Consistency and Indexes of Fuzzy Games

Hui-Chuan Wei, Ai-Tzu Li, Wei-Ni Wang and Yu-Hsien Liao

Abstract—**Within the realm of fuzzy games, we put forward various expansions of the Banzhaf-Coleman index and the Banzhaf-Owen index. To ensure effectiveness, we also explore their efficient extensions. Each efficient index corresponds to a reduction that characterizes its properties.**

Index Terms—**Fuzzy game, the accumulate index, the average index, reduction.**

I. Introduction

Numerous alternative indices have been introduced as mappings through strategical quotas and weights to power. Among these, the extensively utilized index is the Banzhaf-Coleman index, introduced by Banzhaf [3]. This power index calculates the likelihood of altering a voting outcome when voting rights aren't equally distributed among voters. Additionally, other power indices have emerged, such as the Banzhaf-Owen index, driven by demands in fields like accounting, economics, management science, and so on. Pertinent studies on this topic can be found in the works of Banzhaf [3], van den Brink and van der Laan [5], Cheng et al. [7], Dubey and Shapley [9], Haller [10], Hwang and Liao [16], Lehrer [17], Liao et al. [19], Moulin [21], Owen [22], and so on.

The concept of *consistency* plays a vital role under characterizing viable solutions. The notion of consistency revolves around agents' expectations regarding the game and their willingness to have their payments computed based on these expectations. A solution concept is considered consistent if it assigns coincident payments to players in both the initial game and an imaginary reduction. This requirement of consistency ensures the internal "robustness" of compromises. reductions have been extensively used to explore the fundamental properties of solutions in various problem classes. Different versions of reductions have been considered, basing on how the payments to agents outside the subgroup are determined. The literature presents three distinct forms of reductions. Peleg [23] and Sobolev [24] provided axiomatic characterizations for the core, the prekernel, and the prenucleolus, respectively, using consistency with respect to the Davis and Maschler's [8] reduction. Moulin [21] also introduced a specific reduction to analyze allocating notion under the context of quasi-linear cost allocation issues. Hart and Mas-Colell [12] developed different reduction to axiomatize the Shapley value [25], among others.

The concept of fuzzy games and the core of a fuzzy game were introduced by Aubin [1], [2], giving rise to the theory of fuzzy games. Unlike traditional TU games where players are either fully involved or not involved at all, fuzzy games allow for infinitely varied levels of player participation. The field of fuzzy games has seen the development of numerous solution concepts. Extensive research by authors such as Branzei et al. [4], Butnariu [6], Hwang and Liao [13], [14], Molina and Tejada [20], Tsurumi et al. [26], among others, has contributed to the understanding and application of fuzzy game solutions in various domains, including accounting, economics, management science, and political science. Within the framework of fuzzy games, Hwang and Liao [13] have proposed several expansions of Davis-Maschler's [8] reduction and Moulin's [21] reduction to capture the core introduced by Aubin [1], [2]. In this context, Davis-Maschler's [8] reduction is rooted in "maximizing behavior".

In contrast, we propose different results in this study.

1) In section 2, we extend the Banzhaf-Coleman index and the Banzhaf-Owen index to fuzzy games, labeling them as the "accumulate index" and the "average index" respectively.
2) Additionally, we introduce the "sum-reduction" that aligns with "summing behavior" and the "average-reduction" that corresponds to "averaging behavior" in our study.
3) Since these indices lack efficiency, we also explore their efficient extensions. Taking inspiration from Hart and Mas-Colell [12], we present axiomatic characterizations for the accumulate index and the average index, based on consistency in relation to the sum-reduction and the average-reduction respectively.

II. Definitions and notation

Let $\overline{\mathbb{U}}$ denote the set of players. For $i \in \overline{\mathbb{U}}$ and $a_i \in [0,1]$, we define $A_i = [0, a_i]$ as the participating level collection for player i, where 0 represents no participation and $A_i^+ = (0, a_i]$. For $U \subseteq \overline{\mathbb{U}}$, $U \neq \emptyset$, let $a = (a_i)_{i \in U} \in [0,1]^U$ be the vector representing the highest participating grade for each player, at which they can operate. Let $A^U = \prod_{i \in U} A_i$ be the Cartesian product of the participating level collections for players of U. For any $M \subseteq U$, a player coalition $M \subseteq U$ corresponds to the fuzzy coalition $e^M \in A^U$, where $e_t^M = 1$ if $t \in M$ and $e_t^M = 0$ if $t \in U \setminus M$. Let 0_U denote the zero vector in $\mathbb{R}^U$.

A **fuzzy game**is a triple (U, a, m), where $U \neq \emptyset$ is finite collection of players, a is the vector representing the highest participating grade for each player, and $m : A^U \to \mathbb{R}$ is a measuring function matching $m(0_U) = 0$, which assigns a worth to each $\mu = (\mu_t)_{t \in U} \in A^U$ indicating the players'

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potential gain if each player t operates at participating level μ_t. If there is no potential confusion, a game (U, a, m) may be denoted by its measuring function m alone. Given a fuzzy game (U, a, m) and $\mu \in A^U$, (U, μ, m) represents the **fuzzy subgame** formed via restricting m to $\{\eta \in A^U \mid \eta_i \leq \mu_i \text{ for all } i \in U\}$.

