycle ends.

Step 5: Output the final label matrix.

B. Pseudocode

Input	The adjacency matrix of the graph $G(p,q)$
Output	The matrix satisfying the labeling requirements
1	read the adjacency matrix $initAdjust$ of the graph G
2	Calculate the required properties, such as the number of vertices, the number of edges, and the set of adjacent vertices of the same degree.
3	$get\ FinaMartix, flag = true$
4	$while(flag)$
5	$for\ i \leftarrow 0\ to\ VertexNum$
6	$ei++$
7	if (The backward function is satisfied)
8	$ei--$
9	$end\ if$
10	if (The current matrix satisfies the equilibrium condition)
11	$MidAdjust_i \leftarrow currentAdjust$
12	$end\ if$
13	$if(ei == VerNum + EdgeNum \mid\mid equalFun(MidAdjust_i, MidAdjust_{i+1})$
14	$FinalAdjust \leftarrow MidAdjust_i$
15	$break$
16	$end\ if$
17	$end\ for$
18	$if(FinalAdjust.maxLabel == p + q)$
19	$Output\ FinalAdjust$
20	$end\ if$
21	$end\ while$
22	end

C. Analysis of the findings from the experiment

All non-isomorphic graphs within 6-10 vertices were found for the Adjacent Vertex Reducible Total Labeling using this algorithm, and the experimental results were analyzed to produce the following line graph. It can be observed from this figure that the number of AVRTL graphs within finite vertices consistently accounts for more than half of the total number of graphs.

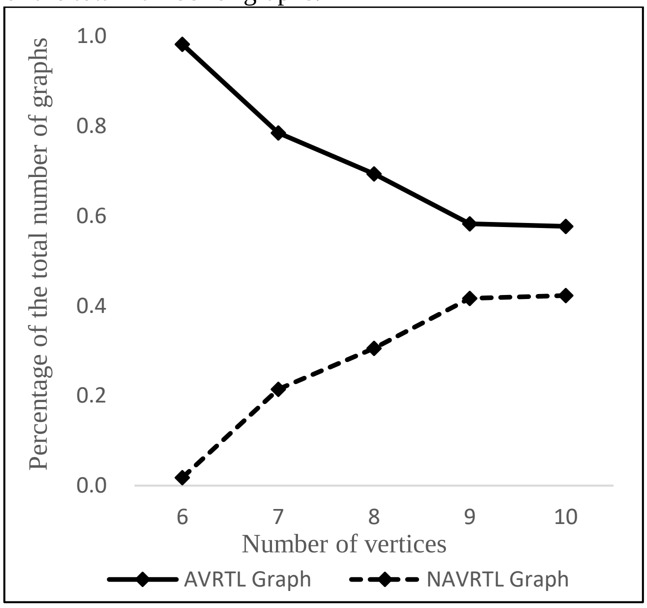
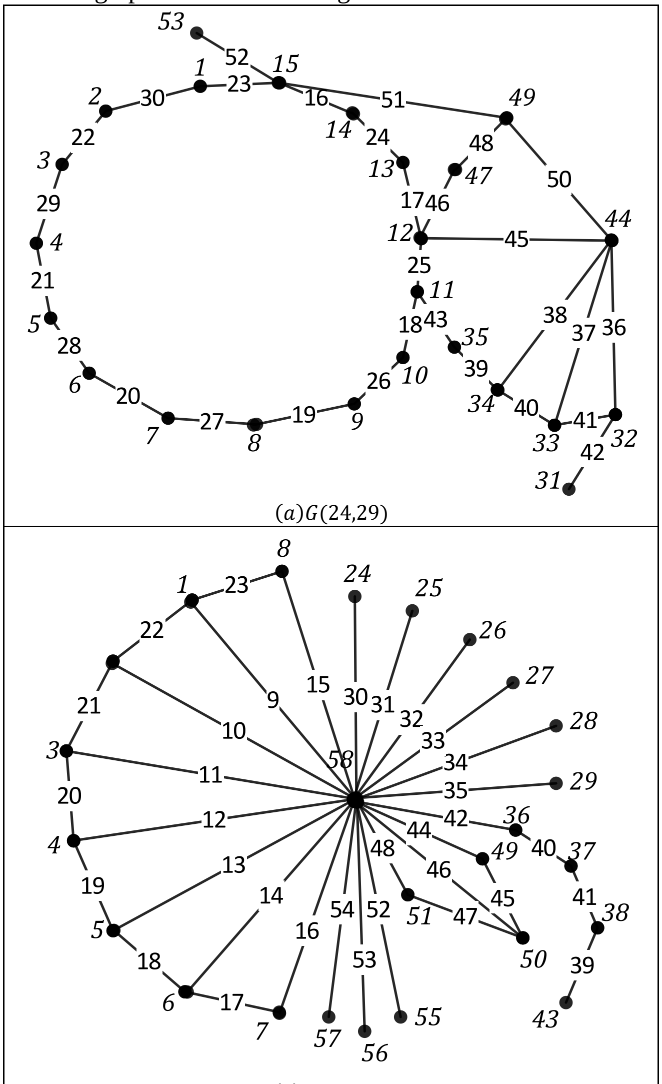


Fig 3. Trends in the percentage of AVRTL graphs versus NAVRTL graphs in the total number of graphs within 6-10 vertices

Adjacent Vertex Reducible Total Labeling results of two random graphs are shown in Figure 4.



$(a)G(24,29)$

$(b)\ G(25,33)$

Fig 4. Labeling results of two random graphs

IV. Theorem and Proof

Theorem 1: AVRTL exists for the road graph P_n when $n \geq 3$.

Proof: Let $\{u_1, u_2, \cdots, u_n\}$ be the set of P_n's vertices, with u_1 and u_n located at the two ends of P_n. P_n contains a total of n vertices and $n-1$ edges, and the following mapping can be used to get f.

$$f(u_i) = \begin{cases} i, 1 \le i \le n-1 \\ 2n-1, i = n \end{cases};$$

$$f(u_i u_{i+1}) = \begin{cases} 2n - \frac{i}{2} - \frac{3}{2}, i \equiv 1 (\mathrm{mod}\ 2) \\ 2n - \left\lfloor \frac{n}{2} \right\rfloor - \frac{i}{2} - 1, i \equiv 0 (\mathrm{mod}\ 2) \end{cases} \quad 1 \le i \le n-1$$

At this point, $f(V(G)) \cup f(E(G)) \rightarrow [1, 2n-1]$, $f(V(G)) \cap f(E(G)) = \emptyset$, and $S(u_i) = f(u_i u_{i+1}) + f(u_i) + f(u_i u_{i-1}) = 4n - \lfloor \frac{n}{2} \rfloor - 2, 2 \le i \le n-1$, satisfying the definition of Adjacent Vertex Reducible Total Labeling, so Theorem 1 holds.

Theorem 2: AVRTL exists for the circle graph $C_n (n \geq 3)$ when $n \equiv 1(mod\ 2)$.

Proof: Let the set of vertices of C_n be $\{u_1, u_2, \cdots, u_n\}$, where u_1 is a vertex adjacent to both u_n and u_n. Given that C_n contains n vertices and n edges, we can get the following mapping about f.

$$f(u_i) = i, 1 \le i \le n;$$

$$f(u_1 u_n) = 2n - \left\lfloor \frac{n}{2} \right\rfloor;$$

$$f(u_i u_{i+1}) = \begin{cases} 2n - \dfrac{i}{2} + \dfrac{1}{2}, i \equiv 1 (\bmod 2) \\ 2n - \left\lfloor \dfrac{n}{2} \right\rfloor - \dfrac{i}{2}, i \equiv 0 (\bmod 2) \end{cases} \quad 1 \le i < n$$

At this point, $f(V(G)) \cup f(E(G)) \to [1,2n]$, $f(V(G)) \cap f(E(G)) = \emptyset$, and $S(u_i) = f(u_i u_{i+1}) + f(u_i u_{i-1}) + f(u_i) = 4n - \lfloor\frac{n}{2}\rfloor + 1, 1 \le i \le n$, satisfying the definition of Adjacent Vertex Reducible Total Labeling, so Theorem 2 holds.

Theorem 3: If the graph S_m represents a star graph with $m + 1 (m \ge 2)$ vertices, then AVRTL exists for S_m.

Theorem 3 obviously holds according to the definition of Adjacent Vertex Reducible Total Labeling.

Theorem 4: If the graph F_n represents a fan graph with $n + 1 (n > 3)$ vertices, then AVRTL exists for F_n.

Proof: Let the center of the fan be u_0, and the set of vertices for F_n is $\{u_0, u_1, \cdots, u_n\}$. F_n contains $n + 1$ vertices and $2n - 1$ edges in total, and a mapping about f can be obtained as follows.

$$f(u_i) = \begin{cases} 3n, i = 0 \\ 3n - 1, i = 1 \\ i - 1, 2 \le i \le n - 1 \\ 3n - 2, i = n \end{cases};$$

$$f(u_0 u_i) = \begin{cases} 3n - 3, i = 1 \\ n + i - 3, 2 \le i \le n - 1; \\ 3n - 4, i = n \end{cases}$$

$$f(u_i u_{i+1}) = 3n - i - 4, 1 \le i \le n - 1$$

At this point, $f(V(G)) \cup f(E(G)) \to [1,3n]$, $f(V(G)) \cap f(E(G)) = \emptyset$, and $S(u_i) = f(u_i u_{i+1}) + f(u_i u_{i-1}) + f(u_i u_0) + f(u_i) = 7n - 11, 2 \le i \le n - 1$, satisfying the definition of Adjacent Vertex Reducible Total Labeling, so Theorem 4 holds.

