Adjacent Vertex Reducible Total Labeling of Graphs

Li Wang, Jingwen Li, Lijing Zhang

Abstract—The Adjacent Vertex Reducible Total Labeling (AVRTL) of a graph $G(V, E)$ is a bijection from $V(G) \cup E(G)$ to the set of consecutive integers $\{1, 2, \ldots, |V(G)| + |E(G)|\}$, and the sum of the labels is the same for any two adjacent vertices in the graph of the same degree, being the label of that vertex plus the labels of all associated edges of that vertex. Combining with real-life problems, a novel algorithm is designed by drawing on the ideas of traditional intelligent algorithms such as genetic algorithms and bee colony algorithms. The algorithm labels the vertices and edges of random graphs with the help of preprocessing function, adjustment function, backward function, etc., in a circular, iterative merit-seeking way and discriminates whether there is an Adjacent Vertex Reducible Total Labeling for all non-isomorphic graphs within finite vertices. By analyzing and summarizing the experimental results, the labeling rules of road graphs, circle graphs, star graphs, wheel graphs, fan graphs, friendship graphs and several joint graphs were obtained, the relevant theorems were summarized, and the corresponding proofs were given.

Index Terms—joint graphs, reducible total labeling, labeling algorithm, graph labeling

I. INTRODUCTION

Graph theory [1] originated from the Seven Bridges of Konigsberg problem proposed in the 18th century, a branch of mathematics that uses graphs as objects for related research. Since 1736, when the Swiss mathematician Euler solved the Seven Bridges of Konigsberg problem and created a new branch of mathematics, graph theory has been studied by many scholars and developed continuously. Graph labeling is one of the important branches in graph theory, starting from a conjecture proposed by Ringel [2] in 1963: any tree is graceful. In the decades following the Graceful Tree Conjecture, researchers have launched on the study and research of graph labeling problems. Later, some scholars proposed the concepts of Graceful Labeling, (p,1)–Total Labeling and Magic Labeling, and the literature [3] introduced the concept of Vertex Magic Total Labeling. Through the study of special graphs and some more easily inscribed graphs, numerous researchers have published many research results on Vertex Magic Total Labeling.

In 2002, the idea of Adjacent Strong Edge Coloring of graphs was put up by Professor Zhongfu Zhang et al. [4], which satisfies that the color sets of any adjacent vertices are different on the basis of proper edge coloring. And later, Zhu et al. further studied and explored Distinguishable Coloring [5]–[7]. Based on the theory of Distinguishable Coloring, Prof. Zhongfu Zhang and colleagues proposed the idea of Reducible Coloring in 2009 [8], and whether graphs have reducible coloring has become one of the research hotspots of great interest. However, the research results in this area are still relatively few [9]–[10].

Suppose the weights of nodes and edges in a transportation network stand for a vehicle site’s vehicle capacity and the amount of transit between sites, respectively. Then the Adjacent Vertex Reducible Total Labeling model may be used to depict a real-life situation where neighboring stations of the same degree are required to maintain about equal capacity, and the diversity of connecting roadways between stations reaches an extreme value. This paper proposes the concept of Adjacent Vertex Reducible Total Labeling based on the research of previous scholars, designs an AVRTL algorithm to solve the Adjacent Vertex Reducible Total Labeling of random graphs within finite points, and obtains some related theorems and gives proofs by analyzing and summarizing the experimental results.

II. BASIC KNOWLEDGE

In this paper, $G(p, q)$ represents a simple connected graph with p vertices and q edges. The central node of the fan graph $F_m$ is $v_0$ and contains m fan vertices. $W_n$ is a wheel graph with a total of n + 1 vertices, and the central node $u_0$ is adjacent to the remaining n vertices.

Definition 1: Assume that $G(V, E)$ is a simple undirected connected graph, if there exists a bijection $f: V(G) \cup E(G) \to \{1, 2, \ldots, |V(G)| + |E(G)|\}$, for any two adjacent vertices $uv \in E(G)$, if $d(u) = d(v)$, we get $S(u) = S(v)$, where $S(u) = f(u) + \sum_{v \in V(G)} f(uv)$. $d(u)$ denotes the degree of vertex $u$, then f is said to be the Adjacent Vertex Reducible Total Labeling of the graph $G$, or AVRTL for short, and graph $G$ is an AVRTL graph. If a graph does not have an AVRTL, it is said to be a NAVRTL graph.

Definition 2: The graph $T(m, n)(n \geq 1, m \geq 3)$ denotes the graph formed by joining a $c_m$ circle graphs with common vertex $u_0$ as shown in Figure 1(a). The joint graph formed by

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bonding the common vertex $u_0$ of $T_{(m,n)}$ to the circle graph $C_r$ is $T_{(m,n)} \uparrow C_r$.

**Definition 3:** Suppose the vertex set of the graph $G_n$ is $\{u_1, u_2, \cdots, u_n\}$, the vertex set of the graph $S_m$ is $\{v_1, v_2, \cdots, v_m\}$, and the central vertex of graph $S_m$ is denoted by $v_0$. $S_m^{G_n}$ denotes the graph obtained by connecting all the $m$ vertices in the star graph except the star center with a $G_n$, as illustrated in Figure 1(b).

![Fig 1. $T_{(5,4)}$ and $S_m^{G_n}$](image)

**Definition 4:** The joint graph $F_n \uparrow C_1 \uparrow S_m$ is the graph obtained by connecting the central vertex $v_0$ of the fan graph $F_n$ with any vertex of $C_1$, and then connecting the connection vertex to the central vertex of the star graph $S_m$, $|V(F_n \uparrow C_1 \uparrow S_m)| = |V(F_n)| + |V(C_1)| + \left| V(S_m) \right| - 2$, $|E(F_n \uparrow C_1 \uparrow S_m)| = |E(F_n)| + |E(C_1)| + |E(S_m)|$.

**Definition 5:** The joint graph $S_m \uparrow F_n \downarrow P_t$ is obtained by merging the center vertex of the star graph $S_m$ with the center vertex of the fan graph $F_n$, setting the merged vertex as $u_0$, and then merging any vertex other than $u_0$ with any vertex of the road graph $P_t$.

**Definition 6:** The graph $I_r(P_m)$ denotes the $r$-crown graph of the road graph $P_m$ and is the graph obtained by bonding $r$ hanging edges at each vertex of the road graph $P_m$.

**Definition 7:** $G_{m,n}$ denotes the graph formed by joining the central vertices of graph $G_m$ and $G_n$ with the common vertex $u_0$, as shown in Figure 2(a). Then the joint graph obtained by bonding $u_0$ of graph $G_{m,n}$ to a road graph $P_t$ is called $G_{m,n} \uparrow P_t$.

**Definition 8:** $F_n^{(2)}$ denotes the graph formed by bonding the central vertices of two fan graphs $F_n$, and the common vertex of this joint graph is $u_0$.

**Definition 9:** The tiller graph $H_n$ is the graph obtained by adding a hanging edge to each vertex $u_i$ of $W_n$ except for the central vertex $u_0$, as shown in Figure 2(b).