The class of all fuzzy games is denoted as Λ. For $(U, a, m) \in \Lambda$, let $P^{U,a} = \{(p, q) \mid p \in U, q \in a_p^+\}$. A **solution** on Λ is a mapping τ that assigns to each $(U, a, m) \in \Lambda$ an element

$$\tau(U, a, m) = \big(\tau_{i,j}(U, a, m)\big)_{(i,j)\in P^{U,a}} \in \mathbb{R}^{P^{U,a}}.$$

Here $\tau_{i,j}(U, a, m)$ represents the value of player i if i operates at participating level j under the game m. For convenience, given $(U, a, m) \in \Lambda$ and a solution τ on Λ, one can define $\tau_{t,0}(U, a, m) = 0$ for each $t \in U$.

For $U \subseteq \overline{\mathbb{U}}$, $i \in U$, and $\mu \in \mathbb{R}^U$, we denote $K(\mu)$ as the collection of players in U whose participating level is not zero, and μ_T represents the restriction of μ to the subset $T \subseteq U$. Additionally, we consider the substitution notation μ_{-i} to refer to $\mu_{U\setminus\{i\}}$, and let $\eta = (\mu_{-i}, j) \in \mathbb{R}^U$ be considered as $\eta_{-i} = \mu_{-i}$ and $\eta_i = j$. Further, let $p \in U$ and $l \in U$, μ_{-ip} denotes $\mu_{U\setminus\{i,p\}}$, and (μ_{-ip}, j, l) denotes $((\mu_{-i}, j)_{-p}, l)$.

Under the context of fuzzy games, we introduce relative generalizations of the Banzhaf-Coleman index and the Banzhaf-Owen index.

Definition 1:

- The **accumulate index**, Θ^{AC}, is an index that associates with $(U, a, m) \in \Lambda$ and all $(i, j) \in P^{U,a}$ the value

$$\begin{aligned}&\Theta^{AC}_{i,j}(U, a, m)\\ =\ &\sum_{\substack{S\subseteq U\\ i\in S}} \Big[m\big((a_{-i}, j)_S, 0_{U\setminus S}\big) - m\big((a_{-i}, 0)_S, 0_{U\setminus S}\big)\Big].\end{aligned} \tag{1}$$

- The **average index**, Θ^{AV}, is an index that associates with $(U, a, m) \in \Lambda$ and all $(i, j) \in P^{U,a}$ the value

$$\begin{aligned}&\Theta^{AV}_{i,j}(U, a, m)\\ =\ &\frac{1}{2^{|U|-1}} \cdot \sum_{\substack{S\subseteq U\\ i\in S}} \Big[m\big((a_{-i}, j)_S, 0_{U\setminus S}\big)\\ &\qquad - m\big((a_{-i}, 0)_S, 0_{U\setminus S}\big)\Big].\end{aligned} \tag{2}$$

Consider a triple $(U, a, m) \in \Lambda$. It can be observed that the accumulate index and the average index are not suitable for sharing the value $m(a)$ of the grand coalition since they lack efficiency. In other words, they may not distribute the value $m(a)$ among the players in U appropriately. Hence, we explore potential efficient extensions for each of these indexes. Let τ be a solution for Λ. We say that τ matches the condition of **efficiency (EFF)** if, for all $(U, a, m) \in \Lambda$, $\sum_{i\in U} \tau_{i,a_i}(U, a, m) = m(a)$. Relative efficient extensions for these indexes are outlined as follows.

Definition 2:

- The **efficient accumulate index**, denoted as $\overline{\Theta^{AC}}$, is an index that associates with $(U, a, m) \in \Lambda$ and each $(i, j) \in P^{U,a}$ the value

$$\begin{aligned}&\overline{\Theta^{AC}_{i,j}}(U, a, m)\\ =\ &\Theta^{AC}_{i,j}(U, a, m) + \tfrac{1}{|U|}\big[m(a) - \sum_{k\in U} \Theta^{AC}_{k,a_k}(U, a, m)\big].\end{aligned} \tag{3}$$

- The **efficient average index**, denoted as $\overline{\Theta^{AV}}$, is an index that associates with $(U, a, m) \in \Lambda$ and each $(i, j) \in P^{U,a}$ the value

$$\begin{aligned}&\overline{\Theta^{AV}_{i,j}}(U, a, m)\\ =\ &\Theta^{AV}_{i,j}(U, a, m) + \tfrac{1}{|U|}\big[m(a) - \sum_{k\in U} \Theta^{AV}_{k,a_k}(U, a, m)\big].\end{aligned} \tag{4}$$

Clearly, $\overline{\Theta^{AC}}$ and $\overline{\Theta^{AV}}$ match the EFF property.