Theorem 5: If the graph W_n represents a wheel graph with $n + 1 (n > 3)$ vertices, then AVRTL exists for W_n.

Proof: Let the set of vertices of W_n be $\{u_0, u_1, \cdots, u_n\}$, and the center of the wheel is u_0. The vertex u_1 is adjacent to both u_2 and u_n, W_n has a total of $n + 1$ vertices and $2n$ edges, and a mapping about f can be obtained as follows.

$$f(u_i) = \begin{cases} 3n + 1, i = 0 \\ i, 1 \le i \le n \end{cases};$$

$$f(u_0 u_i) = \begin{cases} 2n + i + 1, 1 \le i < n \\ 2n + 1, i = n \end{cases};$$

$$f(u_1 u_n) = 2n;$$

$$f(u_i u_{i+1}) = 2n - i, 1 \le i < n$$

At this point, $f(V(G)) \cap f(E(G)) = \emptyset$, $f(V(G)) \cup f(E(G)) \to [1,3n+1]$, and $S(u_i) = f(u_i) + f(u_i u_{i+1}) + f(u_i u_{i-1}) + f(u_i u_0) = 6n + 2, 1 \le i \le n$, satisfying the definition of Adjacent Vertex Reducible Total Labeling, so Theorem 5 holds.

Theorem 6: AVRTL exists for the joint graph $T_{(3,n)} \uparrow C_m$ when $n \ge 1, m \ge 3$.

Proof: Let the set of vertices of $T_{(3,n)} \uparrow C_m$ be $\{u_1, u_2, \cdots, u_{2n}\} \cup \{v_0, v_1, \cdots, v_{m-1}\}$, and the vertex v_0 is adjacent to the vertices v_1, v_{m-1} *and* $u_i (1 \le i \le 2n)$. $T_{(3,n)} \uparrow C_m$ has a total of $2n + m$ vertices and $3n + m$ edges, as shown in Figure 5.

At this point, in the vertex sets $\{u_1, u_2\}, \{u_3, u_4\}, \cdots, \{u_{2n-1}, u_{2n}\}$, any two elements in each set are adjacent and of the same degree, and all are points of degree 2; any two elements of the vertex set $\{v_1, v_2, \cdots, v_{m-1}\}$ are adjacent and of the same degree, both of degree 2 vertices, and a mapping about f can be obtained as follows.

$$f(u_i) = \begin{cases} \dfrac{5}{2} i - \dfrac{3}{2}, i \equiv 1 (\bmod 2) \\ \dfrac{5}{2} i - 3, i \equiv 0 (\bmod 2) \end{cases} \quad 1 \le i \le n;$$

$$f(v_i) = \begin{cases} 5n + i, 1 \le i \le m - 1 \\ 5n + 2m, i = 0 \end{cases};$$

$$f(v_0 u_i) = \begin{cases} \dfrac{5}{2} i + \dfrac{3}{2}, i \equiv 1 (\bmod 2) \\ \dfrac{5}{2} i - 2, i \equiv 0 (\bmod 2) \end{cases} \quad 1 \le i \le n;$$

$$f(u_i u_{i+1}) = \frac{5}{2} i + \frac{5}{2}, i \equiv 1 (\bmod 2), 1 \le i \le n - 1;$$

$$f(v_0 v_{m-1}) = \begin{cases} 5n + m, m \equiv 0 (\bmod 2) \\ 5n + \dfrac{3}{2} m - \dfrac{1}{2}, m \equiv 1 (\bmod 2) \end{cases};$$

$$f(v_i v_{i-1}) = \begin{cases} 5n + 2m - \dfrac{i}{2} - \dfrac{1}{2}, i \equiv 1 (\bmod 2) \\ 5n + 2m - \left\lceil \dfrac{m}{2} \right\rceil - \dfrac{i}{2}, i \equiv 0 (\bmod 2) \end{cases} \quad 1 \le i \le m - 1$$

At this time, $f(V) = \{1, 2\} \cup \{6, 7\} \cup \cdots \cup \{5n - 4, 5n - 3\} \cup \{5n + 1, 5n + 2, \cdots, 5n + m - 1\} \cup \{5n + 2m\}$ and $f(E) = \{3, 4, 5\} \cup \{8, 9, 10\} \cup \cdots \cup \{5n - 2, 5n - 1, 5n\} \cup \{5n + m, 5n + m + 1, \cdots, 5n + 2m - 1\}$, which gives $f(V(G)) \cup f(E(G)) \to [1,5n + 2m]$ and $f(V(G)) \cap f(E(G)) = \emptyset$.

And we have $S(u_i) = f(u_i) + f(u_i u_{i+1}) + f(u_i u_0) = \frac{15}{2} i + \frac{5}{2}, i \equiv 1 (mod\ 2),\ 1 \le i \le n - 1;\ S(u_i) = f(u_i) + f(u_i u_0) + f(u_i u_{i-1}) = \frac{15}{2} i - 5, i \equiv 0 (mod\ 2), 2 \le i \le n$, we can get that in the vertex sets $\{u_1, u_2\}, \{u_3, u_4\}, \cdots, \{u_{n-1}, u_n\}$, the sum of the labels of all elements in each set is the same. $S(v_{m-1}) = f(v_{m-1} v_{m-2}) + f(v_{m-1}) + f(v_{m-1} v_0) = 15n + 4m - \left\lceil\frac{m}{2}\right\rceil - 1;\ S(v_i) = f(v_i v_{i+1}) + f(v_i) + f(v_i v_{i-1}) = 15n + 4m - \left\lceil\frac{m}{2}\right\rceil - 1, 1 \le i \le m - 2$, which give the same sum of labels of all elements in the set $\{v_1, v_2, \cdots, v_{m-1}\}$. The vertex v_0 has no adjacent vertices of the same degree, so it is not necessary to consider the sum of its labels.

To sum up, Theorem 6 holds.

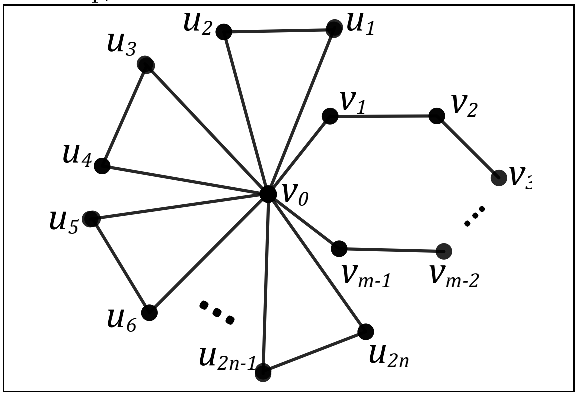


Fig 5. $T_{(3,n)} \uparrow C_m$

Theorem 7: AVRTL exists for the joint graph $F_n^{(2)}$ when $n \geq 4$.

Proof: Let the set of vertices of $F_n^{(2)}$ be $\{u_0, u_1, \cdots, u_n\} \cup \{v_1, v_2, \cdots, v_n\}$, $F_n^{(2)}$ has a total of $2n+1$ vertices and $4n-2$ edges, as shown in Figure 6.

At this point, in the vertex sets $\{u_2, u_3, \cdots, u_{n-1}\}$, $\{v_2, v_3, \cdots, v_{n-1}\}$, any two elements in each set are adjacent and of the same degree, both of degree 3 vertices, and a mapping about f can be obtained as follows.

$$f(u_i) = \begin{cases} 3n-i, 0 \le i \le 1 \\ i-1, 2 \le i \le n-1; \\ 3n-2, i=n \end{cases}$$

$$f(v_i) = \begin{cases} 6n-1, i=1 \\ 3n+i-1, 2 \le i \le n-1; \\ 6n-2, i=n \end{cases}$$

$$f(u_0u_i) = \begin{cases} 3n-3, i=1 \\ n+i-3, 2 \le i \le n-1; \\ 3n-4, i=n \end{cases}$$

$$f(u_iu_{i+1}) = 3n-i-4, 1 \le i \le n-1;$$

$$f(u_0v_i) = \begin{cases} 6n-3, i=1 \\ 4n+i-3, 2 \le i \le n-1; \\ 6n-4, i=n \end{cases}$$

$$f(v_iv_{i+1}) = 6n-i-4, 1 \le i \le n-1$$

At this time, $f(V) = \{1,2,\cdots,n-2\} \cup \{3n-2, 3n-1, \cdots, 4n-2\} \cup \{6n-2, 6n-1\}$ and $f(E) = \{n-1, n, \cdots, 3n-3\} \cup \{4n-1, 4n, \cdots, 6n-3\}$, which gives $f(V(G)) \cup f(E(G)) \to [1, 6n-1]$ and $f(V(G)) \cap f(E(G)) = \emptyset$.

