Formulated Optimal Solution for EOQ Model with Fuzzy Demand

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Abstract—This paper aims to prove that when developing an inventory model with fuzzy demand, the order quantity should be identical for each replenishment cycle in order to achieve an optimal replenishment policy. In 2012, Glock, Schwindl and Jaber published a paper in the International Journal Services and Operations Management. They assumed that the order quantities should be identical for each replenishment cycle. We provide a patch work to prove that to attain the minimum solution, then the order quantities must be identical for each replenishment cycle. Moreover, a formulated solution for the optimal replenishment number is derived in this paper. On the other hand, we discuss two related problems in Analytic Hierarchy Process to present some theoretical derivations to help researchers realize the mathematical structure in those problems.

Index Terms—Fuzzy demand, Formulated optimal solution, Replenishment number, Economic ordering quantity model, Wash criterion, Rank reversal, Analytic hierarchy process, Decision analysis

I. INTRODUCTION

Glock et al. [1] mentioned their inventory model is limited to order lots of equal sizes. Glock et al. [1] also suggested that in possible future research, practitioners could consider inventory models with unequal order sizes. The purpose of this paper is to further develop their inventory model without the restriction of equal order sizes. Up to now, there are nine papers that have cited Glock et al. [1] in their references. After a careful examination of the related papers, it was found that seven of them: Kumar et al. [2], Al Masud et al. [3], Ghosh et al. [4], Jaaron and Backhouse [5], Shrivastava and Gorantiwar [6], Shekarian et al. [7], and Soni and Joshi [8] only mentioned Glock et al. [1] in their introductions without further discussion with respect to Glock et al. [1]. While the other two papers, Andriolo et al. [9] is a review article analyzing the 219 papers published in the one hundred year development of inventory models. Hence, Andriolo et al. [9] just mentioned Glock et al. [1] in a long publishing list, without providing any improvements for Glock et al. [1]. Kim and Glock [10] only mentioned Glock et al. [1] in the section of possible direction for future study. Recently, there is an important paper of Kazemi et al. [11] to incorporate human learning into a fuzzy EOQ inventory model with backorders. Hence, we can conclude that no researcher provided a solution for the open question proposed by Glock et al. [1] to verify whether or not the optimal solution is found when all order quantities are identical for each replenishment cycle. Therefore the first goal of our paper is to prove that the equal order quantity policy is the best strategy for inventory models under fuzzy environments. Moreover, we provide a formulated optimal solution for their inventory model with fuzzy demand.

II. NOTATIONS AND ASSUMPTIONS

To be compatible with Glock et al. [1], we adopt the same notations and assumptions as follows.

Assumptions

1. Lead time is assumed to be negligible.
2. Shortages are not allowed in this inventory system.
3. The planning horizon is finite. Without loss of generality, we assume that the finite planning horizon as one year.
4. The replenishment is instantaneous, that is, the replenishment rate is infinite.
5. A single manufacturer and a single item are considered in this inventory system.

Notation

\( M(n) \) stands for the total cost under fuzzy environment in variable \( n \), with \( Q_1 = \ldots = Q_n = D/n \) assumed by Glock et al. [1]. In this paper, we will prove that \( Q_1 = \ldots = Q_n = D/n \) is the best replenishment strategy.

\( M(Q) \) stands for the total cost under fuzzy environment.

\( TC(Q) \) stands for the total cost under crisp setting.

\( Q \) stands for the order quantity per replenishment cycle.

\( n \) stands for the number of orders in the unit time.

\( b \) stands for the holding cost per unit item per unit time.

\( D \) stands for the demand per unit time.

\( A \) stands for the ordering cost per replenishment cycle.

III. REVIEW OF THE INVENTORY MODEL WITH FUZZY DEMAND OF GLOCK ET AL. [1]

A fuzzy number, \( \bar{A} \), is denoted by its membership function \( \mu_{\bar{A}}(x) \) as

\[
\mu_{\bar{A}}(x) = \begin{cases} 
A_L(x), & a \leq x \leq b \\
\omega, & b \leq x \leq c \\
A_R(x), & c \leq x \leq d \\
0, & \text{o.w.}
\end{cases} \tag{3.1}
\]

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where $0 < \omega \leq 1$ is a constant, $A_x : [a, b] \rightarrow [0, \omega]$ is a non-decreasing and continuous function, and $A_y : [c, d] \rightarrow [0, \omega]$ is a non-increasing and continuous function. If $\omega = 1$, then $\tilde{A}$ is a normal fuzzy number. If $A_x (x)$ and $A_y (x)$ are linear function, then $\tilde{A}$ is a trapezoidal fuzzy number and is denoted by $\tilde{A} = (a, b, c, d; \omega)$. In this case, 

$$A_x (x) = \omega (x - a) / (b - a), \quad (3.2)$$

and 

$$A_y (x) = \omega (d - x) / (d - c). \quad (3.3)$$

In particular, if $b = c$, then the trapezoidal fuzzy number is reduced to a triangular fuzzy number. In this paper, we follow Glock et al. [1] to consider the demand is a normal triangular fuzzy number with $b = c = D$, $b - a = \Delta_1$ and $d - c = \Delta_2$.

