Stability of Nonlinear Semi-Markovian Switched Stochastic Systems with Synchronously Impulsive Jumps Driven by G-Brownian Motion

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Abstract—**This study is devoted to investigate the stability of nonlinear stochastic semi-Markovian switched systems driven by G-Brownian motion. Firstly, we study the existence of global unique solution by using stopping time theory. Secondly, by applying G-Itô formula and ergodic theory, we analyze the stability of the system. Finally, we give an example to verify the stability.**

Index Terms—**Nonlinear semi-Markovian switched stochastic system; almost sure exponential stability; G-Brownian motion; synchronously impulsive jumps**

I. Introduction

Most systems do not satisfy the principle of linear superposition. Thence, except for a small part that can be approximately regarded as linear systems, most of them are nonlinear systems, such as turbulent system of fluids ([7]), simple pendulum systems ([17]) and gravitational three-body systems ([24]). In the physical world, the nonlinear system is the essence and the linear system is the approximation or part of the nonlinear system. Therefore, it is necessary to discuss the property of the nonlinear system. The research of nonlinear system has always been a hot issue in the field of control. For example, Chen et al. ([2]) discussed the finite-horizon two-player zero-sum game for the continuous-time nonlinear system by defining a novel Z-function and proposing a completely model-free reinforcement learning-based method with reduced dimension of the basis functions. Guo et al. ([9]) studied a novel fuzzy logic-aided event-driven-observer-based fault-tolerant control approach for a class of nonlinear systems with time-varying actuator faults and unmatched disturbances. Jin et al. ([10]) investigated the nonlinear characteristics of a jointed beam by using multiple identification methods.

Many practical systems have unpredictable structural changes due to the failure of system components, changes in parameters, changes in the correlation between subsystems, sudden changes in environmental conditions and sudden changes in operating points of nonlinear systems. The changes can be described by the Markov jump system. The residence time of the Markov chain is exponentially distributed and its transition probability is constant, which is conservative to a certain extent. Different from the Markov jumping system, the transition probability of the semi-Markov jumping system can not only obey the exponential distribution, but also obey the Weber distribution, Gaussian distribution. Because its probability distribution condition is loose, the semi-Markov jump system has more extensive applications than the Markov jump system. Some authors have studied the semi-Markov jump system. For instance, Qi et al. ([21]) discussed the problem of sliding mode control design for nonlinear stochastic singular semi-Markov jump systems. Tian et al. ([23]) analyzed the dynamic output-feedback control problem for a class of linear semi-Markov jump systems in discrete-time domain. Wang and Zhu ([25]) investigated the asymptotic stability of semi-Markov switched stochastic systems. Zong et al. ([33]) dealt with L_1 control of positive semi-Markov jump systems with state delay.

In recent years, Peng pioneered the concept of G-expectation and established a corresponding theoretical system ([18]–[20]). Because of the widely applications in the fields of risk measurements, G-Brownian motion has attracted the attention of many scholars ([1], [14], [30]). Furthermore, stochastic systems driven by G-Brownian motion has attracted wide attention. Faizullah et al. ([5]) studied the existence-uniqueness and exponential estimate of solutions for stochastic functional differential equations driven by G-Brownian motion. By applying aperiodically intermittent adaptive control, Li et al. ([13]) investigated the stabilisation of multi-weights stochastic complex networks with time-varying delay driven by G-Brownian motion. Zhu and Huang ([32]) discussed the stability for a class of stochastic delay nonlinear systems driven by G-Brownian motion.

The nonlinear characteristic of the systems make the performance of the systems more complicated, which brings difficulties to the analysis of stability of systems. Stability has always been the most fundamental and core issue in system analysis. In recent years, lots of results about stability has been reported in the literature ([4], [8], [26]). For example, Dong and Zhang ([3]) analyzed the stochastic stability of composite dynamic system for particle swarm optimization. Ngoc ([16]) used a new method to investigate the mean square exponential stability of stochastic delay systems. Song et al. ([22]) dealt with the delay-dependent stability of a class of hybrid neutral stochastic differential equations with multiple delays which is highly nonlinear. Xiao and Zhu ([29]) studied the stability of switched stochastic delay systems with unstable subsystems. However, few literature has reported the almost sure exponential stability of nonlinear semi-Markovian switched stochastic systems

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with synchronously impulsive jumps driven by G-Brownian motion. In this paper, the existence of global unique solution to nonlinear semi-Markovian switched stochastic systems with synchronously impulsive jumps driven by G-Brownian motion is proved. The almost sure exponential stability of the system is investigated by using G-Itô formula, the ergodic property of semi-Markovian process and discrete-time Markov chain.

The rest of this paper is organized as follows. In Section 2, the model is given and some lemmas, definitions, assumptions are introduced. In Section 3, the existence and uniqueness of global solution are derived, the almost sure exponential stability of the system is studied as well. In Section 4, an example is provided. In Section 5, the conclusion and future work are given.