Remark 1: Our approaches offer several advantages. Firstly, these indices for a fuzzy game always exist. Secondly, they provide a different computation approach compared to conventional methods used in fuzzy games. Instead of computing a global value for a player by aggregating their contributions across all participation levels, our methods allow for computing a value specific to a given player operating at a particular level. To illustrate, consider the following example: On each payday of a quarter, each company employee is permited to cherish a percentage $\varepsilon\%$ of its earnings into an official investment project, where $\varepsilon \in [0, 100]$. The whole deposits from total employees are invested in arbitrage funding throughout the quarter. At the end of the quarter, the company and the personnel partake the benefits from the investment project depended upon the personal initial deposit amounts. Each employee can retrieve its deposit at the prime rate if the investment project could not produce benefits that quarter. Based on above instance, a employee has fuzzy participating levels denoted as σ_ε: deposit $\varepsilon\%$ of its earnings into the investment project, with $\varepsilon \in [0, 100]$.

III. Axioms, reductions and characterizations

For these efficient indices, we establish the existence of relative reductions that can be utilized to axiomatize these efficient indices.

Definition 3: Given a solution τ, $(U, a, m) \in \Lambda$ and $S \subseteq U$.

- The **sum-reduction** $(S, a_S, m^{sum}_{S,\tau})$ is defined as for all $\mu \in A^S$,

$$m^{sum}_{S,\tau}(\mu) = \begin{cases} 0 & \mu = 0_S,\\ m(a) - \sum_{i\in U\setminus S} \tau_{i,a_i}(U, a, m) & \mu = a_S,\\ \sum_{Q\subseteq U\setminus S} \big[m(\mu, a_Q, 0_{U\setminus(S\cup Q)}) \\ \qquad - \sum_{i\in Q} \tau_{i,a_i}(U, a, m)\big] & \text{o.w.}\end{cases}$$

- The **average-reduction** $(S, a_S, m^{ave}_{S,\tau})$ is defined as for all $\mu \in A^S$,

$$m^{ave}_{S,\tau}(\mu) = \begin{cases} 0 & \mu = 0_S,\\ m(a) - \sum_{i\in U\setminus S} \tau_{i,a_i}(U, a, m) & \mu = a_S,\\ \frac{1}{2^{|U\setminus S|}} \sum_{Q\subseteq U\setminus S} \big[m(\mu, a_Q, 0_{U\setminus(S\cup Q)}) \\ \qquad - \sum_{i\in Q} \tau_{i,a_i}(U, a, m)\big] & \text{o.w.}\end{cases}$$

For each of these reductions, there is a relative consistency as follows. Let τ be a solution on Λ.

- **Sum-consistency (Sum-CON)**: For each $(U, a, m) \in \Lambda$, for each $S \subseteq U$ and for each $(i, j) \in P^{S,a_S}$, $\tau_{i,j}(U, a, m) = \tau_{i,j}(S, a_S, m^{sum}_{S,\tau})$.

- **Average-consistency (Ave-CON)**: For each $(U, a, m) \in \Lambda$, for each $S \subseteq U$ and for each $(i, j) \in P^{S,a_S}$, $\tau_{i,j}(U, a, m) = \tau_{i,j}(S, a_S, m^{ave}_{S,\tau})$.

The concept of bilateral consistency serves as a significant relaxation of consistency and was originally introduced by Harsanyi [11] as bilateral equilibrium.

- **Bilateral sum-consistency (Bil-Sum-CON)**: For each $(U, a, m) \in \Lambda$, for each $S \subseteq U$ with $|S| = 2$ and for each $(i, j) \in P^{S,a_S}$, $\tau_{i,j}(U, a, m) = \tau_{i,j}(S, a_S, m^{sum}_{S,\tau})$.
- **Bilateral average-consistency (Bil-Ave-CON)**: For each $(U, a, m) \in \Lambda$, for each $S \subseteq U$ with $|S| = 2$ and for each $(i, j) \in P^{S,a_S}$, $\tau_{i,j}(U, a, m) = \tau_{i,j}(S, a_S, m^{ave}_{S,\tau})$.

Subsequently, we demonstrate that the solutions $\overline{\Theta^{AC}}$ and $\overline{\Theta^{AV}}$ matches Bil-Sum-CON and Bil-Ave-CON, respectively.

Lemma 1:

1) The solution $\overline{\Theta^{AC}}$ matches Bil-Sum-CON.
2) The solution $\overline{\Theta^{AV}}$ matches Bil-Ave-CON.