For the initial inventory in Glock et al. [1], they considered the classical EOQ model, 

$$EOQ(Q) = AD \frac{Q}{2} + \frac{Q}{2} h, \quad (3.4)$$

with the crisp demand $D$.

Remark. In this paper, we use $D$ to express two things: (a) The crisp demand, and (b) The center of a normal triangular fuzzy demand.

Glock et al. [1] extended a crisp demand to a triangular fuzzy number, and denoted as $\tilde{D}$, where the membership function of $\tilde{D}$ is expressed as follows,

if $D - \Delta_1 \leq x \leq D$, then 

$$\mu_{\tilde{D}}(x) = \frac{(x - D + \Delta_1)}{\Delta_1}, \quad (3.5)$$

if $D \leq x \leq D + \Delta_2$, then 

$$\mu_{\tilde{D}}(x) = \frac{(D + \Delta_2 - x)}{\Delta_2}, \quad (3.6)$$

otherwise, 

$$\mu_{\tilde{D}}(x) = 0. \quad (3.7)$$

with the center $D$, the left spread $\Delta_1$ and the right spread $\Delta_2$.

We recall the extension principle proposed by Zadeh [12] and then further discussed by Yager [13], Kaufmann and Gupta [14] and Zimmermann [15].

The mapping $f$ is defined on the powers of $X_1 \times X_2 \times \cdots \times X_n$ to the power sets of $Y$, as 

$$f : P(X_1 \times X_2 \times \cdots \times X_n) \rightarrow P(Y), \quad (3.8)$$

where $\tilde{A}_1$, $\tilde{A}_2$, ..., $\tilde{A}_n$ are fuzzy sets defined on $X_1$, $X_2$, ..., $X_n$, then the image $\tilde{B} = f(\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n)$ is a fuzzy set with its membership function, $\mu_{\tilde{B}}(y)$, as 

$$\mu_{\tilde{B}}(y) = \max \left\{ \min \{\mu_{\tilde{A}_1}(x_1), \mu_{\tilde{A}_2}(x_2), ..., \mu_{\tilde{A}_n}(x_n)\} \right\}. \quad (3.9)$$

We will apply the extension principle to find the membership function of the fuzzy inventory model proposed by Glock et al. [1], $(\tilde{AD} / Q) + (Qh / 2)$, we denoted it as $\tilde{B} = (\tilde{AD} / Q) + (Qh / 2). \quad (3.10)$

We have only one membership function, $\mu_{\tilde{D}}(x)$ and the relation of $y = f(x)$ as

$$f(x) = (AD / x) + (xh / 2) \quad (3.11)$$

is one to one such that based on Equation (3.7), we derive the membership function of $\tilde{B}$ as

$$\mu_{\tilde{B}}(y) = \mu_{\tilde{D}}(f^{-1}(y)) = \mu_{\tilde{D}}((2y - hQ)Q / 2A). \quad (3.12)$$

Based on Equations (3.5-3.7), we obtain the membership function as follows,

if $\Omega - \Delta_3 \leq y \leq \Omega$, then 

$$\mu_{\tilde{B}}(y) = (y - \Omega + \Delta_3) / \Delta_3, \quad (3.13)$$

if $\Omega \leq y \leq \Omega + \Delta_4$, then 

$$\mu_{\tilde{B}}(y) = (\Omega + \Delta_4 - y) / \Delta_4, \quad (3.14)$$

otherwise, 

$$\mu_{\tilde{B}}(y) = 0. \quad (3.15)$$

with the center $\Omega = (\tilde{AD} / Q) + (Qh / 2)$, left spread $\Delta_3 = A\Delta_1 / Q$ and the right spread $\Delta_4 = A\Delta_2 / Q$.

