Coalitional Model Predictive Control Scheme: Constraint Satisfaction and Application

Zaiful Ulum, Salmah, Ari Suparwanto, and Solikhatun

Abstract—**This work presents a novel coalitional model predictive control scheme for linear systems composed of disturbed and constrained subsystems. These subsystems are dynamically interconnected and controlled by a group of local model predictive controllers or agents connected by a network. A supervisor periodically activates or deactivates the network links to achieve the best balance between control performance and communication cost. Each coalition, which is a group of agents connected by activated links, then employs a decentralized model predictive control strategy to coordinate their control actions. To guarantee the satisfaction of the system's constraints, we extend the concept of a control invariant set for non-switched constraint systems to switched constraint systems, referred to as a "common control invariant set." This set is adopted as the terminal constraint of the collective coalitional model predictive control optimization problem. A three-reaches irrigation canal system is considered for validating the performance of the proposed control strategy.**

Index Terms—**model predictive control schemes, coalitional control schemes, common control invariant sets, switched systems, irrigation canal systems**

I. Introduction

MODEL predictive control (MPC) is an optimal control technique employing a process model to forecast the future output of the controlled system and the receding horizon principle [1]. At each time step, a finite-horizon optimal control problem, possibly subject to some constraints, is solved to obtain a sequence of future control inputs. Subsequently, the first element of the sequence is implemented as the control input for the current time step. This process is repeated based on the new measurements and the receding horizon principle to determine the control input for the subsequent time steps. Due to its advantages over other control strategies, MPC has been applied in many process industry sectors [2]. However, its application to large-scale systems encounters challenges such as computational complexity, communication limitations, and lack of flexibility. Therefore, to address these challenges, non-centralized MPC schemes have been developed.

In the non-centralized MPC scheme, the computation of control input to the system is divided among multiple MPC agents, where each agent is responsible for calculating control inputs for different subsystems. The scheme is referred to as a decentralized MPC if the agents do not share information in determining their control inputs. Conversely, if information exchange occurs among the agents, it is known as distributed MPC. The distributed MPC scheme offers control performance comparable to that achieved by the centralized MPC strategy due to the information sharing among the agents [3]. The applications of the non-centralized MPC scheme encompass various domains, including microgrids [4], irrigation canal systems [5], power systems [6], supply chain systems[7], road traffic systems[8], and train transportation systems [9], among others.

In terms of the communication network topology among the agents, the distributed control strategy is classified as either partially connected or fully connected [3]. Both approaches assume that the network topology connecting the agents is initially specified and remains constant, regardless of the level of interaction between the agents. In recent years, a novel control strategy which modifies the communication network topology based on the level of interaction among the agents has been proposed, known as a coalitional control strategy [10]. The primary objective of this strategy is to look for a balance between control performance and communication load.

The modification of the network topology in the coalitional control strategy could be determined by either a top-down approach [11], [12], [13], [14] or a bottom-up approach [15]. In the top-down approach, a supervisor is responsible for selecting the optimal network topology based on some information transmitted by all agents. Conversely, in the bottom-up approach, the agents autonomously determine the network topology among themselves. Based on the selected network topology, the agents connected via enabled links establish a *coalition* and coordinate their control actions cooperatively.

In this work, we present a top-down coalitional MPC strategy for systems composed of disturbed and constrained subsystems that are dynamically coupled. Specifically, the network topology selection process follows the approach presented in [11], where a supervisor periodically solves an optimization problem based on the state vector information of all subsystems to find the best network topology. Subsequently, each coalition resulting from the selected network topology independently calculates its control actions using a decentralized MPC scheme instead of the feedback control scheme employed in [11].

Ensuring the satisfaction of the system's constraints in the coalitional MPC scheme is challenging since the dynamics of the controlled system switch among different system modes according to the active network topology. Enforcing constraints on switched systems is challenging because they

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must be satisfied in both the current and potential modes [16]. In the coalitional MPC strategy proposed in [17], constraint satisfaction is guaranteed under the assumption that the optimization problem at the upper control layer has a feasible control sequence for the centralized topology at the initial time step. This assumption implies that the initial topology should be chosen as the centralized topology.

Our proposed coalitional control strategy aims to provide more flexibility in selecting the initial topology while still ensuring the satisfaction of the system's constraints. To achieve this, we extend the existing concept of a control invariant set for non-switched constrained systems to switched constrained systems, referred to as a common control invariant set. In particular, we employ the common control invariant set as the terminal constraint of the collective coalitional MPC optimization problem. Additionally, we present an algorithm for computing the common control invariant set. The computation of the set is straightforward since it is obtained by intersecting the control invariant sets of all system modes.

The rest of this paper is set up as follows: Section II formulates the problem statement. Section III presents the control algorithm, including network topology selection and the computation of control actions. Section IV discusses the recursive feasibility analysis of the proposed control scheme. Section V demonstrates the application of the proposed control scheme in irrigation canal systems. Finally, we present the conclusions in Section VI.

Notation 1. *The Cartesian product of sets $\mathcal{X}_i \subseteq \mathbb{R}^n, i = 1, \ldots, p$ is denoted by $\prod_{i=1}^{p} \mathcal{X}_i$. For $\mathcal{X}, \mathcal{Y} \subseteq \mathbb{R}^n$, $\mathcal{X} \oplus \mathcal{Y}$ and $\mathcal{X} \ominus \mathcal{Y}$ respectively denote the Minkowski sum and the Minksowski difference of $\mathcal{X}$ and $\mathcal{Y}$. The set of non-negative integers is indicated by $\mathbb{Z}_+$. A subset $\mathcal{O} \subseteq \mathcal{X}$ is referred to as a positive invariant set for a system described by $x(k+1) = f(x(k))$, where $x(k) \in \mathcal{X}, \forall k \in \mathbb{Z}_+$, if the initial state $x(0)$ belongs to $\mathcal{O}$, then $x(k)$ remains in $\mathcal{O}$ for all $k \in \mathbb{Z}_+$. A subset $\mathcal{C} \subseteq \mathcal{X}$ is considered a control invariant set for a system specified by $x(k+1) = g(x(k), u(k))$, in which $x(k) \in \mathcal{X}$, $u(k) \in \mathcal{U}, \forall k \in \mathbb{Z}_+$, if $x(k)$ is an element of $\mathcal{C}$, then there exist a feedback control $u(k) = h(x(k)) \in \mathcal{U}$ such that $x(k+1)$ remains in $\mathcal{X}$ for all $k \in \mathbb{Z}_+$. The precursor set to the set $\mathcal{A} \subset \mathbb{R}^n$ with respect to the system $\Omega : x(k+1) = g(x(k), u(k))$, $x(k) \in \mathcal{X}$, $u(k) \in \mathcal{U}, \forall k \in \mathbb{Z}_+$, is denoted by $\mathrm{Pre}_\Omega(\mathcal{A})$, and defined as $\mathrm{Pre}_\Omega(\mathcal{A}) = \{x \in \mathbb{R}^n | \exists u \in \mathcal{U}$ such that $g(x, u) \in \mathcal{A}\}$.*

II. Problem Statement

The system under consideration is a linear discrete-time system that consists of a collection of dynamically interconnected subsystems, denoted as $\mathcal{S} = \{1, 2, \ldots, M\}$. Each subsystem $i \in \mathcal{S}$ is characterized by its specific dynamics and constraints, represented by

$$x_i(k+1) = A_{ii}x_i(k) + B_{ii}u_i(k) + d_i(k), \tag{1a}$$

$$d_i(k) = \sum_{j \in \mathcal{N}_i} (A_{ij}x_j(k) + B_{ij}u_j(k)) + D_{ii}w_i(k), \tag{1b}$$

$$x_i \in \mathbb{X}_i \subset \mathbb{R}^{n_i}, \quad u_i \in \mathbb{U}_i \subset \mathbb{R}^{q_i}, \quad w_i \in \mathbb{W}_i \subset \mathbb{R}^{n_i}, \tag{1c}$$

in which x_i, u_i and w_i are the state, control input, and external disturbance vectors of subsystem i, respectively. Matrices A_{ii}, A_{ij}, B_{ii}, B_{ij} and D_{ii} are real matrices with appropriate dimensions. The set $\mathcal{N}_i$ denotes the set of all neighbours of subsystem i defined as $\mathcal{N}_i = \{j \in \mathcal{S} | A_{ij} \neq \mathbf{0} \vee B_{ij} \neq \mathbf{0}\}$, where $\mathbf{0}$ indicates a zero matrix with suitable dimension. From (1b) and (1c), we have

$$d_i \in \mathbb{D}_i = \bigoplus_{j \in \mathcal{N}_i} \left(A_{ij}\mathbb{X}_j \oplus B_{ij}\mathbb{U}_j\right) \oplus D_{ii}\mathbb{W}_i. \tag{2}$$

The sets $\mathbb{X}_i$, $\mathbb{U}_i$, and $\mathbb{W}_i$ are assumed to satisfy the following assumption, which is a standard requirement in the MPC control design.

Assumption 1. *For all $i \in \mathcal{S}$, $\mathbb{X}_i$, $\mathbb{U}_i$, and $\mathbb{W}_i$ are compact polyhedral sets that contain the origin within their interior.*

Remark 1. *The set $\mathbb{D}_i$ is also a polyhedral compact set with its origin in its interior based on Assumption 1.*

A. Communication Network

The M subsystems are independently controlled by a set of local MPC controllers or agents interconnected through a communication network. The network is represented by an undirected graph $(\mathcal{S}, \mathcal{E})$, where $\mathcal{S}$ represents the group of agents, and $\mathcal{E}$ denotes the collection of links interconnecting them specified by

$$\mathcal{E} \subseteq \mathcal{E}^{\mathcal{S}} = \{\{i, j\} | \{i, j\} \subseteq \mathcal{S}, i, j \in \mathcal{S}, i \neq j\}. \tag{3}$$

In the coalitional approach, each link $\{i, j\} \in \mathcal{E}$ can be either enabled or disabled, where an enabled link is associated with a fixed stage cost $c > 0$. The set of all enabled links is denoted by the symbol g, referred to as the *topology*. It is important to note that there are $2^{|\mathcal{E}|}$ possible topologies that can be established if the cardinality of $\mathcal{E}$ is $|\mathcal{E}|$. All topologies that can be formed are collected in the set $\mathcal{G}$, i.e., $\mathcal{G} = \{g | g \subseteq \mathcal{E}\}$.

Each topology $g \in \mathcal{G}$ partitions the set $\mathcal{S}$ into distinct clusters of agents connected by a path of enabled links. These clusters are referred to as *coalitions*. All coalitions generated by the topology g are collected in the set $\mathcal{P}_g$ defined as $\mathcal{P}_g = \{\mathcal{C} | \mathcal{C} \subseteq \mathcal{S}\}$, where all coalitions $\mathcal{C} \in \mathcal{P}_g$ are nonempty sets, the intersection of any pair $\mathcal{C}, \mathcal{C}' \in \mathcal{P}_g$, is a nonempty set, and the union of all coalitions $\mathcal{C} \in \mathcal{P}_g$ is the set $\mathcal{S}$. The number of the elements of the set $\mathcal{P}_g$, indicated by $|\mathcal{P}_g|$, varies from 1 to M. The case $|\mathcal{P}_g| = 1$ occurs when all communication links are active (i.e., $g = \mathcal{E}$) and $|\mathcal{P}_g| = M$ corresponds to the case no links are active (i.e., $g = \emptyset$).