Proof: Given $(U, a, m) \in \Lambda$ and $S = \{i, k\}$ for some $i, k \in U$, $i \neq k$. To verify 1, for all $(p, q) \in P^{S,a_S}$,

$$\begin{aligned} & \overline{\Theta^{AC}_{p,q}}(S, a_S, m^{sum}_{S,\overline{\Theta^{AC}}}) \\ = \;& \Theta^{AC}_{p,q}(S, a_S, m^{sum}_{S,\overline{\Theta^{AC}}}) \\ & + \tfrac{1}{|S|}\big[m^{sum}_{S,\overline{\Theta^{AC}}}(a_S) - \sum_{t\in S} \Theta^{AC}_{t,a_t}(S, a_S, m^{sum}_{S,\overline{\Theta^{AC}}})\big]. \end{aligned} \tag{5}$$

By definitions of Θ and $m^{sum}_{S,\overline{\Theta^{AC}}}$, for all $j \in a_i^+$,

$$\begin{aligned} & \Theta^{AC}_{i,j}(S, a_S, m^{sum}_{S,\overline{\Theta^{AC}}}) \\ = \;& \sum_{\substack{H\subseteq S\\ i\in H}} \Big[m^{sum}_{S,\overline{\Theta^{AC}}}\big((a_{-i}, j)_H, 0_{S\setminus H}\big) \\ & \qquad - m^{sum}_{S,\overline{\Theta^{AC}}}\big((a_{-i}, 0)_H, 0_{S\setminus H}\big)\Big] \\ = \;& \sum_{\substack{H\subseteq S\\ i\in H}} \Big[\sum_{Q\subseteq U\setminus S} \big[m(((a_{-i}, j)_H, 0_{S\setminus H}), a_Q, 0_{U\setminus(S\cup Q)}) \\ & \qquad - \sum_{i\in Q} \tau_{i,a_i}(U, a, m)\big] \\ & \quad - \sum_{Q\subseteq U\setminus S} \big[m(((a_{-i}, 0)_H, 0_{S\setminus H}), a_Q, 0_{U\setminus(S\cup Q)}) \\ & \qquad - \sum_{i\in Q} \tau_{i,a_i}(U, a, m)\big]\Big] \\ = \;& \sum_{\substack{K\subseteq U\\ i\in K}} \Big[m\big((a_{-i}, j)_K, 0_{U\setminus K}\big) - m\big((a_{-i}, 0)_K, 0_{U\setminus K}\big)\Big] \\ = \;& \Theta^{AC}_{i,j}(U, a, m). \end{aligned} \tag{6}$$

By equations (5), (6) and definitions of $m^{sum}_{S,\overline{\Theta^{AC}}}$ and $\overline{\Theta^{AC}}$,

$$\begin{aligned} & \overline{\Theta^{AC}_{i,j}}(S, a_S, m^{sum}_{S,\overline{\Theta^{AC}}}) \\ = \;& \Theta^{AC}_{i,j}(U, a, m) \\ & + \tfrac{1}{|S|}\big[m^{sum}_{S,\overline{\Theta^{AC}}}(a_S) - \sum_{t\in S} \Theta^{AC}_{t,a_t}(U, a, m)\big] \\ = \;& \Theta^{AC}_{i,j}(U, a, m) \\ & + \tfrac{1}{|S|}\big[m(a) - \sum_{t\in U\setminus S} \overline{\Theta^{AC}_{t,a_t}}(U, a, m) \\ & \qquad - \sum_{t\in S} \Theta^{AC}_{t,a_t}(U, a, m)\big] \\ = \;& \Theta^{AC}_{i,j}(U, a, m) \\ & + \tfrac{1}{|S|}\big[\sum_{t\in S} \overline{\Theta^{AC}_{t,a_t}}(U, a, m) - \sum_{t\in S} \Theta^{AC}_{t,a_t}(U, a, m)\big] \\ & \text{(by EFF of } \overline{\Theta^{AC}}) \\ = \;& \Theta^{AC}_{i,j}(U, a, m) \\ & + \tfrac{1}{|S|}\Big[\tfrac{|S|}{|U|}\big[m(a) - \sum_{t\in U} \Theta^{AC}_{t,a_t}(U, a, m)\big]\Big] \\ & \text{(by Definition 1)} \\ = \;& \Theta^{AC}_{i,j}(U, a, m) \\ & + \tfrac{1}{|U|}\big[m(a) - \sum_{t\in U} \Theta^{AC}_{t,a_t}(U, a, m)\big] \\ = \;& \overline{\Theta^{AC}_{i,j}}(U, a, m). \end{aligned}$$

Hence, the solution $\overline{\Theta^{AC}}$ matches Bil-Sum-CON. The proof of 2 is similar. ■

Inspired by Hart-Mas-Colell [12], we apply a standard approach to axiomatize these efficient indices.

- A solution τ matches **accumulate-standard of games (ACSG)** if for all $(U, a, m) \in \Lambda$ with $|U| \leq 2$, $\tau(U, a, m) = \overline{\Theta^{AC}}(U, a, m)$.
- A solution τ matches **average-standard of games (AVSG)** if for all $(U, a, m) \in \Lambda$ with $|U| \leq 2$, $\tau(U, a, m) = \overline{\Theta^{AV}}(U, a, m)$.