We will follow Glock et al. [1] to defuzzify a fuzzy inventory model to a crisp model by centroid method (center of area, center of gravity) proposed by Sugeno [16] and Lee [17] denoted as $M(Q)$, where

$$M(Q) = \int_{-\infty}^{\infty} y \mu_{\tilde{B}}(y) dy / \int_{-\infty}^{\infty} \mu_{\tilde{B}}(y) dy. \quad (3.16)$$

For a normal triangular fuzzy number of Equations (3.13-3.15), we can show that

$$M(Q) = \Omega + \frac{\Delta_1 - \Delta_3}{3}. \quad (3.17)$$

For the numerator, we compute that

$$\int_{\Omega - \Delta_3}^{\Omega} \frac{y - (\Omega - \Delta_3)}{\Delta_3} dy = \left. \frac{\Delta_1}{2} \left[ \frac{y - (\Omega - \Delta_3)}{\Delta_3} \right]^2 \right|_{\Omega - \Delta_3}^{\Omega} = \Omega - \Delta_3 - \frac{(\Delta_1)^2}{6}, \quad (3.18)$$

and
From Equation (4.9), to solve the system of
\[
Q_1 + Q_2 + \ldots + Q_n = D. \quad (4.1)
\]
Our goal will be to prove
\[
Q_1 = Q_2 = \ldots = Q_n, \quad (4.2)
\]
for the optimal replenishment policy.

For the \(k\)-th replenishment cycle, with order quantity \(Q_k\), the inventory level is
\[
I(t) = Q_k - D\left(t - \sum_{i=1}^{k-1} \frac{Q_i}{D}\right), \quad (4.3)
\]
for \(\sum_{i=1}^{k-1} \frac{Q_i}{D} \leq t \leq \sum_{i=1}^{k} \frac{Q_i}{D}\), to imply that
\[
\int_{\sum_{i=1}^{k-1} \frac{Q_i}{D}}^{\sum_{i=1}^{k} \frac{Q_i}{D}} \frac{1}{2} \frac{h}{2D} Q_k^2. \quad (4.4)
\]
Hence, the holding cost for one year is \(\sum_{k=1}^{n} \frac{h}{2D} Q_k^2\).

Motivated by our above derivation of Equation (3.22), the average cost can be rewritten as
\[
\frac{A + \left(hQ^2 / 2D\right) + A(\Delta_2 - \Delta_1)/3D}{Q/D}. \quad (4.5)
\]
Therefore, the total cost of the centroid for the membership function of inventory model with fuzzy demand with order quantity \(Q_k\) is expressed as
\[
A + \frac{h}{2D} Q_k^2 + \frac{A}{3D}(\Delta_2 - \Delta_1). \quad (4.6)
\]
For the finite planning horizon of one year with \(n\) replenishment with order quantity \(Q_1, Q_2, \ldots, Q_n\) under the restriction \(Q_1 + Q_2 + \ldots + Q_n = D\), the total cost of the centroid for our inventory model with fuzzy demand is expressed as
\[
F(Q_1, \ldots, Q_n) =
\]
\[
na + \sum_{k=1}^{n} \frac{h}{2D} Q_k^2 + \frac{na}{3D}(\Delta_2 - \Delta_1). \quad (4.7)
\]
with \(Q_1 + Q_2 + \ldots + Q_n = D\).

In the following, we begin to verify that \(Q_1 = \ldots = Q_n\) for the optimal replenishment policy. The Lagrange method is applied to consider \(G(Q_1, \ldots, Q_n, \lambda)\) with
\[
G(Q_1, \ldots, Q_n, \lambda) = F(Q_1, \ldots, Q_n) +
\]
\[
\lambda(Q_1 + Q_2 + \ldots + Q_n - D). \quad (4.8)
\]
We find that
\[
\frac{\partial G}{\partial Q_j} = \frac{h}{D} Q_j + \lambda, \quad (4.9)
\]
for \(j = 1, \ldots, n\). From Equation (4.9), to solve the system of
For the finite planning horizon, if the order number is \( n \), we prove that the optimal solution satisfies \( Q_1 = Q_2 = ... = Q_n = D/n \).

After proving that \( Q_1 = Q_2 = ... = Q_n = D/n \) is the optimal solution when the replenishment number, \( n \), is given, thus Equation (4.7) can be rewritten as

\[
F(n) = nA + \frac{DH}{2n} + \frac{nA}{3D} (\Delta_2 - \Delta_1). \tag{4.12}
\]

In the following, the optimal \( n^* \) is solved. First, we extend the domain from discrete natural numbers \( \{1, 2, ..., n\} \) to the positive real number \( \{x > 0\} \) to convert \( F(n) \) to \( F(x) \) as

\[
F(x) = xA + \frac{DH}{2x} + \frac{xA}{3D} (\Delta_2 - \Delta_1), \tag{4.13}
\]

for \( x > 0 \).