B. Coalitions Dynamics

Consider any topology $g \in \mathcal{G}$ and a coalition $\mathcal{C} \in \mathcal{P}_g$. In the coalitional control design, all agents within coalition $\mathcal{C}$ collaborate to compute their control signals and work as a single system. Specifically, the dynamics of coalition $\mathcal{C}$ are derived by combining the dynamics of each subsystem $i \in \mathcal{C}$. In other words, the individual subsystems are assembled to obtain the behavior of coalition $\mathcal{C}$, that is,

$$x_{\mathcal{C}}(k+1) = A^g_{\mathcal{C}}x_{\mathcal{C}}(k) + B^g_{\mathcal{C}}u_{\mathcal{C}}(k) + d_{\mathcal{C}}(k), \tag{4}$$

where $u_{\mathcal{C}} = (u_i)_{i \in \mathcal{C}} \in \mathbb{R}^{q_{\mathcal{C}}}$ and $x_{\mathcal{C}} = (x_i)_{i \in \mathcal{C}} \in \mathbb{R}^{n_{\mathcal{C}}}$ respectively denote the control input and the state vectors of coalition $\mathcal{C}$, with $q_{\mathcal{C}} = \sum_{i \in \mathcal{C}} q_i$ and $n_{\mathcal{C}} = \sum_{i \in \mathcal{C}} n_i$. The matrices $A^g_{\mathcal{C}} = [A_{ij}]_{i,j \in \mathcal{C}}$ and $B^g_{\mathcal{C}} = [B_{ij}]_{i,j \in \mathcal{C}}$ are real matrices with suitable dimension. The vector $d_{\mathcal{C}} = (d_i^{\mathcal{C}})_{i\in\mathcal{C}} \in \mathbb{R}^{n_{\mathcal{C}}}$ captures the effect of the vectors x_j and u_j, $j \in \mathcal{N}_i\backslash\mathcal{C}$, and the vector w_i, $i \in \mathcal{C}$, on the state of coalition $\mathcal{C}$. In addition, the vector $d_i^{\mathcal{C}}$, $i \in \mathcal{C}$, is specified by

$$d_i^{\mathcal{C}}(k) = \big(\sum_{j\in\mathcal{N}_i\backslash\mathcal{C}} A_{ij}x_j(k) + B_{ij}u_j(k)\big) + D_{ii}w_i(k). \quad (5)$$

Considering (1c), the constraints of coalition $\mathcal{C}$ are given by

$$x_{\mathcal{C}} \in \mathbb{X}_{\mathcal{C}}^g, \quad u_{\mathcal{C}} \in \mathbb{U}_{\mathcal{C}}^g, \quad d_{\mathcal{C}} \in \mathbb{D}_{\mathcal{C}}^g, \quad (6)$$

where $\mathbb{X}_{\mathcal{C}}^g = \prod_{i\in\mathcal{C}} \mathbb{X}_i$, $\mathbb{U}_{\mathcal{C}}^g = \prod_{i\in\mathcal{C}} \mathbb{U}_i$, and $\mathbb{D}_{\mathcal{C}}^g = \prod_{i\in\mathcal{C}} \mathbb{D}_i^{\mathcal{C}}$, with

$$d_i^{\mathcal{C}} \in \mathbb{D}_i^{\mathcal{C}} = \big(\bigoplus_{j\in\mathcal{N}_i\backslash\mathcal{C}} A_{ij}\mathbb{X}_j \oplus B_{ij}\mathbb{U}_j\big) \oplus D_{ii}\mathbb{W}_i. \quad (7)$$

Furthermore, based on Assumption 1, $\mathbb{X}_{\mathcal{C}}^g$, $\mathbb{U}_{\mathcal{C}}^g$, and $\mathbb{D}_{\mathcal{C}}^g$ are compact polyhedral with their interiors containing the origin.

Let $x_{\mathcal{S}} = (x_{\mathcal{C}})_{\mathcal{C}\in\mathcal{P}_g}$ and $u_{\mathcal{S}} = (u_{\mathcal{C}})_{\mathcal{C}\in\mathcal{P}_g}$ be the state and control input vectors of the whole system corresponding to the topology g. The state-space form of the whole system is described by

$$x_{\mathcal{S}}(k+1) = A_{\mathcal{S}}^g x_{\mathcal{S}}(k) + B_{\mathcal{S}}^g u_{\mathcal{S}}(k) + D_{\mathcal{S}}^g w_{\mathcal{S}}(k), \quad (8)$$

where $A_{\mathcal{S}}^g = [A_{\mathcal{C}}^g]_{\mathcal{C}\in\mathcal{P}_g}$, $B_{\mathcal{S}}^g = [B_{\mathcal{C}}^g]_{\mathcal{C}\in\mathcal{P}_g}$, $D_{\mathcal{S}}^g = \text{diag}(D_{\mathcal{C}}^g)_{\mathcal{C}\in\mathcal{P}_g}$, and $w_{\mathcal{S}} = (w_{\mathcal{C}})_{\mathcal{C}\in\mathcal{P}_g}$, with $D_{\mathcal{C}}^g = \text{diag}(D_{ii})_{i\in\mathcal{C}}$ and $w_{\mathcal{C}} = (w_i)_{i\in\mathcal{C}}$. In addition, the constraints of System (8) are

$$x_{\mathcal{S}} \in \mathbb{X}_{\mathcal{S}}^g \quad u_{\mathcal{S}} \in \mathbb{U}_{\mathcal{S}}^g, \quad w_{\mathcal{S}} \in \mathbb{W}_{\mathcal{S}}^g \quad (9)$$

where $\mathbb{X}_{\mathcal{S}}^g = \prod_{\mathcal{C}\in\mathcal{P}_g} \mathbb{X}_{\mathcal{C}}^g$, $\mathbb{U}_{\mathcal{S}}^g = \prod_{\mathcal{C}\in\mathcal{P}_g} \mathbb{U}_{\mathcal{C}}^g$ and $\mathbb{W}_{\mathcal{S}}^g = \prod_{\mathcal{C}\in\mathcal{P}_g} \mathbb{W}_{\mathcal{C}}^g$. Since $\mathbb{X}_{\mathcal{C}}^g$, $\mathbb{U}_{\mathcal{C}}^g$, and $\mathbb{W}_{\mathcal{C}}^g$ are compact polyhedral sets with their interiors containing the origin for all $\mathcal{C} \in \mathcal{P}_g$, so do $\mathbb{X}_{\mathcal{S}}^g$, $\mathbb{U}_{\mathcal{S}}^g$, and $\mathbb{W}_{\mathcal{S}}^g$.

The assumption below, commonly utilized in the non-centralized MPC designs such as [18], [19], [20], is employed to stabilize System (8).

Assumption 2. *There exists a block-diagonal matrix $K_{\mathcal{S}}^g \triangleq \text{diag}(K_{\mathcal{C}}^g)_{\mathcal{C}\in\mathcal{P}_g}$ for each topology $g \in \mathcal{G}$, with $K_{\mathcal{C}}^g \in \mathbb{R}^{q_{\mathcal{C}}\times n_{\mathcal{C}}}$, that satisfies: (i) $F_{\mathcal{S}}^g \triangleq A_{\mathcal{S}}^g + B_{\mathcal{S}}^g K_{\mathcal{S}}^g$ is Schur, and (ii) $F_{\mathcal{C}}^g \triangleq A_{\mathcal{C}}^g + B_{\mathcal{C}}^g K_{\mathcal{C}}^g$ is Schur for all $\mathcal{C} \in \mathcal{P}_g$.*

C. Control Goal

The control objective is to determine the sequence of control actions and the sequence of network topologies that steer the state of all subsystems to the origin while reducing communication costs. Let $\sigma : \mathbb{Z}_+ \to \mathcal{G}$ denote the function specifying the active topology at each time step. Then, the following objective function:

$$\sum_{t=0}^{\infty} \sum_{\mathcal{C}\in\mathcal{P}_{\sigma(k+t)}} \ell_{\mathcal{C}}^{\sigma(k+t)}\big(x_{\mathcal{C}}(k+t), u_{\mathcal{C}}(k+t)\big) + c|\sigma(k+t)| \quad (10)$$

subject to (4) and (6) is minimized to achieve the control objective. Here, $\ell_{\mathcal{C}}^g(x_{\mathcal{C}}(k), u_{\mathcal{C}}(k))$ represents the stage cost for coalition $\mathcal{C} \in \mathcal{P}_g$, defined as

$$\begin{aligned}\ell_{\mathcal{C}}^g(x_{\mathcal{C}}(k), u_{\mathcal{C}}(k)) = x_{\mathcal{C}}^T(k) Q_{\mathcal{C}}^g x_{\mathcal{C}}(k) + \\ u_{\mathcal{C}}^T(k) R_{\mathcal{C}}^g u_{\mathcal{C}}(k),\end{aligned} \quad (11)$$

where the weighting matrices $Q_{\mathcal{C}}^g \in \mathbb{R}^{n_{\mathcal{C}}\times n_{\mathcal{C}}}$ and $R_{\mathcal{C}}^g \in \mathbb{R}^{q_{\mathcal{C}}\times q_{\mathcal{C}}}$ are defined as $Q_{\mathcal{C}}^g = \text{diag}(Q_i)_{i\in\mathcal{C}}$ and $R_{\mathcal{C}}^g = \text{diag}(R_i)_{i\in\mathcal{C}}$, and assumed to be positive definite for all $\mathcal{C} \in \mathcal{P}_{g(k)}$.

Consider that the decision variables of Optimization Problem (10) are the sequence of network topologies and the sequence of control inputs. Notice that $\mathcal{G}$ is a finite set established by enabling or disabling the links in $\mathcal{E}$. By associating an enabled link with 1 and a disabled link with 0, we have $\mathcal{G} \subseteq \{0, 1\}^{|\mathcal{E}|}$. For example, the network topology where all links are enabled can be represented as $\{\underbrace{1, 1, \ldots, 1}_{1 \text{ as many as } |\mathcal{E}|}\}$. Conversely, the topology with all links disabled can be represented as $\{\underbrace{0, 0, \ldots, 0}_{0 \text{ as many as } |\mathcal{E}|}\}$. Thus, Optimization Problem (10) belongs to the family of mixed-integer optimization problems, which are generally challenging to solve due to their non-convex nature. To address this issue, we modify the two-layer control approach given in [11] to find the suboptimal solution to Optimization Problem (10). Specifically, we adopt the method suggested in [11] to determine the network topology. However, instead of employing linear feedback control schemes as proposed in [11], we utilize a decentralized MPC scheme to compute the control actions for the overall system.

III. Control Algorithm

This section presents a two-layer control strategy that provides the suboptimal solution to Optimization Problem (10). At the upper layer, a supervisor determines the network topology. Subsequently, at the lower layer, a group of coalitions generated by the selected network topology employs a decentralized MPC strategy to determine the control input for the entire system.

A. Upper Layer

The assumption below is required in the selection of network topologies.

