Intriguing Relationships among Eisenstein Series, Borewein's Cubic Theta Functions, and the Class One Infinite Series

Harekala Chandrashekara Vidya and Badanidiyoor Ashwath Rao*

***Abstract*—This article focuses on the application of the Ramanujan-type Eisenstein series to the formation of numerous differential identities. In this paper, utilizing Alaca's (p,k) parametrization, we explore a few additional interesting links between Eisenstein Series and Borewein's cubic theta functions. Further, we build a set of higher-order nonlinear differential equations that include Ramanujan's function $k(q)$. In addition, we formulate identities relating the Class one infinite series and Ramanujan's k-function, using the relationship derived between Eisenstein and the Class one infinite series.**

***Index Terms*—Eisenstein series, Dedekind η-function, Cubic theta functions, Continued fractions.**

I. Introduction

Differential equations help solve many scientific, engineering, medical and business problems. They are also used to solve several problems in computational fluid dynamics and thermodynamics. In the medical domain, the segmentation of medical images and optimal investment strategies are developed using differential equations. Differential equations are also essential in computational mathematics.

Ramanujan [8] cited several theta function-based differential equations in his book. The importance of constructing differential equations involving η-functions and Eisenstein series was emphasized by B. C. Berndt [4]. Ramanujan stated several formulas involving k-functions on page 56 and other scattered places in his lost notebook [8] and further expressed the Roger's-Ramanujan continued fractions $r(\sqrt{q})$ and $r(q^4)$ in terms of k-functions. In his lost notebook [8], Ramanujan also devised formulas for relating the Class one infinite series $T_{2r}(q)$, $r = 1, 2, ...6$ to the Eisenstein series L, M, and N. E. X. M. Xia and O. X. M. Yao [12] used computers to develop series relations involving cubic theta functions based on Ramanujan's elliptic function theory and the (p, k)-parametrization due to Alaca et al. [1].

Motivated by their work, we devise an expression for the Eisenstein series in terms of the classical Class one infinite series. We use this expression to develop exciting equations relating the Class one infinite series with k-functions. It is interesting to note that in this article, the Ramanujan-type Eisenstein series have been represented in terms of the product of cubic theta functions without the aid of a computer. Section 2 is devoted to documenting some preliminary findings that will aid in achieving the main objectives. In section 3, using (p,k) parametrization, we generate a few fascinating identities relating the Eisenstein series and Borwein's cubic theta functions. In Section 4, we develop some remarkable first and second-order nonlinear differential equations with the help of Eisenstein series of level 10 that involve Ramanujan's k- functions. In the final section, we formulate a relationship between the Eisenstein series and the Class one infinite series. Further, employing the developed identities, we generate new expressions between Class one infinite series and Ramanujan's k-functions.

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Harekala Chandrashekara Vidya is an Assistant Professor (Senior Scale) in the Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, India. (e-mail: vidyaashwath@gmail.com).

Badanidiyoor Ashwath Rao is an Associate Professor in Computer Science and Engineering Department, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, India (corresponding author to provide phone: 9845403148; e-mail:ashwath.rao.b@gmail.com).

II. Priliminaries

Definition 2.1: [3]

Ramanujan, in his notebook [3, p.35], defined general theta function as follows: For any complex q, a and b with $|ab| < 1$,

$$f(a,b) := \sum_{i=-\infty}^{\infty} a^{i(i+1)/2} b^{i(i-1)/2}$$
$$= (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

where

$$(a;q)_\infty := \prod_{i=0}^{\infty}(1-aq^i), \qquad |q|<1.$$

The following are the special cases of theta functions defined by Ramanujan [3, p.35]:

$$\varphi(q) := f(q,q) = \sum_{i=-\infty}^{\infty} q^{i^2}$$
$$= (-q;q^2)^2_\infty(q^2;q^2)_\infty,$$
$$\psi(q) := f(q,q^3) = \sum_{i=0}^{\infty} q^{\frac{i(i+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$$
$$f(-q) := f(-q,-q^2) = \sum_{i=-\infty}^{\infty} (-1)^i q^{i(3i-1)/2}$$
$$= (q;q)_\infty = q^{-1/24}\eta(\tau),$$

where $q = e^{2\pi i\tau}$. We denote $f(-q^n) = f_n$.

J. M. Borewein and P. B. Borewein [6] recorded the following two dimensional theta functions in their work on a cubic counterpart of jacobi and a cubic analogue of arithmetic geometric mean iteration of Legendre and Gauss:

$$a(q) := \sum_{\alpha=-\infty}^{\infty}\sum_{\beta=-\infty}^{\infty} q^{\alpha^2+\alpha\beta+\beta^2},$$
$$b(q) := \sum_{\alpha=-\infty}^{\infty}\sum_{\beta=-\infty}^{\infty} y^{\alpha-\beta} q^{\alpha^2+\alpha\beta+\beta^2},$$
$$c(q) := \sum_{\alpha=-\infty}^{\infty}\sum_{\beta=-\infty}^{\infty} q^{(\alpha+\frac{1}{3})^2+(\alpha+\frac{1}{3})(\beta+\frac{1}{3})+(\beta+\frac{1}{3})^2},$$

where $y = exp(2\pi i/3)$ and $q \in \mathbb{C}$, the set of complex numbers.
Also, note that

$$a(0) = 1, b(0) = 1, c(0) = 1.$$

The (p,k) parametrization of theta functions was initially defined by Alaca et al. [2] in their remarkable study. It is beneficial, especially for establishing duplication and triplication principles and obtaining a certain sum to product identities. The definitions of these variables p and k are as follows:

$$p = p(q) := \frac{\varphi^2(q) - \varphi^2(q^3)}{2\varphi^2(q^3)},$$
$$k = k(q) := \frac{\varphi^2(q^3)}{\varphi(q)}.$$