II. Problem Formulation and Preliminaries

Let $(\{\mathscr{F}_t\}_{t\geq 0})$ be a filtration generated by G-Brownian motion $\{B(t), t \geq 0\}$. Define $M_G^{p,0}([0,t],\mathbb{R}^n,\mathbb{S}) = \{\alpha_t(\omega) = \Sigma_{j=1}^{N-1}\gamma_{ij}\beta_{t_j}(\omega)I_{[t_j,t_{j+1})}; \beta_{t_j} \in L^p_{\mathscr{F}_t}(\Omega;\mathbb{R}^n),$ $t > 0\}$, $M_G^p([0,t],\mathbb{R}^n,\mathbb{S})$:= the completion of $M_G^{p,0}([0,t],\mathbb{R}^n,\mathbb{S})$ under the norm $\|\alpha\|_{M_G^p([0,t],\mathbb{R}^n,\mathbb{S})} = (\int_0^t \widehat{\mathbb{E}}|\alpha_s|^p ds)^{\frac{1}{p}}$ where $L^p_{\mathscr{F}_t}(\Omega;\mathbb{R}^n$:= the family of all $\mathscr{F}_t$ measurable $\mathbb{R}^n$-valued stochastic variables β satisfies $\widehat{\mathbb{E}}|\beta|^p < \infty$.

We consider the following nonlinear semi-Markovian switched stochastic system with impulse effects driven by G-Brownian motion:

$$\begin{cases} dx(t) = f(x(t),r(t))dt + g(x(t),r(t))d<B>(t) \\ \qquad\qquad + h(x(t),r(t))dB(t), t \neq t_k, \\ x(t) = m(x(t^-),\sigma(k)), \quad t = t_k, k \in \mathbb{N}^+, \end{cases} \tag{1}$$

where $B(t)$ is a one-dimensional G-Brownian motion with $G(a) := \frac{1}{2}\widehat{\mathbb{E}}[aB^2(1)] = \frac{1}{2}(\overline{\sigma}^2 a^+ + \underline{\sigma} a^-)$, for $a \in \mathbb{R}$, where $a^+ = \max\{a,0\}$, $a^- = \max\{-a,0\}$, $\overline{\sigma}^2 = \widehat{\mathbb{E}}[B^2(1)]$, $\underline{\sigma}^2 = -\widehat{\mathbb{E}}[-B^2(1)]$, $<B>(t)$ is the quadratic variation process of the G-Brownian motion $B(t)$, $\widehat{\mathbb{E}}$ stands for the G-expectation. $r(t) : [0,\infty) \to \mathcal{S}_S = \{1,2,\cdots,r\}$ is the switching signal, which is a piecewise constant function specifying the index of the active subsystem. The switching time sequence $\mathcal{I}_S = \{T_1, T_2, \cdots\}$ where $r(t)$ chooses a value randomly from $\mathcal{S}_S$ is strictly increasing and satisfies $\lim_{k\to\infty} T_k = \infty$. $\{\sigma(k), k \in \mathbb{N}^+\}$ is a stochastic process representing the type of jump map at the kth impulse arrival time t_k and taking values in the index set $\mathcal{S}_J$. The impulse arrival time sequence $\mathcal{I}_J = \{t_1, t_2, \cdots\}$ where $\sigma(k)$ chooses a value randomly from $\mathcal{S}_J$ is strictly increasing and satisfies $\lim_{k\to\infty} t_k = \infty$. $f : \mathbb{R}^n \times \mathcal{S}_S \to \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathcal{S}_S \to \mathbb{R}^n$, $h : \mathbb{R}^n \times \mathcal{S}_J \to \mathbb{R}^n$.

For $r(t)$, let $\mathcal{M}_S^i(t)$ be the activated number of the ith subsystem in $(0,t]$, $\mathcal{M}_{ij}(t)$ be the number of switching from the ith subsystem to the jth subsystem in $(0,t]$ and $\mathcal{Q}_i(t)$ be the total time for system (1) active on the ith subsystem in $(0,t]$, $i \in \mathcal{S}_S$. For $\{t_k, k \in \mathbb{N}^+\}$, let $\mathcal{M}_J(t)$ be the total number of impulse occurring in $(0,t]$ with $\mathcal{M}_J(0) = 0$. Let $\alpha(k) = t_k - t_{k-1}$ denote the impulsive interval between the $k-1$th impulse arrival time and kth impulse arrival time.

Firstly, we introduce some assumptions, definitions and lemmas which are very important in the proof of main results.

Assumption 1: There exists a positive constant L_K, for $\forall t \geq 0$, $|x_1| \vee |x_2| \leq K$ and $i \in \mathcal{S}_S$, $|f(x_1,i) - f(x_2,i)| \vee |g(x_1,i) - g(x_2,i)| \vee |h(x_1,i) - h(x_2,i)| \leq L_K(|x_1 - x_2|)$.

Assumption 2: There exists nonnegative function $V(x,i) : \mathbb{R}^n \times \mathcal{S}_S \to [0,\infty)$ which is twice differentiable in x and positive constant b_1 such that $\lim_{|x|\to\infty} \inf_{t\geq 0, i\in\mathcal{S}_S} V(x,i) = \infty$ and $\mathcal{L}V(x,i) \leq b_1 V(x,i)$.