Assumption 3. *Suppose the system $x_{\mathcal{S}}(k+1) = A_{\mathcal{S}}^g x_{\mathcal{S}}(k) + B_{\mathcal{S}}^g u_{\mathcal{S}}(k)$ is controlled by a feedback law $u_{\mathcal{S}}(k) = K_{\mathcal{S}}^g x_{\mathcal{S}}(k)$ for any topology $g \in \mathcal{G}$. A positive definite matrix $P_{\mathcal{S}}^g \triangleq \text{diag}(P_{\mathcal{C}}^g)_{\mathcal{C}\in\mathcal{P}_g}$ exists and satisfies*

$$x_{\mathcal{S}}^T(k) P_{\mathcal{S}}^g x_{\mathcal{S}}(k) \geq \sum_{t=0}^{\infty} \ell_{\mathcal{S}}^g(x_{\mathcal{S}}(k+t), u_{\mathcal{S}}(k+t)), \quad (12)$$

where

$$\ell_{\mathcal{S}}^g(x_{\mathcal{S}}(k), u_{\mathcal{S}}(k)) = \sum_{\mathcal{C}\in\mathcal{P}_g} \ell_{\mathcal{C}}^g(x_{\mathcal{C}}(k), u_{\mathcal{C}}(k)).$$

Let $A_{\mathcal{S}} = (A_{ij})_{i,j\in\mathcal{S}}$, $B_{\mathcal{S}} = (B_{ij})_{i,j\in\mathcal{S}}$, $Q_{\mathcal{S}} = \text{diag}(Q_i)_{i\in\mathcal{S}}$, $R_{\mathcal{S}} = \text{diag}(R_i)_{i\in\mathcal{S}}$, $W_{\mathcal{S}}^g = (W_{\mathcal{S}}^g)^T = (W_{ij})_{i,j\in\mathcal{S}}$, and $Y_{\mathcal{S}}^g = (Y_{\mathcal{S}}^g)^T = (Y_{ij})_{i,j\in\mathcal{S}}$, where $W_{ij} \in \mathbb{R}^{n_i\times n_j}$, $Y_{ij} \in \mathbb{R}^{q_i\times n_j}$, and $W_{ij} = 0$, $Y_{ij} = 0$ $\forall i, j \in \mathcal{S}, \mathcal{C} \in \mathcal{P}_g$ such that $i \in \mathcal{C}, j \notin \mathcal{C}$. The following linear matrix inequality (LMI) proposed in [11]

$$\begin{pmatrix} W_{\mathcal{S}}^g & W_{\mathcal{S}}^g A_{\mathcal{S}}^T + (Y_{\mathcal{S}}^g)^T B_{\mathcal{S}}^T & W_{\mathcal{S}}^g Q_{\mathcal{S}}^{1/2} & (Y_{\mathcal{S}}^g)^T R_{\mathcal{S}}^{1/2} \\ A_{\mathcal{S}} W_{\mathcal{S}}^g + B_{\mathcal{S}} Y_{\mathcal{S}}^g & W_{\mathcal{S}}^g & 0 & 0 \\ Q_{\mathcal{S}}^{1/2} W_{\mathcal{S}}^g & 0 & I & 0 \\ R_{\mathcal{S}}^{1/2} Y_{\mathcal{S}}^g & 0 & 0 & I \end{pmatrix} > 0, \quad (13)$$

is solved to obtain matrices $K_{\mathcal{S}}^{g}$ and $P_{\mathcal{S}}^{g}$, respectively satisfying Assumptions 2 and 3. If $W_{\mathcal{S}}^{g}$ and $Y_{\mathcal{S}}^{g}$ satisfying the above LMI exist, then

$$K_{\mathcal{S}}^{g} = (W_{\mathcal{S}}^{g})^{-1}, \quad P_{\mathcal{S}}^{g} = Y_{\mathcal{S}}^{g}(W_{\mathcal{S}}^{g})^{-1}.$$

Assume that the state $x_{\mathcal{S}}(k)$ is given at time step k. The network topology at time step k is determined by solving the following optimization:

$$\min_{g \in \mathcal{G}} r(g, x_{\mathcal{S}}(k)), \tag{14}$$

where

$$r(g, x_{\mathcal{S}}(k)) \triangleq x_{\mathcal{S}}(k)^T P_{\mathcal{S}}^{g} x_{\mathcal{S}}(k) + c|g|. \tag{15}$$

Due to the rapid growth of the cardinality of $\mathcal{G}$ to the cardinality of $\mathcal{E}$, Optimization Problem (14) is evaluated at every multiple of specified time steps to to decrease the computational load.

B. Lower Layer

In this work, we modify the decentralized MPC scheme proposed in [21] to determine the control input to System (8). The control scheme in [21] was initially designed for controlling constrained large-scale systems composed of subsystems interconnected through their state. Additionally, the network communication considered in [21] is assumed to be constant. In this work, we extend the control scheme in [21] to handle disturbed and constrained systems that switch among a finite set of system modes. Particularly, we define and utilize a common control invariant set for the switched systems to ensure the persistent feasibility of the entire system.

The work in [21] relies on the tube-based MPC control strategy suggested in [22]. The control strategy addresses the feasibility and robust stability of constrained linear systems subject to bounded disturbances. Moreover, the MPC optimization problem of this strategy is derived by employing the nominal system that associates with the actual system and the constraints that are tighter than the original ones.

Considering $d_{\mathcal{C}}$ as a disturbance, the nominal system corresponding to System (4) is defined as

$$\hat{x}_{\mathcal{C}}(k+1) = A_{\mathcal{C}}^{g}\hat{x}_{\mathcal{C}}(k) + B_{\mathcal{C}}^{g}\hat{u}_{\mathcal{C}}(k), \tag{16}$$

where $\hat{u}_{\mathcal{C}}$ and $\hat{x}_{\mathcal{C}}$ are the nominal control input and state vectors of coalition $\mathcal{C}$. Furthermore, the coalitional input $u_{\mathcal{C}}(k)$ in (4) is determined by the following feedback control policy:

$$u_{\mathcal{C}}(k) = \hat{u}_{\mathcal{C}}(k) + K_{\mathcal{C}}^{g}(x_{\mathcal{C}}(k) - \hat{x}_{\mathcal{C}}(k)), \tag{17}$$

where the value of $x_{\mathcal{C}}(k)$ is given. Suppose that $\sigma(k) = g$, the value of the pair $(\hat{x}_{\mathcal{C}}(k), \hat{u}_{\mathcal{C}}(k))$ is obtained by solving the following coalitional MPC optimization problem:

$$\min_{(\hat{x}_{\mathcal{C}}(k), \{\hat{u}_{\mathcal{C}}(k+t)\}_{t=0}^{N_p-1})} J_{\mathcal{C}}^{g}(\hat{x}_{\mathcal{C}}(k), \{\hat{u}_{\mathcal{C}}(k+t)\}_{t=0}^{N_p-1}), \tag{18a}$$

subject to

$$x_{\mathcal{C}}(k) - \hat{x}_{\mathcal{C}}(k) \in \mathcal{Z}_{\mathcal{C}}^{g}, \tag{18b}$$

$$\hat{x}_{\mathcal{C}}(k+t+1) = A_{\mathcal{C}}^{g}\hat{x}_{\mathcal{C}}(k+t) + B_{\mathcal{C}}^{g}\hat{u}_{\mathcal{C}}(k+t),\ t = 0, \ldots, N_p - 1, \tag{18c}$$

$$\hat{x}_{\mathcal{C}}(k+t) \in \hat{\mathbb{X}}_{\mathcal{C}}^{g}, \quad t = 0, \ldots, N_p - 1, \tag{18d}$$

$$\hat{u}_{\mathcal{C}}(k+t) \in \hat{\mathbb{U}}_{\mathcal{C}}^{g}, \quad t = 0, \ldots, N_p - 1, \tag{18e}$$

$$\hat{x}_{\mathcal{C}}(k+N_p) \in \hat{\mathbb{T}}_{\mathcal{C}}^{g}, \tag{18f}$$

where N_p is the length of horizon prediction and the objective function $J_{\mathcal{C}}^{g}(.,.)$ is defined as

$$J_{\mathcal{C}}^{g}(\hat{x}_{\mathcal{C}}(k), \{\hat{u}_{\mathcal{C}}(k+t)\}_{t=0}^{N_p-1}) = \sum_{t=0}^{N_p-1} \ell_{\mathcal{C}}^{g}(\hat{x}_{\mathcal{C}}(k+t), \hat{u}_{\mathcal{C}}(k+t)) + V_{\mathcal{C}}^{g}(\hat{x}_{\mathcal{C}}(k+N_p)), \tag{19}$$

where $V_{\mathcal{C}}^{g}(.)$ is defined as

$$V_{\mathcal{C}}^{g}(\hat{x}_{\mathcal{C}}(k+N_p)) = \hat{x}_{\mathcal{C}}^{T}(k+N_p) P_{\mathcal{C}}^{g} \hat{x}_{\mathcal{C}}(k+N_p). \tag{20}$$

Let the pair $(\hat{x}_{\mathcal{C}}^{*}(k), \{\hat{u}_{\mathcal{C}}^{*}(k+t)\}_{t=0}^{N_p-1})$ be the optimal solution to Optimization Problem (18), then the control input for System (4) is determined by:

$$u_{\mathcal{C}}^{*}(k) = \hat{u}_{\mathcal{C}}^{*}(k) + K_{\mathcal{C}}^{g}\big(x_{\mathcal{C}}(k) - \hat{x}_{\mathcal{C}}^{*}(k)\big).$$

The set $\mathcal{Z}_{\mathcal{C}}^{g}$ in (18b) is a robust positively invariant (RPI) set for the system

$$z_{\mathcal{C}}(k+1) = F_{\mathcal{C}}^{g} z_{\mathcal{C}}(k) + d_{\mathcal{C}}(k), \tag{21}$$

where $z_{\mathcal{C}} = x_{\mathcal{C}} - \hat{x}_{\mathcal{C}}$. The existence of the set $\mathcal{Z}_{\mathcal{C}}^{g}$ is guaranteed since $F_{\mathcal{C}}^{g}$ is assumed to be Schur, and the disturbance $d_{\mathcal{C}}$ lies in the bounded set $\mathbb{D}_{\mathcal{C}}^{g}$ [23]. We employ the algorithm proposed in [24] to compute the set $\mathcal{Z}_{\mathcal{C}}^{g}$. Once the set $\mathcal{Z}_{\mathcal{C}}^{g}$ is obtained, the sets $\hat{\mathbb{U}}_{\mathcal{C}}^{g}$ and $\hat{\mathbb{X}}_{\mathcal{C}}^{g}$ are calculated as follows:

$$\hat{\mathbb{U}}_{\mathcal{C}}^{g} \triangleq \mathbb{U}_{\mathcal{C}}^{g} \ominus K_{\mathcal{C}}^{g}\mathcal{Z}_{\mathcal{C}}^{g}, \quad \hat{\mathbb{X}}_{\mathcal{C}}^{g} \triangleq \mathbb{X}_{\mathcal{C}}^{g} \ominus \mathcal{Z}_{\mathcal{C}}^{g}. \tag{22}$$

The set $\hat{\mathbb{T}}_{\mathcal{C}}^{g}$ in (18f) is the terminal constraint that will be clearly specified in the next section.