Lemma 2.1: [1] The demonstrated parametric representations of $a(q^m), b(q^m), c(q^m)(m \in 1, 2, 3, 4, 6, 8, 12)$, as well as $a(-q)$ and $c(-q)$ in terms of the parameters p and k are given by

$$a(-q) = (-2p^2 - 2p + 1)k,$$
$$a(q) = (p^2 + 4p + 1)k,$$
$$a(q^2) = (p^2 + p + 1)k,$$
$$a(q^3) = \frac{(p^2 + 4p + 1)k}{3} + \frac{2^{2/3}(1-p)((1-p)(2+p)(1+2p))^{1/3})k}{3},$$
$$a(q^4) = \frac{(-p^2 + 2p + 2)k}{2},$$
$$a(q^6) = \frac{(p^2+p+1+2^{1/3}((1-p)(2+p)(1+2p))^{2/3})k}{3},$$
$$b(q) = 2^{-1/3}(1 - p)((1 - p)(2 + p)(1 + 2p))^{1/3}k,$$
$$b(q^2) = 2^{-2/3}((1 - p)(2 + p)(1 + 2p))^{2/3}k,$$
$$c(q^6) = \frac{(p^2+p+1-2^{-2/3}((1-p)(2+p)(1+2p))^{2/3})k}{3}.$$

Definition 2.2: [7] Ramanujan recorded infinite series known as Ramanujan-type Eisenstein series:

$$L(q) := 1 - 24\sum_{\alpha=1}^{\infty}\frac{\alpha q^{\alpha}}{1 - q^{\alpha}}$$
$$= 1 + 24q\frac{d}{dq}\sum_{j=1}^{\infty} log(1 - q^j), \quad (1)$$
$$M(q) := 1 + 240\sum_{\alpha=1}^{\infty}\frac{\alpha^3 q^{\alpha}}{1 - q^{\alpha}}$$

For simplicity, we denote $L(q^n) = L_n$ and $M(q^n) = M_n$.

Lemma 2.2: [1] For the above specified Eisenstein series, the representations in terms of the parameters p and k are given by

$$L(-q) - L(q) = -3(8p + 12p^2 + 6p^3 + p^4)k^2,$$
$$L(q) - 2L(q^2) = -(1 + 14p(1 + p^2) + 24p^2 + p^4)k^2,$$
$$L(q) - 3L(q^3) = -(1 + 8p(1 + p^2) + 18p^2 + p^4)k^2,$$
$$L(q) - 4L(q^4) = -(3 + 18p(1 + 2p) + 24p^3)k^2,$$
$$L(q) - 6L(q^6) = -(5 + 22p(1 + p^2) + 36p^2 + 5p^4)k^2,$$
$$L_{1,2}(q) = \frac{(L(-q)-L(q))}{48} = \left(\frac{p}{2} + \frac{3p^2}{4} + \frac{3p^3}{8} + \frac{p^4}{16}\right)k^2,$$
$$L_{1,2}(q^3) = \frac{(L(-q^3) - L(q^3))}{48} = \left(\frac{1}{8}p^3 + \frac{1}{16}p^4\right)k^2.$$

Definition 2.3: [5] The Ramanujan's function $k_1(q)$ is given by

$$k_1(q) = r(q)r^2(q^2)$$
$$= q\prod_{j=1}^{\infty}\frac{(1-q^{10j-9})(1-q^{10j-8})(1-q^{10j-2})(1-q^{10j-1})}{(1-q^{10j-7})(1-q^{10j-6})(1-q^{10j-4})(1-q^{10j-3})},$$

where

$$r(q) := \frac{q^{\frac{1}{5}}}{1}\,\frac{q}{+1}\,\frac{q^2}{+1}\,\frac{q^3}{+1}\,{}_{+\ldots},$$

is known as the Roger's-Ramanujan continued fraction.
Let y_{10} be the logarithmic derivative of k_1,

$$y_{10} = y_{10}(q) = q\frac{d}{dq}logk_1.$$

Definition 2.4: The Class one infinite series introduced by Ramanujan in his lost notebook [8], is given by

$$T_{2l}(q) := 1 + \sum_{r=1}^{\infty}(-1)^r\left[(6r - 1)^{2l}\{q^{\frac{r(3r-1)}{2}} + (6r + 1)^{2l}q^{\frac{r(3r+1)}{2}}\}\right]. \quad (2)$$

Ramanujan further expressed the above infinite series in terms of the Ramanujan-type Eisenstein series for $l = 1, 2, ..., 6$. Also, B. C. Berndt [4] established the relation

$$\frac{T_2(q)}{(q;q)_\infty} = L(q), \quad (3)$$

where

$$(q;q)_\infty = 1 + \sum_{r=1}^{\infty}(-1)^r\{q^{\frac{r(3r-1)}{2}} + q^{\frac{r(3r+1)}{2}}\},$$

is known as a famous pentagonal number theorem [3].

Lemma 2.3: [5] The following relation among Eisenstein series and Ramanujan's k-functions hold:

$$\begin{pmatrix} L(q) \\ L(q^2) \\ L(q^5) \\ L(q^{10}) \end{pmatrix} = \begin{pmatrix} 4 & 1 & -4 & 6 \\ \frac{5}{2} & -2 & \frac{1}{2} & 3 \\ -\frac{4}{5} & 1 & \frac{4}{5} & \frac{6}{5} \\ \frac{1}{10} & \frac{2}{5} & \frac{1}{2} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} \frac{(1+k_1^2)}{(1-k_1^2)}y_{10} \\ \frac{(1+k_1^2)}{(1+k_1-k_1^2)}y_{10} \\ \frac{(1+k_1^2)}{(1-4k_1-k_1^2)}y_{10} \\ k_1\frac{dy_{10}}{dk_1} \end{pmatrix}.$$

Lemma 2.4: [5] The following relation among Eisenstein series of level 10 and k-functions hold:

$$L(q)-2L(q^2)-25L(q^5)+50L(q^{10})=24\tfrac{(1+k_1^2)}{(1-k_1^2)}y_{10},$$
$$L(q)-4L(q^2)-5L(q^5)+20LP(q^{10})=\tfrac{12(1+k_1^2)}{(1+k_1-k_1^2)}y_{10},$$
$$-3L(q)+2L(q^2)-5L(q^5)+30L(q^{10})=\tfrac{24(1+k_1^2)}{(1-4k_1-k_1^2)}y_{10},$$
$$L(q)+4L(q^2)+15L(q^5)-20L(q^{10})=24k_1\tfrac{dy_{10}}{dk_1}.$$