And $S(v_i) = f(v_iu_0) + f(v_iv_{i+1}) + f(v_iv_{i-1}) + f(v_i) = 7n-11, 2 \le i \le n-1$, which gives the same sum of labels of all elements in the set $\{v_2, v_3, \cdots, v_{n-1}\}$. $S(u_i) = f(u_i) + f(u_iu_{i+1}) + f(u_iu_{i-1}) + f(u_iu_0) = 19n-11, 2 \le i \le n-1$, which gives the same sum of labels of all elements in the set $\{u_2, u_3, \cdots, u_{n-1}\}$. The rest of the vertices have no adjacent vertices of the same degree, so it is not necessary to consider the sum of their labels.

To sum up, Theorem 7 holds.

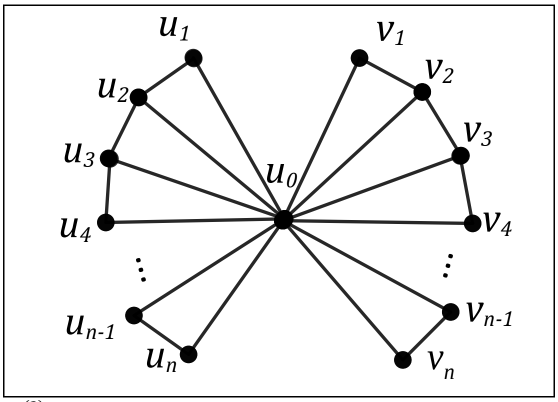


Fig 6. $F_n^{(2)}$

Theorem 8: AVRTL exists for the joint graph $W_{m,n} \uparrow P_t$ when $m = n, m \geq 3, t \geq 4$.

Proof: Let the set of vertices of $W_{m,n} \uparrow P_t$ be $\{u_0, u_1, \cdots, u_{m-1}\} \cup \{v_0, v_1, \cdots, v_{n-1}\} \cup \{w_1, w_2, \cdots, w_t\}$, and $W_{m,n} \uparrow P_t$ has a total of $n+m+t$ vertices and $2n+2m+t-1$ edges, as shown in Figure 7.

Scenario 1: When $n = m = 3, t \geq 4$

At this point, in the vertex sets $\{u_0, u_1, \cdots, u_{m-1}\}$, $\{v_1, v_2, \cdots, v_{n-1}\}$, any two elements in each set are adjacent and of the same degree, and all are points of degree 3; any two elements of the vertex set $\{w_2, w_3, \cdots, w_{t-1}\}$ are adjacent and of the same degree, both of degree 2 vertices, and a mapping about f can be obtained as follows.

$$f(u_i) = \begin{cases} 1, i=0 \\ 2m+1, i=1; \\ m, i=2 \end{cases}$$

$$f(v_i) = \begin{cases} 3m+1, i=0 \\ 3m+2n+1, i=1; \\ 3m+n, i=2 \end{cases}$$

$$f(w_i) = \begin{cases} 3m+3n+i, 1 \le i \le t-1 \\ 3m+3n+2t-1, i=t \end{cases};$$

$$f(u_1u_m) = 2m;$$

$$f(u_0u_i) = \begin{cases} m-1, i=1 \\ 2m+i, 2 \le i \le m \end{cases};$$

$$f(u_iu_{i+1}) = 2m-i, 1 \le i \le m-1;$$

$$f(v_1v_n) = 3m+2n;$$

$$f(v_iv_{i+1}) = 3m+2n-i, 1 \le i \le n-1;$$

$$f(v_0v_i) = \begin{cases} 3m+n-1, i=1 \\ 3m+2n+i, 2 \le i \le n \end{cases};$$

$$f(w_iw_{i+1}) = \begin{cases} 3m+3n+2t-\dfrac{i}{2}-\dfrac{3}{2}, i \equiv 1 (\bmod 2) \\ 3m+3n+2t-\left\lfloor \dfrac{t}{2} \right\rfloor - \dfrac{i}{2} - 1, i \equiv 0 (\bmod 2) \end{cases} \quad 1 \le i \le t-1$$

At this time, $f(V) = \{1,3,7,10,12,16,3m+3n+2t-1\} \cup \{3m+3n+1,\ 3m+3n+2,\ \cdots,\ 3m+3n+t-1\}$ and $f(E) = \{2,4,5,6,8,9\} \cup \{11,13,14,15,17,18\} \cup \{3m+3n+t,\ 3m+3n+t+1,\ \cdots, 3m+3n+2t-2\}$, which gives $f(V(G)) \cup f(E(G)) \rightarrow [1,3n+3m+2t-1]$ and $f(V(G)) \cap f(E(G)) = \emptyset$.

And the $S(u_1) = f(u_1u_2) + f(u_1u_m) + f(u_1u_0) + f(u_1) = 5m+5$; $S(u_0) = f(u_0) + f(u_0u_1) + f(u_0u_2) + f(u_0u_3) = 5m+5$; $S(u_i) = f(u_iu_{i-1}) + f(u_iu_{i+1}) + f(u_iu_0) + f(u_i) = 5m+5, 2 \le i \le m-1$, which give the same sum of labels of all elements in the set $\{u_0, u_1, \cdots, u_{m-1}\}$. $S(v_1) = f(v_1) + f(v_1v_2) + f(v_1v_n) + f(v_1v_0) = 12m+5n+5$; $S(v_0) = f(v_0) + f(v_0v_1) + f(v_0v_2) + f(v_0v_3) = 12m+5n+5$; $S(v_i) = f(v_iv_0) + f(v_iv_{i+1}) + f(v_iv_{i-1}) + f(v_i) = 12m+5n+5, 2 \le i \le n-1$, which give the same sum of labels of all elements in the set $\{v_0, v_1, \cdots, v_{n-1}\}$. $S(w_i) = f(w_i) + f(w_iw_{i+1}) + f(w_iw_{i-1}) = 9(m+n) + 4t - \left\lfloor \frac{t}{2} \right\rfloor - 2, 2 \le i \le t-1$, which gives the same sum of labels of all elements in the set $\{w_2, w_3, \cdots, w_{t-1}\}$. The rest of the vertices have no adjacent vertices of the same degree, so it is not necessary to consider the sum of their labels.

Therefore, AVRTL exists for $W_{m,n} \uparrow P_t (n = m = 3, t \ge 4)$.

Scenario 2: When $n = m, m \ge 4, t \ge 4$

At this point, in the vertex sets $\{u_1, u_2, \cdots, u_{m-1}\}$, $\{v_1, v_2, \cdots, v_{n-1}\}$, any two elements in each set are adjacent and of the same degree, and all are points of degree 3; any two elements of the vertex set $\{w_2, w_3, \cdots, w_{t-1}\}$ are adjacent and of the same degree, both of degree 2 vertices, and a mapping about f can be obtained as follows.

$$f(u_i) = \begin{cases} 3m, i = 0 \\ i, 1 \le i \le m-1 \end{cases};$$

$$f(v_i) = \begin{cases} 3m+3n, i = 0 \\ 3m+i, 1 \le i \le n-1 \end{cases};$$

$$f(w_i) = \begin{cases} 3m+3n+i, 1 \le i \le t-1 \\ 3m+3n+2t-1, i = t \end{cases};$$

$$f(u_1u_m) = 2m-1;$$

$$f(u_0u_i) = 2m+i-1, 1 \le i \le m;$$

$$f(u_iu_{i+1}) = 2m-i-1, 1 \le i \le m-1;$$

$$f(v_1v_n) = 3m+2n-1;$$

$$f(v_iv_{i+1}) = 3m+2n-i-1, 1 \le i \le n-1;$$

$$f(v_0v_i) = 3m+2n+i-1, 1 \le i \le n;$$

$$f(w_iw_{i+1}) = \begin{cases} 3m+3n+2t-\frac{i}{2}-\frac{3}{2}, i \equiv 1 (\mathrm{mod}\ 2) \\ 3m+3n+2t-\left\lfloor \frac{t}{2} \right\rfloor - \frac{i}{2} - 1, i \equiv 0 (\mathrm{mod}\ 2) \end{cases} \quad 1 \le i \le t-1$$

At this time, $f(V) = \{1,2, \cdots, m-1\} \cup \{3m, 3m+1, \cdots, 3m+n-1\} \cup \{3m+3n, 3m+3n+1, \cdots, 3m+3n+t-1\} \cup \{3m+3n+2t-1\}$, $f(E) = \{m, m+1, \cdots, 3m-1\} \cup \{3m+n, 3m+n+1, \cdots, 3m+3n-1\} \cup \{3m+3n+t, 3m+3n+t+1, \cdots, 3m+3n+2t-2\}$, which gives $f(V(G)) \cup f(E(G)) \rightarrow [1,3n+3m+2t-1]$ and $f(V(G)) \cap f(E(G)) = \emptyset$.

And the $S(u_i) = f(u_iu_{i+1}) + f(u_iu_{i-1}) + f(u_iu_0) + f(u_i) = 6m-2, 2 \le i \le m-1$; $S(u_1) = f(u_1u_2) + f(u_1u_m) + f(u_1u_0) + f(u_1) = 6m-2$, which give the same sum of labels of all elements in the set $\{u_1, u_2, \cdots, u_{m-1}\}$. $S(v_1) = f(v_1) + f(v_1v_2) + f(v_1v_n) + f(v_1v_0) = 12m+6n-2$; $S(v_i) = f(v_i) + f(v_iv_{i+1}) + f(v_iv_{i-1}) + f(v_iv_0) = 12m+6n-2, 2 \le i \le n-1$, which give the same sum of labels of all elements in the set $\{v_1, v_2, v_3, \cdots, v_{n-1}\}$. $S(w_i) = f(w_iw_{i+1}) + f(w_iw_{i-1}) + f(w_i) = 9m+9n+4t-\left\lfloor \frac{t}{2} \right\rfloor - 2, 2 \le i \le t-1$, which gives the same sum of labels of all elements in the set $\{w_2, w_3, \cdots, w_{t-1}\}$. The rest of the vertices have no adjacent vertices of the same degree, so it is not necessary to consider the sum of their labels.