Remark 2: Clearly, the accumulate index and the average index match ACSG and AVSG respectively. And it is not difficult to derive that $\tau_{i,a_i}(\{i\}, a_i, m) = m(a_i)$ for all $(\{i\}, a_i, m) \in \Lambda$ if τ matches ACSG (AVSG) and Bil-Sum-CON (Bil-Ave-CON). (Hart and Mas-Colell [12]: pp.599)

Lemma 2: Let τ be a solution on Λ.

1) τ matches EFF if τ matches ACSG and Bil-Sum-CON.
2) τ matches EFF if τ matches AVSG and Bil-Ave-CON.

Proof: To verify 1, suppose that τ matches ACSG and Bil-Sum-CON. Let $(U, a, m) \in \Lambda$. τ matches EFF by Remark 2 if $|U| = 1$. Assume $|U| = 2$. It is trivial that τ matches EFF by ACSG. Suppose $|U| > 2$. Since Bil-Sum-CON and τ matches EFF for one-person cases, it is easy to derive that τ matches EFF. The proof of 2 is similar. ■

Theorem 1:

1) A solution τ on matches ACSG and Bil-Sum-CON if and only if $\tau = \overline{\Theta^{AC}}$.
2) A solution τ on matches AVSG and Bil-Ave-CON if and only if $\tau = \overline{\Theta^{AV}}$.

Proof: By Lemma 1, $\overline{\Theta^{AC}}$ matches Bil-Sum-CON and $\overline{\Theta^{AV}}$ matches Bil-Ave-CON. Absolutely, $\overline{\Theta^{AC}}$ and $\overline{\Theta^{AV}}$ match ACSG and AVSG, respectively.

To present the uniqueness of 1, suppose τ matches ACSG and Bil-Sum-CON. Let $(U, a, m) \in \Lambda$. If $|U| \leq 2$, it is trivial that $\tau(U, a, m) = \overline{\Theta^{M}}(U, a, m)$ by ACSG. The situation $|U| > 2$: Let $i \in U$ and $S = \{i, k\}$ for some $k \in U \setminus \{i\}$.

For each $j \in A_i^+$ and for each $l \in A_k^+$,

$$\begin{aligned}
& \tau_{i,j}(U,a,m) - \tau_{k,l}(U,a,m) \\
= \;& \tau_{i,j}(S,a_S,m^{sum}_{S,\tau}) - \tau_{k,l}(S,a_S,m^{sum}_{S,\tau}) \\
& \text{(by Bil-Sum-CON of } \tau) \\
= \;& \overline{\Theta^{AC}_{i,j}}(S,a_S,m^{sum}_{S,\tau}) - \overline{\Theta^{AC}_{k,l}}(S,a_S,m^{sum}_{S,\tau}) \\
& \text{(by ACSG of } \tau) \\
= \;& \Theta^{AC}_{i,j}(S,a_S,m^{sum}_{S,\tau}) - \Theta^{AC}_{k,l}(S,a_S,m^{sum}_{S,\tau}) \\
& \text{(by Definition 2)} \\
= \;& \big[m^{sum}_{S,\tau}(a_k,j) - m^{sum}_{S,\tau}(a_k,0) \\
& \quad + m^{sum}_{S,\tau}(0,j) - m^{sum}_{S,\tau}(0,0)\big] \\
& - \big[m^{sum}_{S,\tau}(a_i,l) - m^{sum}_{S,\tau}(a_i,0) \\
& \quad + m^{sum}_{S,\tau}(0,l) - m^{sum}_{S,\tau}(0,0)\big] \\
& \text{(by Definition 1)} \\
= \;& \big[m(a_{-i},j) - \sum_{t\in U\setminus S} \tau_{t,a_t}(U,a,m) - m(a_{-i},0) \\
& \quad + \sum_{t\in U\setminus S} \tau_{t,a_t}(U,a,m) + m(a_{-ik},j,0) \\
& \quad - \sum_{t\in U\setminus S} \tau_{t,a_t}(U,a,m)\big] \\
& - \big[m(a_{-k},l) - \sum_{t\in U\setminus S} \tau_{t,a_t}(U,a,m) - m(a_{-k},0) \\
& \quad + \sum_{t\in U\setminus S} \tau_{t,a_t}(U,a,m) + m(a_{-ik},0,l) \\
& \quad - \sum_{t\in U\setminus S} \tau_{t,a_t}(U,a,m)\big] \\
& \text{(by Definition of } m^{sum}_{S,\tau}) \\
= \;& \big[m(a_{-i},j) - m(a_{-i},0) + m(a_{-ik},j,0)\big] \\
& - \big[m(a_{-k},l) - m(a_{-k},0) + m(a_{-ik},0,l)\big].
\end{aligned} \tag{7}$$

Similarly,

$$\begin{aligned}
& \overline{\Theta^{AC}_{i,j}}(U,a,m) - \overline{\Theta^{AC}_{k,l}}(U,a,m) \\
= \;& \big[m(a_{-i},j) - m(a_{-i},0) + m(a_{-ik},j,0)\big] \\
& - \big[m(a_{-k},l) - m(a_{-k},0) + m(a_{-ik},0,l)\big].
\end{aligned} \tag{8}$$