Lemma 2.5: [5] The following relations hold:

$$M(q)=y_{10}^2\left(\frac{(1-4k_1-4k_1^2)^3}{(1-k_1^2)(1+k_1-k_1^2)^2}+\frac{256k_1(1+k_1-k_1^2)^3}{(1-k_1^2)^2(1-4k_1-k_1^2)^2}\right),$$
$$M(q^2)=y_{10}^2\left(\frac{(1-4k_1-4k_1^2)^3}{(1-k_1^2)(1+k_1-k_1^2)^2}+\frac{16k_1(1+k_1-k_1^2)^3}{(1-k_1^2)^2(1-4k_1-k_1^2)^2}\right),$$
$$M(q^5)=y_{10}^2\left(\frac{(1-k_1^2)^3}{(1+k_1-k_1^2)^2(1-4k_1-k_1^2)}+\frac{256k_1^5}{(1-k_1^2)^2(1+k_1-k_1^2)(1-4k_1-k_1^2)^2}\right),$$
$$M(q^{10})=y_{10}^2\left(\frac{(1-k_1^2)^3}{(1+k_1-k_1^2)^2(1-4k_1-k_1^2)}+\frac{16k_1^5}{(1-k_1^2)^2(1+k_1-k_1^2)(1-4k_1-k_1^2)^2}\right).$$

Ramanujan gave a lot of attention to the construction and application of the Eisenstein series in his notebook [3] and provided a few fascinating identities relating these series with theta functions. Using computer, E. X. W. Xia and O. X. M. Yao [12] found several lovely correlations between Eisenstein series and cubic theta functions of Borewein, as well as the use of these identities in the explicit evaluation of convolution sum. H. C. Vidya and B. Ashwath Rao [11] recently built new identities associating Boreweins cubic theta functions to Eisenstein series and used them to evaluate specific convolution identities. Interestingly, without the help of a computer, in this article, a few new sum-to-product identities involving $L(q^n), n=1,2,3,6,$ and $L(-q^m)$ for $m=1,3$ have been generated. Further, B. C. Berndt [4], D. Anuradha et al. [9], H. C. Vidya, and B. R. Srivatsa Kumar [10] developed several differential equations and stressed the need of using it in the construction of incomplete elliptic integrals. Inspired by their work, in this article, a few nonlinear differential identities have been constructed by implementing Eisenstein relations of level 10. These relations have been described in terms of Class one infinite series.

III. Relation among Eisenstein Series and Cubic theta functions

Theorem 3.1: Few new identities relating Ramanujan type Eisenstein series and Borewein's cubic theta functions are listed as follows:

$$(i)1+12\sum_{\alpha=1}^{\infty}\left[\frac{2(1+3l)\alpha(-q)^\alpha}{1-(-q)^\alpha}+\frac{(1-2l)\alpha q^\alpha}{1-q^\alpha}-\frac{(5+12l)\alpha q^{2\alpha}}{1-q^{2\alpha}}-\frac{3\alpha q^{3\alpha}}{1-q^{3\alpha}}+\frac{8l\alpha q^{4\alpha}}{1-q^{4\alpha}}+\frac{3\alpha q^{6\alpha}}{1-q^{6\alpha}}\right]=\frac{a(-q)b^2(q)}{b(q^2)}.$$

$$(ii)1-4\sum_{\alpha=1}^{\infty}\left[\frac{2(2+84l_1-9l_2)\alpha(-q)^\alpha}{1-(-q)^\alpha}-\frac{(7+114l_1+18l_2)\alpha q^\alpha}{1-q^\alpha}-\frac{108(l_1-l_2)\alpha q^{2\alpha}}{1-q^{2\alpha}}+\frac{9(1+2l_1)\alpha q^{3\alpha}}{1-q^{3\alpha}}-\frac{72l_2\alpha q^{4\alpha}}{1-q^{4\alpha}}-\frac{108l_1\alpha q^{6\alpha}}{1-q^{6\alpha}}\right]=(a(q^3)-\frac{2}{3}b(q))a(q).$$

$$(iii)\frac{1}{3}+4\sum_{\alpha=1}^{\infty}\left[\frac{6l\alpha(-q)^\alpha}{1-(-q)^\alpha}+\frac{6l\alpha q^\alpha}{1-q^\alpha}-\frac{(1-36l)\alpha q^{2\alpha}}{1-q^{2\alpha}}+\frac{24l\alpha q^{3\alpha}}{1-q^{3\alpha}}-\frac{3\alpha q^{4\alpha}}{1-q^{4\alpha}}+\frac{3\alpha q^{6\alpha}}{1-q^{6\alpha}}\right]=(a(q^6)-\frac{2}{3}b(q^2))a(q^2).$$

$$(iv)1+2\sum_{\alpha=1}^{\infty}\left[\frac{4(1+3l)\alpha(-q)^\alpha}{1-(-q)^\alpha}+\frac{(1+36l)\alpha q^\alpha}{1-q^\alpha}-\frac{6(1+36l)\alpha q^{2\alpha}}{1-q^{2\alpha}}+\frac{9\alpha q^{3\alpha}}{1-q^{3\alpha}}+\frac{144l\alpha q^{4\alpha}}{1-q^{4\alpha}}-\frac{18\alpha q^{6\alpha}}{1-q^{6\alpha}}\right]=(c(q^6)+\frac{1}{3}b(q^2))a(q).$$

$$(v)1+4\sum_{\alpha=1}^{\infty}\left[\frac{(18l-1)\alpha(-q)^\alpha}{1-(-q)^\alpha}-\frac{(6l-1)\alpha q^\alpha}{1-q^\alpha}-\frac{3(12l-1)\alpha q^{2\alpha}}{1-q^{2\alpha}}+\frac{24\alpha q^{4\alpha}}{1-q^{4\alpha}}-\frac{8\alpha q^{6\alpha}}{1-q^{6\alpha}}\right]=(3a(q^6)-2b(q^2))^2.$$

