Minimum Covering Maximum Reverse Degree Energy Of Graphs

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Abstract—**In this paper, we have introduced minimum covering maximum reverse degree energy of simple graphs. Few properties on minimum covering maximum reverse degree eigenvalues and bounds for minimum covering maximum reverse degree energy of a graph are achieved. Further minimum covering maximum reverse degree energy of some families of graphs are computed.**

Index Terms—**Minimum covering maximum reverse degree matrix, Minimum covering maximum reverse degree eigenvalues, Minimum covering maximum reverse degree energy.**

I. Introduction

Let $G = (V, E)$ be a graph with $V = \{v_1, v_2, \dots, v_n\}$ as its vertex set and $E = \{e_1, e_2, \dots, e_n\}$ as its edge set. Let $A = a_{ij}$ be the adjacency matrix of G. Then $|A - \lambda I| = 0$ is called characteristic equation of G. $\lambda_1, \lambda_2, \dots, \lambda_n$ of A, are called eigenvalues of G which are assumed to be in non increasing order. As A is real symmetric matrix, the eigenvalues of G are real with sum equal to zero. The energy of G is defined to be sum of absolute values of the eigenvalues of G. i.,e $E(G) = \sum\limits_{i=1}^{n} |\lambda_i|$. In theoretical chemistry, the π-electron energy of a conjugated carbon molecule, computed using Huckle theory, coincides with the energy as defined above. Hence, results on graph energy assume special importance in graph theory. Because of the numerous implications of graph energy, many researchers have defined multiple energies with regard to a graph. For more on energy of graphs, one can refer [1]–[8].

Adiga and Smitha defined Maximum degree matrix $M(G)$ of a graph G as follows:

Definition I.1. *[9] Let G be a simple graph with n vertices $\{v_1, v_2, \dots, v_n\}$ and d_i be the degree of v_i for $i = 1, 2, \dots, n$. Then maximum degree matrix $M(G) = (d_{ij})$, is defined as*

$$d_{ij} = \begin{cases} max\{d_i, d_j\}, & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0, & \text{otherwise.} \end{cases}$$

A subset C of V is called a covering set of G, if every edge of G is incident to atleast one vertex of C. Any covering set with minimum cardinality is called minimum covering set. For more on minimum covering energy of graphs, one can refer [10], [11].

Definition I.2. *[12] Let C be the minimum covering set of a graph G. The minimum covering maximum degree matrix*

$$A_c[M(G)] = a_{ij} = \begin{cases} 1, & \text{if } i = j, v_i \in C \\ max\{d_{v_i}, d_{v_j}\}, & \text{if } v_i \sim v_j \in E \\ 0, & \text{otherwise.} \end{cases}$$

Let $\Delta(G)$ denote the maximum degree among the vertices of G. The reverse vertex degree of a vertex v_i in G is defined as $c_{v_i} = \Delta(G) - d(v_i) + 1$, where $d(v_i)$ is degree of vertex v_i.

Definition I.3. *[13] Let G be a simple graph with n vertices and size m. Let c_{v_i} be the reverse vertex degree of the vertex v_i. Then maximum reverse degree matrix is defined as $M_R(G) = (r_{ij})$, where*

$$r_{ij} = \begin{cases} max\{c_{v_i}, c_{v_j}\}, & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0, & \text{otherwise.} \end{cases}$$

In this paper, we have introduced minimum covering maximum reverse degree energy of graphs.

Definition I.4. *Let G be a simple graph with n vertices and size m. Let c_{v_i} be the reverse vertex degree of the vertex v_i. Then minimum covering maximum reverse degree matrix is defined as $A_C[M_R(G)] = (r_{ij})$, where*

$$r_{ij} = \begin{cases} max\{c_{v_i}, c_{v_j}\}, & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 1, & \text{if } i = j \text{ and } v_i \in C \\ 0, & \text{otherwise.} \end{cases}$$

The characteristic polynomial of minimum covering maximum reverse degree of a graph G is defined by $\phi\{A_c[M_R(G)]\} = |\lambda I - A_C[M_R(G)]|$ and minimum covering maximum reverse degree energy of G is denoted by $EA_c[M_R(G)]$, is defined as $\sum\limits_{i=1}^{n} |\lambda_i|$ where $\lambda_i's$ are minimum covering maximum reverse degree eigenvalues of G.