A state $x_{\mathcal{C}}(k) \in \mathbb{X}_{\mathcal{C}}^{g}$ is said to be feasible for Optimzation Problem (18) if

$$\begin{aligned}\mathbb{U}_{\mathcal{C}}^{g,N_p}(x_{\mathcal{C}}(k)) \triangleq \big\{ &\{\hat{u}_{\mathcal{C}}(k+t)\}_{t=0}^{N_p-1} | x_{\mathcal{C}} - \hat{x}_{\mathcal{C}}(k) \in \mathcal{Z}_{\mathcal{C}}^{g}, \\ &\hat{x}_{\mathcal{C}}(k+t+1) = A_{\mathcal{C}}^{g}\hat{x}_{\mathcal{C}}(k+t) + B_{\mathcal{C}}^{g}\hat{u}_{\mathcal{C}}(k+t), \\ &\hat{u}_{\mathcal{C}}(k+t) \in \hat{\mathbb{U}}_{\mathcal{C}}^{g}, \hat{x}_{\mathcal{C}}(k+t) \in \hat{\mathbb{X}}_{\mathcal{C}}^{g}, \\ &\hat{x}_{\mathcal{C}}(k+N_p) \in \hat{\mathbb{T}}_{\mathcal{C}}^{g}, \forall t \in \{0, \ldots, N_p - 1\} \big\} \neq \emptyset.\end{aligned} \tag{23}$$

We denote the set of all feasible states for Optimization Problem (18) as $\mathbb{X}_{\mathcal{C}}^{g,N_p}$, which is described by

$$\mathbb{X}_{\mathcal{C}}^{g,N_p} \triangleq \{x_{\mathcal{C}} \in \mathbb{X}_{\mathcal{C}}^{g} | \text{ if } x_{\mathcal{C}}(k) = x_{\mathcal{C}} \text{ then } \mathbb{U}_{\mathcal{C}}^{g,N_p}(x_{\mathcal{C}}(k)) \neq \emptyset\}. \tag{24}$$

Let $\hat{\mathbb{X}}_{\mathcal{C}}^{g,N_p}$ be the set of all states $\hat{x}_{\mathcal{C}} \in \hat{\mathbb{X}}_{\mathcal{C}}^{g}$ for which the following optimization problem

$$\min_{(\{\hat{u}_{\mathcal{C}}(k+t)\}_{t=0}^{N_p-1})} J_{\mathcal{C}}^{g}(\hat{x}_{\mathcal{C}}(k), \{\hat{u}_{\mathcal{C}}(k+t)\}_{t=0}^{N_p-1}), \tag{25a}$$

subject to

$$\hat{x}_{\mathcal{C}}(k) = \hat{x}_{\mathcal{C}}, \tag{25b}$$

$$\hat{x}_{\mathcal{C}}(k+t+1) = A_{\mathcal{C}}^{g}\hat{x}_{\mathcal{C}}(k+t) + B_{\mathcal{C}}^{g}\hat{u}_{\mathcal{C}}(k+t),\ t = 0, \ldots, N_p - 1, \tag{25c}$$

$$\hat{x}_{\mathcal{C}}(k+t) \in \hat{\mathbb{X}}_{\mathcal{C}}^{g}, \quad t = 0, \ldots, N_p - 1, \tag{25d}$$

$$\hat{u}_{\mathcal{C}}(k+t) \in \hat{\mathbb{U}}_{\mathcal{C}}^{g}, \quad t = 0, \ldots, N_p - 1, \tag{25e}$$

$$\hat{x}_{\mathcal{C}}(k+N_p) \in \hat{\mathbb{T}}_{\mathcal{C}}^{g}, \tag{25f}$$

is feasible. Mathematically, $\hat{\mathbb{X}}_{\mathcal{C}}^{g,N_p}$ is represented by

$$\hat{\mathbb{X}}_{\mathcal{C}}^{g,N_p} \triangleq \{\hat{x}_{\mathcal{C}} \in \hat{\mathbb{X}}_{\mathcal{C}}^{g} | \text{ if } \hat{x}_{\mathcal{C}}(k) = \hat{x}_{\mathcal{C}} \text{ then } \hat{\mathbb{U}}_{\mathcal{C}}^{g,N_p}(\hat{x}_{\mathcal{C}}(k)) \neq \emptyset\}, \tag{26}$$

where

$$\begin{aligned} \hat{\mathbb{U}}_{\mathcal{C}}^{g,N_p}(\hat{x}_{\mathcal{C}}(k)) \triangleq \{&\{\hat{u}_{\mathcal{C}}(k+t)\}_{t=0}^{N_p-1} | \hat{x}_{\mathcal{C}}(k+t+1) = \\ & A_{\mathcal{C}}^{g}\hat{x}_{\mathcal{C}}(k+t) + B_{\mathcal{C}}^{g}\hat{u}_{\mathcal{C}}(k+t), \hat{u}_{\mathcal{C}}(k+t) \in \hat{\mathbb{U}}_{\mathcal{C}}^{g}, \\ & \hat{x}_{\mathcal{C}}(k+t) \in \hat{\mathbb{X}}_{\mathcal{C}}^{g}, \forall t \in \{0,\ldots,N_p-1\}, \\ & \hat{x}_{\mathcal{C}}(k+N_p) \in \hat{\mathbb{T}}_{\mathcal{C}}^{g}\}. \end{aligned} \tag{27}$$

The relationship between $\mathbb{X}_{\mathcal{C}}^{g,N_p}$ and $\hat{\mathbb{X}}_{\mathcal{C}}^{g,N_p}$ is explained in the following proposition.

Proposition 4. *Suppose $\mathbb{X}_{\mathcal{C}}^{g,N_p}$ and $\hat{\mathbb{X}}_{\mathcal{C}}^{g,N_p}$ are the feasible regions for Optimization Problems (18) and (25). Then, $\mathbb{X}_{\mathcal{C}}^{g,N_p} = \hat{\mathbb{X}}_{\mathcal{C}}^{g,N_p} \oplus \mathcal{Z}_{\mathcal{C}}^{g}$.*

Proof: Based on the definitions of $\mathbb{X}_{\mathcal{C}}^{g,N_p}$ and $\hat{\mathbb{X}}_{\mathcal{C}}^{g,N_p}$, we have

$$\begin{aligned} \mathbb{X}_{\mathcal{C}}^{g,N_p} &= \{x_{\mathcal{C}} \in \mathbb{X}_{\mathcal{C}} | \exists \hat{x}_{\mathcal{C}} \in \hat{\mathbb{X}}_{\mathcal{C}}^{g,N_p} \text{ such that } x_{\mathcal{C}} = \hat{x}_{\mathcal{C}} \oplus \mathcal{Z}_{\mathcal{C}}^{g}\} \\ &= \hat{\mathbb{X}}_{\mathcal{C}}^{g,N_p} \oplus \mathcal{Z}_{\mathcal{C}}^{g} \end{aligned}$$

■

The algorithm below is used to calculate the set $\mathbb{X}_{\mathcal{C}}^{g,N_p}$.

Algorithm 1: The computation of the set $\mathbb{X}_{\mathcal{C}}^{g,N_p}$

Require: system $\Omega_{\mathcal{C}}^{g}: \ \hat{x}_{\mathcal{C}}(k+1) = A_{\mathcal{C}}^{g}\hat{x}_{\mathcal{C}}(k) + B_{\mathcal{C}}^{g}\hat{u}_{\mathcal{C}}(k)$, subject to $\hat{x}_{\mathcal{C}}(k) \in \hat{\mathbb{X}}_{\mathcal{C}}^{g}$, $\hat{u}_{\mathcal{C}}(k) \in \hat{\mathbb{U}}_{\mathcal{C}}^{g}$, $\forall k \in \mathbb{Z}_+$, and the sets $\hat{\mathbb{T}}_{\mathcal{C}}^{g}$ and $\mathcal{Z}_{\mathcal{C}}^{g}$.
Initialize $\Psi_{\mathcal{C}}^{g}(0) = \hat{\mathbb{T}}_{\mathcal{C}}^{g}$
for $k = 1 : N_p$ **do**
$\Psi_{\mathcal{C}}^{g}(k) = \mathrm{Pre}_{\Omega_{\mathcal{C}}^{g}}(\Psi_{\mathcal{C}}^{g}(k-1)) \cap \hat{\mathbb{X}}_{\mathcal{C}}^{g}$
end for
$\hat{\mathbb{X}}_{\mathcal{C}}^{g,N_p} = \Psi_{\mathcal{C}}^{g}(N_p)$
$\mathbb{X}_{\mathcal{C}}^{g,N_p} = \hat{\mathbb{X}}_{\mathcal{C}}^{g,N_p} \oplus \mathcal{Z}_{\mathcal{C}}^{g}$

Remark 2. *If $\hat{\mathbb{X}}_{\mathcal{C}}^{g}$ and $\hat{\mathbb{U}}_{\mathcal{C}}^{g}$ are polyhedral sets represented by*

$$\hat{\mathbb{X}}_{\mathcal{C}}^{g} = \{\hat{x}_{\mathcal{C}} | H_{\mathcal{C}}\hat{x}_{\mathcal{C}} \leq h_{\mathcal{C}}\},$$

$$\hat{\mathbb{U}}_{\mathcal{C}}^{g} = \{\hat{u}_{\mathcal{C}} | L_{\mathcal{C}}\hat{u}_{\mathcal{C}} \leq l_{\mathcal{C}}\},$$

where $H_{\mathcal{C}} \in \mathbb{R}^{m_{\mathcal{C}} \times n_{\mathcal{C}}}$, $L \in \mathbb{R}^{p_{\mathcal{C}} \times q_{\mathcal{C}}}$, $h_{\mathcal{C}} \in \mathbb{R}^{m_{\mathcal{C}}}$, and $l_{\mathcal{C}} \in \mathbb{R}^{p_{\mathcal{C}}}$, then the precursor set to the set $\hat{\mathbb{X}}_{\mathcal{C}}^{g}$ with respect to the system $\Omega_{\mathcal{C}}^{g}$, i.e. $\mathrm{Pre}_{\Omega_{\mathcal{C}}^{g}}(\hat{\mathbb{X}}_{\mathcal{C}}^{g})$, is obtained by projecting the following set

$$\left\{\begin{bmatrix}\hat{x}_{\mathcal{C}} \\ \hat{u}_{\mathcal{C}}\end{bmatrix} \in \mathbb{R}^{n_{\mathcal{C}}+q_{\mathcal{C}}} : \begin{bmatrix}H_{\mathcal{C}}A_{\mathcal{C}}^{g} & H_{\mathcal{C}}B_{\mathcal{C}}^{g} \\ 0 & L_{\mathcal{C}}\end{bmatrix}\begin{bmatrix}\hat{x}_{\mathcal{C}} \\ \hat{u}_{\mathcal{C}}\end{bmatrix} \leq \begin{bmatrix}h_{\mathcal{C}} \\ l_{\mathcal{C}}\end{bmatrix}\right\}$$

to the space of the state $\hat{x}_{\mathcal{C}}$.

Let $\mathcal{T} \triangleq \{mT | m \in \mathbb{Z}_+\}$, for some positive integer T, represent the set of time steps at which the network topology is updated. The two-layer control strategy proposed in this work is outlined in the following algorithm.

Algorithm 2: The two-layer control strategy

1) a) For $k \in \mathcal{T}$, i.e., $k = m\mathcal{T}$ for some $m \in \mathbb{Z}_+$, all agents share their state vector, and the supervisor solves the Optimization Problem (14). The optimal solution to Optimization Problem (14) is set as the topology from time step k to time step $(m+1)T - 1$.
   b) Otherwise, every agent only informs their state vector to other members of the same coalition.
2) Every coalition $\mathcal{C} \in \mathcal{P}_{\sigma(k)}$ determines its control input $u_{\mathcal{C}}(k)$ by solving Optimization Problem (18).