Assumption 3: $f(0,i) \equiv 0, \quad g(0,i) \equiv 0, \quad h(0,i) \equiv 0$, $\forall i \in \mathcal{S}_S$.

Assumption 4: $T_k = t_k$, $\mathcal{S}_J = \mathcal{S}_S \times \mathcal{S}_S$, $\sigma_k = r_{k-1}, r_k$.

Definition 1: If there exists a constant $\lambda > 0$ satisfying

$$\lim_{t\to\infty} \sup \frac{1}{t} \log(|x(t;x_0,r_0)|) < -\lambda,$$

for any $x_0 \in \mathbb{R}^n$ and $r_0 \in \mathcal{S}_S$. The system (1) is almost sure exponential stability.

Given $V(x,i) : \mathbb{R}^n \times \mathcal{S}_S \to [0,\infty)$, we define the operator $\mathcal{L}V$ by

$$\begin{aligned} & \mathcal{L}V(x,i) \\ = \; & V_x(x,i)f(x,y,t,i) \\ + \; & G(2V_x(x,i)g(x,i) + h^T(x,i)V_{xx}(x,i)h(x,i)) \end{aligned}$$

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Definition 2: If $r(t) = r(T_k) = r_k$, $t \in [T_k, T_{k+1})$, $k \in \mathbb{N}$ satisfies: (1) The process $\{r_k, k \in \mathbb{N}\}$ is a Markov chain with $P = [p_{ij}]_{r\times r}$, $i,j \in \mathcal{S}_S$ in which $p_{ij} = \mathbb{P}\{r(T_{k+1}) = j | r(T_k) = i\}$; (2) The distribution function of $\delta_{k+1} = T_{k+1} - T_k$ is

$$\begin{aligned} F_{ij}(s) \quad & = \mathbb{P}\{\delta_{k+1} \leq s | r(k) = i, r(k+1) = j\}, \\ & i,j \in \mathcal{S}_S, s \geq 0, \end{aligned}$$

which has the continuous differentiable density $f_{ij}(t)$ and does not depend on k.

Then, the switching signal $r(t)$ is said to be semi-Markovian switching.

Let δ_i be the sojourn time on the ith subsystem, $i \in \mathcal{S}_S$. Thus, its distribution function can be defined as follows:

$$\begin{aligned} & F_i(s) \\ & = \mathbb{P}\{\delta_i \leq s\} = \mathbb{P}\{T(k+1) - T(k) \leq s | r_k = i\} \\ & = \sum_{i,j\in\mathcal{S}_S} F_{ij}(s)p_{ij}, s \geq 0, k \in \mathbb{N}. \end{aligned}$$

It is assumed that $p_{ii} = 0$ and $\{r_k, k \in \mathbb{N}\}$ is irreducible. Therefore, $\{r_k, k \in \mathbb{N}\}$ has a stationary distribution π satisfying $\pi P = \pi$ and $\sum_{i\in\mathcal{S}_S} \pi_i = 1$.

Lemma 1: [20] For any $0 \leq s \leq t < \infty$,

$$\begin{aligned} & \widehat{\mathbb{E}}[|\int_0^t \nu_s d<B>(s)|] \\ & \leq \overline{\sigma}^2 \widehat{\mathbb{E}}[\int_0^t |\nu_s| d(s)], \forall \nu_s \in M_G^1([0,t],\mathbb{R}^n,\mathbb{S}), \end{aligned}$$

$$\begin{aligned} & \widehat{\mathbb{E}}[(\int_0^t \nu_s dB(s))^2] \\ & = \widehat{\mathbb{E}}[\int_0^t |\nu_s^2| d<B>(s)], \forall \nu_s \in M_G^2([0,t],\mathbb{R}^n,\mathbb{S}), \end{aligned}$$

$$\widehat{\mathbb{E}}[(\int_0^t |\nu_s|^2 ds)^2] \leq \int_0^t \widehat{\mathbb{E}}[|\nu_s|^2] ds, \forall \nu_s \in M_G^2([0,t],\mathbb{R}^n,\mathbb{S}),$$

Lemma 2: [12] For $0 \leq s \leq t < \infty$,

$$\widehat{\mathbb{E}}[\sup_{s\leq d\leq t} |\int_s^d \nu_s d<B>(s)|^2] \leq \overline{\sigma}^4|t-s|^2 \int_s^t \widehat{\mathbb{E}}[|\nu_s|^2]d_s, \forall \nu_s \in M_G^1([0,t],\mathbb{R}^n,\mathbb{S}).$$

Lemma 3: [11]

$$\widehat{\mathbb{E}}[\sup_{s\leq d\leq t} |\int_s^d \nu_s dB(s)|^2] \leq 4\overline{\sigma}^4|t-s|^2 \int_s^t \widehat{\mathbb{E}}[|\nu_s|^2]d_s, \forall \nu_s \in M_G^2([0,t],\mathbb{R}^n,\mathbb{S}).$$