Therefore, AVRTL exists for $W_{m,n} \uparrow P_t (n = m, m \ge 3, t \ge 4)$.

To sum up, Theorem 8 holds.

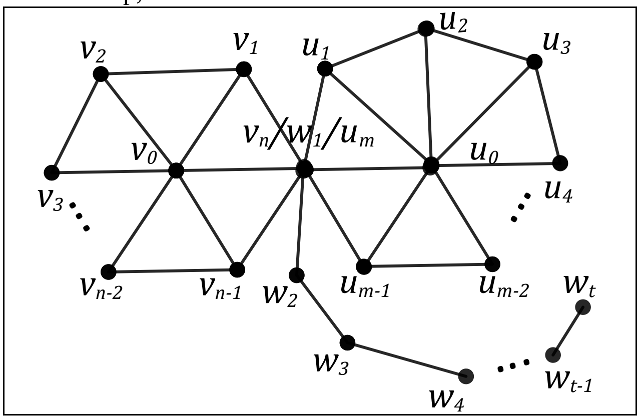


Fig 7. $W_{m,n} \uparrow P_t$

Theorem 9: When $n \equiv 1 (mod\ 2), n \ge 3$, or when $n \equiv 0 (mod\ 2), 4 \le n \le 6$, AVRTL exists for the tiller graph H_n.

Proof: Let the set of vertices of H_n be $\{u_0, u_1, \cdots, u_n\} \cup \{v_1, v_2, \cdots, v_n\}$, and the vertex u_1 is adjacent to the vertices v_1, u_0, u_2 *and* u_n. H_n has a total of $2n+1$ vertices and $3n$ edges, as shown in Figure 8.

Scenario 1: When $n \equiv 1 (mod\ 2), n \ge 3$

At this point, any two elements of the vertex set $\{u_1, u_2, \cdots, u_{n-1}, u_n\}$ are adjacent and of the same degree, both of degree 4 vertices, and a mapping about f can be obtained as follows.

$$f(u_i) = \begin{cases} 5n+1, i = 0 \\ i, 1 \le i \le n \end{cases};$$

$$f(v_i) = 4n+i, 1 \le i \le n;$$

$$f(u_1u_n) = 2n - \left\lfloor \frac{n}{2} \right\rfloor;$$

$$f(u_iv_i) = 2n+i, 1 \le i \le n;$$

$$f(u_0u_i) = 4n-i+1, 1 \le i \le n;$$

$$f(u_iu_{i+1}) = \begin{cases} 2n - \left\lfloor \frac{i}{2} \right\rfloor, i \equiv 1 (\mathrm{mod}\ 2) \\ 2n - \left\lfloor \frac{n}{2} \right\rfloor - \frac{i}{2}, i \equiv 0 (\mathrm{mod}\ 2) \end{cases} \quad 1 \le i < n;$$

At this time, $f(V) = \{1,2, \cdots, n\} \cup \{4n+1, 4n+2, \cdots, 5n+1\}$ and $f(E) = \{n+1, n+2, \cdots, 4n\}$, which gives $f(V(G)) \cup f(E(G)) \rightarrow [1,5n+1]$, $f(V(G)) \cap f(E(G)) = \emptyset$.

And $S(u_1) = f(u_1u_2) + f(u_1u_n) + f(u_1v_1) + f(u_1) + f(u_1u_0) = \frac{19}{2}n + \frac{5}{2}$; $S(u_n) = f(u_1u_n) + f(u_{n-1}u_n) + f(u_nv_n) + f(u_n) + f(u_nu_0) = \frac{19}{2}n + \frac{5}{2}$; $S(u_i) = f(u_i) + f(u_iu_{i+1}) + f(u_iu_{i-1}) + f(u_iv_i) + f(u_iu_0) = \frac{19}{2}n + \frac{5}{2}$, $2 \le i \le n-1$, which give the same sum of labels of all elements in the set $\{u_1, u_2, \cdots, u_n\}$. The rest of the vertices have no adjacent vertices of the same degree, so it is not necessary to consider the sum of their labels.

Therefore, AVRTL exists for $H_n (n \equiv 1 (mod\ 2), n \ge 3)$.

Scenario 2: When $n = 4$

At this point, any two elements of the vertex set $\{u_0, u_1, u_2, \cdots, u_n\}$ are adjacent and of the same degree, both of degree 4 vertices, and a mapping about f can be obtained as follows.

$$f(u_i) = \begin{cases} 1, i = 0 \\ 4, i = 1 \\ i, 2 \le i \le n-1 \\ 9, i = n \end{cases};$$

$$f(v_i) = 4n + i + 1, 1 \le i \le n;$$

$$f(u_1u_n) = n + 1;$$

$$f(u_iu_{i+1}) = \begin{cases} 2n+2, i = 1 \\ 2n-1, i = 2 \\ n+2, i = 3 \end{cases};$$

$$f(u_iv_i) = 3n + i + 1, 1 \le i \le n;$$

$$f(u_0u_i) = \begin{cases} 3n - i + 1, 1 \le i \le n-2 \\ 3n+1, i = n-1 \\ 2n, i = n \end{cases}$$

At this time, $f(V) = \{1, 2, \cdots, n\} \cup \{4n+2, 4n+3, \cdots, 5n+1\}$ and $f(E) = \{n+1, n+2, \cdots, 4n+1\}$, which gives $f(V(G)) \cup f(E(G)) \to [1, 5n+1]$ and $f(V(G)) \cap f(E(G)) = \emptyset$.

And $S(u_1) = f(u_1u_2) + f(u_1) + f(u_1u_n) + f(u_1v_1) + f(u_0u_1) = 11n + 1$; $S(u_n) = f(u_{n-1}u_n) + f(u_1u_n) + f(u_nv_n) + f(u_n) + f(u_0u_n) = 11n + 1$; $S(u_0) = f(u_0) + \sum_1^n f(u_0u_i) = 11n + 1$; $S(u_i) = f(u_iu_{i+1}) + f(u_iu_{i-1}) + f(u_iv_i) + f(u_i) + f(u_0u_i) = 11n + 1, 2 \le i \le n-1$, which give the same sum of labels of all elements in the set $\{u_0, u_1, \cdots, u_n\}$. The rest of the vertices have no adjacent vertices of the same degree, so it is not necessary to consider the sum of their labels.

Therefore, AVRTL exists for $H_n (n = 4)$.

Scenario 3: When $n = 6$

At this point, any two elements of the vertex set $\{u_1, u_2, \cdots, u_n\}$ are adjacent and of the same degree, both of degree 4 vertices, and a mapping about f can be obtained as follows.

$$f(u_i) = \begin{cases} 5n+1, i = 0 \\ i, 1 \le i \le 2 \\ i+1, 3 \le i \le n-1 \\ 3, i = n \end{cases};$$

$$f(v_i) = 4n + i, 1 \le i \le n;$$

$$f(u_1u_n) = 4n - 1;$$

$$f(u_iu_{i+1}) = \begin{cases} 4n, i = 1 \\ 4n-2, i = 2 \\ 3n-1, i = 3 \\ 3n+1, i = 4 \\ 4n-3, i = 5 \end{cases};$$

$$f(u_iv_i) = \begin{cases} n+3, i = 1 \\ n+i-1, 2 \le i \le 3 \\ 2n, i = 4 \\ 2n+i-7, n-1 \le i \le n \end{cases};$$

$$f(u_0u_i) = \begin{cases} 2n+2, i = 1 \\ 3n-2, i = 2 \\ 3n+2, i = 3 \\ 3n, i = 4 \\ 2n+3, i = 5 \\ 2n+1, i = 6 \end{cases}$$

At this time, $f(V) = \{1, 2, \cdots, n\} \cup \{4n+1, 4n+2, \cdots, 5n+1\}$ and $f(E) = \{n+1, n+2, \cdots, 4n\}$, which gives $f(V(G)) \cup f(E(G)) \to [1, 5n+1]$, $f(V(G)) \cap f(E(G)) = \emptyset$.

And $S(u_1) = f(u_1) + f(u_1u_2) + f(u_1u_n) + f(u_1v_1) + f(u_0u_1) = 12n - 1$; $S(u_n) = f(u_{n-1}u_n) + f(u_1u_n) + f(u_n) + f(u_nv_n) + f(u_0u_n) = 12n - 1$; $S(u_i) = f(u_i) + f(u_iu_{i+1}) + f(u_iu_{i-1}) + f(u_iv_i) + f(u_0u_i) = 12n - 1$, $2 \le i \le n-1$, which give the same sum of labels of all elements in the set $\{u_1, u_2, \cdots, u_n\}$. The rest of the vertices have no adjacent vertices of the same degree, so it is not necessary to consider the sum of their labels.

Therefore, AVRTL exists for $H_n (n = 6)$.

To sum up, Theorem 9 holds.