By equations (7) and (8), for all $(i,j),(k,l) \in P^{U,a}$ with $i \neq k$,

$$\begin{aligned}
& \tau_{i,j}(U,a,m) - \tau_{k,l}(U,a,m) \\
= \;& \overline{\Theta^{AC}_{i,j}}(U,a,m) - \overline{\Theta^{AC}_{k,l}}(U,a,m).
\end{aligned}$$

Thence, there exists $\omega \in \mathbb{R}$ such that

$$\tau_{i,j}(U,a,m) - \overline{\Theta^{AC}_{i,j}}(U,a,m) = \omega$$

for all $(i,j) \in P^{U,a}$. By EFF of τ and $\overline{\Theta^{AC}}$,

$$\begin{aligned}
|U|\cdot\omega &= \sum_{t\in U} [\tau_{t,a_t}(U,a,m) - \overline{\Theta^{AC}_{t,a_t}}(U,a,m)] \\
&= m(a) - m(a) \\
&= 0.
\end{aligned}$$

Hence, $\omega = 0$. Therefore, $\tau_{i,j}(U,a,m) = \overline{\Theta^{AC}_{i,j}}(U,a,m)$ for all $(i,j) \in P^{U,a}$. The proof of 2 is similar. ■

The following instances demonstrate that each of the axioms utilized in Theorem 1 is logically independent of the remaining axioms.

Example 1: Consider a solution τ on Λ as for each $(U,a,m) \in \Lambda$ and for each $(i,j) \in P^{U,a}$,

$$\tau_{i,j}(U,a,m) = 0.$$

Absolutely, τ matches Bil-Sum-CON and Bil-Ave-CON, but it violates ACSG and AVSG.

Example 2: Consider a solution τ on Λ as for each $(U,a,m) \in \Lambda$ and for each $(i,j) \in P^{U,a}$,

$$\tau_{i,j}(U,a,m) = \begin{cases} \overline{\Theta^{AC}_{i,j}}(U,a,m) & \text{, if } |U| \le 2 \\ \overline{\Theta^{AC}_{i,j}}(U,a,m) - \delta & \text{, otherwise.} \end{cases}$$

where $\delta \in \mathbb{R} \setminus \{0\}$. Absolutely, τ matches ACSG, but it violates Bil-Sum-CON.

Example 3: Consider a solution τ on Λ as for each $(U,a,m) \in \Lambda$ and for each $(i,j) \in P^{U,a}$,

$$\tau_{i,j}(U,a,m) = \begin{cases} \overline{\Theta^{AV}_{i,j}}(U,a,m) & \text{, if } |U| \le 2 \\ \overline{\Theta^{AV}_{i,j}}(U,a,m) - \delta & \text{, otherwise.} \end{cases}$$

where $\delta \in \mathbb{R} \setminus \{0\}$. Absolutely, τ matches AVSG, but it violates Bil-Ave-CON.

Based on the allocating conception of the EANSC, Liao [18] presented an extended EANSC and related results as follows.

Definition 4: The **efficient marinal index (Liao [18])**, denoted as $\overline{\Theta^M}$, is an index that associates with $(U,a,m) \in \Lambda$ and each $(i,j) \in P^{U,a}$ the value

$$\begin{aligned}
& \overline{\Theta^M_{i,j}}(U,a,m) \\
= \;& \Theta^M_{i,j}(U,a,m) + \tfrac{1}{|U|}\big[m(a) - \sum_{k\in U} \Theta^M_{k,a_k}(U,a,m)\big],
\end{aligned} \tag{9}$$

where $\Theta^M_{i,j} = m\big(a_{U\setminus\{i\}},j\big) - m\big(a_{U\setminus\{i\}},0\big)$ is the **marginal index** of the player i with participating grade j.

Liao [18] showed that the efficient marginal index matches efficiency. Based on related axiomatic conceptions due to Moulin [21], Liao [18] also considered an extended reduction to axiomatize the efficient marginal index. Given a solution τ, $(U,a,m) \in \Lambda$ and $S \subseteq U$. The **complement-reduction** $(S,a_S,m^{com}_{S,\tau})$ **(Liao [18])** is defined as for all $\mu \in A^S$,

$$\begin{aligned}
& m^{com}_{S,\tau}(\mu) \\
= \;& \begin{cases} 0 & \mu = 0_S, \\ m\big(\mu,a_{U\setminus S}\big) - \sum_{i\in U\setminus S} \tau_{i,a_i}(U,a,m) & \text{o.w.} \end{cases}
\end{aligned}$$

For the marginal index and its efficient extension, relative properties of consistency and standard are considered by Liao [18] as follows. Let τ be a solution on Λ.