IV. Recursive Feasibility Analysis

Define matrices $\tilde{A}_{\mathcal{S}}^{g} \triangleq \mathrm{diag}(A_{\mathcal{C}}^{g})_{\mathcal{C}\in\mathcal{P}_g}$ and $\tilde{B}_{\mathcal{S}}^{g} \triangleq \mathrm{diag}(B_{\mathcal{C}}^{g})_{\mathcal{C}\in\mathcal{P}_g}$ and matrices $\bar{A}_{\mathcal{S}}^{g}$ and $\bar{B}_{\mathcal{S}}^{g}$ as $\bar{A}_{\mathcal{S}}^{g} \triangleq A_{\mathcal{S}}^{g} - \tilde{A}_{\mathcal{S}}^{g}$ and $\bar{B}_{\mathcal{S}}^{g} \triangleq B_{\mathcal{S}}^{g} - \tilde{B}_{\mathcal{S}}^{g}$. Then, the following system

$$x_{\mathcal{S}}(k+1) = \tilde{A}_{\mathcal{S}}^{g}x_{\mathcal{S}}(k) + \tilde{B}_{\mathcal{S}}^{g}u_{\mathcal{S}}(k) + d_{\mathcal{S}}(k), \tag{28}$$

where

$$d_{\mathcal{S}}(k) = (d_{\mathcal{C}}(k))_{\mathcal{C}\in\mathcal{P}_g} = \bar{A}_{\mathcal{S}}^{g}x_{\mathcal{S}}(k) + \bar{B}_{\mathcal{S}}^{g}u_{\mathcal{S}}(k) + D_{\mathcal{S}}^{g}w_{\mathcal{S}}(k)$$

is equivalent to System (8). In addition, the constraint of vector $d_{\mathcal{S}}$ is provided by

$$d_{\mathcal{S}} \in \mathbb{D}_{\mathcal{S}}^{g} = \prod_{\mathcal{C}\in\mathcal{P}_(g)} \mathbb{D}_{\mathcal{C}}^{g}. \tag{29}$$

Considering vector $d_{\mathcal{S}}$ as an additive disturbance, we have the nominal system that corresponds to System (28) as follows:

$$\hat{x}_{\mathcal{S}}(k+1) = \tilde{A}_{\mathcal{S}}^{g}\hat{x}_{\mathcal{S}}(k) + \tilde{B}_{\mathcal{S}}^{g}\hat{u}_{\mathcal{S}}(k), \tag{30}$$

in which $\hat{x}_{\mathcal{S}} = (\hat{x}_{\mathcal{C}})_{\mathcal{C}\in\mathcal{P}_g}$ and $\hat{u}_{\mathcal{S}} = (\hat{u}_{\mathcal{C}})_{\mathcal{C}\in\mathcal{P}_g}$. Furthermore, the constraints of System (30) are given by

$$\hat{x}_{\mathcal{S}} \in \hat{\mathbb{X}}_{\mathcal{S}}^{g} = \prod_{\mathcal{C}\in\mathcal{P}_g} \hat{\mathbb{X}}_{\mathcal{C}}^{g}, \ \hat{u}_{\mathcal{S}} \in \hat{\mathbb{U}}_{\mathcal{S}}^{g} = \prod_{\mathcal{C}\in\mathcal{P}_g} \hat{\mathbb{U}}_{\mathcal{C}}^{g}. \tag{31}$$

In line with Equation (17), the feedback control policy for System (28) is provided by

$$u_{\mathcal{S}}(k) = \hat{u}_{\mathcal{S}}(k) + K_{\mathcal{S}}^{g}(x_{\mathcal{S}}(k) - \hat{x}_{\mathcal{S}}(k)). \tag{32}$$

By defining $\mathcal{Z}_{\mathcal{S}}^{g} \triangleq \prod_{\mathcal{C}\in\mathcal{P}_g} \mathcal{Z}_{\mathcal{C}}^{g}$, $\hat{\mathbb{X}}_{\mathcal{S}}^{g} \triangleq \prod_{\mathcal{C}\in\mathcal{P}_g} \hat{\mathbb{X}}_{\mathcal{C}}^{g}$, $\hat{\mathbb{U}}_{\mathcal{S}}^{g} \triangleq \prod_{\mathcal{C}\in\mathcal{P}_g} \hat{\mathbb{U}}_{\mathcal{C}}^{g}$, and $\hat{\mathbb{T}}_{\mathcal{S}}^{g} \triangleq \prod_{\mathcal{C}\in\mathcal{P}_g} \hat{\mathbb{T}}_{\mathcal{C}}^{g}$, we can express the coalitional MPC optimization problem for all coalitions in $\mathcal{P}_g$ as a collective MPC optimization problem as follows:

$$\min_{(\hat{x}_{\mathcal{S}}(k),\{\hat{u}_{\mathcal{S}}(k+t)\}_{t=0}^{N_p-1})} J_{\mathcal{S}}^{g}(\hat{x}_{\mathcal{S}}(k), \{\hat{u}_{\mathcal{S}}(k+t)\}_{t=0}^{N_p-1}) \tag{33a}$$

subject to

$$x_{\mathcal{S}}(k) - \hat{x}_{\mathcal{S}}(k) \in \mathcal{Z}_{\mathcal{S}}^{g} \tag{33b}$$

$$\hat{x}_{\mathcal{S}}(k+t+1) = \tilde{A}_{\mathcal{S}}^{g}\hat{x}_{\mathcal{S}}(k+t) + \tilde{B}_{\mathcal{S}}^{g}\hat{u}_{\mathcal{S}}(k+t), \tag{33c}$$

$$\hat{x}_{\mathcal{S}}(k+t) \in \hat{\mathbb{X}}_{\mathcal{S}}^{g}, \ t = 0,\ldots,N_p-1, \tag{33d}$$

$$\hat{u}_{\mathcal{S}}(k+t) \in \hat{\mathbb{U}}_{\mathcal{S}}^{g}, \ t = 0,\ldots,N_p-1, \tag{33e}$$

$$\hat{x}_{\mathcal{S}}(k+N_p) \in \hat{\mathbb{T}}_{\mathcal{S}}^{g}, \tag{33f}$$

where

$$\begin{aligned} J_{\mathcal{S}}^{g}(\hat{x}_{\mathcal{S}}(k), \{\hat{u}_{\mathcal{S}}(k+t)\}_{t=0}^{N_p-1}) = \\ \sum_{\mathcal{C}\in\mathcal{P}_g} J_{\mathcal{C}}^{g}(\hat{x}_{\mathcal{C}}(k), \{\hat{u}_{\mathcal{C}}(k+t)\}_{t=0}^{N_p-1}). \end{aligned} \tag{34}$$

Extending the right-hand side of Equation (34) leads us to

$$J_{\mathcal{S}}^{g}(\hat{x}_{\mathcal{S}}(k), \{\hat{u}_{\mathcal{S}}(k+t)\}_{t=0}^{N_p-1}) = \sum_{t=0}^{N_p-1} \Big(x_{\mathcal{S}}^{T}(k+t) Q_{\mathcal{S}}^{g} (x_{\mathcal{S}}(k+t) + u_{\mathcal{S}}^{T}(k+t) R_{\mathcal{S}}^{g} (u_{\mathcal{S}}(k+t) \Big) + x_{\mathcal{S}}^{T}(k+N_p) P_{\mathcal{S}}^{g} (x_{\mathcal{S}}(k+N_p), \tag{35}$$

where $Q_{\mathcal{S}}^{g} = \mathrm{diag}(Q_{\mathcal{C}}^{g})_{\mathcal{C}\in\mathcal{P}_g}$ and $R_{\mathcal{S}}^{g} = \mathrm{diag}(R_{\mathcal{C}}^{g})_{\mathcal{C}\in\mathcal{P}_g}$.

Remark 3. *The set $\mathcal{Z}_{\mathcal{S}}^{g}$ in Constraint (33b) is robustly positive invariant for the system*

$$z_{\mathcal{S}}(k+1) = F_{\mathcal{S}}^{g} z_{\mathcal{S}}(k) + d_{\mathcal{S}}(k), \tag{36}$$

where $z_{\mathcal{S}} = (z_{\mathcal{C}})_{\mathcal{C}\in\mathcal{P}_g}$.

Remark 4. *Suppose $\mathbb{X}_{\mathcal{S}}^{g,N_p}$ denotes the set of all states $x_{\mathcal{S}}$ for which Optimization Problem (33) is feasible. Then, the set $\mathbb{X}_{\mathcal{S}}^{g,N_p}$ is given by $\mathbb{X}_{\mathcal{S}}^{g,N_p} = \prod_{\mathcal{C}\in\mathcal{P}_g} \mathbb{X}_{\mathcal{C}}^{g,N_p}$.*

The subsequent assumption is required to ensure that the Optimization Problem (33) is recursively feasible.

Assumption 5. *For any topology $g \in \mathcal{G}$, a set $\hat{\mathbb{X}}_{F}^{g} \subseteq \hat{\mathbb{X}}_{\mathcal{S}}^{g}$ that is positive invariant for the system*

$$\Omega_{\mathcal{S}}^{g}:\ \hat{x}_{\mathcal{S}}(k+1) = \big(\tilde{A}_{\mathcal{S}}^{g} + \tilde{B}_{\mathcal{S}}^{g} K_{\mathcal{S}}^{g}\big)\hat{x}_{\mathcal{S}}(k) \tag{37}$$

exists and satisfies

$$\hat{u}_{\mathcal{S}}(k) = K_{\mathcal{S}}^{g}\hat{x}_{\mathcal{S}}(k) \in \hat{\mathbb{U}}_{\mathcal{S}}^{g},\ \forall \hat{x}_{\mathcal{S}}(k) \in \hat{\mathbb{X}}_{F}^{g}.$$

In other words, $\hat{\mathbb{X}}_{F}^{g}$ is a control invariant set for System (30) subject to Constraints (31). The following algorithm provides the computation of $\hat{\mathbb{X}}_{F}^{g}$, which is adopted from [25].

Algorithm 3: The computation of the control invariant set $\hat{\mathbb{X}}_{F}^{g}$

Require: System (37) and Constraints (31).
Initialization: $\Psi(0) \leftarrow \hat{\mathbb{X}}_{\mathcal{S}}^{g}$, $k \leftarrow -1$
repeat
$k \leftarrow k+1$
$\Psi(k+1) \leftarrow \mathrm{Pre}_{\Omega_{\mathcal{S}}^{g}}(\Psi(k)) \cap \Psi_k$
until $\Psi(k+1) = \Psi(k)$
$\hat{\mathbb{X}}_{F}^{g} \leftarrow \Psi(k)$

Remark 5. *If $\hat{\mathbb{X}}_{\mathcal{S}}^{g}$ and $\hat{\mathbb{U}}_{\mathcal{S}}^{g}$ are polyhedral sets represented by*

$$\hat{\mathbb{X}}_{\mathcal{S}}^{g} = \{\hat{x}_{\mathcal{S}} | H\hat{x}_{\mathcal{S}} \le h\},$$

$$\hat{\mathbb{U}}_{\mathcal{S}}^{g} = \{\hat{u}_{\mathcal{S}} | L\hat{u}_{\mathcal{S}} \le l\},$$

then the precursor to the set $\hat{\mathbb{X}}_{\mathcal{S}}^{g}$ with respect to System (37) is given by the following polyhedron

$$\mathrm{Pre}_{\Omega_{\mathcal{S}}^{g}}(\hat{\mathbb{X}}_{\mathcal{S}}^{g}) = \left\{ \hat{x}_{\mathcal{S}} \in \mathbb{R}^{n} | \begin{bmatrix} H(\tilde{A}_{\mathcal{S}}^{g} + \tilde{B}_{\mathcal{S}}^{g} K_{\mathcal{S}}^{g}) \\ LK_{\mathcal{S}}^{g} \end{bmatrix} \hat{x}_{\mathcal{S}} \le \begin{bmatrix} h \\ l \end{bmatrix} \right\},$$

where $H \in \mathbb{R}^{m\times n}$, $L \in \mathbb{R}^{p\times q}$, $h \in \mathbb{R}^{m}$, and $l \in \mathbb{R}^{p}$.

In addition, to guarantee the recursive feasibility of Optimization Problem (33), we require that the terminal set $\hat{\mathbb{T}}_{\mathcal{S}}^{g}$ is a common control invariant set. The definition of a common control invariant set is provided below.