![Fig 2. $F_{k,5}$ and $H_8$](image)

**III. ALGORITHM**

A. **Fundamentals of the algorithm**

The restrictions are built in accordance with the definition of the Adjacent Vertex Reducible Total Labeling, drawing on the design principles of conventional intelligent algorithms. The vertices and edges of graph $G(p,q)$ are labeled using an iterative loop once all vertices have been classified by degree and whether or not they are adjacent. Finally, the labeling matrix satisfying the requirements is obtained to finish the labeling process of the graph.

Step 1: The preprocessing function reads the graph set file to get the graph's adjacency matrix, as well as the necessary attribute values, like the number of vertices and edges and the set of nearby vertices with the same degree.

Step 2: The construction of the adjustment function, according to the adjustment criteria, determines the current matrix of all positions in the adjustment order, as well as the adjustment span of the label value (generally set to 1).

Step 3: Determine whether the difference between the sum of labels of adjacent points of the same degree is greater than 2 after adjustment or whether the attribute values in the set of labels are discontinuous. The matrix that meets the requirement is recorded as the middle matrix $MidAdjust$.

Step 4: If the label's value in the matrix reaches the maximum or the intermediate matrix does not change, the final intermediate matrix is recorded as the final matrix, and the cycle ends.

Step 5: Output the final label matrix.
B. Pseudocode

<table>
<thead>
<tr>
<th>Input</th>
<th>The adjacency matrix of the graph $G(p, q)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Output</td>
<td>The matrix satisfying the labeling requirements</td>
</tr>
<tr>
<td>1</td>
<td>Read the adjacency matrix $\text{initAdjust}$ of the graph $G$</td>
</tr>
<tr>
<td>2</td>
<td>Calculate the required properties, such as the number of vertices, the number of edges, and the set of adjacent vertices of the same degree.</td>
</tr>
<tr>
<td>3</td>
<td>$\text{get FinalMatrix, flag = true}$</td>
</tr>
<tr>
<td>4</td>
<td>While (flag)</td>
</tr>
<tr>
<td>5</td>
<td>For $i \leftarrow 0$ to $\text{VertexNum}$</td>
</tr>
<tr>
<td>6</td>
<td>$e_i + +$</td>
</tr>
<tr>
<td>7</td>
<td>If (The backward function is satisfied)</td>
</tr>
<tr>
<td>8</td>
<td>$e_i \leftarrow -$</td>
</tr>
<tr>
<td>9</td>
<td>End if</td>
</tr>
<tr>
<td>10</td>
<td>If (The current matrix satisfies the equilibrium condition)</td>
</tr>
<tr>
<td>11</td>
<td>$\text{MidAdjust}_i \leftarrow \text{currentAdjust}_i$</td>
</tr>
<tr>
<td>12</td>
<td>End if</td>
</tr>
<tr>
<td>13</td>
<td>If ($e_i == \text{VerNum} + \text{EdgeNum}$</td>
</tr>
<tr>
<td>14</td>
<td>FinalAdjust $\leftarrow$ MidAdjust$_{i}$</td>
</tr>
<tr>
<td>15</td>
<td>Break</td>
</tr>
<tr>
<td>16</td>
<td>End if</td>
</tr>
<tr>
<td>17</td>
<td>End for</td>
</tr>
<tr>
<td>18</td>
<td>If (FinalAdjust, maxLabel $==$ $p + q$)</td>
</tr>
<tr>
<td>19</td>
<td>Output FinalAdjust</td>
</tr>
<tr>
<td>20</td>
<td>End if</td>
</tr>
<tr>
<td>21</td>
<td>End while</td>
</tr>
<tr>
<td>22</td>
<td>End</td>
</tr>
</tbody>
</table>

C. Analysis of the findings from the experiment

All non-isomorphic graphs within 6-10 vertices were found for the Adjacent Vertex Reducible Total Labeling using this algorithm, and the experimental results were analyzed to produce the following line graph. It can be observed from this figure that the number of AVRTL graphs within finite vertices consistently accounts for more than half of the total number of graphs.

IV. THEOREM AND PROOF

**Theorem 1:** AVRTL exists for the road graph $P_n$ when $n \geq 3$.

**Proof:** Let $\{u_1, u_2, \ldots, u_n\}$ be the set of $P_n$’s vertices, with $u_1$ and $u_n$ located at the two ends of $P_n$. $P_n$ contains a total of $n$ vertices and $n - 1$ edges, and the following mapping can be used to get $f$,

$$f(u_i) = \begin{cases} i, & 1 \leq i \leq n-1; \\ 2n-1-i, & i = n; \end{cases}$$

$$f(u_i, u_{i+1}) = \begin{cases} 2n-i-3, & i \equiv 1 \pmod{2}; \\ 2n-\frac{n}{2}-i-1, & i \equiv 0 \pmod{2}; \end{cases}$$

At this point, $f(V(G)) \cup f(E(G)) = \{1,2n-1\}$, $f(V(G)) \cap f(E(G)) = \emptyset$, and $S(u_i) = f(u_i, u_{i+1}) + f(u_i) + f(u_{i+1}) = 4n-\left\lfloor \frac{n}{2} \right\rfloor - 2$, $2 \leq i \leq n-1$, satisfying the definition of Adjacent Vertex Reducible Total Labeling, so Theorem 1 holds.

**Theorem 2:** AVRTL exists for the circle graph $C_n (n \geq 3)$ when $n \equiv 1 \pmod{2}$. 

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**Fig 4.** Labeling results of two random graphs

**Fig 3.** Trends in the percentage of AVRTL graphs versus NAVRTL graphs in the total number of graphs within 6-10 vertices
Proof: Let the set of vertices of $C_n$ be $\{u_1, u_2, \ldots, u_n\}$, where $u_i$ is a vertex adjacent to both $u_{i-1}$ and $u_{i+1}$. Given that $C_n$ contains $n$ vertices and $n$ edges, we can get the following mapping about $f$.

$$f(u_i) = i, 1 \leq i \leq n;$$

$$f(u_{i+1}) = \begin{cases} 2n - \frac{n}{2} + 1, & i = 1 \text{ (mod 2)} \\ 2n - \frac{n}{2} - 1, & i = 0 \text{ (mod 2)} \end{cases}$$

At this point, $f(V(G)) \cup f(E(G)) = \{1, 3n, 1\}$ and $S(u_i) = f(u_i) + f(u_{i+1}) + f(u_{i-1}) + f(u_{i+2}) = 6n + 2i, 1 \leq i \leq n$, satisfying the definition of Adjacent Vertex Reducible Total Labeling, so Theorem 5 holds.

**Theorem 6:** AVRTL exists for the joint graph $T_{(3,n)} \uparrow C_m$ when $n \geq 1, m \geq 3$.

Proof: Let the set of vertices of $T_{(3,n)} \uparrow C_m$ be $\{u_1, u_2, \ldots, u_{2n}\}$ and the vertex $v_0$ adjacent to the vertices $v_1, v_{m-1}$ and $u_1 (1 \leq i \leq 2n)$. $T_{(3,n)} \uparrow C_m$ has a total of $2n + m$ vertices and $3n + m$ edges, as shown in Figure 5.