$$(vi)1+12\sum_{\alpha=1}^{\infty}\left[\frac{2\alpha(-q)^\alpha}{1-(-q)^\alpha}+\frac{(1+2l)\alpha q^\alpha}{1-q^\alpha}-\frac{12l\alpha q^{2\alpha}}{1-q^{2\alpha}}-\frac{3\alpha q^{3\alpha}}{1-q^{3\alpha}}+\frac{8l\alpha q^{4\alpha}}{1-q^{4\alpha}}\right]=(3a(q^3)-2b(q))^2.$$

$$(vii)1+2\sum_{\alpha=1}^{\infty}\left[\frac{3(1+3l)\alpha(-q)^\alpha}{1-(-q)^\alpha}-\frac{(7-12l)\alpha q^\alpha}{1-q^\alpha}+\frac{(3-24l)\alpha q^{2\alpha}}{1-q^{2\alpha}}+\frac{9\alpha q^{3\alpha}}{1-q^{3\alpha}}+\frac{27l\alpha q^{4\alpha}}{1-q^{4\alpha}}-\frac{18\alpha q^{6\alpha}}{1-q^{6\alpha}}\right]=a(-q)a(q^4).$$

Proof. Consider the relation

$$\begin{aligned}&C_1(L(-q)-L(q))+C_2(L(q)-2L(q^2))\\&+C_3(L(q)-3L(q^3))+C_4(L(q)-4L(q^4))\\&+C_5(L(q)-6L(q^6))=\frac{a(-q)b^2(q)}{b(q^2)}.\end{aligned} \tag{4}$$

We formulate a system by incorporating Lemma 2.2 and expressing the above relation in terms of (p,k) parametrization and then equalizing the coefficients of $k^2, pk^2, p^2k^2, p^3k^2$ and p^4k^2 on either side,

$$\begin{pmatrix}0&-1&-2&-3&-5\\24&-14&-16&-18&-22\\36&-24&-36&-36&-36\\18&-14&-16&-24&-22\\3&-1&-2&0&-5\end{pmatrix}\begin{pmatrix}C_1\\C_2\\C_3\\C_4\\C_5\end{pmatrix}=\begin{pmatrix}1\\-4\\3\\2\\-2\end{pmatrix}.$$

Note that, the above linear system possess infinitely many solutions,

$$\begin{aligned}&C_1=-1-l, C_2=-3l-\frac{5}{4}, C_3=-\frac{1}{2},\\&C_4=l, C_5=\frac{1}{4},\end{aligned}$$

where $k\in\mathbb{R}$.
The result follows immediately by substituting the above statistic in (4) and simplifying using Definition 2.2.
Likewise, utilizing the same technique, the following identities are deduced.

$$\begin{aligned}&\left(\frac{2}{9}-l_2+\frac{28}{3}l_1\right)(L(-q)-L(q))\\&-(9l_1-3l_2)(L(q)-2L(q^2))\\&-\left(\frac{1}{6}+7l_1\right)(L(q)-3L(q^3))\\&+l_2(L(q)-4L(q^4))+l_1(L(q)-6L(q^6))\\&=(a(q^3)-\frac{2}{3}b(q))a(q).\end{aligned}$$

$$\begin{aligned}&-l(L(-q)-L(q))\\&+\left(\frac{1}{12}-3l\right)(L(q)-2L(q^2))\\&+l(L(q)-4L(q^4))-\frac{1}{12}(L(q)-6L(q^6))\\&=(a(q^6)-\frac{2}{3}b(q^2))a(q^2).\end{aligned}$$

$$\begin{aligned}&-\left(\frac{1}{9}+l\right)(L(-q)-L(q))\\&-\left(\frac{1}{12}+3l\right)(L(q)-2L(q^2))\\&+\frac{1}{12}(L(q)-L(q^3))\\&+l(L(q)-4L(q^4))-\frac{1}{12}(L(q)-6L(q^6))\\&=(c(q^6)+\frac{1}{3}b(q^2))a(q).\end{aligned}$$

$$\begin{aligned}&\left(\frac{1}{6}-l\right)(L(-q)-L(q))\\&+\left(\frac{1}{4}-3l\right)(L(q)-2L(q^2))\\&+l(L(q)-4L(q^4))-\frac{1}{4}(L(q)-6L(q^6))\\&=(3a(q^6)-2b(q^2))^2\\&-l(L(-q)-L(q))\\&-3l(L(q)-2L(q^2)-\frac{1}{2}(L(q)-3L(q^3))\\&+l(L(q)-4L(q^4))\\&=(3a(q^3)-2b(q))^2.\\&-\left(l+\frac{1}{3}\right)(L(-q)-L(q))\\&-\left(\frac{3}{8}-3l(L(q)-2L(q^2)\right)\\&+\frac{1}{4}(L(q)-3L(q^3))\\&+l(L(q)-4L(q^4))\\&-\frac{3}{8}(L(q)-6L(q^6))=a(-q)a(q^4).\end{aligned}$$

Applying the Definition 2.2 to the above-mentioned relations yields the identities $(ii)-(vi)$.