Definition 6. *Consider the switched system*

$$\hat{x}_{\mathcal{S}}(k+1) = \tilde{A}_{\mathcal{S}}^{\sigma(k)}\hat{x}_{\mathcal{S}}(k) + \tilde{B}_{\mathcal{S}}^{\sigma(k)}\hat{u}_{\mathcal{S}}(k), \tag{38a}$$

$$\hat{x}_{\mathcal{S}}(k) \in \hat{\mathbb{X}}_{\sigma(k)}, \hat{u}_{\mathcal{S}}(k) \in \hat{\mathbb{U}}_{\sigma(k)}, \tag{38b}$$

that switches between $|\mathcal{G}|$ system modes according to switching signal σ. A set $\mathbb{T}$ is called a common control invariant set for System (38) if it is control invariant for all system modes.

The computation of the common control invariant set is presented in the algorithm below.

Algorithm 4: The computation of the common control invariant set $\mathbb{T}$

1) For each topology $g \in \mathcal{G}$, compute $\hat{\mathbb{X}}_{F}^{g}$ using Algorithm 3.
2) Compute $\mathbb{T} = \cap_{g\in\mathcal{G}} \hat{\mathbb{X}}_{F}^{g}$.

Remark 6. *Given the common control invariant set $\mathbb{T}$. The set $\hat{\mathbb{T}}_{\mathcal{C}}^{g}$ in Optimization Problem (18) is obtained by projecting the set $\mathbb{T}$ onto the state space of the coalition $\mathcal{C}$.*

The following remark explains the order of the execution of the algorithms presented above. It is important to note that Algorithms 1, 3, and 4 are executed offline, while Algorithm 2 is performed online.

Remark 7. *First, Algorithm 3 is executed to obtain the control invariant set $\hat{\mathbb{X}}_{F}^{g}$, for all $g \in \mathcal{G}$. Subsequently, Algorithm 4 is executed to determine the common invariant set $\mathbb{T}$ and the sets $\hat{\mathbb{T}}_{\mathcal{C}}^{g}$, for all $\mathcal{C} \in \mathcal{P}_g$, for all $g \in \mathcal{G}$. Then, algorithm 1 is executed to compute the feasible region $\mathbb{X}_{\mathcal{C}}^{g,N_p}$ for all $\mathcal{C} \in \mathcal{P}_g$, for all $g \in \mathcal{G}$. Finally, Algorithm 2 is executed utilizing these results.*

The assumption below is also needed to ensure that Optimization Problem (33) is recursively feasible.

Assumption 7. *For any consecutive time steps $mT \in \mathcal{T}$ and $(m+1)T \in \mathcal{T}$, it holds that*

$$\mathcal{Z}_{\mathcal{S}}^{\sigma(mT)} \subseteq \mathcal{Z}_{\mathcal{S}}^{\sigma((m+1)T)},$$

for any $m \in \mathbb{Z}_{+}$.

The main result of this work is presented in the theorem below.

Theorem 8. *Suppose all assumptions above hold, $N_p \le T$, and $\hat{\mathbb{T}}_{\mathcal{S}}^{g} = \mathbb{T}$ for all $g \in \mathcal{G}$. Then, if Optimization Problem (33) has a feasible solution at time step k, it also has a feasible solution at time step $k+1$ for all $k \in \mathbb{Z}_{+}$.*

Proof: In this proof, we investigate two scenarios. In the first scenario, we consider the topology at time steps k and $k+1$ unchanged. Then, in the second case, we examine the situation in which the topology at time steps k and $k+1$ is modified.

Let $\left(\hat{x}_{\mathcal{S}}^{*}(k), \{\hat{u}_{\mathcal{S}}^{*}(k+t)\}_{t=0}^{N_p-1}\right)$ be the feasible solution to Optimization Problem (33) for the state $x_{\mathcal{S}}(k)$. Denote the state trajectory for Nominal System (30) corresponding to the solution as $\{\hat{x}_{\mathcal{S}}^{*}(k+t)\}_{t=0}^{N_p}$. Substituting

$$u_{\mathcal{S}}^{*}(k) = \hat{u}_{\mathcal{S}}^{*}(k) + K_{\mathcal{S}}^{\sigma(k)}\left(x_{\mathcal{S}}(k) - \hat{x}_{\mathcal{S}}^{*}(k)\right)$$

into (28), we have $x_{\mathcal{S}}(k+1)$ for any $d_{\mathcal{S}}(k) \in \mathbb{D}_{\mathcal{S}}^{\sigma(k)}$.
(**Scenario 1.** $\sigma(k+1) = \sigma(k)$) Consider the tuple

$$\left(\hat{x}_{\mathcal{S}}^*(k+1), \{\hat{u}_{\mathcal{S}}^*(k+t+1)\}_{t=0}^{N_p-1}\right),$$

with $\hat{u}_{\mathcal{S}}^*(k+N_p) = K_{\mathcal{S}}^{\sigma(k)}\hat{x}_{\mathcal{S}}^*(k+N_p)$, to be a solution to Optimization Problem (33) corresponding to the state $x_{\mathcal{S}}(k+1)$. Then, the state trajectory for Nominal System (30) associated with the later solution is

$$\{\hat{x}_{\mathcal{S}}^*(k+t+1)\}_{t=0}^{N_p},$$

where

$$\hat{x}_{\mathcal{S}}^*(k+N_p+1) = \left(\tilde{A}_{\mathcal{S}}^{\sigma(k)} + \tilde{B}_{\mathcal{S}}^{\sigma(k)}K_{\mathcal{S}}^{\sigma(k)}\right)\hat{x}_{\mathcal{S}}^*(k+N_p).$$

From the feasibility at time step k, it follows that

$$\hat{x}_{\mathcal{S}}^*(k+t+1) \in \hat{\mathbb{X}}_{\mathcal{S}}^{\sigma(k)} = \hat{\mathbb{X}}_{\mathcal{S}}^{\sigma(k+1)}, \text{ for } t = 0, \ldots, N_p - 1,$$

$$\hat{u}_{\mathcal{S}}^*(k+t+1) \in \hat{\mathbb{U}}_{\mathcal{S}}^{\sigma(k)} = \hat{\mathbb{U}}_{\mathcal{S}}^{\sigma(k+1)} \text{ for } t = 0, \ldots, N_p - 2,$$

and

$$x_{\mathcal{S}}(k) - \hat{x}_{\mathcal{S}}^*(k) \in \mathcal{Z}_{\mathcal{S}}^{\sigma(k)}.$$

Consequently, based on Proposition 1 in [22], we have

$$x_{\mathcal{S}}(k+1) - \hat{x}_{\mathcal{S}}^*(k+1) \in \mathcal{Z}_{\mathcal{S}}^{\sigma(k)} = \mathcal{Z}_{\mathcal{S}}^{\sigma(k+1)}.$$

Moreover, due to

$$\hat{x}_{\mathcal{S}}^*(k+N_p) \in \hat{\mathbb{T}}_{\mathcal{S}}^{\sigma(k)} = \hat{\mathbb{T}}_{\mathcal{S}}^{\sigma(k+1)} = \mathbb{T},$$

it follows that

$$\hat{x}_{\mathcal{S}}^*(k+N_p+1) \in \hat{\mathbb{T}}_{\mathcal{S}}^{\sigma(k+1)},$$

and

$$\hat{u}_{\mathcal{S}}^*(k+N_p) \in \hat{\mathbb{U}}_{\mathcal{S}}^{\sigma(k+1)},$$

according to Assumption 5. This proves that $(\hat{x}_{\mathcal{S}}^*(k+1), \{\hat{u}_{\mathcal{S}}^*(k+t+1)\}_{t=0}^{N_p-1})$ is a feasible solution to (33) at time step $k+1$.
(**Scenario 2.** $\sigma(k+1) \neq \sigma(k)$) Since $N_p \leq T$, then

$$\hat{x}_{\mathcal{S}}^*(k+t) \in \hat{\mathbb{T}}_{\mathcal{S}}^{\sigma(k)} = \hat{\mathbb{T}}_{\mathcal{S}}^{\sigma(k+1)} = \mathbb{T}, \text{ for all } t = 0, \ldots, N_p.$$

Consequently, by defining $\hat{x}_{\mathcal{S}}^0(k+1) = \hat{x}_{\mathcal{S}}^*(k+1)$, we obtain

$$\hat{u}_{\mathcal{S}}^0(k+1) = K_{\mathcal{S}}^{\sigma(k+1)}\hat{x}_{\mathcal{S}}^0(k+1) \in \hat{\mathbb{U}}_{\mathcal{S}}^{\sigma(k+1)},$$

$$\hat{x}_{\mathcal{S}}^0(k+2) = (\tilde{A}_{\mathcal{S}}^{\sigma(k+1)} + \tilde{B}_{\mathcal{S}}^{\sigma(k+1)}K_{\mathcal{S}}^{\sigma(k+1)})\hat{x}_{\mathcal{S}}^0(k+1) \in \mathbb{T}_{\mathcal{S}}^{\sigma(k+1)},$$

according to Assumption 5. In the same way, we also have

$$\hat{u}_{\mathcal{S}}^0(k+2) = K_{\mathcal{S}}^{\sigma(k+1)}\hat{x}_{\mathcal{S}}^0(k+2) \in \hat{\mathbb{U}}_{\mathcal{S}}^{\sigma(k+1)},$$

$$\hat{x}_{\mathcal{S}}^0(k+3) = (\tilde{A}_{\mathcal{S}}^{\sigma(k+1)} + \tilde{B}_{\mathcal{S}}^{\sigma(k+1)}K_{\mathcal{S}}^{\sigma(k+1)})\hat{x}_{\mathcal{S}}^0(k+2) \in \mathbb{T}_{\mathcal{S}}^{\sigma(k+1)}.$$

Repeating this procedure recursively, we get the following sequences:

$$\{\hat{u}_{\mathcal{S}}^0(k+t+1)\}_{t=0}^{N_p-1},$$

and

$$\{\hat{x}_{\mathcal{S}}^0(k+t+1)\}_{t=0}^{N_p}\},$$

where $\hat{u}_{\mathcal{S}}^0(k+t+1) \in \hat{\mathbb{U}}_{\mathcal{S}}^{\sigma(k+1)}$, for all $t = 0, \ldots, N_p-1$ and $\hat{x}_{\mathcal{S}}^0(k+t+1) \in \hat{\mathbb{X}}_{\mathcal{S}}^{\sigma(k+1)}$, for all $t = 0, \ldots, N_p$. Furthermore, according to the proof in Scenario 1, it follows that

$$x_{\mathcal{S}}(k+1) - \hat{x}_{\mathcal{S}}^*(k+1) \in \mathcal{Z}_{\mathcal{S}}^{\sigma(k)}.$$

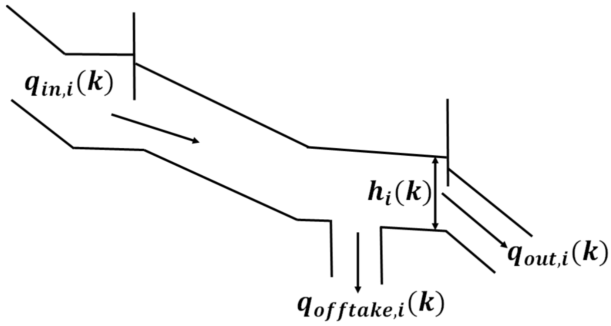

Fig. 1. Illustration of canal reach i

Since $\mathcal{Z}_{\mathcal{S}}^{\sigma(k)} \subseteq \mathcal{Z}_{\mathcal{S}}^{\sigma(k+1)}$, we have

$$x_{\mathcal{S}}(k+1) - \hat{x}_{\mathcal{S}}^0(k+1) \in \mathcal{Z}_{\mathcal{S}}^{\sigma(k+1)}.$$

Thus, the pair $\left(\hat{x}_{\mathcal{S}}^0(k+1), \{\hat{u}_{\mathcal{S}}^0(k+t+1)\}_{t=0}^{N_p-1}\right)$ is a feasible solution to (33) at time step $k+1$. ∎

V. SIMULATION

To validate the performance of the proposed control scheme, we apply it to an irrigation canal system comprising three canal reaches. As shown in Fig. 1, each canal reach i is assumed to have an inflow, denoted by $q_{in,i} \in \mathbb{R}_{\geq 0}(m^3/s)$, originating from an upstream canal reach, and an outflow, indicated by $q_{out,i} \in \mathbb{R}_{\geq 0}(m^3/s)$, directed towards a downstream canal reach. Furthermore, the canal reach i is also assumed to have a local outflow, represented by $q_{offtake,i} \in \mathbb{R}_{\geq 0}(m^3/s)$, resulting from factors such as farmer usage.