At this point, in the vertex sets of $\{u_1, u_2, \ldots, u_{2n}\}$, any two elements in each set are adjacent and of the same degree, and all are points of degree 2; any two elements of the vertex set $\{v_1, v_2, \ldots, v_{m-1}\}$ are adjacent and of the same degree, both of degree 2 vertices, and a mapping about $f$ can be obtained as follows.

$$f(v_1) = \begin{cases} 5n + i, 1 \leq i \leq m - 1 \\ 5n + 2m, i = 0 \end{cases}$$

$$f(v_{m-1}) = \begin{cases} 5n + m + m = 0 \text{ (mod 2)} \\ 5n + m + m - 1 \text{ (mod 2)} \end{cases}$$

At this point, $f(V(G)) \cup f(E(G)) = [1, 15n, 1, 15n + 2n, 1, 15n + 2n + 1, \ldots, 15n + 2n + m - 1]$, which gives $f(V(G)) \cup f(E(G)) = \emptyset$ and $f(V(G)) \cup f(E(G)) = \{1, 3n, 1\}$ and $S(u_i) = f(u_i) + f(u_{i+1}) + f(u_{i-1}) + f(u_{i+2}) = 6n + 2i, 1 \leq i \leq n$, we can get that in the vertex sets of $\{u_1, u_2, \ldots, u_{2n}\}$, the sum of the labels of all elements in each set is the same. $S(v_{m-1}) = f(v_{m-1}) + f(v_{m-2}) = 15n + 4m - \frac{m}{2} - 1; S(v_i) = f(v_{i+1}) + f(v_{i-1}) = 15n + 4m - \frac{m}{2} - 1, 1 \leq i \leq m - 2, which give the same sum of labels of all elements in the set $\{v_1, v_2, \ldots, v_{m-1}\}$. The vertex $v_0$ has no adjacent vertices of
the same degree, so it is not necessary to consider the sum of its labels.

To sum up, Theorem 6 holds.

Fig 5. \(T_{3,N} \uparrow C_m\)

**Theorem 7:** AVRTL exists for the joint graph \(F_n^{(2)}\) when \(n \geq 4\).

**Proof:** Let the set of vertices of \(F_n^{(2)}\) be \(\{u_0,u_3,\ldots,u_{2n}\} \cup \{v_0,v_3,\ldots,v_{2n}\}\). \(F_n^{(2)}\) has a total of \(2n+1\) vertices and \(4n-2\) edges, as shown in Figure 6.

At this point, in the vertex sets \(\{u_2,u_5,\ldots,u_{2n-1}\}\), \(\{v_2,v_5,\ldots,v_{2n-1}\}\), any two elements in each set are adjacent and of the same degree, both of degree 3 vertices, and a mapping about \(f\) can be obtained as follows.

\[
\begin{align*}
3n-i, 0 \leq i \leq 1 \\
f(u_i) = & \begin{cases} 
-1, 2 \leq i \leq n-1; \\
3n-2, i = n \\
6n-1, i = 1 \\
f(v_i) = & \begin{cases} 
3n+i+1, 2 \leq i \leq n-1; \\
3n-2, i = n \\
6n-3, i = 1 \\
f(u_0u_i) = & \begin{cases} 
3n-i, 3 \leq i \leq n-1; \\
3n-4, i = n \\
f(u_0u_{i+1}) = & \begin{cases} 
6n-3, i = 1 \\
4n+i-3, 2 \leq i \leq n-1; \\
6n-4, i = n \\
f(v_i) = & \begin{cases} 
6n-i-4, 1 \leq i \leq n-1 \\

At this time, \(f(V) = \{1,2,\ldots,n-2\} \cup \{3n-2,3n-1,\ldots,4n-2\} \cup \{6n-2,6n-1\}\) and \(f(V) = \{n-1,n,\ldots,3n-3\} \cup \{4n-1,4n,\ldots,6n-3\}\), which gives \(f(V(G))\) and \(f(E(G)) = \{1,6n-1\}\) and \(f(V(G)) \cap f(E(G)) = \emptyset\).

And \(S(v_i) = f(u_0v_i) + f(v_iu_i) + f(v_iu_{i+1}) + f(v_iu_{i-1}) = 7n-11, 2 \leq i \leq n-1\), which gives the same sum of labels of all elements in the set \(\{v_2,v_3,\ldots,v_{n-1}\}\). \(S(u_i) = f(u_i) + f(u_0u_i) + f(u_0u_{i+1}) + f(u_0u_{i-1}) + f(u_0u_{i+1}) = 19n-11, 2 \leq i \leq n-1\), which gives the same sum of labels of all elements in the set \(\{u_2,u_3,\ldots,u_{n-1}\}\). The rest of the vertices have no adjacent vertices of the same degree, so it is not necessary to consider the sum of their labels.

To sum up, Theorem 7 holds.

Fig 6. \(F_n^{(2)}\)

**Theorem 8:** AVRTL exists for the joint graph \(W_{m,n} \uparrow P_t\) when \(m = n, m \geq 3, t \geq 4\).

**Proof:** Let the set of vertices of \(W_{m,n} \uparrow P_t\) be \(\{u_0,u_1,\ldots,u_{m-1}\} \cup \{v_0,v_1,\ldots,v_{n-1}\}\) and \(W_{m,n} \uparrow P_t\) has a total of \(n+m+t-1\) edges, as shown in Figure 7.

**Scenario 1:** When \(n = m = 3, t \geq 4\)

At this point, in the vertex sets \(\{u_0,u_1,\ldots,u_{m-1}\}\), \(\{v_1,v_2,\ldots,v_{n-1}\}\), any two elements in each set are adjacent and of the same degree, and all are points of degree 3; any two elements of the vertex set \(\{w_2,w_3,\ldots,w_{n-1}\}\) are adjacent and of the same degree, both of degree 2 vertices, and a mapping about \(f\) can be obtained as follows.

\[
\begin{align*}
1, i = 0 \\
f(u_i) = & \begin{cases} 
2m+1, i = 1; \\
m, i = 2 \\
3m+1, i = 0 \\
f(v_i) = & \begin{cases} 
3m+2n+1, i = 1; \\
3m+n, i = 2 \\
3m+3n+i, 1 \leq i \leq t-1 \\
3m+3n+2t-1, i = t \\
f(u_0u_i) = & \begin{cases} 
2m+i, i = 1; \\
2m+i, 2 \leq i \leq m \\
f(u_0u_i) = 2m-i, 1 \leq i \leq m-1; \\
f(v_iu_i) = & \begin{cases} 
3m+n-1, i = 1 \\
3m+2n+i, 2 \leq i \leq n; \\
f(v_0v_i) = & \begin{cases} 
3m+3n+2t-i, 1 \leq i \leq t-1 \\
3m+3n+2t-i, 1 \equiv 0 \pmod{2} \\

At this time, \(f(V) = \{1,3,7,10,12,16,3m+3n+2t-1\} \cup \{3m+3n+1, 3m+3n+2, \ldots, 3m+3n+t-1\}\) and \(f(E) = \{2,4,5,6,8,9\} \cup \{11,13,14,15,17,18\}\) and \(f(E) = \{2,4,5,6,8,9\} \cup \{11,13,14,15,17,18\}\) and \(f(E) = \{2,4,5,6,8,9\} \cup \{11,13,14,15,17,18\}\).