Theorem 3.2: One has

$$\begin{aligned}&(i)\sum_{\alpha=1}^{\infty}\left[\frac{(1+4l)\alpha(-q)^{\alpha}}{1-(-q)^{\alpha}}+\frac{3\alpha(-q)^{3\alpha}}{1-(-q)^{3\alpha}}\right.\\&-\frac{(1-4l)\alpha q^{\alpha}}{1-q^{\alpha}}-\frac{24l\alpha q^{2\alpha}}{1-q^{2\alpha}}-\frac{3\alpha q^{3\alpha}}{1-q^{3\alpha}}\\&\left.+\frac{16l\alpha(q)^{4\alpha}}{1-q^{4\alpha}}\right]=\frac{(a(-q)-a(q))a(q^2)}{6}.\\&(ii)\sum_{\alpha=1}^{\infty}\left[\frac{(-1+4l)\alpha(-q)^{\alpha}}{1-(-q)^{\alpha}}+\frac{3\alpha(-q)^{3\alpha}}{1-(-q)^{3\alpha}}\right.\\&+\frac{(1+4l)\alpha q^{\alpha}}{1-q^{\alpha}}-\frac{24l\alpha q^{2\alpha}}{1-q^{2\alpha}}-\frac{3\alpha q^{3\alpha}}{1-q^{3\alpha}}\\&\left.+\frac{16l\alpha q^{4\alpha}}{1-q^{4\alpha}}\right]=(a(q^4)(a(q)-a(-q)).\\&(iii)(1+3l)+12\sum_{\alpha=1}^{\infty}\left[\frac{2l\alpha(-q)^{\alpha}}{1-(-q)^{\alpha}}\right.\\&+\frac{(1+2l)\alpha(-q)^{3\alpha}}{1-(-q)^{3\alpha}}-\frac{12l\alpha q^{2\alpha}}{1-q^{2\alpha}}\\&\left.-\frac{3\alpha q^{3\alpha}}{1-q^{3\alpha}}+\frac{2l\alpha q^{4\alpha}}{1-q^{4\alpha}}\right]=a^2(q).\\&(iv)1+\sum_{\alpha=1}^{\infty}\left[\frac{12(1+2l)\alpha(-q)^{\alpha}}{1-(-q)^{\alpha}}\right.\\&-\frac{36\alpha(-q)^{3\alpha}}{1-(-q)^{3\alpha}}+\frac{(36-l)\alpha q^{\alpha}}{1-q^{\alpha}}\\&-\frac{144l\alpha q^{2\alpha}}{1-q^{2\alpha}}-\frac{36\alpha q^{3\alpha}}{1-q^{3\alpha}}\\&\left.+\frac{96l\alpha q^{4\alpha}}{1-q^{4\alpha}}\right]=a^2(-q).\end{aligned}$$

Proof. Consider the relation

$$C_1L_{1,2}(q)+C_2L_{1,2}(q^3)$$
$$+C_3(L(q)-2L(q^2))+C_4(L(q)-3L(q^3))$$
$$+C_5(L(q)-4L(q^4))=\frac{(a(-q)-a(q))a(q^2)}{6}. \quad (5)$$

Now, the following system is generated by incorporating Lemma 2.2 and comparing the coefficients of $k^2, pk^2, p^2k^2, p^3k^2$ and p^4k^2 on either sides,

$$\begin{pmatrix} 0 & 0 & -1 & -2 & -3 \\ 1/2 & 0 & -14 & -16 & -18 \\ 3/4 & 0 & -24 & -36 & -36 \\ 3/8 & 1/8 & -14 & -16 & -24 \\ 1/16 & 1/16 & -1 & -2 & 0 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{pmatrix}=\begin{pmatrix} 0 \\ -6 \\ -9 \\ -9 \\ -3 \end{pmatrix}.$$

The infinitely many solutions obtained after solving the system are given by,

$$C_1=-12-48l, C_2=-36, C_3=-3l,$$
$$C_4=0, C_5=l,$$

where $l\in\mathbb{R}$.
We complete the proof by replacing the above values in (5) and then using Definition 2.2.
Likewise, utilizing the same technique, the following identities are deduced.

$$12(1-4l)L_{1,2}(q)-36L_{1,2}(q^3)$$
$$-3l(L(q)-2L(q^2))+l(L(q)-6L(q^6))$$
$$=a(q^4)(a(q)-a(-q)).$$

$$-48L_{1,2}(q)-3l(L(q)-2L(q^2))$$
$$-\frac{1}{2}(L(q)-3L(q^3))+l(L(q)-4L(q^4))$$
$$=a^2(q).$$

$$-24(1+2l)L_{1,2}(q)+72L_{1,2}(q^3)$$
$$-3l(L(q)-2L(q^2))-\frac{1}{2}(L(q)-3L(q^3))$$
$$+l(L(q)-4L(q^4))=a^2(-q).$$

Applying the Definition 2.2 to the above-mentioned relations yields the identities $(ii)-(iv)$.

IV. Construction of Differential equations

Theorem 4.1: If

$$S(q)=\varphi^2(-q^5)-\varphi^2(-q),$$

then the following differential identity holds:

$$i) q\frac{dS}{dq}-\frac{1}{2}\left[\frac{u(2u^2-5v-10)}{v(v+1)(v-4)}y_{10}+w\right]S=0,$$

$$ii) q^2\frac{d^2S}{dq^2}-\frac{q^2}{S}\left(\frac{dS}{dq}\right)^2+q\frac{dS}{dq}$$
$$-\frac{1}{2}\left[\left(\frac{w}{2}+\frac{u(2vu^2-5v-10)}{v(v+1)(v-4)}y_{10}\right)w\right.$$
$$+\frac{1}{144}\left(\frac{2u^2(275u^4+2640u^2+13361)}{v^2(v+1)^2(v-4)^2}\right.$$
$$\left.\left.+\frac{v(1600v^4-567v^2-1381)-12164}{v^2(v+1)^2(v-4)^2}\right)y_{10}^2\right]S=0,$$

where $k_1+\frac{1}{k_1}=u$, $-k_1+\frac{1}{k_1}=v$ and $k_1\frac{dy_{10}}{dk_1}=w$.

Proof: We note from [5, pp. 525] that

$$S(q)=\varphi^2(-q^5)-\varphi^2(-q)=4\frac{\eta_1\eta_{10}^3}{\eta_2\eta_5}.$$

Using theta functions, $S(q)$ may be reformulated as

$$S(q)=4q\frac{f_1f_{10}^3}{f_2f_5}.$$

Employing Definition 2.1 and logarithmically differentiating, we arrive at

$$\frac{1}{S}\frac{dS}{dq}=\frac{1}{q}\left[1-\sum_{r=1}^{\infty}\frac{rq^r}{1-q^r}+\sum_{r=1}^{\infty}\frac{2rq^{2r}}{1-q^{2r}}\right.$$
$$\left.+\sum_{r=1}^{\infty}\frac{5rq^{5r}}{1-q^{5r}}-3\sum_{r=1}^{\infty}\frac{10rq^{10r}}{1-q^{10r}}\right].$$