This paper is organised as follows. In section 2, the properties of minimum covering maximum reverse degree energy of graphs are studied. In section 3, bounds for minimum covering maximum reverse degree energy of graphs are established. In section 4, minimum covering maximum reverse degree energy of some families of graphs are computed.

Throughout this paper, x_i refers to the number of vertices in the neighbourhood of v_i whose reverse vertex degree is less than c_{v_i} and y_i refers to the number of vertices $v_j (j > i)$ in the neighbourhood of v_i whose reverse vertex degree is equal to c_{v_i}.

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II. Properties of Minimum Covering Maximum Reverse Degree Energy of Graphs

Theorem II.1. *Let G be a simple graph with n vertices and m edges. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ represent minimum covering maximum reverse degree eigenvalues of G, then*

1) $\sum_{i=1}^{n} \lambda_i = |C|$.
2) $\sum_{i=1}^{n} \lambda_i^2 = |C| + 2\sum_{i=1}^{n}(x_i + y_i)c_{v_i}^2$.

Proof:

1) Sum of eigenvalues of $A_C[M_R(G)]$ is equal to trace of $A_C[M_R(G)]$,
$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} r_{ii} = |C|$.
2) The sum of squares of eigenvalues of $A_C[M_R(G)]$ is the trace of $A_C[M_R^2(G)]$.

$$i.e., \sum_{i=1}^{n} \lambda_i^2 = \sum_{i=1}^{n}\sum_{i=1}^{n} r_{ij}r_{ji}$$
$$= \sum_{i=1}^{n} r_{ii}^2 + \sum_{i\neq j} r_{ij}r_{ji}$$
$$\sum_{i=1}^{n} \lambda_i^2 = |C| + 2\sum_{i=1}^{n}(x_i + y_i)c_{v_i}^2.$$

■

Theorem II.2. *Let $G = (V, E)$ be a graph. Let $\phi\{A_C[M_R(G)], \lambda\} = a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + a_3\lambda^{n-3} + \ldots + a_n$ be the minimum covering maximum reverse degree characteristic polynomial of graph G. Then,*

1) $a_0 = 1$.
2) $a_1 = -|C|$.
3) $a_2 = \binom{|C|}{2} - \sum_{i=1}^{n}(x_i + y_i)c_{v_i}^2$.

Proof:

1) From the definition of $\phi\{A_C[M_R(G)], \lambda\}$, it follows that $a_0 = 1$.
2) Sum of diagonal elements of $A_C[M_R(G)]$ is equal to cardinality of the set C.
Hence, $(-1)a_1 = -trace\{A_C[M_R(G)]\} = -|C|$.
3) We have

$$(-1)^2 a_2 = \sum_{1\leq i<j\leq n} \begin{vmatrix} r_{ii} & r_{ij} \\ r_{ji} & r_{jj} \end{vmatrix}$$
$$= \sum_{1\leq i<j\leq n} r_{ii}r_{jj} - \sum_{1\leq i<j\leq n} r_{ji}r_{ij}$$
$$a_2 = \binom{|C|}{2} - \sum_{i=1}^{n}(x_i + y_i)c_{v_i}^2.$$

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III. Bounds for Minimum Covering Maximum Reverse Degree Energy of Graphs

Theorem III.1. *Let G be a graph and C be minimum covering set of G. Then $\sqrt{(|C| + \beta)} \leq EA_C[M_R(G)] \leq \sqrt{n(|C| + \beta)}$.*

Proof: Taking $a_i = 1$, $b_i = |\lambda_i|$ in Cauchy Schwarz inequality, we get

$$\left(\sum_{i=1}^{n} \lambda_i\right)^2 \leq \left(\sum_{i=1}^{n} 1\right)\left(\sum_{i=1}^{n} \lambda_i^2\right)$$

$$(EA_C[M_R(G)])^2 \leq n\left(|C| + 2\sum_{i=1}^{n}(x_i + y_i)c_{v_i}^2\right).$$