Lemma 4: [19] (G-Itô formula): Let $\varphi \in \mathcal{C}^{1,2}(\mathbb{R}^n \times \mathbb{R}_+;\mathbb{R}_+)$ and

$$X_t = X_0 + \int_0^t f_s ds + \int_0^t g_s d<B>_s + \int_0^t h_s dB_s,$$

where $f,g,h \in M_G^2(0,T;\mathbb{R}^n)$. Then, for $\forall t>0$,

$$\begin{aligned}&\varphi(X_t,t)-\varphi(X_0,t)\\ &= \int_0^t [\partial_s\varphi(X_s,s)+\partial_x\varphi(X_s,s)f_s+G(2\partial_x\varphi(X_s,s)g_s)\\ &+\partial_{xx}\varphi(X_s,s)h_s^2]ds\\ &+\int_0^t \partial_x\varphi(X_s,s)h_s dB_s + \int_0^t [\partial_x\varphi(X_s,s)g_s\\ &+\frac{1}{2}\partial_{xx}\varphi(X_s,s)h_s^2]d<B>_s\\ &-\int_0^t G(2\partial_x\varphi(X_s,s)g_s+\partial_{xx}\varphi(X_s,s)h_s^2)ds.\end{aligned}$$

Lemma 5: [27] Let $\{r(t), t\geq 0\}$ be a semi-Markovian process and $\{r_k, k\in\mathbb{N}\}$ is its associated embedded Markov chain. Thus,

$$\lim_{t\to\infty}\frac{Q_i(t)}{t} = \overline{\pi}_i, \quad a.s., i\in\mathcal{S}_S,$$
$$\lim_{t\to\infty}\frac{\mathcal{M}_S^i(t)}{t} = \frac{\overline{\pi}_i}{\widehat{\mathbb{E}}[\delta_i]}, \quad a.s., i\in\mathcal{S}_S,$$

where $\overline{\pi}$ is the stationary distribution of $r(t)$.

III. Main Results

In the following theorem, we proved the existence of the global unique solution under the linear growth condition and local Lipschitz condition.

Theorem 1: When Assumptions 1-3 hold, the system (1) has a global unique solution $\{x(t), t\geq 0\}$.

Proof: Let the initial value $|x_0|\leq\xi$. For $k\geq\xi$, $k\in\mathbb{N}$, we suppose that

$$\begin{aligned}&f^{(k)}(x,i) = f(\frac{|x|\wedge k}{|x|}x,i), \quad g^{(k)}(x,i) = g(\frac{|x|\wedge k}{|x|}x,i),\\ &h^{(k)}(x,i) = h(\frac{|x|\wedge k}{|x|}x,i),\end{aligned} \tag{2}$$

where $(\frac{|x|\wedge k}{|x|}x) = 0$ when $x=0$.

We obtain that $f^{(k)}$, $g^{(k)}$ and $h^{(k)}$ satisfy the linear growth condition and local Lipschitz condition. Thus,

$$\begin{aligned}&dx_k(t) = f^{(k)}(x_k(t),r(t))dt + g^{(k)}(x_k(t),r(t))d<B>(t)\\ &+h^{(k)}(x_k(t),r(t))dB(t),\end{aligned} \tag{3}$$

has the global unique solution.

Let

$$\eta_k = \inf\{t\geq 0: |x_k(t)|\geq k\}, \tag{4}$$

where $k\in\mathbb{N}$, $\inf\phi=\infty$.

When $0\leq t\leq\eta_k$, $x_k(t) = x_{k+1}$. Then, $\{\eta_k\}$ is increasing. Thus, there exists a stopping time η such that

$$\eta = \lim_{k\to\infty}\eta_k. \tag{5}$$

Let

$$x(t) = \lim_{k\to\infty} x_k(t), \quad 0\leq t<\eta. \tag{6}$$

It is easy to check that when $0\leq t<\eta$, $x(t)$ is the unique solution of system (1).

By using G-Itô formula, for $t\geq 0$, we obtain

$$\begin{aligned}&V(x_k(t\wedge\eta_k), r(t\wedge\eta_k))\\ &= V(\xi(0),r_0) + \int_0^{t\wedge\eta_k}\mathcal{L}^{(k)}V(x_k(s),r(s))ds\\ &+\int_0^{t\wedge\eta_k} V_x(x_k(s),r(s))h(x_k(s),r(s))dB(s)\\ &+\int_0^{t\wedge\eta_k}[V_x(x_k(s),r(s))g(x_k(s),r(s))\\ &+\frac{1}{2}h^T(x_k(s),r(s))V_{xx}(x_k(s),r(s))\\ &h(x_k(s),r(s))]d<B>(s)\\ &-\int_0^{t\wedge\eta_k} G(2V_x(x_k(s),r(s))g(x_k(s),r(s))\\ &+h^T(x_k(s),r(s))V_{xx}(x_k(s),r(s))\\ &h(x_k(s),r(s)))ds\end{aligned}$$

where $\mathcal{L}^{(k)}V(x_k(s),r(s)) = \mathcal{L}V(x_k(s),r(s))$ when $0\leq s\leq t\wedge\eta_k$.