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which gives \( f(V(G)) \cup f(E(G)) \rightarrow [1, 3n + 3m + 2t - 1] \) and \( f(V(G)) \cap f(E(G)) = \emptyset \).

And the \( S(u_i) = f(u_iu_2) + f(u_iu_m) + f(u_iu_0) + f(u_i) = 5m + 5; S(u_0) = f(u_0) + f(u_0u_1) + f(u_0u_2) + f(u_0u_{m-1}) = 5m + 5; S(u_{m-1}) = f(u_{m-1}u_1) + f(u_{m-1}u_{m-2}) + f(u_{m-1}u_0) + f(u_{m-1}) = 5m + 5, 2 \leq i \leq m - 1, \) which give the same sum of labels of all elements in the set \( \{u_0, u_1, \ldots, u_{m-1}\} \). \( S(v_i) = f(v_i) + f(v_{i+1}) + f(v_{i+2}) \) and \( f(v_i) = 12m + 5n + 5; S(v_0) = f(v_0) + f(v_0u_1) + f(v_0u_2) + f(v_0u_{m-1}) = 12m + 5n + 5, 2 \leq i \leq n - 1, \) which give the same sum of labels of all elements in the set \( \{v_0, v_1, \ldots, v_{n-1}\} \). \( S(w_i) = f(w_i) + f(w_{i+1}) + f(w_{i+2}) \) and \( f(w_i) = 9(m + n) + 4t - \left[ \frac{3}{2} \right] - 2, 2 \leq i \leq t - 1, \) which give the same sum of labels of all elements in the set \( \{w_0, w_1, \ldots, w_{t-1}\} \). The rest of the vertices have no adjacent vertices of the same degree, so it is not necessary to consider the sum of their labels.

Therefore, AVRTL exists for \( W_{m,n} \uparrow P_t(n = m, m \geq 3, t \geq 4) \).

Scenario 2: When \( n = m, m \geq 4, t \geq 4 \)

At this point, in the vertex sets \( \{u_1, u_2, \ldots, u_{m-1}\} \) \( \{v_1, v_2, \ldots, v_{n-1}\} \), any two elements in each set are adjacent and of the same degree, and all are points of degree 3; any two elements of the vertex set \( \{w_2, w_3, \ldots, w_{t-1}\} \) are adjacent and of the same degree, both of degree 2 vertices, and a mapping about \( f \) can be obtained as follows.

\[
\begin{align*}
\mathbf{f(u_i)} = & 3m + i, 0 \leq i \leq m - 1; \\
\mathbf{f(v_i)} = & 3m + 3n + i, 0 \leq i \leq n - 1; \\
\mathbf{f(w_i)} = & 3m + 3n + i + 1, t \leq i \leq t - 1; \\
\mathbf{f(u_0u_m)} = & 2m - 1; \\
\mathbf{f(u_0u_t)} = & 2m + i - 1, 1 \leq i \leq m; \\
\mathbf{f(u_0u_{t+1})} = & 2m - i + 1, 1 \leq i \leq m - 1; \\
\mathbf{f(v_0v_i)} = & 3m + 2n + i; \\
\mathbf{f(v_0v_{i+1})} = & 3m + 2n - i - 1, 1 \leq i \leq n - 1; \\
\mathbf{f(w_0w_{i+1})} = & 3m + 3n + 2t - i - \frac{3}{2}, 1 \leq i \leq t - 1
\end{align*}
\]

At this time, \( f(V(G)) \) \( \cup f(E(G)) \) \( \rightarrow [1, 3n + 3m + 2t - 1] \) and \( f(V(G)) \cap f(E(G)) = \emptyset \).

And the \( S(u_i) = f(u_iu_{i+1}) + f(u_iu_{i-1}) + f(u_iu_0) + f(u_i) = 6m - 2, 2 \leq i \leq m - 1; S(u_1) = f(u_1u_2) + f(u_1u_0) + f(u_1) = 6m - 2, \) which give the same sum of labels of all elements in the set \( \{u_1, u_2, \ldots, u_{m-1}\} \). \( S(v_i) = f(v_{i-1}) + f(v_{i+1}) + f(v_{i+2}) \) and \( f(v_i) = 12m + 6n - 2; S(v_0) = f(v_0) + f(v_0u_1) + f(v_0u_{m-1}) + f(v_0u_0) = 12m + 6n - 2, 2 \leq i \leq n - 1, \) which give the same sum of labels of all elements in the set \( \{v_0, v_1, v_2, \ldots, v_{n-1}\} \). \( S(w_i) = f(w_i) + f(w_iu_{i+1}) + f(w_iu_{i-1}) + f(w_iu_0) + f(w_i) = 9m + 9n + 4t - \left[ \frac{3}{2} \right] - 2, 2 \leq i \leq t - 1, \) which give the same sum of labels of all elements in the set \( \{w_0, w_1, \ldots, w_{t-1}\} \). The rest of the vertices have no adjacent vertices of the same degree, so it is not necessary to consider the sum of their labels.

Therefore, AVRTL exists for \( W_{m,n} \uparrow P_t(n = m, m \geq 3, t \geq 4) \).

To sum up, Theorem 8 holds.

**Theorem 9:** When \( n \equiv 1(\text{mod} 2), n \geq 3 \), or when \( n \equiv 0(\text{mod} 2), 4 \leq n \leq 6 \), AVRTL exists for the tiler graph \( H_n \).

**Proof:** Let the set of vertices of \( H_n \) be \( \{u_0, u_1, \ldots, u_n\} \) \( \cup \{v_1, v_2, \ldots, v_n\} \) and the vertex \( u_1 \) is adjacent to the vertices \( v_1, u_0, u_2 \) and \( u_n. H_n \) has a total of \( 2n + 1 \) vertices and \( 3n \) edges, as shown in Figure 8.

Scenario 1: When \( n \equiv 1(\text{mod} 2), n \geq 3 \)

At this point, any two elements of the vertex set \( \{u_1, u_2, \ldots, u_{n-1}, u_n\} \) are adjacent and of the same degree, both of degree 4 vertices, and a mapping about \( f \) can be obtained as follows.

\[
\begin{align*}
\mathbf{f(u_i)} = & 5n + i, 0 \leq i \leq n; \\
\mathbf{f(v_i)} = & 4n + i, 1 \leq i \leq n; \\
\mathbf{f(u_0u_n)} = & 2n - \left[ \frac{n}{2} \right]; \\
\mathbf{f(u_0u_1)} = & 2n + i, 1 \leq i \leq n; \\
\mathbf{f(u_0u_{t+1})} = & 4n - i - 1, 1 \leq i \leq n; \\
\mathbf{f(u_1u_{i+1})} = & 2n - \left[ \frac{i}{2} \right] + 1, 1 \leq i \leq n; \\
\mathbf{f(u_1u_{i-1})} = & 2n - \left[ \frac{i}{2} \right], 1 \equiv 1(\text{mod} 2)
\end{align*}
\]

At this time, \( f(V(G)) \) \( \cup f(E(G)) \) \( \rightarrow [1, 3n + 3m + 2t - 1] \) and \( f(V(G)) \cap f(E(G)) = \emptyset \).