- **Com-consistency (Com-CON)**: For each $(U,a,m) \in \Lambda$, for each $S \subseteq U$ and for each $(i,j) \in P^{S,a_S}$, $\tau_{i,j}(U,a,m) = \tau_{i,j}(S,a_S,m^{com}_{S,\tau})$.
- **Bilateral com-consistency (Bil-Com-CON)**: For each $(U,a,m) \in \Lambda$, for each $S \subseteq U$ with $|S| = 2$ and for each $(i,j) \in P^{S,a_S}$, $\tau_{i,j}(U,a,m) = \tau_{i,j}(S,a_S,m^{com}_{S,\tau})$.
- **Marginal-standard of games (MASG)**: For all $(U,a,m) \in \Lambda$ with $|U| \le 2$, $\tau(U,a,m) = \overline{\Theta^M}(U,a,m)$.

Liao [18] presented several axiomatic results for the efficient marginal index.

Theorem 2:

- The solution $\overline{\Theta^M}$ matches EFF.
- The solution $\overline{\Theta^M}$ matches Bil-Com-CON.
- A solution τ matches EFF if τ matches MASG and Bil-Com-CON.
- A solution τ on matches MASG and Bil-Com-CON if and only if $\tau = \overline{\Theta^M}$.

Liao [18] adopted the following instances to demonstrate that each of the axioms utilized in Theorem 2 is logically independent of the remaining axioms.

Example 4: Consider a solution τ on Λ as for each $(U,a,m) \in \Lambda$ and for each $(i,j) \in P^{U,a}$,

$$\tau_{i,j}(U,a,m) = 0.$$

Absolutely, τ matches Bil-Com-CON, but it violates MASG.

Example 5: Consider a solution τ on Λ as for each $(U,a,m) \in \Lambda$ and for each $(i,j) \in P^{U,a}$,

$$\tau_{i,j}(U,a,m) = \begin{cases} \overline{\Theta^{M}_{i,j}}(U,a,m) & \text{, if } |U| \le 2 \\ \overline{\Theta^{M}_{i,j}}(U,a,m) - \delta & \text{, otherwise.} \end{cases}$$

where $\delta \in \mathbb{R} \setminus \{0\}$. Absolutely, τ matches MASG, but it violates Bil-Com-CON.

Remark 3: A **standard game** is a pair (U,M) where U is a coalition and M is a mapping such that $M : 2^U \longrightarrow \mathbb{R}$ and $U(\emptyset) = 0$. Denote the family of whole standard games as $\mathbb{G}$. In the context of standard games, the EANSC, the Banzhaf-Coleman index and the Banzhaf-Owen index are defined as follows.

Definition 5:

- The **EANSC**, denoted by $\overline{\Theta^E}$, is the solution on $\mathbb{G}$ which associates with $(U,M) \in \mathbb{G}$ and each player $i \in U$ the value
$$\begin{aligned} & \overline{\Theta^E_i}(U,M) \\ = \; & \Theta^E_i(U,M) + \tfrac{1}{|U|}\big[M(U) - \sum_{k\in N} \Theta^E_k(U,M)\big], \end{aligned}$$
where $\Theta^E_i(U,M) = M(U) - M(U \setminus \{i\})$ for all $i \in U$.
- The **Banzhaf-Coleman index**, denoted by $\overline{\Theta^{BC}}$, is the solution on $\mathbb{G}$ which associates with $(U,M) \in \mathbb{G}$ and each player $i \in U$ the value
$$\begin{aligned} & \overline{\Theta^{BC}_i}(U,M) \\ = \; & \Theta^{BC}_i(U,M) + \tfrac{1}{|U|}\big[M(U) - \sum_{k\in N} \Theta^{BC}_k(U,M)\big], \end{aligned}$$
where $\Theta^{BC}_i(U,M) = \sum_{\substack{S\subseteq U \\ i\in S}} \big[M(S) - M(S \setminus \{i\})\big]$ for all $i \in U$.
- The **Banzhaf-Owen index**, denoted by $\overline{\Theta^{BO}}$, is the solution on $\mathbb{G}$ which associates with $(U,M) \in \mathbb{G}$ and each player $i \in U$ the value
$$\begin{aligned} & \overline{\Theta^{BO}_i}(U,M) \\ = \; & \Theta^{BO}_i(U,M) + \tfrac{1}{|U|}\big[M(U) - \sum_{k\in N} \Theta^{BO}_k(U,M)\big], \end{aligned}$$
where $\Theta^{BO}_i(U,M) = \frac{1}{2^{|U|-1}} \sum_{\substack{S\subseteq U \\ i\in S}} \big[M(S) - M(S \setminus \{i\})\big]$ for all $i \in U$.