From ([20]), we know that

$$\widehat{\mathbb{E}}[\int_0^{t\wedge\eta_k} V_x(x_k(s),r(s))h(x_k(s),r(s))dB(s)] = 0,$$

and

$$\begin{aligned}&\widehat{\mathbb{E}}[\int_0^{t\wedge\eta_k}[V_x(x_k(s),r(s))g(x_k(s),r(s))\\ &+\frac{1}{2}h^T(x_k(s),r(s))V_{xx}(x_k(s),r(s))\\ &h(x_k(s),r(s))]d<B>(s)\\ &-\int_0^{t\wedge\eta_k} G(2V_x(x_k(s),r(s))g(x_k(s),r(s))\\ &+h^T(x_k(s),r(s))V_{xx}(x_k(s),r(s))\\ &h(x_k(s),r(s)))ds]\leq 0.\end{aligned}$$

Then, it can be checked that

$$\begin{aligned}&\widehat{\mathbb{E}}[V(x_k(t\wedge\eta_k),r(t\wedge\eta_k))]\\ &\leq\widehat{\mathbb{E}}[V(\xi(0),r_0)] + \widehat{\mathbb{E}}[\int_0^{t\wedge\eta_k}\mathcal{L}^{(k)}V(x_k(s),r(s))ds]\\ &\leq\widehat{\mathbb{E}}[V(\xi(0),r_0)] + b_1\int_0^{t\wedge\eta_k}\widehat{\mathbb{E}}[V(x_k(s),r(s))]ds.\end{aligned}$$

According to the Gronwall's inequality, we obtain

$$\begin{aligned}&\widehat{\mathbb{E}}[V(x_k(t\wedge\eta_k),r(t\wedge\eta_k))]\\ &\leq\widehat{\mathbb{E}}[V(\xi(0),r_0)]e^{b_1(t\wedge\eta_k)}.\end{aligned} \tag{7}$$

Furthermore, as

$$\begin{aligned}&\mathbb{P}\{\eta_k \le t\} \inf_{|x|\ge n, i\in\mathcal{S}_S} V(x,i)\\ &\le \int_{\eta_k\le t} V(x_k(t\wedge\eta_k), r(t\wedge\eta_k))dP\\ &\le \widehat{\mathbb{E}}V(x_k(t\wedge\eta_k), r(t\wedge\eta_k)),\end{aligned}$$

we have

$$\mathbb{P}\{\eta_k \le t\} \le \frac{\widehat{\mathbb{E}}[V(\xi(0), r_0)]e^{b_1(t\wedge\eta_k)}}{\inf_{|x|\ge n, i\in\mathcal{S}_S} V(x,i)}. \tag{8}$$

When $t\to\infty$,

$$\mathbb{P}\{\eta \le t\} = 0. \tag{9}$$

Thus,

$$\mathbb{P}\{\eta = \infty\} = 1. \tag{10}$$

The proof is complete. ■

In the following theorem, the almost sure exponential stability of system (1) is discussed.

Theorem 2: If there exists a function $V(x,i):\mathbb{R}^n\times\mathcal{S}_S\to[0,\infty)$ and some positive constants $a_i, c_i, d_i, e_i, l_{ij}, \beta, p$ satisfies

$$V_x(x,i)f(x,i) \le a_iV(x,i), \tag{11}$$

$$h^T(x,i)V_{xx}(x,i)h(x,i) \le c_iV(x,i), \tag{12}$$

$$V_x(x,i)g(x,i) \le d_iV(x,i), \tag{13}$$

$$|V_x(x,i)h(x,i)|^2 \ge e_iV^2(x,i), \tag{14}$$

$$\beta|x|^p \le V(x,i), \tag{15}$$

$$V(m(x,i,j),j) \le l_{ij}V(x,i), \tag{16}$$

for $\forall(x,i)\in(\mathbb{R}^n\times\mathcal{S}_S)$, and $\Sigma_{i,j\in\mathcal{S}_S}\overline{\pi}_i(\frac{p_{ij}}{\widehat{\mathbb{E}}[\delta_i]}\log l_{i,j}+\overline{\sigma}^2(a_i+\frac{c_i}{2}+d_i)-\frac{1}{2}\underline{\sigma}^2e_i)<0$, the system (1) is almost sure exponential stability.

Proof: By using G-Itô formula, for $\forall i\in\mathcal{S}_S$, $t>0$, we get

$$\begin{aligned}&\log V(x(t),i)\\ &= \log V(\xi(0),i) + \int_0^t \frac{(V_x(x(s),i)f(x(s),i))}{V(x(s),i)}ds\\ &+\int_0^t \frac{h^T(x(s),i)V_{xx}(x(s),i)h(x(s),i)}{2V(x(s),i)}d<B>(S)\\ &+\int_0^t \frac{V_x(x(s),i)g(x(s),i)}{V(x(s),i)}d<B>(S)\\ &-\frac{1}{2}\int_0^t \frac{[V_x(x(s),i)h(x(s),i)]^2}{V^2(x(s),i)}d<B>(S)\\ &+\int_0^t \frac{V_x(x(s),i)h(x(s),i)}{V(x(s),i)}dB(s).\end{aligned}$$