And the \( S(u_i) = f(u_iu_{i+1}) + f(u_iu_{i-1}) + f(u_iu_0) + f(u_i) = 6m - 2, 2 \leq i \leq m - 1; S(u_1) = f(u_1u_2) + f(u_1u_0) + f(u_1) = 6m - 2, \) which give the same sum of labels of all elements in the set \( \{u_1, u_2, \ldots, u_{m-1}\} \). \( S(v_i) = f(v_{i-1}) + f(v_{i+1}) + f(v_{i+2}) \) and \( f(v_i) = 12m + 6n - 2; S(v_0) = f(v_0) + f(v_0u_1) + f(v_0u_{m-1}) + f(v_0u_0) = 12m + 6n - 2, 2 \leq i \leq n - 1, \) which give the same sum of labels of all elements in the set \( \{v_0, v_1, v_2, \ldots, v_{n-1}\} \). \( S(w_i) = f(w_i) + f(w_iu_{i+1}) + f(w_iu_{i-1}) + f(w_iu_0) + f(w_i) = 9m + 9n + 4t - \left[ \frac{3}{2} \right] - 2, 2 \leq i \leq t - 1, \) which give the same sum of labels of all elements in the set \( \{w_0, w_1, \ldots, w_{t-1}\} \). The rest of the vertices have no adjacent vertices of the same degree, so it is not necessary to consider the sum of their labels.

Therefore, AVRTL exists for \( W_{m,n} \uparrow P_t(n = m, m \geq 3, t \geq 4) \).

To sum up, Theorem 8 holds.
And \( S(u) = f(u_1u_2) + f(u_2u_n) + f(u_nu_1) + f(u_i) = \frac{19}{2} n + \frac{5}{2} \); \( S(u) = f(u_1u_2) + f(u_2u_n) + f(u_nu_1) + f(u_i) = \frac{19}{2} n + \frac{5}{2} \), \( 2 \leq i \leq n - 1 \), which give the same sum of labels of all elements in the set \( \{ u_0, u_2, \ldots, u_n \} \). The rest of the vertices have no adjacent vertices of the same degree, so it is not necessary to consider the sum of their labels.

Therefore, AVRTL exists for \( H_n(n \equiv 1(\text{mod} \ 2), n \geq 3) \).

### Scenario 2: When \( n = 4 \)

At this point, any two elements of the vertex set \( \{ u_0, u_2, u_2, \ldots, u_n \} \) are adjacent and of the same degree, both of degree 4 vertices, and a mapping about \( f \) can be obtained as follows.

\[
f(u_i) = \begin{cases} 
1, i = 0 \\
4, i = 1 \\
2n + 2, i = 1 \\
2n - 1, i = 2 \\
2n + 3, i = 3 \\
2n + 4, i = n \end{cases} \]

At this time, \( f(V) = \{ 1, 2, \ldots, n \} \cup \{ 4n + 2, 4n + 3, \ldots, 5n + 1 \} \) and \( f(E) = \{ n + 1, n + 2, \ldots, 4n + 1 \} \), which gives \( f(V(G)) \cup f(E(G)) \rightarrow [1, 5n + 1] \) and \( f(V(G)) \cap f(E(G)) = \emptyset \).

And \( S(u_i) = f(u_1u_2) + f(u_iu_2) + f(u_iu_n) + f(u_{i+1}) + f(u_{n+1}) + f(u_{n+1}) + f(u_{n+2}) + f(u_{n+2}) + f(u_{n+2}) \) and \( S(u_i) = f(u_1u_2) + f(u_iu_2) + f(u_iu_n) + f(u_{i+1}) + f(u_{n+1}) + f(u_{n+1}) + f(u_{n+2}) + f(u_{n+2}) + f(u_{n+2}) \), \( 2 \leq i \leq n - 1 \), which give the same sum of labels of all elements in the set \( \{ u_0, u_2, \ldots, u_n \} \). The rest of the vertices have no adjacent vertices of the same degree, so it is not necessary to consider the sum of their labels.

Therefore, AVRTL exists for \( H_n(n = 4) \).

### Scenario 3: When \( n = 6 \)

At this point, any two elements of the vertex set \( \{ u_1, u_2, \ldots, u_n \} \) are adjacent and of the same degree, both of degree 4 vertices, and a mapping about \( f \) can be obtained as follows.

\[
f(u_i) = \begin{cases} 
5n + 1, i = 0 \\
3, i = n \end{cases} \]

Conjecture 1: AVRTL exists for the tiller graph \( H_n \) when \( n \geq 3 \).

**Theorem 10:** AVRTL exists for the joint graph \( F_n \uparrow C_l \uparrow S_m \) when \( n \geq 3, l \geq 3, m \geq 2 \).

**Proof:** Let the set of vertices of \( F_n \uparrow C_l \uparrow S_m \) be \( \{ \{ v_1, v_2, \ldots, v_l \} \cup \{ w_1, w_2, \ldots, w_m \} \} \). Then, \( v_i \) is the common vertex of \( F_n, C_l, S_m \), \( F_n \uparrow C_l \uparrow S_m \) has a total of \( n + l + m \) vertices and \( 2n + l + m - 1 \) edges, as shown in Figure 9.

![Figure 9](image-url)
Scenario 1: When $n = 3, l \geq 3, m \geq 2$

At this point, any two elements of the vertex set $(v_1, v_2, \ldots, v_{l-1})$ are adjacent and of the same degree, both of degree 2 vertices, and a mapping about $f$ can be obtained as follows.

$$f(u_i) = 2l + i, 1 \leq i \leq n;$$
$$f(v_i) = i, 1 \leq i \leq l - 1;$$
$$f(w_i) = 2l + 3n + i - 1, 1 \leq i \leq m;$$
$$f(v_1v_l) = 2l - 1;$$
$$f(u_iu_{i+1}) = 2l + 2n + i - 1, 1 \leq i \leq n - 1;$$
$$f(u_iw_i) = 2l + n + i, 1 \leq i \leq l - 1;$$
$$f(w_i) = 2l + 3n + m + i - 1, 1 \leq i \leq m;$$
$$f(v_1v_{l-1}) = \begin{cases} 2l - \frac{i - 1}{2}, i \equiv 1 \mod 2 \\ 2l - \frac{l}{2} - \frac{i}{2} - 1, i \equiv 0 \mod 2 \end{cases} 2 \leq i \leq l$$

At this time, $f(V) = \{1, 2, \ldots, n - 2\} \cup \{3n + 1, \ldots, 3n + 1 - 2\} \cup \{2l + 3n - 1, 2l + 3n - 2\} \cup \{2l + 3n + 3, 2l + 3n + 2\} \cup \{2l + 3n + 4, 2l + 3n + 3\} \cup \{3n + 1, 3n + 2, \ldots, 2l + 3n - 2\}$, which gives $f(V(G)) \cup f(E(G)) \rightarrow \{1, 2l + 3n + 2m - 1\}$ and $f(V(G)) \cap f(E(G)) = \emptyset$.