The water level in the canal reaches is influenced by the amount of water that flows into and out of the reaches. In addition, the surface of the reaches also contributes to the change in the water levels. However, for our analysis, we focus solely on the water levels at the downstream end of each reach since, in this location, the water offtakes occur, and the water level should be maintained close to the reference level.

Let $h_i(m)$ and $h_{i,ref}(m)$ represent the water level and reference level at the downstream end of the canal reach i, respectively. We define $e_i = h_i - h_{i,ref}$ as the difference between the water level and the reference level, and $A_{s,i}$ as the surface area of the canal reach i. The change in water level in the canal reach i from time step k to time step $k+1$ is specified by [26]

$$\begin{aligned} e_i(k+1) =& e_i(k) + \frac{T_s}{A_{s,i}}q_{in,i}(k) - \frac{T_s}{A_{s,i}}q_{out,i}(k) \\ &- \frac{T_s}{A_{s,i}}q_{offtake,i}(k), \end{aligned} \tag{39}$$

or

$$\begin{aligned} \Delta e_i(k+1) =& \Delta e_i(k) + \frac{T_s}{A_{s,i}}\Delta q_{in,i}(k) - \frac{T_s}{A_{s,i}}\Delta q_{out,i}(k) \\ &- \frac{T_s}{A_{s,i}}\Delta q_{offtake,i}(k), \end{aligned} \tag{40}$$

where $\Delta e_i(k)$, $\Delta q_{in,i}(k)$, $\Delta q_{out,i}(k)$, and $\Delta q_{offtake,i}(k)$ denote the difference of the corresponding variables at time steps k and $k-1$, and T_s indicates the sampling time.

Define state, control input, and disturbance input vectors as $x_i(k) = \begin{pmatrix}\Delta e_i(k)\\ e_i(k)\end{pmatrix}$, $u_i(k) = \Delta q_{in,i}(k)$, $u_{i+1}(k) = \Delta q_{out,i}(k)$, and $w_i(k) = \Delta q_{offtake,i}(k)$. Then, System (40) can be put in the state-space form as follows:

$$x_i(k+1) = A_{ii}x_i(k) + B_{ii}u_i(k) + B_{i(i+1)}u_{i+1}(k) + D_{ii}w_i(k), \tag{41}$$

where

$$A_{ii} = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}, B_{ii} = \begin{pmatrix}\frac{T_s}{A_{s,i}}\\ \frac{T_s}{A_{s,i}}\end{pmatrix},$$

$$B_{i(i+1)} = \begin{pmatrix}-\frac{T_s}{A_{s,i}}\\ -\frac{T_s}{A_{s,i}}\end{pmatrix}, D_{ii} = \begin{pmatrix}-\frac{T_s}{A_{s,i}}\\ -\frac{T_s}{A_{s,i}}\end{pmatrix}.$$

Therefore, the dynamics of canal reaches 1, 2, and 3 are given by :

$$x_1(k+1) = A_{11}x_1(k) + B_{11}u_1(k) + B_{12}u_2(k) + D_{11}w_1(k), \tag{42a}$$

$$x_2(k+1) = A_{22}x_2(k) + B_{22}u_2(k) + B_{23}u_3(k) + D_{22}w_2(k), \tag{42b}$$

$$x_3(k+1) = A_{33}x_3(k) + B_{33}u_3(k) + D_{33}w_3(k). \tag{42c}$$

In this simulation, we set the constraints of the state, control input, and disturbance input for canal reaches 1,2, and 3 as follows:

$$\mathbb{X}_1 = \left\{x_1 : \begin{pmatrix}-1\\ -1\end{pmatrix} \le x_1 \le \begin{pmatrix}1\\ 1\end{pmatrix}\right\}, \tag{43a}$$

$$\mathbb{U}_1 = \{u_1 : -1.3 \le u_1 \le 1.3\}, \tag{43b}$$

$$\mathbb{W}_1 = \{d_1 : -0.05 \le d_1 \le 0.05\}, \tag{43c}$$

$$\mathbb{X}_2 = \left\{x_2 : \begin{pmatrix}-1\\ -1\end{pmatrix} \le x_2 \le \begin{pmatrix}1\\ 1\end{pmatrix}\right\}, \tag{44a}$$

$$\mathbb{U}_2 = \{u_2 : -0.6 \le u_2 \le 0.6\}, \tag{44b}$$

$$\mathbb{W}_2 = \{d_2 : -0.05 \le d_2 \le 0.05\}, \tag{44c}$$

$$\mathbb{X}_3 = \left\{x_3 : \begin{pmatrix}-1\\ -1\end{pmatrix} \le x_3 \le \begin{pmatrix}1\\ 1\end{pmatrix}\right\}, \tag{45a}$$

$$\mathbb{U}_3 = \{u_3 : -0.3 \le u_3 \le 0.3\}, \tag{45b}$$

$$\mathbb{W}_3 = \{d_3 : -0.05 \le d_3 \le 0.05\}. \tag{45c}$$

Furthermore, the values of the parameters used in this simulation are presented in Table I.

TABLE I
THE VALUES OF THE PARAMETERS

No.	Parameter	value
1.	T_s	240 s
2.	$A_{s,1}$	397 m^2
3.	$A_{s,2}$	653 m^2
4.	$A_{s,3}$	503 m^2
5.	Q_i	diag(100, 100), $\forall i = 1, 2, 3$
6.	R_i	10, $\forall i = 1, 2, 3$
7.	$x_i(0)$	$\begin{bmatrix}0.08 & 0.08\end{bmatrix}^T, \forall i = 1, 2, 3$
8.	N_p	5 time steps
9.	T	5 time steps
10.	c	0.1

Each canal reach is independently controlled by a set of agents denoted by $\{A1, A2, A3\}$. Each agent $Ai, i = 1, 2, 3$, is responsible for maintaining the deviation between the water and reference levels close to zero. These agents are interconnected through a communication network consisting of two links, as illustrated in Fig.2. Due to both links I and II can be enabled or disabled, four different network topologies could be established. All topologies that can appear and their corresponding coalitions generated by them are presented in Table II.

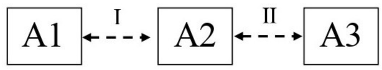

Fig. 2. Communication Network of Agents

TABLE II
ALL POSSIBLE TOPOLOGIES AND THEIR CORRESPONDING PARTITIONS

No	Topology	The collection of coalitions
1.	$g_1 = \emptyset$	$\mathcal{P}_{g_1} = \{\{A1\}, \{A2\}, \{A3\}\}$
2.	$g_2 = \{I\}$	$\mathcal{P}_{g_2} = \{\{A1, A2\}, \{A3\}\}$
3.	$g_3 = \{II\}$	$\mathcal{P}_{g_3} = \{\{A1\}, \{A2, A3\}\}$
4.	$g_4 = \{I, II\}$	$\mathcal{P}_{g_4} = \{A1, A2, A3\}$

For each topology $g \in G$, the RPI set $\mathcal{Z}_{\mathcal{C}}^g$ is computed for all coalitions $\mathcal{C} \in \mathcal{P}_g$ using the algorithm described in [24]. For this purpose, we solve LMI (13) to obtain matrices $K_{\mathcal{S}}^g = \text{diag}(K_{\mathcal{C}}^g)_{\mathcal{C}\in\mathcal{P}_g}$ and $P_{\mathcal{S}}^g = \text{diag}(P_{\mathcal{C}}^g)_{\mathcal{C}\in\mathcal{P}_g}$. For example, for topology g_1, we obtain the following matrices:

$$K_{\mathcal{C}_1}^{g_1} = \begin{pmatrix}-1.6440 - 0.9413\end{pmatrix},$$

$$P_{\mathcal{C}_1}^{g_1} = \begin{pmatrix}127.7995 & 16.2370\\ 16.2370 & 176.7849\end{pmatrix},$$

$$K_{\mathcal{C}_2}^{g_1} = \begin{pmatrix}-2.3363 & -0.7581\end{pmatrix},$$

$$P_{\mathcal{C}_2}^{g_1} = \begin{pmatrix}996.7954 & 297.3096\\ 297.3096 & 324.2800\end{pmatrix},$$

$$K_{\mathcal{C}_3}^{g_1} = \begin{pmatrix}-1.1868 & -0.3849\end{pmatrix},$$

$$P_{\mathcal{C}_3}^{g_1} = \begin{pmatrix}1024.6 & 308.3\\ 308.3 & 328.6\end{pmatrix}.$$

The RPI sets $\mathcal{Z}_{\mathcal{C}_i}^{g_1}$, $i = 1, 2, 3$, are displayed in Fig. 3. Then, the sets $\hat{\mathbb{X}}_{\mathcal{C}}^g$ and $\hat{\mathbb{U}}_{\mathcal{C}}^g$ for all coalitions $\mathcal{C} \in \mathcal{P}_g$ are computed by using (22). Subsequently, the common control invariant set $\mathbb{T}$ is computed using Algorithm 4. Then, by projecting the sets $\mathbb{T}$ onto the state space of the coalition $\mathcal{C}$, we obtain $\hat{\mathbb{T}}_{\mathcal{C}}^g$ for all coalitions $\mathcal{C} \in \mathcal{P}_g$. The sets $\hat{\mathbb{T}}_{\mathcal{C}}^g$ corresponding to the topology g_1 are displayed in Fig. 4. Finally, by using Algorithm 1, we compute the feasible regions $\hat{\mathbb{X}}_{\mathcal{C}}^{g,N_p}$ and $\hat{\mathbb{X}}_{\mathcal{C}}^{g,N_p}$ for all coalitions $\mathcal{C} \in \mathcal{P}_g$, $g \in G$. The sets $\hat{\mathbb{X}}_{\mathcal{C}}^{g,N_p}$ and $\hat{\mathbb{X}}_{\mathcal{C}}^{g,N_p}$ that corresponds to the topology g_1 are shown in Fig.5.

In this simulation, we compare our coalitional control scheme to centralized MPC and decentralized MPC schemes. The network topology g_4 is used in the centralized MPC strategy, while the network topology g_1 is exploited in the decentralized MPC strategy. Furthermore, the values of disturbance vectors d_i for all $i \in \{1, 2, 3\}$ are generated using the *rand* command in Matlab.