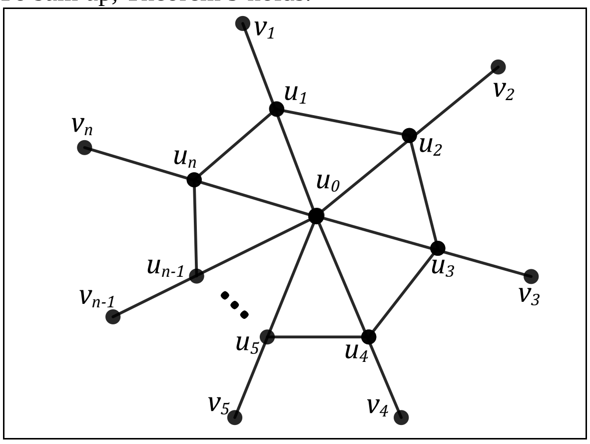


Fig 8. H_n

Conjecture 1: AVRTL exists for the tiller graph H_n when $n \ge 3$.

Theorem 10: AVRTL exists for the joint graph $F_n \uparrow C_l \uparrow S_m$ when $n \ge 3, l \ge 3, m \ge 2$.

Proof: Let the set of vertices of $F_n \uparrow C_l \uparrow S_m$ be $\{u_1, u_2, \cdots, u_n\} \cup \{v_1, v_2, \cdots, v_l\} \cup \{w_1, w_2, \cdots, w_m\}$, and v_l is the common vertex of F_n, C_l, S_m. $F_n \uparrow C_l \uparrow S_m$ has a total of $n + l + m$ vertices and $2n + l + m - 1$ edges, as shown in Figure 9.

Scenario 1: When $n=3, l\ge 3, m\ge 2$

At this point, any two elements of the vertex set $\{v_1, v_2, \cdots, v_{l-1}\}$ are adjacent and of the same degree, both of degree 2 vertices, and a mapping about f can be obtained as follows.

$$f(u_i)=2l+i, 1\le i\le n;$$
$$f(v_i)=i, 1\le i\le l-1;$$
$$f(w_i)=2l+3n+i-1, 1\le i\le m;$$
$$f(v_1v_l)=2l-1;$$
$$f(u_iu_{i+1})=2l+2n+i, 1\le i\le n-1;$$
$$f(u_iv_l)=2l+n+i, 1\le i\le n;$$
$$f(w_iv_l)=2l+3n+m+i-1, 1\le i\le m;$$
$$f(v_iv_{i-1})=\begin{cases}2l-\dfrac{i}{2}-\dfrac{1}{2}, i\equiv 1(\mathrm{mod}\,2)\\ 2l-\left\lceil\dfrac{l}{2}\right\rceil-\dfrac{i}{2}, i\equiv 0(\mathrm{mod}\,2)\end{cases} \quad 2\le i\le l$$

At this time, $f(V)=\{1,2,\cdots,l-1\}\cup\{2l,2l+1,\cdots,2l+3\}\cup\{2l+3n,2l+3n+1,\cdots,2l+3n+m-1\}$ and $f(E)=\{l,l+1,\cdots,2l-1\}\cup\{2l+4,2l+5,\cdots,2l+3n-1\}\cup\{2l+3n+m,\cdots,2l+3n+2m-1\}$, which gives $f(V(G))\cup f(E(G))\to[1,2l+3n+2m-1]$ and $f(V(G))\cap f(E(G))=\emptyset$.

And $S(v_i)=f(v_i)+f(v_iv_{i+1})+f(v_iv_{i-1})=4l-1-\left\lceil\frac{l}{2}\right\rceil, 2\le i\le l-1$; $S(v_1)=f(v_1v_2)+f(v_1v_l)+f(v_1)=4l-\left\lceil\frac{l}{2}\right\rceil-1$, which give the same sum of labels of all elements in the set $\{v_1, v_2, \cdots, v_{l-1}\}$. The rest of the vertices have no adjacent vertices of the same degree, so it is not necessary to consider the sum of their labels.

Therefore, AVRTL exists for $F_n\uparrow C_l\uparrow S_m$ $(n=3, l\ge 3, m\ge 2)$.

Scenario 2: When $n\ge 4, l\ge 3, m\ge 2$

At this point, in the vertex sets $\{v_1, v_2, \cdots, v_{l-1}\}$ and $\{u_2, u_3, \cdots, u_{n-1}\}$, any two elements in each set are adjacent and of the same degree, and all are points of degree 2, and a mapping about f can be obtained as follows.

$$f(u_i)=\begin{cases}3n-1, i=1\\ i-1, 2\le i\le n-1;\\ 3n-2, i=n\end{cases}$$
$$f(v_i)=\begin{cases}3n+i-1, 1\le i\le l-1\\ 3n+2l-1, i=l\end{cases};$$
$$f(w_i)=2l+3n+i-1, 1\le i\le m;$$
$$f(v_1v_l)=3n+2l-1;$$
$$f(u_iu_{i+1})=3n-i-4, 1\le i\le n-1;$$
$$f(u_iv_l)=\begin{cases}3n-4, i=1\\ n+i-3, 2\le i\le n-1;\\ 3n-3, i=n\end{cases}$$
$$f(w_iv_l)=2l+3n+m+i-1, 1\le i\le m;$$
$$f(v_iv_{i-1})=\begin{cases}3n+2l-\dfrac{i}{2}-\dfrac{3}{2}, i\equiv 1(\mathrm{mod}\,2)\\ 3n+2l-\left\lceil\dfrac{l}{2}\right\rceil-\dfrac{i}{2}-1, i\equiv 0(\mathrm{mod}\,2)\end{cases} \quad 2\le i\le l$$

At this time, $f(V)=\{1,2,\cdots,n-2\}\cup\{3n,3n+1,\cdots,3n+l-2\}\cup\{2l+3n-1,2l+3n,\cdots,2l+3n+m-1\}$ and $f(E)=\{n-1,n,\cdots,3n-1\}\cup\{2l+3n+m,2l+3n+m+1,\cdots,2l+3n+2m-1\}\cup\{3n+l-1,3n+l,\cdots,2l+3n-2\}$, which gives $f(V(G))\cup f(E(G))\to[1,2l+3n+2m-1]$ and $f(V(G))\cap f(E(G))=\emptyset$.

And the $S(u_i)=f(u_iu_{i+1})+f(u_iu_{i-1})+f(u_iv_l)+f(u_i)=7n-11, 2\le i\le n-1$, which gives the same sum of labels of all elements in the set $\{u_2, u_3, \cdots, u_{n-1}\}$. $S(v_1)=f(v_1)+f(v_1v_2)+f(v_1v_l)=9n+4l-\left\lceil\frac{l}{2}\right\rceil-1$; $S(v_i)=f(v_i)+f(v_iv_{i+1})+f(v_iv_{i-1})=9n+4l-\left\lceil\frac{l}{2}\right\rceil-1, 2\le i\le l-1$, which gives the same sum of labels of all elements in the set $\{v_1, v_2, \cdots, v_{l-1}\}$. The rest of the vertices have no adjacent vertices of the same degree, so it is not necessary to consider the sum of their labels.

Therefore, AVRTL exists for $F_n\uparrow C_l\uparrow S_m$ $(n\ge 4, l\ge 3, m\ge 2)$.

To sum up, Theorem 10 holds.

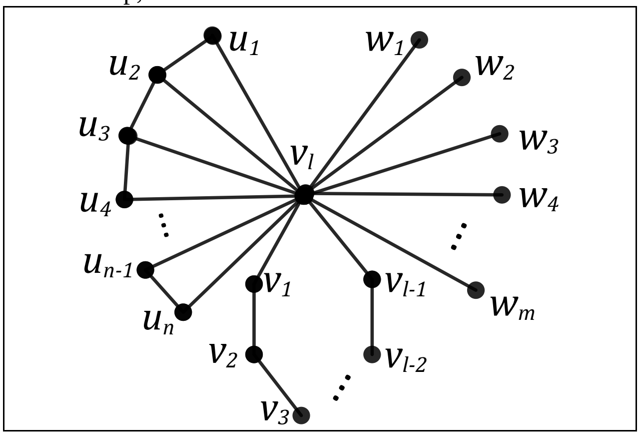


Fig 9. $F_n\uparrow C_l\uparrow S_m$

Theorem 11: AVRTL exists for the graph $I_r(P_n)$ when $n\ge 4, r\ge 2$.

Proof: Let the set of vertices of $I_r(P_n)$ be $\{u_{ij}|1\le i\le n, 0\le j\le r\}$, and $u_{i0}(1\le i\le n)$ is the common vertex of the graph P_n and r hanging edges. $I_r(P_n)$ has a total of $n+nr$ vertices and $n+nr-1$ edges, as shown in Figure 10.