And the $S(u_i) = f(u_iu_{i+1}) + f(u_iw_i) + f(u_{i+1}) = 7n - 11, 2 \leq i \leq n - 1$, which gives the same sum of labels of all elements in the set $(u_2, u_3, \ldots, u_{n-1})$. $S(v_l) = f(v_1v_2) + f(v_2v_3) = 9n + 4l - \frac{l}{2} - 1; S(v_1) = f(v_1v_2) + f(v_1v_3) + f(v_1v_{l-1}) = 9n + 4l - \frac{2l}{2} - 1, 2 \leq i \leq l - 1$, which gives the same sum of labels of all elements in the set $(v_1, v_2, \ldots, v_{l-1})$. The rest of the vertices have no adjacent vertices of the same degree, so it is not necessary to consider the sum of their labels.

Therefore, AVRTL exists for $F_n \uparrow G_1 \uparrow S_m$ $(n \geq 4, l \geq 3, m \geq 2)$.

Scenario 2: When $n \geq 4, l \geq 3, m \geq 2$

At this point, in the vertex sets $(v_1, v_2, \ldots, v_{l-1})$ and $(u_2, u_3, \ldots, u_{n-1})$, any two elements in each set are adjacent and of the same degree, and all are points of degree 2, and a mapping about $f$ can be obtained as follows.

$$f(u_i) = \begin{cases} 3n - 1, i = 1 \\ i - 1, 2 \leq i \leq n - 1 \\ 3n - 2, i = n \end{cases}$$
$$f(v_i) = \begin{cases} 3n + i - 1, 1 \leq i \leq l - 1 \\ 3n + 2l - i, i = l \end{cases}$$
$$f(w_i) = 2l + 3n + i - 1, 1 \leq i \leq m;$$
$$f(v_1v_l) = 2l - 1;$$
$$f(u_iu_{i+1}) = 3n - i - 4, 1 \leq i \leq n - 1;$$
$$f(u_iw_i) = n + i - 3, 2 \leq i \leq n - 1;$$
$$f(w_i) = 2l + 3n + m + i - 1, 1 \leq i \leq m;$$
$$f(v_1v_{l-1}) = \begin{cases} 2l + 3n + 2l - i, i \equiv 1 \mod 2 \\ 3n + 2l - \frac{l}{2} - \frac{i}{2} - 1, 2 \leq i \equiv 0 \mod 2 \end{cases}$$

To sum up, Theorem 10 holds.

**Theorem 11:** AVRTL exists for the graph $I_r(P_n)$ when $n \geq 4, r \geq 2$.

**Proof:** Let the set of vertices of $I_r(P_n)$ be $\{u_{ij}| 1 \leq i \leq n, 0 \leq j \leq r\}$, and $u_{ij}$ $(1 \leq i \leq n)$ is the common vertex of the graph $P_n$ and $r$ hanging edges. $I_r(P_n)$ has a total of $n + nr$ vertices and $n + nr - 1$ edges, as shown in Figure 10.

Scenario 1: When $n \geq 4, r \geq 2, r \equiv 0 \mod 2$

At this point, any two elements of the vertex set $(u_{20}, u_{30}, \ldots, u_{(n-1)0})$ are adjacent and of the same degree, both of degree $r + 2$ vertices, and a mapping about $f$ can be obtained as follows.

$$f(u_{ij}) = \begin{cases} i, 1 \leq i \leq n - 1 \\ 2n - i, i = n \end{cases}$$

At this time, $f(V) = \{1, 2, \ldots, n - 2\} \cup \{3n + 1, \ldots, 3n + 1 - 2\} \cup \{2l + 3n - 1, 2l + 3n - 2\} \cup \{2l + 3n + 3, 2l + 3n + 2\} \cup \{2l + 3n + 4, 2l + 3n + 3\} \cup \{3n + 1, 3n + 2, \ldots, 2l + 3n - 2\}$, and $f(E) = \{n - 1, n, \ldots, 3n - 1\} \cup \{2l + 3n + m, 2l + 3n + m + 1, \ldots, 2l + 3n + 2m - 1\} \cup \{3n + 1, 3n + 2, \ldots, 2l + 3n - 2\}$, which gives $f(V(G)) \cup f(E(G)) \rightarrow \{1, 2l + 3n + 2m - 1\}$ and $f(V(G)) \cap f(E(G)) = \emptyset$.
which gives \( f(V(G)) \cup f(E(G)) \to [1,2n + 2nr - 1] \) and \( f(V(G)) \cap f(E(G)) = \emptyset \).

And \( S(u_0) = f(u_{i_0}u_{i-1}) + f(u_{i_0}u_{i+1}) + f(u_{i_0}) + \sum_{j} f(u_{i_0}u_{j+1}) = 4n - \left\lfloor \frac{n}{2} \right\rfloor + \frac{n^2}{2} - \frac{r}{2} + 2nr - 2, \quad 2 \leq i \leq n - 1, \) which gives the same sum of labels of all elements in the set \( \{u_{20}, u_{30}, \ldots , u_{(n-1)0}\} \). The rest of the vertices have no adjacent vertices of the same degree, so it is not necessary to consider the sum of their labels.

Therefore, AVRTL exists for \( I_r(P_n) \). Let the set of vertices of \( S_2, u_{30}, \ldots , u_{n-1}\) are adjacent and of the same degree, both of degree 3 vertices, and a mapping about \( f \) can be obtained as follows.

\[
\begin{align*}
&f(u_i) = \begin{cases} 
  i + 0, & 1 \leq i \leq n; \\
  (r + j + 1)n + i - 1, & 1 \leq i \leq n - 1; \\
  2n - i, & 1 \leq i \leq n - 1; \\
  1 \leq j \leq r - 1, & 2n + i - 1 = r.
\end{cases}
\end{align*}
\]

At this time, \( f(V(G)) \cup f(E(G)) \to [1,2n + 2nr - 1] \) and \( f(V(G)) \cap f(E(G)) = \emptyset \).

And \( S(u_0) = f(u_{i_0}u_{i-1}) + f(u_{i_0}u_{i+1}) + f(u_{i_0}) + \sum_{j} f(u_{i_0}u_{j+1}) = \frac{n^2}{2} + 2nr - \frac{r}{2} + \frac{1}{2} + 2n \leq i \leq n - 1, \) which gives the same sum of labels of all elements in the set \( \{u_{20}, u_{30}, \ldots , u_{(n-1)0}\} \). The rest of the vertices have no adjacent vertices of the same degree, so it is not necessary to consider the sum of their labels.

Therefore, AVRTL exists for \( I_r(P_n) \). Let the set of vertices of \( S_2, u_{30}, \ldots , u_{n-1}\) are adjacent and of the same degree, both of degree 3 vertices, and a mapping about \( f \) can be obtained as follows.

\[
\begin{align*}
&f(u_i) = \begin{cases} 
  3n, & 0 \\
  i + 0, & 1 \leq i \leq n; \\
  3n - 3, & 2 \leq i \leq n - 1; \\
  3n - 3, & 0 \leq i \leq n - 1; \\
  3n - 3, & 2 \leq i \leq n - 1; \\
  3n, & 0 \leq i \leq n - 1; \\
  3n - 3, & 2 \leq i \leq n - 1; \\
  3n, & 0 \leq i \leq n - 1; \\
  3n - 3, & 2 \leq i \leq n - 1; \\
  3n, & 0 \leq i \leq n - 1; \\
  3n - 3, & 2 \leq i \leq n - 1; \\
  3n, & 0 \leq i \leq n - 1;
\end{cases}
\end{align*}
\]

At this time, \( f(V(G)) \cup f(E(G)) \to [3n - 3, 1, 2, 1, 3n - 3, 0] \) and \( f(V(G)) \cap f(E(G)) = \emptyset \).