Then, we have

$$\begin{aligned}&\log V(x(t),r(t))\\ &= \log V(x(T_k),r_k) + \int_{T_k}^t \frac{(V_x(x(s),i)f(x(s),i))}{V(x(s),i)}ds\\ &+\int_{T_k}^t \frac{h^T(x(s),i)V_{xx}(x(s),i)h(x(s),i)}{2V(x(s),i)}d<B>(S)\\ &+\int_{T_k}^t \frac{V_x(x(s),i)g(x(s),i)}{V(x(s),i)}d<B>(S)\\ &-\frac{1}{2}\int_{T_k}^t \frac{[V_x(x(s),i)h(x(s),i)]^2}{V^2(x(s),i)}d<B>(S)\\ &+\int_{T_k}^t \frac{V_x(x(s),i)h(x(s),i)}{V(x(s),i)}dB(s), \quad t\in[T_k,T_{k+1}),\\ &k\in\mathbb{N},\end{aligned}$$

and

$$\begin{aligned}&\log V(x(T_k^-),r_{k-1})\\ &= \log V(x(T_{k-1}),r_{k-1})\\ &+\int_{T_{k-1}}^{T_k} \frac{(V_x(x(s),i)f(x(s),i))}{V(x(s),i)}ds\\ &+\int_{T_{k-1}}^{T_k} \frac{h^T(x(s),i)V_{xx}(x(s),i)h(x(s),i)}{2V(x(s),i)}d<B>(S)\\ &+\int_{T_{k-1}}^{T_k} \frac{V_x(x(s),i)g(x(s),i)}{V(x(s),i)}d<B>(S)\\ &-\frac{1}{2}\int_{T_{k-1}}^{T_k} \frac{[V_x(x(s),i)h(x(s),i)]^2}{V^2(x(s),i)}d<B>(S)\\ &+\int_{T_{k-1}}^{T_k} \frac{V_x(x(s),i)h(x(s),i)}{V(x(s),i)}dB(s), \quad k\in\mathbb{N}^+.\end{aligned}$$

Then, we obtain

$$\begin{aligned}&\log V(x(T_k),r_k)\\ &= \log V(m(x(T_k^-),r_{k-1},r_k),r_k)\\ &\le \log l_{r_{k-1},r_k} + \log V(x(T_k^-),r_{k-1}), k\in\mathbb{N}^+.\end{aligned}$$

Thus, for any $t\in[T_k,T_{k+1}), k\in\mathbb{N}^+$, we have

$$\begin{aligned}&\log V(x(t),r(t))\\ &\le \log l_{r_{k-1},r_k} + \log V(x(T_{k-1}),r_{k-1})\\ &+\int_{T_{k-1}}^t \frac{(V_x(x(s),i)f(x(s),i))}{V(x(s),i)}ds\\ &+\int_{T_{k-1}}^t \frac{h^T(x(s),i)V_{xx}(x(s),i)h(x(s),i)}{2V(x(s),i)}d<B>(S)\\ &+\int_{T_{k-1}}^t \frac{V_x(x(s),i)g(x(s),i)}{V(x(s),i)}d<B>(S)\\ &-\frac{1}{2}\int_{T_{k-1}}^t \frac{[V_x(x(s),i)h(x(s),i)]^2}{V^2(x(s),i)}d<B>(S)\\ &+\int_{T_{k-1}}^t \frac{V_x(x(s),i)h(x(s),i)}{V(x(s),i)}dB(s).\end{aligned}$$

Then, for any $t \in [T_k, T_{k+1}), k \in \mathbb{N}^+$, we obtain

$$\begin{aligned}
&\log V(x(t), r_k) \\
&\leq \Sigma_{\mu=1}^{k} \log l_{r_{\mu-1}, r_\mu} + \log V(x_0, r_0) \\
&+ \int_0^t \frac{(V_x(x(s), i) f(x(s), i))}{V(x(s), i)} ds \\
&+ \int_0^t \frac{h^T(x(s), i) V_{xx}(x(s), i) h(x(s), i)}{2V(x(s), i)} d < B > (S) \\
&+ \int_0^t \frac{V_x(x(s), i) g(x(s), i)}{V(x(s), i)} d < B > (S) \\
&- \frac{1}{2} \int_0^t \frac{[V_x(x(s), i) h(x(s), i)]^2}{V^2(x(s), i)} d < B > (S) \\
&+ \int_0^t \frac{V_x(x(s), i) h(x(s), i)}{V(x(s), i)} dB(s),
\end{aligned}$$

where $\int_0^t \frac{V_x(x(s),i)h(x(s),i)}{V(x(s),i)} dB(s)$ is a continuous local martingale.