Based on \( F''(x) = \frac{DH}{x^3} > 0 \), we know that \( F(x) \) is convex up, for \( x > 0 \) such that the trajectory of \( \{(1, F(1)), (2, F(2)), ...\} \) is also spotted as a concave up dotted curve. Hence, the optimal replenishment number, \( n^* \), satisfies

\[
n^* = \min \{n : F(n) \leq F(n+1)\}. \tag{4.14}
\]

We compute that

\[
F(n+1) - F(n) = A \left( \frac{D + \frac{\Delta_2 - \Delta_1}{3}}{3D} - \frac{DH}{2n(n+1)} \right). \tag{4.15}
\]

Based on Equation (4.15), we obtain that \( F(n+1) \geq F(n) \) is equivalent to

\[
n(n+1) \geq \frac{1}{\Omega}, \tag{4.16}
\]

where we assume an abbreviation denoted as \( \Omega \), with

\[
\Omega = \frac{3D^2h}{2A(3D + \Delta_2 - \Delta_1)}. \tag{4.17}
\]

The inequality of (4.16) can be rewritten as

\[
n^2 + n + 0.25 \geq \Omega + 0.25 \tag{4.18}
\]

to imply that

\[
n^* = \left\lfloor \sqrt{\Omega + 0.25} - 0.5 \right\rfloor \tag{4.19}
\]

where \( \left\lfloor x \right\rfloor \) is the smallest integer that satisfies \( \left\lfloor x \right\rfloor \geq x \). We obtain a theoretical result for the optimal solution.

We summarize our results in the next theorem.

**Theorem 1.** For the finite planning horizon, if the order number is \( n \), we prove that the optimal solution satisfies \( Q_1 = Q_2 = ... = Q_n = D/n \).

**Theorem 2.** For inventory models with fuzzy demand, we derive the optimal solution \( n^* = \left\lfloor \sqrt{\Omega + 0.25} - 0.5 \right\rfloor \), where

\[
\Omega = \frac{3D^2h}{2A(3D + \Delta_2 - \Delta_1)}. \tag{4.17}
\]

Glock et al. [1] did not offer us any analytical work for the optimal solution for this kind of inventory models. We provide an improvement to find the formulated optimal solution.

**V. NUMERICAL EXAMPLES**

To be compatible with Glock et al. [1], we adopt the same numerical examples with the following data: \( A = 75 \), \( D = 1000 \), \( h = 5 \), \( \Delta_1 = 100 \) and \( \Delta_2 = 500 \) for Example 1, and \( \Delta_1 = 10 \), and \( \Delta_2 = 50 \) for Example 2.

We list our results in the next table. For comparison with Glock et al. [1], their findings also listed in Table 1.

From Table 1, we know that Glock et al. [1] and our paper both derived the same optimal number of orders. However, for the total cost, we find that there are some typographical errors in their computation.

To support our derivation, we run more detailed computations for Examples 1 and 2 to list them in the next Table 2.

After we know that \( TC(n) \) is a convex function in discrete variable \( n \), we observe the tendency of Examples 1 and 2, then the minimum point and minimum value are marked by bold to show that our derivation in Table 1 is the optimal solution.

Moreover, by our Theorem 2, we can directly locate the optimal solution, \( n^* \), to illustrate that our formulated optimal solution works for inventory models with fuzzy demand.

**VI. A RELATED PROBLEM**

In this section, we will examine a related problem and then provide our improvements. We recall the wash criterion question that have been studied by Lin et al. [18], Saaty and Vargas [19], Liberatore and Nydick [20], and Finan and Hurley [21] to consider whether or not a wash criterion can be removed to simplify the computation manual labor to construct comparison matrices during the analytic hierarchy process. Finan and Hurley [21] pointed out that they provided a strongest challenge for the analytic hierarchy process developed by Saaty [22, 23]. Saaty and Vargas [19] and Liberatore and Nydick [20] have presented responses for this challenge. Jung et al. [24] provided an example to show that working through by the following three various procedures: (i) Lin et al. [18] and Finan and Hurley [21] without wash...

VII. LITERATURE REVIEW

Finan and Hurley [21] considered wash criteria to develop several theorems to show that a two-level analytic hierarchy process system with a perfectly consistent corresponding comparison matrix, and then deleting a wash criterion will not influence the ordering for alternatives. On the other hand, Finan and Hurley [21] developed a three-level comparison matrix to show the occurrence of rank reversal after deleting a wash criterion and then they raised a severe question on the well-defined problem of the analytic hierarchy process. Liberator and Nydick [20] mentioned that the relative weights among alternatives, or alternatives to criteria will be estimated again such that the rank reversal phenomenon will not happen. Saaty and Vargas [19] showed that a wash criterion cannot be arbitrarily canceled. In this section, the debate about deleting a wash criterion will be further examined.