And \( S(u_0) = f(u_{i_0}u_{i-1}) + f(u_{i_0}u_{i+1}) + f(u_{i_0}) + \sum_{j} f(u_{i_0}u_{j+1}) = \frac{n^2}{2} + 2nr - \frac{r}{2} + \frac{1}{2} + 2n \leq i \leq n - 1, \) which gives the same sum of labels of all elements in the set \( \{u_{20}, u_{30}, \ldots , u_{(n-1)0}\} \). The rest of the vertices have no adjacent vertices of the same degree, so it is not necessary to consider the sum of their labels.

Therefore, AVRTL exists for \( S_2, u_{30}, \ldots , u_{n-1}\) are adjacent and of the same degree, both of degree 3 vertices, and a mapping about \( f \) can be obtained as follows.

\[
\begin{align*}
&f(u_i) = \begin{cases} 
  3n, & 0 \\
  i + 0, & 1 \leq i \leq n; \\
  3n - 3, & 2 \leq i \leq n - 1; \\
  3n - 3, & 0 \leq i \leq n - 1; \\
  3n - 3, & 2 \leq i \leq n - 1; \\
  3n, & 0 \leq i \leq n - 1; \\
  3n - 3, & 2 \leq i \leq n - 1; \\
  3n, & 0 \leq i \leq n - 1; \\
  3n - 3, & 2 \leq i \leq n - 1; \\
  3n, & 0 \leq i \leq n - 1; \\
  3n - 3, & 2 \leq i \leq n - 1; \\
  3n, & 0 \leq i \leq n - 1;
\end{cases}
\end{align*}
\]

At this time, \( f(V(G)) \cup f(E(G)) \to [3n - 3, 1, 2, 1, 3n - 3, 0] \) and \( f(V(G)) \cap f(E(G)) = \emptyset \).

And \( S(u_0) = f(u_{i_0}u_{i-1}) + f(u_{i_0}u_{i+1}) + f(u_{i_0}) + \sum_{j} f(u_{i_0}u_{j+1}) = \frac{n^2}{2} + 2nr - \frac{r}{2} + \frac{1}{2} + 2n \leq i \leq n - 1, \) which gives the same sum of labels of all elements in the set \( \{u_{20}, u_{30}, \ldots , u_{(n-1)0}\} \). The rest of the vertices have no adjacent vertices of the same degree, so it is not necessary to consider the sum of their labels.

Therefore, AVRTL exists for \( S_2, u_{30}, \ldots , u_{n-1}\) are adjacent and of the same degree, both of degree 3 vertices, and a mapping about \( f \) can be obtained as follows.

\[
\begin{align*}
&f(u_i) = \begin{cases} 
  3n, & 0 \\
  i + 0, & 1 \leq i \leq n; \\
  3n - 3, & 2 \leq i \leq n - 1; \\
  3n - 3, & 0 \leq i \leq n - 1; \\
  3n - 3, & 2 \leq i \leq n - 1; \\
  3n, & 0 \leq i \leq n - 1; \\
  3n - 3, & 2 \leq i \leq n - 1; \\
  3n, & 0 \leq i \leq n - 1; \\
  3n - 3, & 2 \leq i \leq n - 1; \\
  3n, & 0 \leq i \leq n - 1; \\
  3n - 3, & 2 \leq i \leq n - 1; \\
  3n, & 0 \leq i \leq n - 1;
\end{cases}
\end{align*}
\]
\[
\begin{align*}
f(w_{i+1}) &= \begin{cases} 
2t - \frac{i - 5}{2}, & i \equiv 1 \text{ (mod 2)} \\
2t - \frac{t + 2}{2}, & i \equiv 0 \text{ (mod 2)} 
\end{cases} \quad 1 \leq i \leq t - 1
\end{align*}
\]

At this time, \( f(V) = \{1, 2, \ldots, t - 1\} \cup \{2t - 2, 2t - 1, \ldots, 2t + n - 1\} \cup \{2t + 3n - 1, 2t + 3n - 2, \ldots, 2t + 3n + m - 2\} \) and \( f(E) = \{(t, t + 1, \ldots, 2t - 2) \cup \{2t + n, 2t + n + 1, \ldots, 2t + 3n - 2\} \cup \{2t + 3n + m - 1, 2t + 3n + m, \ldots, 2t + 3n + m - 2\} \), which gives \( f(V(G)) \cup f(E(G)) \rightarrow \{1, 2, 3, 4\} \cup f(E(G)) = \emptyset \).

And \( S(w_i) = f(w_i) + f(w_{i+1}) + f(w_{i+1}) = 4t - \frac{t^2}{2} - 2, \) so it is not necessary to consider the sum of their labels.

Therefore, AVRTL exists for \( S_m \uparrow F_n \downarrow P_t (n \geq 5) \) when \( t \geq 4, m \geq 2 \).

To sum up, Theorem 12 holds.

**Theorem 13:** When \( n \geq 3, m \geq 2, m \neq 3 \), or when \( 3 \leq n \leq 5, m = 3 \), AVRTL exists for the graph \( S_m^n \).

**Proof:** Let the set of vertices of \( S_m^n \) be \( \{u_i | 1 \leq i \leq n - 1, 1 \leq i \leq m\} \cup \{v_0, v_1, \ldots, v_m\} \) and \( v_i (1 \leq i \leq m) \) is any common vertex of the graph \( S_m^n \) and \( C_n \). \( S_m^n \) has a total of \( nm + 1 \) vertices and \( nm + m \) edges, as shown in Figure 12.

Scenario 1: When \( n \geq 3, m \geq 2, m \neq 3 \).

At this point, in the vertex sets \( \{u_1, u_2, \ldots, u_{n-1}\}, \{u_1, u_2, \ldots, u_{n-1}\}, \ldots, \{u_1, u_2, \ldots, u_{n-1}\}, \ldots, \{u_1, u_2, \ldots, u_{n-1}\} \), any two elements in each set are adjacent and of the same degree, and all are points of degree 2, and a mapping about \( f \) can be obtained as follows.

\[
\begin{align*}
f(u_0) &= 2m + m + 1 + 1 \leq n - 1, 1 \leq i \leq n - 1; \\
f(u) &= 2n - 2 + i, 1 \leq i \leq n - 1; \\
f(v_0) &= 2m + m + 1 + 1 \leq n - 1, 1 \leq i \leq n - 1; \\
f(l_0) &= 2n - 2 + i, 1 \leq i \leq n - 1; \\
f(l_1) &= 2n - 2 + i, 1 \leq i \leq n - 1; \\
f(l_2) &= 2n - 2 + i, 1 \leq i \leq n - 1;
\end{align*}
\]

At this time, \( f(V) = \{1, 2, \ldots, n - 1\} \cup \{2n + 1, 2n + 2, \ldots, 3n - 1\} \cup \{2m - 2 + 1, \ldots, 2m - 2 + 2m + 2\} \cup \{2m + m + 1, 2m + m + 2, 2m + m + 2 + 2t - 3\} \), which gives \( f(V(G)) \cup f(E(G)) \rightarrow \{1, 2m + m + 1\} \) and \( f(E(G)) = \emptyset \).