Hence, it can be checked that

$$\begin{aligned}
&\log V(x(t), r(t)) \\
&\leq \Sigma_{i,j \in \mathcal{S}_S} \mathcal{M}_{ij}(t) \log l_{i,j} + \log V(x_0, r_0) \\
&+ \int_0^t \frac{(V_x(x(s), i) f(x(s), i))}{V(x(s), i)} ds \\
&+ \int_0^t \frac{h^T(x(s), i) V_{xx}(x(s), i) h(x(s), i)}{2V(x(s), i)} d < B > (S) \\
&+ \int_0^t \frac{V_x(x(s), i) g(x(s), i)}{V(x(s), i)} d < B > (S) \\
&- \frac{1}{2} \int_0^t \frac{[V_x(x(s), i) h(x(s), i)]^2}{V^2(x(s), i)} d < B > (S) \\
&+ \int_0^t \frac{V_x(x(s), i) h(x(s), i)}{V(x(s), i)} dB(s), \quad t \geq 0.
\end{aligned}$$

According to Lemma 2.6 in ([6]), for $\forall \varepsilon \in (0, 1)$ and all $\omega \in \Omega$, there exists an integer n_0, when $n \geq n_0$, we have

$$\begin{aligned}
&\int_0^t \frac{V_x(x(s), i) h(x(s), i)}{V(x(s), i)} dB(s) \\
&\leq \frac{2}{\varepsilon} \log(n) \\
&+ \frac{\varepsilon}{2} \int_0^t \frac{[V_x(x(s), i) h(x(s), i)]^2}{V^2(x(s), i)} d < B > (S).
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
&\log V(x(t), r(t)) \\
&\leq \Sigma_{i,j \in \mathcal{S}_S} \mathcal{M}_{ij}(t) \log l_{i,j} + \log V(x_0, r_0) \\
&+ \int_0^t \frac{(V_x(x(s), i) f(x(s), i))}{V(x(s), i)} ds \\
&+ \int_0^t \frac{h^T(x(s), i) V_{xx}(x(s), i) h(x(s), i)}{2V(x(s), i)} d < B > (S) \\
&+ \int_0^t \frac{V_x(x(s), i) g(x(s), i)}{V(x(s), i)} d < B > (S) \\
&- \frac{1}{2} \int_0^t \frac{[V_x(x(s), i) h(x(s), i)]^2}{V^2(x(s), i)} d < B > (S) \\
&+ \frac{2}{\varepsilon} \log(n) \\
&+ \frac{\varepsilon}{2} \int_0^t \frac{[V_x(x(s), i) h(x(s), i)]^2}{V^2(x(s), i)} d < B > (S).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\log V(x(t), r(t)) \\
&\leq \Sigma_{i,j \in \mathcal{S}_S} \mathcal{M}_{ij}(t) \log l_{i,j} + \log V(x_0, r_0) + \frac{2}{\varepsilon} \log(n) \\
&+ \overline{\sigma}^2 \int_0^t (a_i + \frac{c_i}{2} + d_i) ds - \underline{\sigma}^2 \int_0^t \frac{(1-\varepsilon)}{2} e_i ds \\
&= \Sigma_{i,j \in \mathcal{S}_S} \mathcal{M}_{ij}(t) \log l_{i,j} + \log V(x_0, r_0) + \frac{2}{\varepsilon} \log(n) \\
&+ \overline{\sigma}^2 \Sigma_{i \in \mathcal{S}_S} (a_i + \frac{c_i}{2} + d_i) \mathcal{Q}_i(t) \\
&- \underline{\sigma}^2 \Sigma_{i \in \mathcal{S}_S} \frac{(1-\varepsilon)}{2} e_i \mathcal{Q}_i(t).
\end{aligned}$$

Thus, it can be checked that

$$\begin{aligned}
&\frac{1}{t} \log V(x(t), r(t)) \\
&\leq \Sigma_{i,j \in \mathcal{S}_S} \frac{\mathcal{M}_{ij}(t)}{t} \log l_{i,j} + \frac{1}{t} \log V(x_0, r_0) \\
&+ \frac{1}{t} \frac{2}{\varepsilon} \log(n) \\
&+ \overline{\sigma}^2 \Sigma_{i \in \mathcal{S}_S} (a_i + \frac{c_i}{2} + d_i) \frac{\mathcal{Q}_i(t)}{t} \\
&- \underline{\sigma}^2 \Sigma_{i \in \mathcal{S}_S} \frac{(1-\varepsilon)}{2} e_i \frac{\mathcal{Q}_i(t)}{t}.
\end{aligned}$$

Since

$$\lim_{t \to \infty} \frac{\mathcal{M}_{ij}(t)}{t} = \lim_{t \to \infty} \frac{p_{ij}(\mathcal{M}_S^i(t) - I(r(t) = i))}{t} = \frac{p_{ij} \overline{\pi}_i}{\widehat{\mathbb{E}}[\delta_i]}, \tag{17}$$

where $I(\cdot)$ denotes the indicator function.

Therefore,

$$\begin{aligned}
&\limsup_{t \to \infty} \frac{1}{t} \log V(x(t), r(t)) \\
&\leq \Sigma_{i,j \in \mathcal{S}_S} \overline{\pi}_i (\frac{p_{ij}}{\widehat{\mathbb{E}}[\delta_i]} \log l_{i,j} + \overline{\sigma}^2 (a_i + \frac{c_i}{2} + d_i) \\
&- \frac{(1-\varepsilon)}{2} \underline{\sigma}^2 e_i).
\end{aligned}$$

When $\varepsilon \to 0$, we obtain

$$\begin{aligned}
&\limsup_{t \to \infty} \frac{1}{t} \log V(x(t), r(t)) \\
&\leq \Sigma_{i,j \in \mathcal{S}_S} \overline{\pi}_i (\frac{p_{ij}}{\widehat{\mathbb{E}}[\delta_i]} \log l_{i,j} + \overline{\sigma}^2 (a_i + \frac{c_i}{2} + d_i) - \frac{1}{2} \underline{\sigma}^2 e_i) \\
&< 0.
\end{aligned}$$

Therefore, the system (1) is almost sure exponential stability.