VIII. OUR IMPROVEMENTS

Let us recall the solution approach of Finan and Hurley [21]. When deleting the wash criterion, $J_0$, with weight 0.6, Finan and Hurley [21] considered the weights of $J_1$ and $J_2$ as 0.2 and 0.2, then they normalized them to obtain the relative weights of $J_1$, 

\[
\frac{0.2}{0.2 + 0.2} = 0.5, \quad (8.1)
\]

and of $J_2$, 

\[
\frac{0.2}{0.2 + 0.2} = 0.5, \quad (8.2)
\]

so after the wash criterion $J_0$ is deleted, and then the weights of $J_1$ and $J_2$ becomes 0.5 and 0.5.

It means that the comparison matrix for $J_0$, $J_1$ and $J_2$ with respect to $J$ is assumed as $[a_{i,j}]_{3 \times 3}$ then the normalized eigenvector will satisfy

\[
\begin{bmatrix}
1 & a_{i,2} & a_{i,3} \\
a_{2,1} & 1 & a_{2,3} \\
a_{3,1} & a_{3,2} & 1
\end{bmatrix}
\begin{bmatrix}
0.6 \\
0.2 \\
0.2
\end{bmatrix}
= \lambda
\begin{bmatrix}
0.6 \\
0.2 \\
0.2
\end{bmatrix},
\]

where $\lambda$ is the maximum eigenvalue corresponding to the matrix of Equation (8.3).

Owing to $a_{i,j}$ is defined as the comparison of $J_{j-1}$ with $J_{j-1}$ with respect to criterion $J$ in the upper level, its value will not influence by deleting the wash criterion whether or not. Hence, after deleting the wash criterion $J_0$, then the first row and the first column of $[a_{i,j}]_{3 \times 3}$ is disappeared.

According to every 2 by 2 comparison matrix is consistent, so the eigenvector problem becomes

\[
\begin{bmatrix}
1 & a_{2,3} \\
a_{3,2} & 1
\end{bmatrix}
\begin{bmatrix}
\alpha_1 \\
\alpha_2
\end{bmatrix}
= 2
\begin{bmatrix}
\alpha_1 \\
\alpha_2
\end{bmatrix},
\]

with $\alpha_1 + \alpha_2 = 1$.

Finan and Hurley [21] did not try to solve Equation (2), instead, they directly based on the eigenvector of Equation (1), to obtain the relative ratio between $J_1$ and $J_2$, and then normalized them.

We recall that Finan and Hurley [21] evaluated a comparison matrix $[b_{i,j}]_{n \times n}$ to derive the normalized eigenvector, say $[v_{i}]_{n \times 1}$ corresponding to the maximum eigenvalue, and then Finan and Hurley [21] deleted the first row and the first column of $[b_{i,j}]_{n \times n}$ to construct a matrix $[c_{i,j}]_{n \times n}$ with $c_{i,j} = b_{i+1,j+1}$ for $1 \leq i \leq n$ and $1 \leq j \leq n$.

In their Proposition 1, they claimed that the normalized eigenvector, corresponding to the maximum eigenvalue will be

\[
\begin{bmatrix}
\frac{v_{i+1}}{1-v_{i+1}}
\end{bmatrix}_{n \times 1},
\]

From the above discussion, we may say that Finan and Hurley [21] approach is reasonable that is verified by Proposition 1 of Finan and Hurley [21].

However, in Lin et al. [18], they have pointed out that Proposition 1 contained questionable results. When $[b_{i,j}]_{n \times n}$ is a perfect consistent matrix then their claim assertion is corrected. For the general case, their assertion is false.

Based on our previous discussion, we need to consider the following question:

\[
\begin{bmatrix}
1 & a_{i,2} & a_{i,3} \\
a_{2,1} & 1 & a_{2,3} \\
a_{3,1} & a_{3,2} & 1
\end{bmatrix}
\begin{bmatrix}
0.6 \\
0.2 \\
0.2
\end{bmatrix}
= \lambda
\begin{bmatrix}
0.6 \\
0.2 \\
0.2
\end{bmatrix},
\]

with $a_{i,2}, a_{i,3}, a_{2,3} \in \{1,2,...,9\}$ the 1-9 scale proposed by Saaty [22], under the restrictions of $a_{2,1} = \frac{1}{a_{i,2}}$, $a_{3,1} = \frac{1}{a_{i,3}}$, $a_{3,2} = \frac{1}{a_{2,3}}$ and $\lambda$ is the maximum eigenvalue.