Scenario 1: When $n\ge 4, r\ge 2, r\equiv 0(mod\ 2)$

At this point, any two elements of the vertex set $\{u_{20}, u_{30}, \cdots, u_{(n-1)0}\}$ are adjacent and of the same degree, both of degree $r+2$ vertices, and a mapping about f can be obtained as follows.

$$f(u_{ij})=\begin{cases}\begin{cases}i, 1\le i\le n-1\\ 2n-1, i=n\end{cases} j=0\\ (r+j+1)n+i-1, 1\le i\le n, 1\le j\le r\end{cases};$$
$$f(u_{i0}u_{ij})=\begin{cases}(1+j)n+i-1, j\equiv 1(\mathrm{mod}\,2)\\ (2+j)n-i, j\equiv 0(\mathrm{mod}\,2)\end{cases} 1\le i\le n, 1\le j\le r;$$
$$f(u_{i0}u_{(i+1)0})=\begin{cases}2n-\dfrac{i}{2}-\dfrac{3}{2}, i\equiv 1(\mathrm{mod}\,2)\\ 2n-\left\lfloor\dfrac{n}{2}\right\rfloor-\dfrac{i}{2}-1, i\equiv 0(\mathrm{mod}\,2)\end{cases} \quad 1\le i\le n-1$$

At this time, $f(V)=\{1,2,\cdots,n-1\}\cup\{2n-1\}\cup\{2n+nr,2n+nr+1,\cdots,2n+2nr-1\}$ and $f(E)=\{n,n+1,\cdots,2n-2\}\cup\{2n,2n+1,\cdots,2n+nr-1\}$, which gives $f(V(G)) \cup f(E(G)) \to [1,2n+2nr-1]$ and $f(V(G)) \cap f(E(G)) = \emptyset$.

And $S(u_{i0}) = f(u_{i0}u_{(i-1)0}) + f(u_{i0}u_{(i+1)0}) + f(u_{i0}) + \sum_1^r f(u_{i0}u_{ij}) = 4n - \left\lfloor \frac{n}{2} \right\rfloor + \frac{n}{2}r^2 - \frac{r}{2} + 2nr - 2,\ 2 \le i \le n-1$, which gives the same sum of labels of all elements in the set $\{u_{20}, u_{30}, \cdots, u_{(n-1)0}\}$. The rest of the vertices have no adjacent vertices of the same degree, so it is not necessary to consider the sum of their labels.

Therefore, AVRTL exists for $I_r(P_n)(n \ge 4, r \ge 2, r \equiv 0(mod\ 2))$.

Scenario 2: When $n \ge 4, r \ge 2, r \equiv 1(mod\ 2)$

At this point, any two elements of the vertex set $\{u_{20}, u_{30}, \cdots, u_{(n-1)0}\}$ are adjacent and of the same degree, both of degree $r+2$ vertices, and a mapping about f can be obtained as follows.

$$f(u_{ij}) = \begin{cases} i, j=0 \\ (r+j+1)n+i-1, 1 \le j \le r \end{cases} 1 \le i \le n;$$

$$f(u_{i0}u_{(i+1)0}) = 2n-i, 1 \le i \le n-1;$$

$$f(u_{i0}u_{ij}) = \begin{cases} \begin{cases} (2+j)n+i-1, j \equiv 1(\mathrm{mod}\ 2) \\ (3+j)n-i, j \equiv 0(\mathrm{mod}\ 2) \end{cases} 1 \le j \le r-1 \\ 2n+i-1, j=r \end{cases} 1 \le i \le n$$

At this time, $f(V) = \{1,2,\cdots,n\} \cup \{2n+nr, 2n+nr+1, \cdots, 2n+2nr-1\} \cup \{2n-1\}$ and $f(E) = \{n+1, n+2, \cdots, nr+2n-1\}$, which gives $f(V(G)) \cup f(E(G)) \to [1,2n+2nr-1]$ and $f(V(G)) \cap f(E(G)) = \emptyset$.

And $S(u_{i0}) = f(u_{i0}u_{(i-1)0}) + f(u_{i0}u_{(i+1)0}) + f(u_{i0}) + \sum_1^r f(u_{i0}u_{ij}) = \frac{n}{2}r^2 + 2nr - \frac{r}{2} + \frac{7}{2}n + \frac{1}{2}, 2 \le i \le n-1$, which gives the same sum of labels of all elements in the set $\{u_{20}, u_{30}, \cdots, u_{(n-1)0}\}$. The rest of the vertices have no adjacent vertices of the same degree, so it is not necessary to consider the sum of their labels.

Therefore, AVRTL exists for $I_r(P_n)(n \ge 4, r \ge 2, r \equiv 1(mod\ 2))$.

To sum up, Theorem 11 holds.

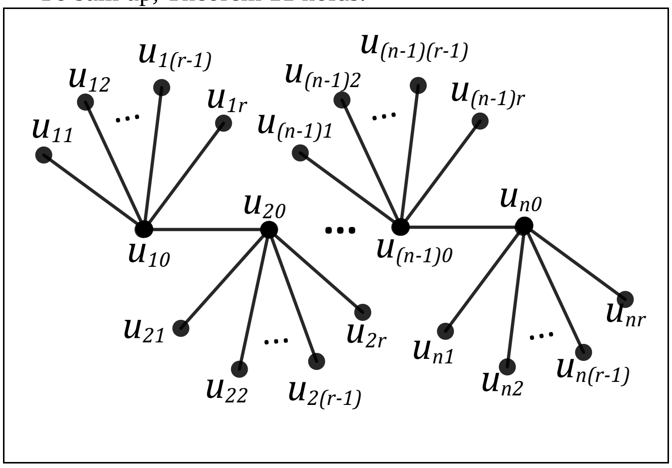


Fig 10. $I_r(P_n)$

Theorem 12: AVRTL exists for the joint graph $S_m \uparrow F_n \downarrow P_t$ when $m \ge 2, n \ge 2, t \ge 2$.

Proof: Let the set of vertices of $S_m \uparrow F_n \downarrow P_t$ be $\{u_0, u_1, \cdots, u_n\} \cup \{v_1, v_2, \cdots, v_m\} \cup \{w_2, w_3, \cdots, w_t\}$, and u_0 is the common vertex of F_n and S_m. $S_m \uparrow F_n \downarrow P_t$ has a total of $n+t+m$ vertices and $2n+t+m-2$ edges, as shown in Figure 11.

Scenario 1: When $m \ge 2, 2 \le t < 4$ and $2 \le n \le 4$

Since there are no adjacent vertices of the same degree in the graph, Scenario 1 obviously holds according to the definition of Adjacent Vertex Reducible Total Labeling.

Therefore, AVRTL exists for $S_m \uparrow F_n \downarrow P_t$ $(2 \le n \le 4, 2 \le t < 4, m \ge 2)$.

Scenario 2: When $m \ge 2, 2 \le t < 4$ and $n \ge 5$

At this point, any two elements of the vertex set $\{u_3, u_4, \cdots, u_{n-1}\}$ are adjacent and of the same degree, both of degree 3 vertices, and a mapping about f can be obtained as follows.

$$f(u_i) = \begin{cases} 3n, i=0 \\ 3n-1, i=1 \\ i-1, 2 \le i \le n-1 \\ 3n-2, i=n \end{cases};$$

$$f(v_i) = 3n+i, 1 \le i \le m;$$

$$f(w_i) = 3n+2m+i-1, 2 \le i \le t;$$

$$f(u_iu_{i+1}) = 3n-i-4, 1 \le i \le n-1;$$

$$f(u_0u_i) = \begin{cases} 3n-3, i=1 \\ n+i-3, 2 \le i \le n-1; \\ 3n-4, i=n \end{cases}$$

$$f(u_0v_i) = 3n+m+i, 1 \le i \le m;$$

$$f(w_iw_{i+1}) = 3n+2m+t+i-1, 1 \le i \le t-1$$

At this time, $f(V) = \{1,2,\cdots,n-2\} \cup \{3n-2, 3n-1, \cdots, 3n+m\} \cup \{3n+2m+1, \cdots 3n+2m+t-1\}$ and $f(E) = \{n-1, n, \cdots, 3n-3\} \cup \{3n+m+1, 3n+m+2, \cdots, 3n+2m\} \cup \{3n+2m+t, 3n+2m+t+1, \cdots, 3n+2m+2t-2\}$, which gives $f(V(G)) \cup f(E(G)) \to [1,3n+2m+2t-2]$, $f(V(G)) \cap f(E(G)) = \emptyset$.

And the $S(u_i) = f(u_iu_{i+1}) + f(u_iu_{i-1}) + f(u_iu_0) + f(u_i) = 7n-11, 3 \le i \le n-1$, which gives the same sum of labels of all elements in the set $\{u_3, u_4, \cdots, u_{n-1}\}$. The rest of the vertices have no adjacent vertices of the same degree, so it is not necessary to consider the sum of their labels.

Therefore, AVRTL exists for $S_m \uparrow F_n \downarrow P_t$ $(n \ge 5, 2 \le t < 4, m \ge 2)$.

Scenario 3: When $m \ge 2, t \ge 4$ and $2 \le n \le 4$

At this point, any two elements of the vertex set $\{w_2, w_3, \cdots, w_{t-1}\}$ are adjacent and of the same degree, both of degree 2 vertices, and a mapping about f can be obtained as follows.

$$f(u_i) = 2t+i-1, 0 \le i \le n;$$

$$f(v_i) = 2t+3n+i-2, 1 \le i \le m;$$

$$f(w_i) = \begin{cases} i-1, 2 \le i \le t-1 \\ 2t-2, i=t \end{cases};$$

$$f(u_iu_{i+1}) = 2t+2n+i-1, 1 \le i \le n-1;$$

$$f(u_0u_i) = 2t+n+i-1, 1 \le i \le n;$$

$$f(u_0v_i) = 2t+3n+m+i-2, 1 \le i \le m;$$

$$f(w_iw_{i+1})=\begin{cases}2t-\frac{i}{2}-\frac{5}{2}, i\equiv 1(\bmod 2)\\ 2t-\left\lfloor\frac{t}{2}\right\rfloor-\frac{i}{2}-2, i\equiv 0(\bmod 2)\end{cases}\quad 1\le i\le t-1$$

At this time, $f(V)=\{1,2,\cdots,t-1\}\cup\{2t-2,2t-1,\cdots,2t+n-1\}\cup\{2t+3n-1,2t+3n,\cdots,2t+3n+m-2\}$ and $f(E)=\{t,t+1,\cdots,2t-2\}\cup\{2t+n,2t+n+1,\cdots,2t+3n-2\}\cup\{2t+3n+m-1,2t+3n+m,\cdots,2t+3n+2m-2\}$, which gives $f(V(G))\cup f(E(G))\rightarrow[1,3n+2t+2m-2]$ and $f(V(G))\cap f(E(G))=\emptyset$.