Let $(U,a,m) \in \Lambda$. One would define the **standard-duality game** (U,M^D) to be for all $S \subseteq U$,

$$M^D(S) = m\big(a_S, 0_{U\setminus S}\big).$$

It is easy to verify that

- $\Theta^E_i(U,M^D) = \Theta^M_{i,a_i}(U,a,m)$ and $\overline{\Theta^E_i}(U,M^D) = \overline{\Theta^M_{i,a_i}}(U,a,m)$.
- $\Theta^{BC}_i(U,M^D) = \Theta^{AC}_{i,a_i}(U,a,m)$ and $\overline{\Theta^{BC}_i}(U,M^D) = \overline{\Theta^{AC}_{i,a_i}}(U,a,m)$.
- $\Theta^{BO}_i(U,M^D) = \Theta^{AV}_{i,a_i}(U,a,m)$ and $\overline{\Theta^{BO}_i}(U,M^D) = \overline{\Theta^{AV}_{i,a_i}}(U,a,m)$.

Based on the definitions provided above, the EANSC, Banzhaf-Coleman index, and Banzhaf-Owen index are all based on the participation or non-participation of participants to assign corresponding values. Furthermore, combining the statements above with the findings of this study, the marginal index, accumulate index, and average index not only assign values based on the fuzzy participation behavior of participants, but also encompass the values assigned to participants in the EANSC, Banzhaf-Coleman index, and Banzhaf-Owen index in standard game settings.

Remark 4: In this section, we discuss six different types of indexes within the framework of fuzzy games, including the marginal index and its efficient extension, the accumulate index and its efficient extension, and the average index and its efficient extension. To illustrate the application of these indexes, we consider an example of an operational organization with departments, a representative assembly, and various committees. In the operational organization, there are different executive departments responsible for operational policies and related activities. The representative assembly, composed of representatives elected by the departments, determines the operational policies and procedures of the organization. Each department is guaranteed at least one representative, and there is an upper limit on the total number of representatives in the assembly. The remaining seats, apart from the guaranteed representatives, are allocated among the departments in proportion to their contributions to the company. The representative assembly also includes various committees, such as the policy committee, personnel committee, resource allocation committee, etc., which are formed through voting by the representatives based on their preferences.

Measuring interactions based on individuals, groups, and their related behaviors is more realistic, as interactive behaviors are inherently complex and variable. Considering fuzzy games is therefore reasonable. The concept of the marginal index arises when relative contributions are measured solely based on the differences of individuals or groups participating or not participating in a fixed environment. The accumulate index, on the other hand, measures relative contributions based on the cumulative differences of individuals or groups participating or not participating in all relevant environments. Lastly, the average index quantifies relative contributions based on the average differences of individuals or groups participating or not participating in all relevant environments.

These three concepts of relative contributions have their own situational considerations, and there is no direct comparison of superiority or inferiority among them. In a unit where individuals or groups have non-overlapping and independent operations that do not affect each other, the concept of the marginal index is suitable for measuring relative contributions based on the difference between participation and non-participation of individuals or groups in that unit. However, in a unit where everyone's operations intersect and have an impact on each other, to measure relative contributions, the concept of the accumulate index is more appropriate, which measures the cumulative difference between participation and non-participation of individuals or groups in all the environments they have been involved in. The concept of the average index is applicable in a similar context as the accumulate index, but it presents the average difference, which is the cumulative difference relative to the expected value of participation in all environments.

However, the cumulative measure of relative contributions in these three concepts is not efficient for all individuals or groups. Whether in surplus or deficit, there are often differences between the cumulative measure and the overall resources. Since organizational members are interconnected, these differences should be collectively borne by all individuals or groups, giving rise to three different types of efficient extensions.

Based on the aforementioned, the proportions of seats other than the guaranteed representatives can be determined by utilizing the values generated by the three different types of efficient extensions as proportions.

IV. Conclusions

By simultaneously incorporating players and their participial levels within the context of fuzzy behavior, this study introduces various power indices and their related characterizations in fuzzy games.

- The accumulate index and the average index are proposed by employing relevant concepts from the Banzhaf-Coleman index and the Banzhaf-Owen index, respectively.
- Furthermore, the sum-reduction aligned with and the average-reduction are introduced to axiomatize the accumulate index and the average index.

Hwang and Liao [14], [15] and Liao [18] conducted research simultaneously applying players and their participial levels under fuzzy behavior, presenting several solutions. A comparison should be made between the results of Hwang and Liao [14], [15] and Liao [18] and the related findings of this paper.

- Hwang and Liao [14], [15] extended the core and the Shapley value [25] to fuzzy games, while Liao [18] extended the equal allocation non-separable costs (EANSC) to fuzzy games. This paper extends the Banzhaf-Coleman index and the Banzhaf-Owen index to fuzzy games.
- Hwang and Liao [14], [15] extended the reductions introduced by Davis and Maschler [8], Moulin [21] and Hart and Mas-Colell [12] to axiomatize the extended core and the extended Shapley value. Liao [18] extended the reduction of Moulin [21] to axiomatize the extended EANSC. Unlike existing results under the context of fuzzy games, this study adopts the sum-reduction and the average-reduction to axiomatize the accumulate index and the average index.

As mentioned above, the following question arises:

- Are there any additional power indices and related results that can be explored within the framework of fuzzy behavior?

To the best of our knowledge, these issues remain open questions.

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