And the \( S(u_i) = f(u_{i+1}) + f(u_{i+1}) + f(u_{i+1}) \), \( S(u_i) = 7n - 11, 3 \leq i \leq n - 1 \), which gives the same sum of labels of all elements in the set \( \{u_3, u_4, \ldots, u_{n-1}\} \). The sum of the labels of all elements in each set is the same. The rest of...
the vertices have no adjacent vertices of the same degree, so it is not necessary to consider the sum of their labels.

Therefore, AVRTL exists for $S_m^n$ ($n \geq 3, m \geq 2, m \neq 3$).

Scenario 2: When $3 \leq n \leq 5, m = 3$

At this point, in the vertex sets $\{u_1, u_2, \ldots, u_{n-1}\}$, $\{2u_1, 2u_2, \ldots, 2u_{n-1}\}$, $\{3u_1, 3u_2, \ldots, 3u_{n-1}\}$, any two elements in each set are adjacent and of the same degree, and all are points of degree 2; any two elements of the vertex set $\{v_1, v_2, \ldots, v_m\}$ are adjacent and of the same degree, both of degree 3 vertices, and the labeling situation can be divided into three kinds.

(1) When $n = 3, m = 3$, a mapping about $f$ can be obtained as follows.

$$f(v_i) = 3m - i + 1,0 \leq i \leq m;$$

$$f(lu_i) = \begin{cases} 1, & l = 1 \\ 5m + l + 1, & 1 \leq l \leq m; \end{cases}$$

$$f(v_0v_i) = 5m - i - 1,1 \leq i \leq m;$$

$$f(lu_1lu_i) = 7m + l - 2,1 \leq l \leq m;$$

$$f(lu_1lu_{i+1}) = \begin{cases} 5m + l - 2, & i = 1 \\ m + l, & 1 \leq l \leq m \end{cases}$$

At this time, $f(V) = \{1, 2, 3, 7, 8, 9, 10, 17, 18, 19\}$ and $f(E) = \{11, 12, \ldots, 16\} \cup \{4, 5, 6, 20, 21, 22\}$, which gives $f(V(G)) \cup f(E(G)) = \{1, 2, 3, 7, 8, 9, 10, 17, 18, 19\}$ and $f(V(G)) \cap f(E(G)) = \emptyset$.

And the $S(lu_i) = f(lu_i) + f(lu_1lu_2) + f(lu_1lu_{n-1}) = 11m + 3l - 1,1 \leq i \leq m; S(lu_i) = f(lu_i) + f(lu_iu_{i+1}) + f(lu_iu_{i+1}) = 11m + 3l - 1,2 \leq i \leq n - 1; S(lu_i) = f(lu_i) + f(lu_iu_{i+1}) + f(lu_1lu_{i+1}) = 11m + 3l - 2,1 \leq i \leq n - 1; S(lu_i) = f(lu_i) + f(lu_iu_{i+1}) + f(lu_iu_{i+1}) = 11m + 3l - 2,1 \leq i \leq n - 1$, the sum of the labels of all elements in each set is the same. And $S(v_i) = f(v_i) + f(v_iu_1) + f(v_iu_{i+1}) + f(v_iu_{i+1}) = 16m - 2,1 \leq i \leq m; S(v_0v_i) = f(v_0v_i) + f(v_0v_{i+1}) + f(v_0v_{i+1}) = 16m - 2,1 \leq i \leq m - 2$, which give the same sum of labels of all elements in the set $\{v_1, v_2, \ldots, v_m\}$.

Therefore, AVRTL exists for $S_m^n$ ($n = 3, m = 3$).

(2) When $n = 4, m = 3$, a mapping about $f$ can be obtained as follows.

$$f(v_i) = \begin{cases} 4m + 1, & i = 0 \\ 5m - i + 2, & 1 \leq i \leq m; \end{cases}$$

$$f(lu_i) = \begin{cases} 9m + l - 2, & i = 1 \\ 3i + 3l - 2, & 1 \leq i \leq n - 1; \end{cases}$$

$$f(v_0v_i) = 4m - i + 1,1 \leq i \leq m;$$

$$f(lu_1lu_i) = 1,1 \leq i \leq m;$$

$$f(lu_1lu_{i+1}) = \begin{cases} 5m + l + 3,1 \leq i \leq n - 2 \\ 6m + l - 2, & i = n - 1 \end{cases}$$

At this time, $f(V) = \{4, 5, \ldots, 9\} \cup \{13, 14, 15, 16, 26, 27, 28\}$ and $f(E) = \{20, 21, \ldots, 25\} \cup \{1, 2, 3, 10, 11, 12, 17, 18, 19\}$, which gives $f(V(G)) \cup f(E(G)) = \{1, 2, 3, \ldots, 25\}$ and $f(V(G)) \cap f(E(G)) = \emptyset$.

And the $S(lu_i) = f(lu_i) + f(lu_1lu_2) + f(lu_1lu_{n-1}) = 15m + 3l - 1,1 \leq i \leq m; S(lu_i) = f(lu_i) + f(lu_iu_{i+1}) + f(lu_iu_{i+1}) = 15m + 3l - 1,2 \leq i \leq n - 1; S(lu_i) = f(lu_i) + f(lu_iu_{i+1}) + f(lu_1lu_{i+1}) = 15m + 3l - 2,1 \leq i \leq m$, we can get that in the vertex sets $\{u_1, u_2, \ldots, u_{n-1}\}$, $\{2u_1, 2u_2, \ldots, 2u_{n-1}\}$, $\{3u_1, 3u_2, \ldots, 3u_{n-1}\}$, the sum of the labels of all elements in each set is the same. And $S(v_i) = f(v_i) + f(v_iu_1) + f(v_iu_{i+1}) + f(v_iu_{i+1}) = 24m - 2,1 \leq i \leq m; S(v_0v_i) = f(v_0v_i) + f(v_0v_{i+1}) + f(v_0v_{i+1}) + f(v_0v_{i+1}) = 24m - 2,1 \leq i \leq m - 2$, which give the same sum of labels of all elements in the set $\{v_1, v_2, \ldots, v_m\}$.

Therefore, AVRTL exists for $S_m^n$ ($n = 4, m = 3$).

To sum up, Theorem 13 holds.

Conjecture 2: AVRTL exists for the graph $S_m^n$ when $n \geq 3, m \geq 2$.
V. CONCLUSION

This paper proposes the notion of Adjacent Vertex Reducible Total Labeling based on the Vertex Magic Total Labeling and Reducible Coloring concept. And a new related algorithm is designed based on the properties and constraints of this labeling for the real problem that the Adjacent Vertex Reducible Total Labeling model can describe. The algorithm solves the Adjacent Vertex Reducible Total Labeling of any simple connected graph within a finite number of points in a circular, iterative merit-seeking way. By analyzing the labeling cases from the experimental result set, the labeling rules of road graphs, circle graphs, star graphs, wheel graphs, fan graphs, friendship graphs, and several joint graphs were found, and the corresponding theorems and proofs, as well as two conjectures, were given.

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