Let

$$2\sum_{i=1}^{n}(x_i + y_i)c_{v_i}^2 = \beta.$$

$$EA_C[M_R(G)] \leq \sqrt{n(|C| + \beta)}.$$

Also,

$$\left(\sum_{i=1}^{n} \lambda_i\right)^2 \geq \sum_{i=1}^{n} \lambda_i^2$$

$$(EA_C[M_R(G)])^2 \geq |C| + 2\sum_{i=1}^{n}(x_i + y_i)c_{v_i}^2$$

$$EA_C[M_R(G)] \geq \sqrt{(|C| + \beta)}.$$

■

Theorem III.2. *Let G be a graph on n vertices. Then $EA_C[M_R(G)] \geq \sqrt{|C| + \beta + n(n-1)P^{\frac{2}{n}}}$, where $P = |M_{R_p}(G \oplus S)|$.*

Proof: Using arithmetic and geometric mean inequality,

$$\frac{1}{n(n-1)}\sum_{i\neq j}|\lambda_i||\lambda_j| \geq \left(\prod_{i\neq j}|\lambda_i||\lambda_j|\right)^{\dfrac{1}{n(n-1)}}$$
$$= \left(\prod_{i=1}^{n}|\lambda_i|^{2(n-1)}\right)^{\dfrac{1}{n(n-1)}}$$
$$= \left(\prod_{i=1}^{n}|\lambda_i|\right)^{\dfrac{2}{n}}$$
$$= P^{\frac{2}{n}},$$

where $P = |A_C[M_R(G)]|$.

$$\sum_{i\neq j}|\lambda_i||\lambda_j| \geq n(n-1)P^{\frac{2}{n}}.$$

Now,

$$(EA_C[M_R(G)])^2 = \left(\sum_{i=1}^{n}|\lambda_i|\right)^2$$

$$(EA_C[M_R(G)])^2 = \sum_{i=1}^{n}|\lambda_i|^2 + \sum_{i\neq j}|\lambda_i||\lambda_j|$$

$$EA_C[M_R(G)] \geq \sqrt{|C| + \beta + n(n-1)P^{\frac{2}{n}}}.$$

■

Theorem III.3. *Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ represent minimum covering maximum reverse degree eigenvalues of G. Then $EA_C[M_R(G)] \leq |\lambda_1| + \sqrt{(n-1)(|C| + \beta - |\lambda_1|^2)}$.*

Proof: Applying Cauchy-Schwartz inequality for $n-1$ terms,

$$\left(\sum_{i=2}^{n} \lambda_i\right)^2 \leq \left(\sum_{i=2}^{n} 1\right)\left(\sum_{i=2}^{n} \lambda_i^2\right).$$

$$(EA_C[M_R(G)] - |\lambda_1|)^2 \leq (n-1)(|C| + \beta - |\lambda_1|^2)$$

$$(EA_C[M_R(G)] - |\lambda_1|) \leq \sqrt{(n-1)(|C| + \beta - |\lambda_1|^2)}$$

$$EA_C[M_R(G)] \leq |\lambda_1| + \sqrt{(n-1)(|C| + \beta - |\lambda_1|^2)}.$$

■

Theorem III.4. *Let $G = (V, E)$ be a graph and $\rho(G) = \max_{1\leq i\leq n} |\lambda_i|$ be the minimum covering maximum reverse degree spectral radius of G. Then*

$$\sqrt{\frac{|C| + \beta}{n}} \leq \rho(G) \leq \sqrt{|C| + \beta}.$$

Proof: Consider,

$$\begin{aligned}
\rho^2(G) &= \max_{1\leq i\leq n} \{|\lambda_i|\} \\
&\leq \sum_{j=1}^{n} \lambda_j^2 \\
&= |C| + 2\sum_{i=1}^{n}(x_i + y_i)c_{v_i}^2 \\
\rho(G) &\leq \sqrt{|C| + \beta},
\end{aligned}$$

where $\beta = 2\sum_{i=1}^{n}(x_i + y_i)c_{v_i}^2$.