We find that there are three possible solutions as follows and their corresponding reduced 2 by 2 comparison matrix with its maximum eigenvector:
\[
\begin{align*}
&\begin{bmatrix}
1 & 1 & 9 \\
1 & 1 & 1/3 \\
1/9 & 3 & 1
\end{bmatrix}
\begin{bmatrix}
0.6 \\
0.2 \\
0.2
\end{bmatrix}
= \begin{bmatrix}
13/3 \\
0.2 \\
0.2
\end{bmatrix}, \\
&\begin{bmatrix}
1 & 3/1 \\
1/3 & 1 \\
1/9 & 1
\end{bmatrix}
\begin{bmatrix}
0.25 \\
0.75 \\
0.6
\end{bmatrix}
= \begin{bmatrix}
0.25 \\
0.25 \\
0.75
\end{bmatrix}, \\
&\begin{bmatrix}
1 & 3 \\
1/3 & 1 \\
1/9 & 3
\end{bmatrix}
\begin{bmatrix}
0.6 \\
0.2 \\
0.2
\end{bmatrix}
= \begin{bmatrix}
13/3 \\
0.2 \\
0.2
\end{bmatrix}, \\
&\begin{bmatrix}
1 & 1 \\
1/3 & 1 \\
1/9 & 1
\end{bmatrix}
\begin{bmatrix}
0.5 \\
0.5 \\
0.6
\end{bmatrix}
= \begin{bmatrix}
0.5 \\
0.5 \\
0.5
\end{bmatrix}.
\end{align*}
\]

Based on the above computations, we show that there are three possible solutions for the relative weights for \(J_1\) and \(J_2\) as \(J_1 : J_2 = 1:3, 3:1, \) or \(1:1\), it is actually derived from the 2 by 2 pairwise comparison matrix elements that also satisfy the 1-9 scale proposed by Saaty [22]. We recall that Finan and Hurley [21] approach is directly derived from the relative weights from the 3 by 3 pairwise comparison matrix.

Our findings reveal that the solution procedure of Finan and Hurley [21] to claim the solution must be 1:1 which is only a partial solution of the all three possible answers. Furthermore, we verify that the solution process proposed by Finan and Hurley [21] is not complete such that their procedure sometimes cannot obtain the correct relative weights.

**IX. A CONSISTENT PROBLEM WITH WASH CRITERION IN ANALYTIC HIERARCHY PROCESS**

In this section, we will discuss a famous problem in Analytic Hierarchy Process of was criterion. This problem had been studied by Finan and Hurley [21], Liberatore and Nydick [20], Saaty and Vargas [19], Lin et al. [18], Jung et al. [24], and Ke et al. [25].

We will provide a theoretical explanation why their counterexample cannot be constructed.

**X. THEIR PROPOSED COUNTEREXAMPLE**

We consider the hierarchy structure studied by Liberatore and Nydick [20] and Saaty and Vargas [19] with the following relative weights where the criterion, "\(I_{11}\)" is a wash criterion.

We compute the synthesized weight of alternative \(A_1\) as follows. We know that

\[
w(A_1) = w\left(\frac{A_1}{l_1}\right)w(J_1) + w\left(\frac{A_2}{l_2}\right)w(J_2),
\]

\[
w\left(\frac{A_1}{l_1}\right) = w\left(\frac{A_1}{l_1}\right)w\left(\frac{11}{l_1}\right) + w\left(\frac{A_1}{l_1}\right)w\left(\frac{12}{l_1}\right),
\]

and

\[
w\left(\frac{A_2}{l_2}\right) = w\left(\frac{A_2}{l_2}\right)w\left(\frac{11}{l_2}\right) + w\left(\frac{A_2}{l_2}\right)w\left(\frac{12}{l_2}\right).
\]

Liberatore and Nydick [20] claimed that “criteria I_2” lost \(I_{11}\) with 60% weight, then the relative weight between \(I_1\) and \(I_2\) should be changed. However, Liberatore and Nydick [20] did not inform readers how to modify relatives. Based on Equations (10.1-10.3), we derive that

\[
w(A_1) = 0.477.
\]

Similarly, we also know that

\[
w(A_2) = w\left(\frac{A_2}{l_2}\right)w(J_1) + w\left(\frac{A_2}{l_2}\right)w(J_2),
\]

\[
w\left(\frac{A_2}{l_1}\right) = w\left(\frac{A_2}{l_1}\right)w\left(\frac{11}{l_1}\right) + w\left(\frac{A_2}{l_1}\right)w\left(\frac{12}{l_1}\right),
\]

and

\[
w\left(\frac{A_2}{l_2}\right) = w\left(\frac{A_2}{l_2}\right)w\left(\frac{11}{l_2}\right) + w\left(\frac{A_2}{l_2}\right)w\left(\frac{12}{l_2}\right).
\]

Based on Equations (10.5-10.7), we derive that

\[
w(A_2) = 0.523.
\]

Based on our findings of Equations (10.4) and (10.8), it follows that the restriction of

\[
w(A_1) + w(A_2) = 1
\]

is satisfied.