Using (1), we get

$$\frac{q}{S}\frac{dS}{dq}=\frac{1}{24}[L_1-2L_2-5L_5+30L_{10}]. \quad (6)$$

Incorporating Lemma 2.3 and simplifying, we obtain

$$L_1-2L_2-5L_5+30L_{10}=6\left[\frac{(1+k_1^2)y_{10}}{(1-k_1^2)}\right.$$
$$\left.+\frac{2(1+k_1^2)y_{10}}{1+k_1-k_1^2}+\frac{(1+k_1^2)y_{10}}{1-4k_1-4k_1^2}+2k\frac{dy_{10}}{dk_1}\right]. \quad (7)$$

Substituting (7) in (6) and simplifying, we obtain (i).
Applying $q\frac{d}{dq}$ on either sides of (6), we have

$$q\frac{d}{dq}\left[\frac{q}{S}\frac{dS}{dq}\right]=\frac{1}{24}q\frac{d}{dq}\left[L_1-2L_2-5L_5+30L_{10}\right]. \quad (8)$$

We note that, Ramanujan[3] recorded a very useful differential equation,

$$q\frac{d}{dq}L_1=\frac{L_1^2-M_1}{12}. \quad (9)$$

Using the above in (8), we get

$$\frac{q^2}{S}\frac{d^2S}{dq^2}-\frac{q^2}{S^2}\left(\frac{dS}{dq}\right)^2+\frac{q}{S}\frac{dS}{dq}$$
$$=\frac{1}{288}\left[(L_1^2-M_1)-4(L_2^2-M_2)\right.$$
$$\left.-25(L_5^2-M_5)+300(L_{10}^2-M_{10})\right].$$

By Lemma 2.3 and 2.5, each of $L^2(q^s)$ and $M(q^s)$, s=1,2 are expressed in terms of k_1, y_{10} and $k_1\frac{dy_{10}}{dk_1}$. After simplification, the claimed result follows.

■

Theorem 4.2: If

$$S(q)=5\varphi^2(-q^5)-\varphi^2(-q),$$

then the following differential identity holds:

$$i) q\frac{dS}{dq} - \frac{1}{2}\left[\frac{3v-2}{v(v+1)(v-4)}y_{10} + w\right]S = 0.$$

$$ii) q^2\frac{d^2S}{dq^2} - \frac{q^2}{S}\left(\frac{dS}{dq}\right)^2 + q\frac{dS}{dq} - \frac{1}{2}\Bigg[\left(\frac{w}{2} - \frac{u(-3v+2)}{v(v+1)(v-4)}y_{10}\right)w + \frac{1}{18}\left(\frac{9u^2(31u^2+200)}{v^2(v+1)^2(v-4)^2} - \frac{4v(9v^4+243v^2+236)-4652}{v^2(v+1)^2(v-4)^2}\right)y_{10}^2\Bigg]S = 0,$$

where $k_1 + \frac{1}{k_1} = u$, $-k_1 + \frac{1}{k_1} = v$ and $k_1\frac{dy_{10}}{dk_1} = w$.

Proof: From [5, pp. 525], note that

$$S(q) = \varphi^2(-q^5) - \varphi^2(-q) = 4\frac{\eta_2^3\eta_5}{\eta_1\eta_{10}} = 4\frac{f_2^3 f_5}{f_1 f_{10}}.$$

Employing Definition 2.1 and logarithmically differentiating on either sides, we arrive at

$$\frac{1}{S}\frac{dS}{dq} = \frac{1}{q}\left[\sum_{r=1}^{\infty}\frac{rq^r}{1-q^r} - \sum_{r=1}^{\infty}\frac{6rq^{2r}}{1-q^{2r}} - \sum_{r=1}^{\infty}\frac{5rq^{5r}}{1-q^{5r}} + \sum_{r=1}^{\infty}\frac{10rq^{10r}}{1-q^{10r}}\right].$$

Employing Definition 2.2, we get

$$\frac{q}{S}\frac{dS}{dq} = \frac{1}{24}[-L_1 + 6L_2 + 5L_5 - 10L_{10}]. \tag{10}$$

Incorporating Lemma 2.3 to the above and simplifying, we find that

$$-L_1 + 6L_2 + 5L_5 - 10L_{10} = 6\left[\frac{-(1+k_1^2)y_{10}}{(1-k_1^2)} + \frac{2(1+k_1^2)y_{10}}{1+k_1-k_1^2} - \frac{(1+k_1^2)y_{10}}{1-4k_1-4k_1^2} - 2k_1\frac{dy_{10}}{dk_1}\right]. \tag{11}$$

Substituting the above in (10) and simplifying, we obtain (i). Applying $q\frac{d}{dq}$ on either sides of (10) and further using (9), we deduce

$$\frac{q^2}{S}\frac{d^2S}{dq^2} - \frac{q^2}{S^2}\left(\frac{dS}{dq}\right)^2 + \frac{q}{S}\frac{dS}{dq} = \frac{1}{288}\left[(L_1^2 - M_1) - 12(L_2^2 - M_2) - 25(L_5^2 - M_5) + 100(L_{10}^2 - M_{10})\right].$$

Now, each of $L^2(q^s)$ and $M(q^s)$, s=1, 2 are expressed in terms of k_1, y_{10} and $k_1\frac{dy_{10}}{dk_1}$ by using Lemma 2.3 and 2.5. Hence the proof. ■

Theorem 4.3: If

$$S(q) = q^{1/4}\psi^2(q) - q^{5/4}\psi^2(q^5),$$

then the following differential identity holds:

$$i) q\frac{dS}{dq} + \frac{1}{4}\left[\frac{u(u^2-4v-8)}{v(v+1)(v-4)}y_{10} - \frac{1}{2}w\right]S = 0.$$

$$ii) \ q^2\frac{d^2S}{dq^2} - \frac{q^2}{S}\left(\frac{dS}{dq}\right)^2 + q\frac{dS}{dq} + \frac{1}{4}\Bigg[\left(w + \frac{u(u^2-32v+28)}{v(v+1)(v-4)}y_{10}\right)w + \frac{1}{36}\left(\frac{u^2(-25u^4+132u^2+2352)}{v^2(v+1)^2(v-4)^2} \frac{-4v(5v^4+396v^2+2592)-16750}{v^2(v+1)^2(v-4)^2}y_{10}^2\right)\Bigg]S = 0,$$

where $k_1 + \frac{1}{k_1} = u$, $-k_1 + \frac{1}{k_1} = v$ and $k_1\frac{dy_{10}}{dk_1} = w$.