Next,

$$\begin{aligned}
n\rho^2(G) &\geq \max_{1\leq i\leq n} \{|\lambda_i|\} \\
&\geq |C| + \beta \\
\rho(G) &\geq \sqrt{\frac{|C| + \beta}{n}}
\end{aligned}$$

$$\sqrt{\frac{|C| + \beta}{n}} \leq \rho(G) \leq \sqrt{|C| + \beta}.$$

■

IV. Minimum covering maximum reverse degree energy of some families of graphs

Theorem IV.1. *Minimum covering maximum reverse degree energy of K_n is given by,*

$$EA_C[M_R(K_n)] = \sqrt{(n-1)^2 + 4(n-1)}.$$

Proof: Let K_n be complete graph of order n and $C = \{1, 2, \ldots, n-1\}$. Then,

$$A_C[M_R(K_n)] = \begin{bmatrix} J_{n-1} & J_{n-1\times 1} \\ J_{1\times n-1} & 0_1 \end{bmatrix}_n,$$

where J is matrix of all 1's, is the minimum covering maximum reverse degree matrix of K_n. The result is proved by showing $A_C[M_R(K_n)]Z = \lambda Z$ for certain vector Z and by making use of fact that the geometric multiplicity and algebraic multiplicity of each eigenvalue λ is same, as $A_C[M_R(K_n)]$ is real and symmetric.

Let $Z = \begin{bmatrix} X \\ Y \end{bmatrix}$ be an eigenvector of order n partitioned conformally with $A_C[M_R(K_n)]$.

Consider,

$$[A_C[M_R(K_n)] - \lambda I]\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} [J - \lambda I]X + JY \\ JX + [-\lambda I]Y \end{bmatrix}_n. \quad (1)$$

Case 1. Let $X = X_j = e_1 - e_j$, $j = 2, 3, \ldots, n-1$ and $Y = 0_1$. Using equation (1), $[J - \lambda I]X_j + J(0) = -\lambda X_j$, then $\lambda = 0$ is the eigenvalue with multiplicity of at least $n-2$ since there are $n-2$ independent vectors of the form X_j.

Case 2. Let $X = 1_{n-1}$ and $Y = (\lambda - (n-1))\, 1_1$, where λ is any root of the equation,

$$\lambda^2 - (n-1)\lambda - (n-1) = 0.$$

From equation (1),

$$\begin{aligned}
&(J)_{1\times(n-1)} 1_{(n-1)} - \lambda I\, (\lambda - (n-1))\, 1_1 \\
&= (n-1)1_1 - \lambda\, (\lambda - (n-1))\, 1_1 \\
&= \{(n-1) - \lambda^2 + \lambda(n-1)\}1_1 \\
&= \{\lambda^2 - \lambda(n-1) - (n-1)\}1_1
\end{aligned}$$

So,

$$\lambda = \frac{(n-1) + \sqrt{(n-1)^2 + 4(n-1)}}{2}$$

and

$$\lambda = \frac{(n-1) - \sqrt{(n-1)^2 + 4(n-1)}}{2}$$

are the eigenvalues with multiplicity of at least one.

The spectrum of $A_C[M_R(K_n)]$ is given by,

$$\begin{pmatrix} 0 & \lambda_1 & \lambda_2 \\ n-2 & 1 & 1 \end{pmatrix},$$

where $\lambda_1 = \frac{(n-1)+\sqrt{(n-1)^2+4(n-1)}}{2}$,
$\lambda_2 = \frac{(n-1)-\sqrt{(n-1)^2+4(n-1)}}{2}$.

Therefore,

$$EA_C[M_R(K_n)] = \sqrt{(n-1)^2 + 4(n-1)}.$$

■

Theorem IV.2. *Minimum covering maximum reverse degree energy of complete bipartite graph is given by, $EA_C[M_R(K_{m,n})] = (m-1) + \sqrt{1 + 4mn(n-m+1)^2}$.*

Proof: Let $K_{m,n}$ be complete bipartite graph of order $m+n$ with $m < n$, then $C = \{1, 2, \ldots, m\}$. Then,

$$A_C[M_R(K_{m,n})] = \begin{bmatrix} I_m & (n-m+1)_{m\times n} \\ (n-m+1)_{n\times m} & 0_n \end{bmatrix}_{m+n},$$

is the minimum covering maximum reverse degree matrix of $K_{m,n}$. The result is proved by showing $A_C[M_R(K_{m,n})]Z = \lambda Z$ for certain vector Z and by making use of fact that the geometric multiplicity and algebraic multiplicity of each eigenvalue λ is same, as $A_C[M_R(K_{m,n})]$ is real and symmetric.

Let $Z = \begin{bmatrix} X \\ Y \end{bmatrix}$ be an eigenvector of order $m+n$ partitioned conformally with $A_C[M_R(K_{m,n})]$.