Our goal is to find parameters: a, b, c, d, e, f, and g in the unit interval such that the relative ratio between alternatives \(A_1\) and \(A_2\) are altered.

First, with the wash criterion, \(I_{11}\), we compute the synthesized weight of the alternative \(A_1\), and the alternative \(A_2\), then

\[
w(A_1) = \frac{ab}{2} + ace + a(1-b-c)f + (1-a)dg + (1-a)(1-d)h,
\]

and

\[
w(A_2) = \frac{ab}{2} + ac(1-e) + a(1-b-c)(1-f) + (1-a)d(1-g) + (1-a)(1-d)(1-h),
\]

Next, without the wash criterion, \(I_{11}\), we compute the synthesized weight of the alternative \(A_1\), and the alternative \(A_2\), then

\[
\tilde{w}(A_1) = ace + a(1-c)f + (1-a)dg + (1-a)(1-d)h,
\]

and

\[
\tilde{w}(A_2) = ac(1-e) + a(1-c)(1-f) + (1-a)d(1-g) + (1-a)(1-d)(1-h).
\]

Our purpose is find parameters: a, b, c, d, e, f, and g such that

\[
|w(A_1) - w(A_2)|/(\tilde{w}(A_1) - \tilde{w}(A_2)) < 0.
\]

The inequality of Equation (10.14) indicates the rank of \(A_1\) and \(A_2\) are altered, after the wash criterion is removed. Based on our results of Equations (10.10-10.13), we can
simplify the condition of Equation (10.14) as follows,
\[ [\Delta - ab(2f-1)]\Delta < 0, \]  
(10.15)
where \( \Delta \) is an abbreviation to simplify the expression, with
\[ \Delta = ac(2e-1) + a(1-c)(2f-1) + (1-a)d(2g-1) + (1-a)(1-d)(2h-1). \]  
(10.16)

Based on Equation (10.15), we list the following conditions:
\[ 2e - 1 > 0, \]  
(10.17)
\[ 2f - 1 > 0, \]  
(10.18)
\[ 2g - 1 > 0, \]  
(10.19)
\[ 2h - 1 > 0, \]  
(10.20)
and
\[ \Delta < ab(2f-1). \]  
(10.21)
Motivated by conditions of (10.17-10.21), we select
\[ e = 0.6, f = g = h, \]  
(10.22)
to derive that
\[ \Delta = 0.2. \]  
(10.23)

We know that conditions of (10.17-10.20) are satisfied, and we simplify Equation (10.21) in the following,
\[ 0.2 < ab(0.2). \]  
(10.23)
For later discussion, we further simplify Equation (10.23) as follows,
\[ 1 < ab. \]  
(10.24)
For a reasonable hierarchy structure, we know that
\[ 0 \leq a \leq 1, \]  
(10.25)
and
\[ 0 \leq b \leq 1. \]  
(10.26)
Based on Equations (10.25) and (10.26), we know that the inequality in Equation (10.24) cannot find example.

Based on our above detailed examination for the example proposed by Liberatore and Nydick [20], we demonstrate the assertion of Liberatore and Nydick [20] may contain questionable results that deserves further investigation.