Proof: We note from [5, pp. 525] that

$$S(q) = q^{1/4}\psi^2(q) - q^{5/4}\psi^2(q^5) = \frac{\eta_2\eta_5^3}{\eta_1\eta_{10}} = q^{1/4}\frac{f_2 f_5^3}{f_1 f_{10}}.$$

Employing Definition 2.1 and logarithmically differentiating on either sides, and further using Definition 2.2, we arrive at

$$\frac{q}{S}\frac{dS}{dq} = \frac{1}{24}[-L_1 + 2L_2 + 15L_5 - 10L_{10}]. \tag{12}$$

Incorporating Lemma 2.3 in the above and then substituting in (12) and simplifying, we obtain

$$-L_1 + 2L_2 + 15L_5 - 10L_{10} = 3\left[\frac{4(1+k_1^2)y_{10}}{(1-k_1^2)} - \frac{2(1+k_1^2)y_{10}}{1+k_1-k_1^2} - \frac{4(1+k_1^2)y_{10}}{1-4k_1-4k_1^2} - k_1\frac{dy_{10}}{dk_1}\right]. \tag{13}$$

Applying $q\frac{d}{dq}$ on either sides of (12) and further using (9), we deduce that

$$\frac{q^2}{S}\frac{d^2S}{dq^2} - \frac{q^2}{S^2}\left(\frac{dS}{dq}\right)^2 + \frac{q}{S}\frac{dS}{dq} = -\frac{1}{288}\left[(L_1^2 - M_1) - 4(L_2^2 - M_2) - 75(L_5^2 - M_5) + 100(L_{10}^2 - M_{10})\right].$$

Using Lemma 2.3 and 2.5, expressing each of $L^2(q^s)$ and $M(q^s)$, s=1, 2 in terms of k, y_{10} and $k\frac{dy_{10}}{dk}$, we attain the proof. ■

Theorem 4.4: If

$$S(q) = q^{1/4}\psi^2(q) - 5q^{5/4}\psi^2(q^5),$$

then the following differential identity holds:

$$i) q\frac{dS}{dq} - \frac{1}{4}\left[\frac{u(u^2-12v-12)}{v(v+1)(v-4)}y_{10} + w\right]S = 0.$$

$$ii) q^2\frac{d^2S}{dq^2} - \frac{q^2}{S}\left(\frac{dS}{dq}\right)^2 + q\frac{dS}{dq} - \frac{1}{4}\Bigg[\left(\frac{w}{3} + \frac{u(u^2-12v-12)}{v(v+1)(v-4)}y_{10}\right)w - \frac{1}{36}\left(\frac{u^2(-3u^4+338u^2+2200)}{v^2(v+1)^2(v-4)^2} - \frac{v(14v^4+464v^2+567)-1896}{36v^2(v+1)^2(v-4)^2}\right)y_{10}^2\Bigg]S = 0,$$

where $k_1+\frac{1}{k_1}=u$, $-k_1+\frac{1}{k_1}=v$ and $k_1\frac{dy_{10}}{dk_1}=w$.

Proof: From [5, pp. 525], we have

$$S(q)=q^{1/4}\psi^2(q)-5q^{5/4}\psi^2(q^5)$$
$$=\frac{\eta_1^3\eta_{10}}{\eta_2\eta_5}=q^{1/4}\frac{f_1^3f_{10}}{f_2f_5}.$$

Using Definition 2.1 and then logarithmically differentiating and employing Definition 2.2 , we see that

$$\frac{q}{S}\frac{dS}{dq}=\frac{1}{24}[3L_1-2L_2-5L_5+10L_{10}]. \quad (14)$$

Incorporating Lemma 2.3 in the above, we get

$$3L_1-2L_2-5L_5+10L_{10}=6\left[\frac{2(1+k_1^2)y_{10}}{(1-k_1^2)}+\frac{(1+k_1^2)y_{10}}{1+k_1-k_1^2}-\frac{2(1+k_1^2)y_{10}}{1-4k_1-4k_1^2}-k_1\frac{dy_{10}}{dk_1}\right]. \quad (15)$$

substituting (15) in (14) and on simplifying, we obtain (i). Applying $q\frac{d}{dq}$ on either sides of (14) and further using (9), we can deduce that

$$\frac{q^2}{S}\frac{d^2S}{dq^2}-\frac{q^2}{S^2}\left(\frac{dS}{dq}\right)^2+\frac{q}{S}\frac{dS}{dq}$$
$$=-\frac{1}{288}\left[3(L_1^2-M_1)-4(LP_2^2-M_2)-25(L_5^2-M_5)+100(L_{10}^2-M_{10})\right].$$

Using Lemma 2.3 and 2.5, expressing each of $L^2(q^s)$ and $M(q^s)$, s=1, 2 in terms of k_1, y_{10} and $k_1\frac{dy_{10}}{dk_1}$, we complete the proof. ■

V. Relations among class one infinite series and k-functions

Theorem 5.1: For every $n\in\mathbb{N}$ with $n\geq 2$, the following relation among the series holds:

$$L(q^n)=1+nq^{n-1}\left[\frac{T_2(q^n)+1}{(q^n;q^n)_\infty}-1\right]. \quad (16)$$

Proof: First, we prove the result for $n=2$. Replacing q to q^2 in Definition 2.2, we obtain

$$L(q^2)=1+24q^2\frac{d}{dq}log(q^2;q^2)$$
$$=1+24q^2\frac{1}{(q^2;q^2)_\infty}\frac{d}{dq}(q^2;q^2)_\infty.$$