Consider,

$$[A_C[M_R(K_{m,n}] - \lambda I] \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} (I-\lambda I)X + (n-m+1)Y \\ (n-m+1)X - \lambda I_n Y \end{bmatrix}_{m+n} \tag{2}$$

Case 1. Let $X = X_j = e_1 - e_j$, $j = 2, 3, \ldots, m$ and $Y = 0_n$. Using equation (2), $[1-\lambda]IX_j + (n-m+1)0_n = (1-\lambda)X_j$, then $\lambda = 1$ is the eigenvalue with multiplicity of at least $m-1$ since there are $m-1$ independent vectors of the form X_j.

Case 2. Let $X = 0_m$ and $Y = Y_j$, $j = 2, 3, \ldots, n$. Using equation (2), $-\lambda I Y_j$, then $\lambda = 0$ is the eigenvalue with multiplicity of at least $n-1$ since there are $n-1$ independent vectors of the form Y_j.

Case 3. Let $X = 1_m$ and $Y = \frac{(n-m+1)m}{\lambda} 1_n$, where λ is any root of the equation,

$$\lambda^2 - \lambda - mn(n-m+1)^2 = 0.$$

From equation (2),

$$\begin{aligned} &(1-\lambda)_m + (n-m+1)_{m\times n} \frac{(n-m+1)m}{\lambda} 1_n \\ &= \left\{ (1-\lambda) + (n-m+1)n \frac{(n-m+1)m}{\lambda} \right\} 1_m \\ &= \frac{\lambda^2 - \lambda - mn(n-m+1)^2}{\lambda} 1_m. \end{aligned}$$

So,
$\lambda = \dfrac{1 + \sqrt{1 + 4mn(n-m+1)^2}}{2}$ and
$\lambda = \dfrac{1 - \sqrt{1 + 4mn(n-m+1)^2}}{2}$ are the eigenvalues with multiplicity of at least one.

Thus, the spectrum of $A_C[M_R(K_{m,n})]$ is given by,

$$\begin{pmatrix} 1 & 0 & \lambda_1 & \lambda_2 \\ m-1 & n-1 & 1 & 1 \end{pmatrix},$$

where $\lambda_1 = \dfrac{1 + \sqrt{1 + 4mn(n-m+1)^2}}{2}$,
$\lambda_2 = \dfrac{1 - \sqrt{1 + 4mn(n-m+1)^2}}{2}$.

Therefore,

$$EA_C[M_R(K_{m,n})] = (m-1) + \sqrt{1 + 4mn(n-m+1)^2}.$$

∎

Corollary IV.3. *Minimum covering maximum reverse degree energy of star graph $K_{1,n-1}$ is*

$$E_p M_R(K_{1,n-1}) = \sqrt{1 + 4(n-1)^3}.$$

Proof:

Let $K_{1,n-1}$ be star graph, then the minimum covering set C consists of the non-pendant vertex. Then, substituting $m = 1$ and $n = n-1$ in theorem (IV.2), we get

$$E_p M_R(K_{1,n-1}) = \sqrt{1 + 4(n-1)^3}.$$

∎

Theorem IV.4. *Minimum covering maximum reverse degree energy of cocktail party graph $K_{n\times 2}$ is given by $EA_C M_R(K_{n\times 2}) = (2n-3) + \sqrt{(3-2n)^2 - 16(1-n)}$.*

Proof: Let $K_{n\times 2}$ be cocktail party graph of order $2n$ and let $C = \{1, 2, \ldots, n-1, n+1, n+2, \ldots, 2n-1\}$. Then,

$$A_C M_R(K_{n\times 2}) = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 0 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & 1 & \cdots & 0 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 & 1 & 1 & 1 & \cdots & 1 & 0 \\ 0 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 1 & \cdots & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 0 & \cdots & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 0 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 & 1 & 1 & 1 & \cdots & 1 & 0 \end{bmatrix}_{2n}$$