XI. DIRECTIONS FOR FUTURE RESEARCH

We reference several related articles to assist researchers in identifying potential directions for future research. Belton and Gear [26] and Saaty [27, 28] are mentioned. Belton and Gear [26] present a significant challenge to the analytic hierarchy process by constructing an example that results in rank reversal among alternatives when an additional alternative is introduced. Saaty [27] is a well-known textbook that introduces the analytic hierarchy process to researchers. Saaty [28] provides explanations for some of the challenges associated with the rank reversal problem that occurs when a new alternative is added during the decision-making process. Additionally, there are several recently published papers that are also of great importance. We cite four papers: Yen [29, 30], Yang and Chen [31], and Wang and Chen [32]. Yen [29] explores intuitive algebraic methods for solving inventory models with two back-ordered costs and poses an open question regarding the solution procedure of Chang and Schonfeld [30]. Yen [31] first points out questionable results in Çalışkan [32, 33] and subsequently demonstrates that the challenge proposed by Çalışkan [34] to Wee et al. [35] is invalid. Finally, Yen [31] provides a patchwork solution for Minner [36]. Yang and Chen [37] address the open question raised by Yen [29] and Yen [31] independently. Furthermore, Yang and Chen [37] identify questionable findings in Çalışkan [39, 40]. Wang and Chen [41] make improvements to Aguaron and Moreno-Jime 42, rendering their stability interval inadequate for solving the inconsistency problems in the analytic hierarchy process. Wang and Chen [41] also demonstrate that the intuitive approach developed by Yen [29] represents only one possible selection among various choices. We also refer to the works of Nurcahayani et al. [43], Em [44], Ren et al. [45], and Che et al. [46]. Nurcahayani et al. [43] study combined estimators with nonparametric regression bi-response functions. Em [44] derives a sufficient criterion using multiple reaction and diffusion models. Ren et al. [45] develop a new iterative decoding algorithm for two-dimensional problems, and Che et al. [46] examine a novel entropy system for estimating wind velocity within short time periods. Finally, we quote Thangkenpau and Panday [47], Hwang et al. [48], Zhao et al. [49], and Chanmanee et al. [50]. Thangkenpau and Panday [47] obtain the optimal solution for nonlinear systems with an eight-dimensional family without derivatives. Hwang et al. [48] discover analytical findings for multi-condition operational systems. Zhao et al. [49] apply a discrete approach with naked mole rats to derive a locally optimal solution for the traveling salesman problem. Chanmanee et al. [50] study UP algebra to uncover new findings for internal direct multiplications. The articles mentioned above serve to inspire researchers and shed light on the current trends in hop research.

XII. CONCLUSION

For inventory models under fuzzy environment, it is proved that the equal order quantity for each replenishment cycle is the optimal strategy. Meanwhile, we derived the formulated optimal solution for the number of order that will help researchers locate the optimal solution. Moreover, we examine two problems in Analytic Hierarchy Process. First, we provide a reason why Finan and Hurley [21] only consider one-third of possible results. Second, we provide a theoretical development to explain why Liberatore and Nydick [20] cannot construct a counterexample to demonstrate that after deleting a wash criterion, the ranking of alternatives will be altered. Our findings will help researchers to deal operational research problems with analytical approach under a solid mathematical background.

<table>
<thead>
<tr>
<th>Table I</th>
<th>THE COMPARISONS BETWEEN OUR DERIVATIONS WITH THEIRS OF GLOCK ET AL. [1]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Findings of Glock et al. [1]</td>
<td>Our results</td>
</tr>
<tr>
<td>----------------------------------</td>
<td>----------------------------------</td>
</tr>
<tr>
<td>Ex. 1</td>
<td>Ex. 2</td>
</tr>
<tr>
<td>Ex. 1</td>
<td>Ex. 2</td>
</tr>
<tr>
<td>( n_{opt} )</td>
<td>( Q_{opt} )</td>
</tr>
<tr>
<td>5</td>
<td>200</td>
</tr>
<tr>
<td>6</td>
<td>166.67</td>
</tr>
<tr>
<td>( TC(n^*) )</td>
<td></td>
</tr>
<tr>
<td>885.00</td>
<td>867.67</td>
</tr>
<tr>
<td></td>
<td>925</td>
</tr>
<tr>
<td></td>
<td>872.67</td>
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Table II
Detailed computations for Examples 1 and 2

<table>
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<tr>
<th>TC(n)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ex. 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2385</td>
<td>1420</td>
<td>1088</td>
<td>965</td>
<td>925</td>
<td>926.7</td>
<td>952.1</td>
<td>992.5</td>
<td>1043</td>
<td></td>
</tr>
<tr>
<td>Ex. 2</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
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<tr>
<td>2576</td>
<td>1402</td>
<td>1061</td>
<td>929</td>
<td>880</td>
<td>872.7</td>
<td>889.1</td>
<td>920.5</td>
<td>961.8</td>
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</table>

Table III.
Our proposed hierarchy with relative weights among criteria and alternatives

<table>
<thead>
<tr>
<th>J_1</th>
<th>J_2</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>1-a</td>
</tr>
<tr>
<td>J_{11}</td>
<td>J_{12}</td>
</tr>
<tr>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>A_1</td>
<td>0.5</td>
</tr>
<tr>
<td>A_2</td>
<td>0.5</td>
</tr>
</tbody>
</table>

**REFERENCES**


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P. V. L. Em, "Sufficient Condition for Synchronization in Complete Networks of n Reaction-Diffusion Systems of Hindmarsh-Rose Type with Nonlinear Coupling," *Engineering Letters*, vol. 31, no.1, , 2023, pp. 413-418.


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