Further simplifying, we arrive at

$$(q^2;q^2)_\infty L(q^2)=(q^2;q^2)_\infty$$
$$+24q^2\tfrac{d}{dq}\left[1+\textstyle\sum_{i=1}^{\infty}(-1)^i\{q^{i(3i-1)}+q^{i(3i+1)}\}\right]$$
$$=(q^2;q^2)_\infty+24q\textstyle\sum_{i=1}^{\infty}(-1)^i\left[i(3i-1)q^{i(3i-1)}+i(3i+1)q^{i(3i+1)}\right]$$
$$=(q^2;q^2)_\infty+2q\textstyle\sum_{i=1}^{\infty}(-1)^i\left[((6i-1)^2-1)q^{i(3i-1)}\right]$$
$$+\sum_{i=1}^{\infty}(-1)^i\{q^{i(3i-1)}-2q(q^2;q^2)_\infty+2q$$
$$=(q^2;q^2)_\infty+2qT_2(q^2)-2q(q^2;q^2)_\infty+2q.$$

Dividing throughout by $(q^2;q^2)_\infty$ and further rearranging the terms, we deduce the result for $n=2$. Similarly, the proof of $n\in\mathbb{N}$, $n>2$ follows by replacing q to q^n in (1) and using the series (2). ■

Theorem 5.2: The following series expansions hold:

$$\frac{T_2(q)}{f_1}-4q\frac{T_2(q^2)}{f_2}-125q^4\frac{T_2(q^5)}{f_5}+500q^9\frac{T_2(q^{10})}{f_{10}}+4q\left(1-\frac{1}{f_2}\right)+125q^4\left(1-\frac{1}{f_5}\right)-500q^9\left(1-\frac{1}{f_{10}}\right)-24\left(\frac{u}{v}\right)y_{10}+23=0,$$

$$\frac{T_2(q)}{f_1}-8q\frac{T_2(q^2)}{f_2}-25q^4\frac{T_2(q^5)}{f_5}+200q^9\frac{T_2(q^{10})}{f_{10}}+8q\left(1-\frac{1}{f_2}\right)+25q^4\left(1-\frac{1}{f_5}\right)-200q^9\left(1-\frac{1}{f_{10}}\right)-12\left(\frac{u}{v+1}\right)y_{10}+11=0,$$

$$3\frac{T_2(q)}{f_1}-4q\frac{T_2(q^2)}{f_2}+25q^4\frac{T_2(q^5)}{f_5}-300q^9\frac{T_2(q^{10})}{f_{10}}+4q\left(1-\frac{1}{f_2}\right)-25q^4\left(1-\frac{1}{f_5}\right)+300q^9\left(1-\frac{1}{f_{10}}\right)+24\left(\frac{u}{v-4}\right)y_{10}-27=0,$$

$$\frac{T_2(q)}{f_1}+8q\frac{T_2(q^2)}{f_2}+75q^4\frac{T_2(q^5)}{f_5}-200q^9\frac{T_2(q^{10})}{f_{10}}-4q\left(1-\frac{1}{f_2}\right)-75q^4\left(1-\frac{1}{f_5}\right)+200q^9\left(1-\frac{1}{f_{10}}\right)-24w-1=0,$$

where $k+\frac{1}{k}=u$, $-k+\frac{1}{k}=v$ and $k\frac{dy_{10}}{dk}=w$.

Proof: Using (3), putting $n=2,5,10$ in (16) and further substituting these in Lemma 2.4, we achieve the desired outcome. ■

Theorem 5.3: The following series expansions hold:

$$\frac{T_2(q)}{f_1}-4q\frac{T_2(q^2)}{f_2}-25q^4\frac{T_2(q^5)}{f_5}+300q^9\frac{T_2(q^{10})}{f_{10}}+4q\left(1-\frac{1}{f_2}\right)+25q^4\left(1-\frac{1}{f_5}\right)-300q^9\left(1-\frac{1}{f_{10}}\right)-12\left(\frac{u(2u^2-5v-10)}{v(v+1)(v-4)}y_{10}+w\right)+23=0,$$

$$
\begin{aligned}
&\frac{T_2(q)}{f_1} - 12q\frac{T_2(q^2)}{f_2} - 25q^4\frac{T_2(q^5)}{f_5} \\
&+ 100q^9\frac{T_2(q^{10})}{f_{10}} + 12q\left(1 - \frac{1}{f_2}\right) \\
&+ 25q^4\left(1 - \frac{1}{f_5}\right) + 100q^9\left(1 - \frac{1}{f_{10}}\right) \\
&+ 12\left(\frac{(3v-2)}{v(v+1)(v-4)}y_{10} - \frac{w}{2}\right) - 1 = 0, \\
&\frac{T_2(q)}{f_1} - 4q\frac{T_2(q^2)}{f_2} - 75q^4\frac{T_2(q^5)}{f_5} \\
&+ 100q^9\frac{T_2(q^{10})}{f_{10}} + 4q\left(1 - \frac{1}{f_2}\right) \\
&+ 75q^4\left(1 - \frac{1}{f_5}\right) - 100q^9\left(1 - \frac{1}{f_{10}}\right) \\
&- 6\left(\frac{u(u^2-4v-8)}{v(v+1)(v-4)}y_{10} - \frac{w}{2}\right) - 7 = 0, \\
&3\frac{T_2(q)}{f_1} - 4q\frac{T_2(q^2)}{f_2} - 25q^4\frac{T_2(q^5)}{f_5} \\
&+ 100q^9\frac{T_2(q^{10})}{f_{10}} + 4q\left(1 - \frac{1}{f_2}\right) \\
&+ 25q^4\left(1 - \frac{1}{f_5}\right) - 100q^9\left(1 - \frac{1}{f_{10}}\right) \\
&- 6\left(\frac{u(u^2-12v-12)}{v(v+1)(v-4)}y_{10} + w\right) + 3 = 0,
\end{aligned}
$$

where $k_1 + \frac{1}{k_1} = u$, $-k_1 + \frac{1}{k_1} = v$ and $k_1\frac{dy_{10}}{dk_1} = w$.

Proof: The required result is obtained easily by using (3), replacing $n = 2, 5, 10$ in (16) and then substituting these in (7), (11), (13) and (15). ■

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