The proof is complete. ■

Remark 1: Different from the Markov jumping system, the transition probability of the semi-Markov jumping system can not only obey the exponential distribution, but also obey the Weber distribution, Gaussian distribution. Because its probability distribution condition is loose, the semi-Markov jump system has more extensive applications than the Markov jump system. Theorem 2 presents the relationship between system parameters and stability, and explicitly shows the relationship between semi-Markov switching and stability.

IV. Example

$B(t) \sim N(0, [\underline{\sigma}^2, \overline{\sigma}^2])$ is a G-Brownian motion.

Consider the following nonlinear semi-Markovian switched stochastic systems with synchronously impulsive jumps driven by G-Brownian motion:

$$\begin{cases} dx(t) = f(x(t), r(t))dt + g(x(t), r(t))d < B > (t) \\ \quad + h(x(t), r(t))dB(t), t \neq t_k, \\ x(t) = m(x(t^-), \sigma(k)), \quad t = t_k, k \in \mathbb{N}^+, \end{cases}$$

where

$$f(x(t), 1) = -3x(t), \quad g(x(t), 1) = \frac{1}{8}x(t),$$
$$h(x(t), 1) = \frac{1}{4}x(t),$$

$$f(x(t), 2) = -2x(t), \quad g(x(t), 2) = \frac{1}{6}x(t),$$
$$h(x(t), 2) = \frac{1}{2}x(t),$$

$$m(x(t), 1, 2) = \frac{1}{2}x(t),$$

$$m(x(t), 2, 1) = \frac{1}{4}x(t).$$

Let $V(x, i) = x^2, i = 1, 2$, we have

$$V_x(x, 1)f(x, 1) \leq 0.1V(x, 1),$$
$$h^T(x, 1)V_{xx}(x, 1)h(x, 1) \leq 0.25V(x, 1),$$
$$V_x(x, 1)g(x, 1) \leq 0.5V(x, 1),$$
$$|V_x(x, 1)h(x, 1)|^2 \geq 0.125V^2(x, 1),$$
$$\frac{1}{2}|x|^2 \leq V(x, 1),$$
$$V(m(x, 1, 2), 2) \leq 0.5V(x, 1),$$
$$V_x(x, 2)f(x, 2) \leq 0.1V(x, 2),$$
$$h^T(x, 2)V_{xx}(x, 2)h(x, 2) \leq 0.6V(x, 2),$$
$$V_x(x, 2)g(x, 2) \leq 0.4V(x, 2),$$
$$|V_x(x, 2)h(x, 2)|^2 \geq 0.2V^2(x, 2),$$
$$\frac{1}{2}|x|^2 \leq V(x, 2),$$
$$V(m(x, 2, 1), 1) \leq 0.125V(x, 2),$$

where $\beta = \frac{1}{2}$, $a_1 = 0.1$, $c_1 = 0.25$, $d_1 = 0.5$, $e_1 = 0.125$, $l_{1,2} = 0.5$,$a_2 = 0.1$, $c_2 = 0.6$, $d_2 = 0.4$, $e_2 = 0.2$, $l_{2,1} = 0.125$

Let the transition probability matrix of r_k as follows:

$$P = \begin{pmatrix} 0 & 0.5 \\ 0.3 & 0 \end{pmatrix}$$

and $\widehat{\mathbb{E}}[\delta_1] = 0.3$, $\widehat{\mathbb{E}}[\delta_2] = 0.4$.

Thus, we can obtain $\overline{\pi} = (0.1839, 0.8161)$.

Let $\overline{\sigma}^2 = 1$ and $\underline{\sigma}^2 = -1$, we obtain

$$\Sigma_{i,j \in \mathcal{S}_S} \overline{\pi}_i \left(\frac{p_{ij}}{\widehat{\mathbb{E}}[\delta_i]} \log l_{i,j} + \overline{\sigma}^2 (a_i + \frac{c_i}{2} + d_i) - \frac{1}{2}\underline{\sigma}^2 e_i \right)$$
$$= -0.6393 < 0.$$

V. Conclusions

In this paper, we have studied the almost sure exponential stability of nonlinear semi-Markovian switched stochastic systems with synchronously impulsive jumps driven by G-Brownian motion. The existence of the global unique solution has been proved by using G-Itô formula and Gronwall's inequality. The almost sure exponential stability of the system has been analyzed by applying the ergodic property of semi-Markovian process and discrete-time Markov chain. Future work will include stability of nonlinear semi-Markovian switched stochastic systems with asynchronously impulsive jumps driven by G-Brownian motion.

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