Figs. 6-8 demonstrate the simulation results in 25 time steps. Fig. 6 shows the control actions of all subsystems for all schemes, while Fig. 7 and Fig. 8 display the states of

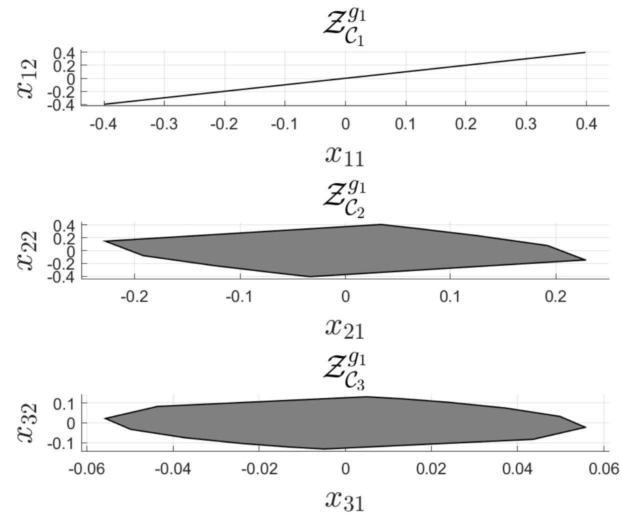

Fig. 3. The RPI sets corresponding to the topology g_1

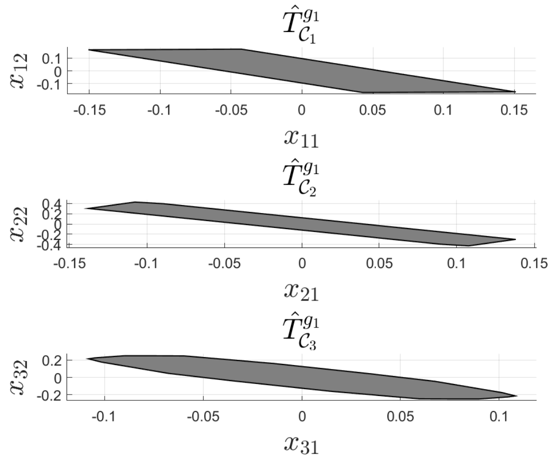

Fig. 4. The target sets $\hat{\mathbb{T}}_{\mathcal{C}}^{g_1}$ for all $\mathcal{C} \in \mathcal{P}_{g_1}$

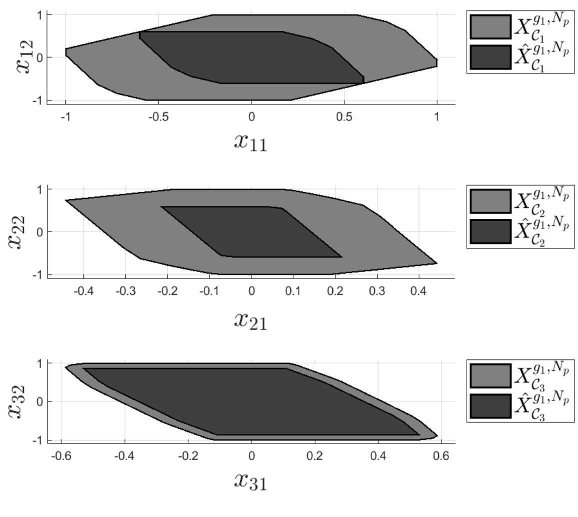

Fig. 5. The sets $\hat{\mathbb{X}}_{\mathcal{C}}^{g_1,N_p}$ and $\mathbb{X}_{\mathcal{C}}^{g_1,N_p}$ for all $\mathcal{C} \in \mathcal{P}_{g_1}$

all subsystems for all schemes. We can observe from Figs. 6-8 that the state and input constraints of all subsystems are satisfied by all schemes. Furthermore, based on Fig. 7 and Fig. 8, the behavior of the states of all subsystems resulting from our proposed scheme is similar to that resulting from the centralized MPC scheme. This is confirmed by the sum of the mean squared error (MSE) of the state vector of all subsystems presented in Table III. We can see from Table III that the sum of the MSE of our coalitional scheme is similar to the MSE given by the centralized scheme. In addition, our proposed control scheme has the lowest cost, as shown in Table IV.

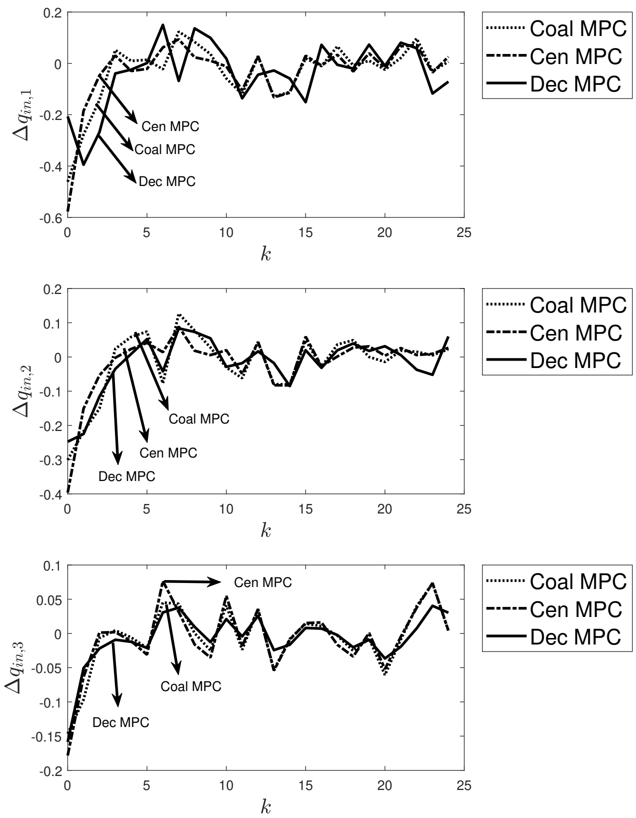

Fig. 6. The dynamic of the control input $\Delta q_{in,i}$

TABLE III
THE COMPARISON OF MEAN SQUARED ERRORS OF STATES

	Coal. MPC	Cen. MPC	Dec. MPC
x_1	8.3675×10^{-4}	7.3511×10^{-4}	0.0045
x_2	0.0011	7.6962×10^{-4}	0.0018
x_3	8.6288×10^{-4}	7.0785×10^{-4}	0.0010
Sum	0.0028	0.0022	0.0073

TABLE IV
COMPARISON OF TOTAL COST

	Coal MPC	Cen MPC	Dec MPC
Total cost	9.5381	11.1597	11.1780

Fig. 9 depicts the dynamic of the network topology resulting from our coalitional control scheme. In this simulation, we use g_2 as the initial network topology and update the network topology at intervals of $T = 5$. Therefore, the network topology update occurs at time steps $k = 5, 10, 15, 20$.

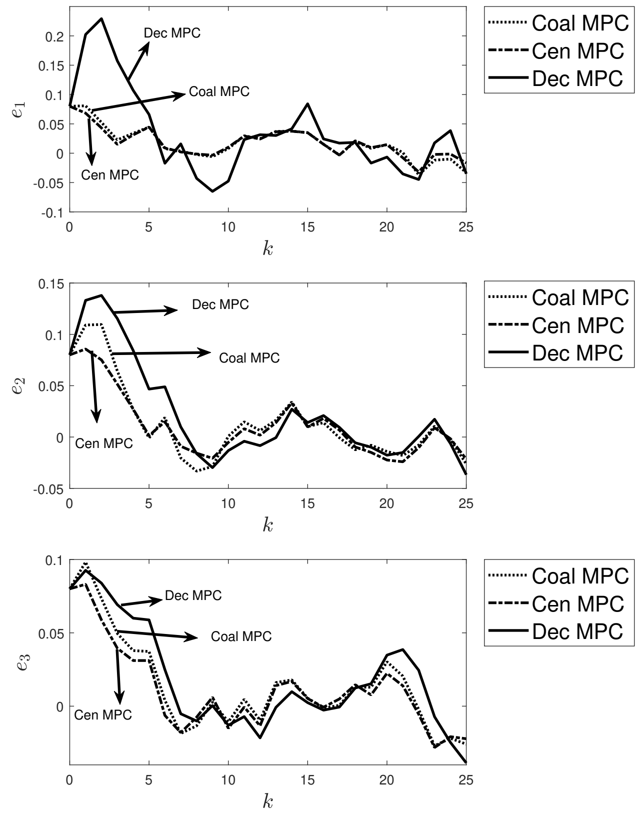

Fig. 7. The dynamic of the state e_i

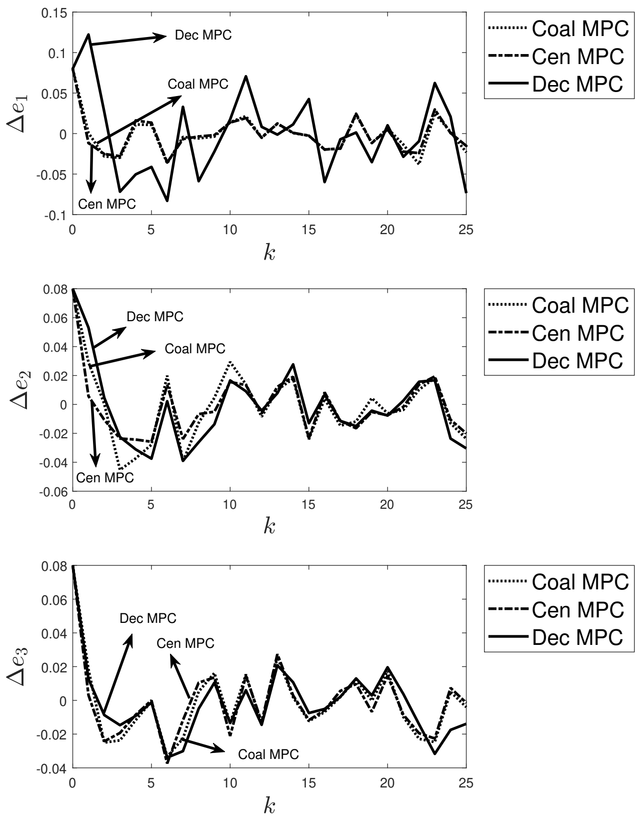

Fig. 8. The dynamic of the state Δe_i

It can be observed from Fig. 9 that $g(5) = g_2$, $g(10) = g_4$, $g(15) = g_2$, and $g(20) = g_3$.

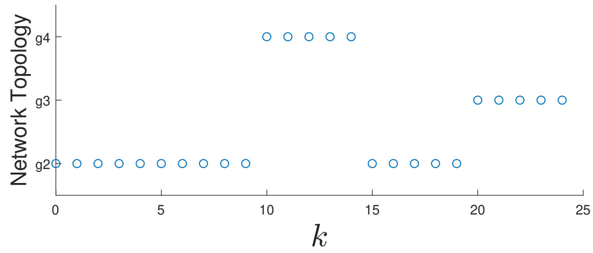

Fig. 9. The dynamic of network topologies

VI. Conclusion

This paper investigates the design of a top-down coalitional MPC scheme for a system consisting of disturbed and constrained subsystems interconnected with each other. The proposed scheme involves a supervisor that periodically selects the optimal network topology at the upper layer. At the bottom layer, each coalition generated by the optimal network topology independently calculates its control actions using a decentralized MPC strategy. The issue of recursive feasibility is addressed by utilizing a common control invariant set as the terminal constraint of the collective coalitional MPC optimization problem. The simulation results demonstrate that our proposed control scheme guarantees constraint satisfaction of the system. Furthermore, the results show that our proposed control scheme has the lowest cost compared to the centralized MPC and decentralized MPC schemes.

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