And $S(w_i)=f(w_i)+f(w_iw_{i+1})+f(w_iw_{i-1})=4t-\lfloor\frac{t}{2}\rfloor-5, 2\le i\le t-1$, which gives the same sum of labels of all elements in the set $\{w_2,w_3,\cdots,w_{t-1}\}$. The rest of the vertices have no adjacent vertices of the same degree, so it is not necessary to consider the sum of their labels.

Therefore, AVRTL exists for $S_m\uparrow F_n\downarrow P_t$ $(2\le n\le 4, t\ge 4, m\ge 2)$.

Scenario 4: When $m\ge 2, t\ge 4$ and $n\ge 5$

At this point, any two elements of the vertex set $\{u_3,u_4,\cdots,u_{n-1}\}$ are adjacent and of the same degree, both of degree 3 vertices; any two elements of the vertex set $\{w_2,w_3,\cdots,w_{t-1}\}$ are adjacent and of the same degree, both of degree 2 vertices, and a mapping about f can be obtained as follows.

$$f(u_i)=\begin{cases}3n, i=0\\ 3n-1, i=1\\ i-1, 2\le i\le n-1\\ 3n-2, i=n\end{cases};$$

$$f(v_i)=3n+i, 1\le i\le m;$$

$$f(w_i)=\begin{cases}3n+2m+i-1, 2\le i\le t-1\\ 3n+2m+2t-2, i=t\end{cases};$$

$$f(u_iu_{i+1})=3n-i-4, 1\le i\le n-1;$$

$$f(u_0u_i)=\begin{cases}3n-3, i=1\\ n+i-3, 2\le i\le n-1;\\ 3n-4, i=n\end{cases}$$

$$f(u_0v_i)=3n+m+i, 1\le i\le m;$$

$$f(w_iw_{i+1})=$$

$$\begin{cases}3n+2t+2m-\frac{i}{2}-\frac{5}{2}, i\equiv 1(\bmod 2)\\ 3n+2t+2m-\left\lfloor\frac{t}{2}\right\rfloor-\frac{i}{2}-2, i\equiv 0(\bmod 2)\end{cases}\quad 1\le i\le t-1$$

At this time, $f(V)=\{1,2,\cdots,n-2\}\cup\{3n-2,3n-1,\cdots,3n+m\}\cup\{3n+2m+1,3n+2m+2,\cdots,3n+2m+t-2\}\cup\{3n+2m+2t-2\}$ and $f(E)=\{n-1,n,\cdots,3n-3\}\cup\{3n+m+1,3n+m+2,\cdots,3n+2m\}\cup\{3n+2m+t-1,3n+2m+t,\cdots,3n+2m+2t-3\}$, which gives $f(V(G))\cup f(E(G))\rightarrow[1,3n+2m+2t-2]$ and $f(V(G))\cap f(E(G))=\emptyset$.

And the $S(u_i)=f(u_iu_{i+1})+f(u_iu_{i-1})+f(u_iu_0)+f(u_i)=7n-11, 3\le i\le n-1$, which gives the same sum of labels of all elements in the set $\{u_3,u_4,\cdots,u_{n-1}\}$. $S(w_i)=f(w_i)+f(w_iw_{i+1})+f(w_iw_{i-1})=9n+6m+4t-\lfloor\frac{t}{2}\rfloor-5, 2\le i\le t-1$, which gives the same sum of labels of all elements in the set $\{w_2,w_3,\cdots,w_{t-1}\}$. The rest of the vertices have no adjacent vertices of the same degree, so it is not necessary to consider the sum of their labels.

Therefore, AVRTL exists for $S_m\uparrow F_n\downarrow P_t$ $(n\ge 5, t\ge 4, m\ge 2)$.

To sum up, Theorem 12 holds.

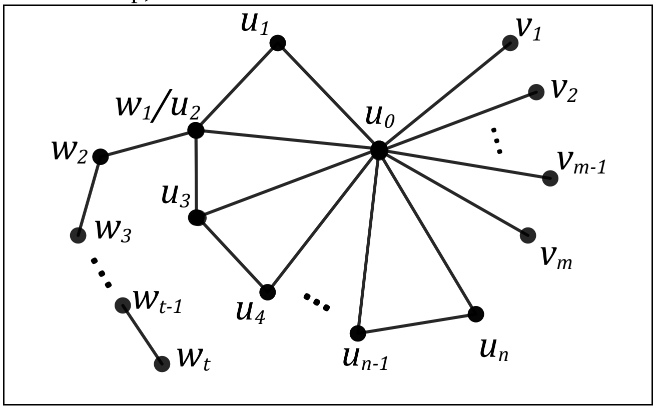


Fig 11. $S_m\uparrow F_n\downarrow P_t$

Theorem 13: When $n\ge 3, m\ge 2, m\ne 3$, or when $3\le n\le 5, m=3$, AVRTL exists for the graph $S_m^{C_n}$.

Proof: Let the set of vertices of $S_m^{C_n}$ be $\{lu_i|1\le i\le n-1, 1\le l\le m\}\cup\{v_0,v_1,\cdots,v_m\}$ and $v_i(1\le i\le m)$ is the common vertex of the graph S_m and C_n. $S_m^{C_n}$ has a total of $nm+1$ vertices and $nm+m$ edges, as shown in Figure 12.

Scenario 1: When $n\ge 3, m\ge 2, m\ne 3$

At this point, in the vertex sets $\{u_1,u_2,\cdots,u_{n-1}\}$, $\{2u_1,2u_2,\cdots,2u_{n-1}\}$, $\cdots$, $\{mu_1,\cdots,mu_{n-1}\}$, any two elements in each set are adjacent and of the same degree, and all are points of degree 2, and a mapping about f can be obtained as follows.

$$f(v_0)=2mn+m+1;$$

$$f(lu_i)=\begin{cases}2nl-2n+i, 1\le i\le n-1\\ 2nl, i=n\end{cases}\quad 1\le l\le m;$$

$$f(v_0v_i)=2mn+i, 1\le i\le m;$$

$$f(lu_1lu_n)=2nl-1, 1\le l\le m;$$

$$f(lu_ilu_{i-1})=\begin{cases}2nl-\frac{i}{2}-\frac{1}{2}, i\equiv 1(\bmod 2)\\ 2nl-\left\lceil\frac{n}{2}\right\rceil-\frac{i}{2}, i\equiv 0(\bmod 2)\end{cases}\quad 2\le i\le n, 1\le l\le m$$

At this time, $f(V)=\{1,2,\cdots,n-1\}\cup\{2n+1,2n+2,\cdots,3n-1\}\cup\cdots\cup\{2mn-2n+1,\cdots,2mn-n-1\}\cup\{2n,4n,\cdots,2mn\}\cup\{2mn+m+1\}$ and $f(E)=\{n,n+1,\cdots,2n-1\}\cup\{3n,3n+1,\cdots,4n-1\}\cup\cdots\cup\{2mn-n,2mn-n+1,\cdots,2mn-1\}\cup\{2mn+1,\cdots,2mn+m\}$, which gives $f(V(G))\cup f(E(G))\rightarrow[1,2mn+m+1]$ and $f(V(G))\cap f(E(G))=\emptyset$.

And the $S(lu_i)=f(lu_i)+f(lu_ilu_{i+1})+f(lu_ilu_{i-1})=6nl-2n-\left\lceil\frac{n}{2}\right\rceil-1, 2\le i\le n-1, 1\le l\le m$; $S(lu_1)=f(lu_1)+f(lu_1lu_2)+f(lu_1lu_n)=6nl-2n-\left\lceil\frac{n}{2}\right\rceil-1$, $1\le l\le m$, we can get that in the vertex sets $\{u_1,u_2,\cdots,u_{n-1}\},\{2u_1,2u_2,\cdots,2u_{n-1}\},\cdots,\{mu_1,\cdots,mu_{n-1}\}$, the sum of the labels of all elements in each set is the same. The rest of the vertices have no adjacent vertices of the same degree, so it is not necessary to consider the sum of their labels.

Therefore, AVRTL exists for $S_m^{C_n}$ $(n \geq 3, m \geq 2, m \neq 3)$.

Scenario 2: When $3 \leq n \leq 5, m = 3$

At this point, in the vertex sets $\{u_1, u_2, \cdots, u_{n-1}\}$, $\{2u_1, 2u_2, \cdots, 2u_{n-1}\}$, $\cdots$, $\{mu_1, \cdots, mu_{n-1}\}$, any two elements in each set are adjacent and of the same degree, and all are points of degree 2; any two elements of the vertex set $\{v_0, v_1, \cdots, v_m\}$ are adjacent and of the same degree, both of degree 3 vertices, and the labeling situation can be divided into three kinds.

(1) When $n = 3, m = 3$, a mapping about f can be obtained as follows.

$$f(v_i) = 3m - i + 1, 0 \leq i \leq m;$$

$$f(lu_i) = \begin{cases} l, i = 1 \\ 5m + l + 1, i = 2 \end{cases} 1 \leq l \leq m;$$

$$f(v_0v_i) = 5m - i - 1, 1 \leq i \leq m;$$

$$f(lu_1lu_n) = 7m + l - 2, 1 \leq l \leq m;$$

$$f(lu_ilu_{i+1}) = \begin{cases} 5m + l - 2, i = 1 \\ m + l, i = 2 \end{cases} 1 \leq l \leq m$$

At this time, $f(V) = \{1,2,3,7,8,9,10,17,18,19\}$ and $f(E) = \{11,12,\cdots,16\} \cup \{4,5,6,20,21,22\}$, which gives $f(V(G)) \cup f(E(G)) \to [1, 2mn + m + 1]$ and $f(V(G)) \cap f(E(G)) = \emptyset$.

And the $S(lu_1) = f(lu_1) + f(lu_1lu_2) + f(lu_1lu_n) = 11m + 3l - 1, 1 \leq l \leq m; S(lu_i) = f(lu_i) + f(lu_ilu_{i+1}) + f(lu_ilu_{i-1}) = 11m + 3l - 1, 2 \leq i \leq n - 1$ and $1 \leq l \leq m$, we can get that in the vertex sets $\{u_1, u_2, \cdots, u_{n-1}\}$, $\{2u_1, 2u_2, \cdots, 2u_{n-1}\}$, $\{3u_1, 3u_2, \cdots, 3u_{n-1}\}$, the sum of the labels of all elements in each set is the same. And $S(v_i) = f(v_i) + f(v_iv_0) + f(v_iiu_1) + f(v_iiu_{n-1}) = 16m - 2, 1 \leq i \leq m$; $S(v_0) = f(v_0) + f(v_1v_0) + f(v_2v_0) + f(v_3v_0) = 16m - 2$, which give the same sum of labels of all elements in the set $\{v_0, v_1, \cdots, v_m\}$.

Therefore, AVRTL exists for $S_m^{C_n}$ $(n = 3, m = 3)$.

(2) When $n = 4, m = 3$, a mapping about f can be obtained as follows.

$$f(v_i) = \begin{cases} 4m + 1, i = 0 \\ 5m - i + 2, 1 \leq i \leq m \end{cases};$$

$$f(lu_i) = \begin{cases} 9m + l - 2, i = 1 \\ 3i + l - 3, 2 \leq i \leq n - 1 \end{cases} 1 \leq l \leq m;$$

$$f(v_0v_i) = 4m - i + 1, 1 \leq i \leq m;$$

$$f(lu_1lu_n) = l, 1 \leq l \leq m;$$

$$f(lu_ilu_{i+1}) = \begin{cases} 5m + l + 3i + 1, 1 \leq i \leq n - 2 \\ 6m + l - 2, i = n - 1 \end{cases} 1 \leq l \leq m$$

At this time, $f(V) = \{4,5,\cdots,9\} \cup \{13,14,15,16,26,27,28\}$ and $f(E) = \{20,21,\cdots,25\} \cup \{1,2,3,10,11,12,17,18,19\}$, which gives $f(V(G)) \cup f(E(G)) \to [1, 2mn + m + 1]$ and $f(V(G)) \cap f(E(G)) = \emptyset$.

And the $S(lu_1) = f(lu_1) + f(lu_1lu_2) + f(lu_1lu_n) = 15m + 3l - 1, 1 \leq l \leq m; S(lu_i) = f(lu_i) + f(lu_ilu_{i+1}) + f(lu_ilu_{i-1}) = 15m + 3l - 1, 2 \leq i \leq n - 1$ and $1 \leq l \leq m$, we can get that in the vertex sets $\{u_1, u_2, \cdots, u_{n-1}\}$, $\{2u_1, 2u_2, \cdots, 2u_{n-1}\}$, $\{3u_1, 3u_2, \cdots, 3u_{n-1}\}$, the sum of the labels of all elements in each set is the same. And $S(v_i) = f(v_i) + f(v_iv_0) + f(v_iiu_1) + f(v_iiu_{n-1}) = 16m - 2, 1 \leq i \leq m$; $S(v_0) = f(v_0) + f(v_1v_0) + f(v_2v_0) + f(v_3v_0) = 16m - 2$, which give the same sum of labels of all elements in the set $\{v_0, v_1, \cdots, v_m\}$.

Therefore, AVRTL exists for $S_m^{C_n}$ $(n = 4, m = 3)$.

(3) When $n = 5, m = 3$, a mapping about f can be obtained as follows.

$$f(v_i) = \begin{cases} 6m + 1, i = 0 \\ 7m - i + 2, 1 \leq i \leq m \end{cases};$$

$$f(lu_i) = \begin{cases} 4m + l, i = 1 \\ 7m + 3i + l - 2, 2 \leq i \leq n - 1 \end{cases} 1 \leq l \leq m;$$

$$f(v_0v_i) = 6m - i + 1, 1 \leq i \leq m;$$

$$f(lu_1lu_n) = 8m + l - 2, 1 \leq l \leq m;$$

$$f(lu_ilu_{i+1}) = \begin{cases} m + l, i = 1 \\ 3m + l, i = 2 \\ l, i = 3 \\ 2m + l, i = 4 \end{cases} 1 \leq l \leq m$$

At this time, $f(V) = \{26,27,\cdots,34\} \cup \{13,14,15,19,20,21,22\}$ and $f(E) = \{1,2,\cdots,12\} \cup \{16,17,18,23,24,25\}$, which gives $f(V(G)) \cup f(E(G)) \to [1, 2mn + m + 1]$ and $f(V(G)) \cap f(E(G)) = \emptyset$.

And the $S(lu_1) = f(lu_1) + f(lu_1lu_2) + f(lu_1lu_n) = 13m + 3l - 2, 1 \leq l \leq m; S(lu_i) = f(lu_i) + f(lu_ilu_{i+1}) + f(lu_ilu_{i-1}) = 13m + 3l - 2, 2 \leq i \leq n - 1$ and $1 \leq l \leq m$, we can get that in the vertex sets $\{u_1, u_2, \cdots, u_{n-1}\}$, $\{2u_1, 2u_2, \cdots, 2u_{n-1}\}$, $\{3u_1, 3u_2, \cdots, 3u_{n-1}\}$, the sum of the labels of all elements in each set is the same. And $S(v_i) = f(v_i) + f(v_iv_0) + f(v_iiu_1) + f(v_iiu_{n-1}) = 24m - 2, 1 \leq i \leq m$; $S(v_0) = f(v_0) + f(v_1v_0) + f(v_2v_0) + f(v_3v_0) = 24m - 2$, which give the same sum of labels of all elements in the set $\{v_0, v_1, \cdots, v_m\}$.

Therefore, AVRTL exists for $S_m^{C_n}$ $(n = 5, m = 3)$.

To sum up, Theorem 13 holds.

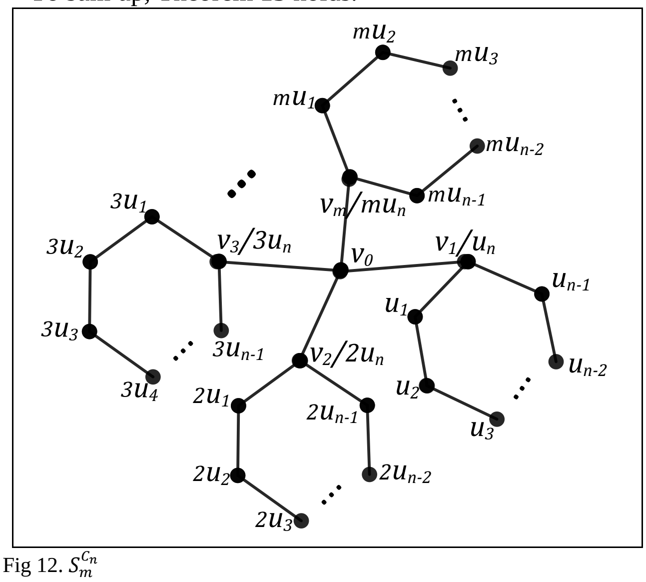


Fig 12. $S_m^{C_n}$

Conjecture 2: AVRTL exists for the graph $S_m^{C_n}$ when $n \geq 3, m \geq 2$.

V. Conclusion

This paper proposes the notion of Adjacent Vertex Reducible Total Labeling based on the Vertex Magic Total Labeling and Reducible Coloring concept. And a new related algorithm is designed based on the properties and constraints of this labeling for the real problem that the Adjacent Vertex Reducible Total Labeling model can describe. The algorithm solves the Adjacent Vertex Reducible Total Labeling of any simple connected graph within a finite number of points in a circular, iterative merit-seeking way. By analyzing the labeling cases from the experimental result set, the labeling rules of road graphs, circle graphs, star graphs, wheel graphs, fan graphs, friendship graphs, and several joint graphs were found, and the corresponding theorems and proofs, as well as two conjectures, were given.

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