$|EA_C M_R(K_{n\times 2}) - \lambda I|$ is given by,

$$\begin{vmatrix} 1-\lambda & 1 & 1 & \cdots & 1 & 1 & 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1-\lambda & 1 & \cdots & 1 & 1 & 1 & 0 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1-\lambda & \cdots & 1 & 1 & 1 & 1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1-\lambda & 1 & 1 & 1 & 1 & \cdots & 0 & 1 \\ 1 & 1 & 1 & \cdots & 1 & -\lambda & 1 & 1 & 1 & \cdots & 1 & 0 \\ 0 & 1 & 1 & \cdots & 1 & 1 & 1-\lambda & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 1 & \cdots & 1 & 1 & 1 & 1-\lambda & 1 & \cdots & 1 & 1 \\ 1 & 1 & 0 & \cdots & 1 & 1 & 1 & 1 & 1-\lambda & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 0 & 1 & 1 & 1 & 1 & \cdots & 1-\lambda & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 & 1 & 1 & 1 & \cdots & 1 & -\lambda \end{vmatrix}_{2n} \tag{3}$$

Step 1: On replacing R_i by $R_i - R_i + 1$, for $i = 1, 2, \ldots, n-1, n+1, \ldots, 2n-1$ and replacing C_i by $C_i + C_{i-1} + \cdots + C_2 + C_1$, for $i = n, n-1, \ldots, 2, 1$ and C_j by $C_j + C_{j-1} + \cdots + C_2 + C_1$, for $j = 2n, 2n-1, \ldots, n+2, n+1$ in equation (3) a new determinant say $det(D)$ is obtained.

Step 3: On multiplying and dividing $C_{n+1}, C_{n+2}, \ldots, C_{2n-1}$ by (λ) and replacing $C_{n+1} \longrightarrow C_{n+1} - C_1, C_{n+2} \longrightarrow C_{n+2} - C_2, \ldots, C_{2n-1} \longrightarrow C_{2n-1} - C_{n-1}$ in $det(D)$ we get a new determinant say, $det(E)$.

Step 4: On expanding $det(E)$ along the rows from R_1 to

R_{n-2} it reduces to,

$$\begin{vmatrix} -\lambda & 1 & 0 & \cdots & 0 & 0 \\ n-1 & (n-1)-\lambda & \lambda-1 & \cdots & (n-1)(\lambda-1) & n-1 \\ 0 & 0 & -\lambda^2+1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & 0 & \cdots & -\lambda^2+1 & 1 \\ (n-1) & (n-1) & \lambda-1 & \cdots & (n-1)(\lambda-1) & (n-1)-\lambda \end{vmatrix}_{n+2} \quad (4)$$

Step 5: Then expanding equation (4) along rows and on simplifying, we get
$\phi\{A_C M_R(K_{n\times 2})\} = \lambda(1-\lambda)^{n-2}(\lambda-1)(\lambda^2+(3-2n)\lambda+4(1-n)) = 0.$

The spectrum of $A_C M_R(K_{n\times 2})$ is given by,

$$\begin{pmatrix} 0 & 1 & -1 & \lambda_1 & \lambda_2 \\ 1 & n-1 & n-2 & 1 & 1 \end{pmatrix},$$

where $\lambda_1 = \dfrac{-(3-2n)+\sqrt{(3-2n)^2-16(1-n)}}{2}$,

$\lambda_2 = \dfrac{-(3-2n)-\sqrt{(3-2n)^2-16(1-n)}}{2}$.

Therefore,

$$EA_C M_R(K_{n\times 2}) = (2n-3)+\sqrt{(3-2n)^2-16(1-n)}.$$

■

Theorem IV.5. *Minimum covering maximum reverse degree energy of crown graph (S_n^0) is $EA_C M_R(S_n^0) = \frac{1+\sqrt{5}}{2}(n-1)+\frac{1-\sqrt{5}}{2}(n-1)+\sqrt{1+4(n-1)^2}$.*

Proof: Let S_n^0 be crown graph of order $2n$ and let $C = \{1,2,\ldots,n\}$. Then,

$$A_C[M_R(S_n^0)] = \begin{bmatrix} I_n & (J-I)_n \\ (J-I)_n & 0_n \end{bmatrix}_{2n},$$

is the minimum covering maximum reverse degree matrix of S_n^0. The result is proved by showing $A_C[M_R(S_n^0)]Z = \lambda Z$ for certain vector Z and by making use of fact that the geometric multiplicity and algebraic multiplicity of each eigenvalue λ is same, as $A_C[M_R(S_n^0)]$ is real and symmetric.

Let $Z = \begin{bmatrix} X \\ Y \end{bmatrix}$ be an eigenvector of order $2n$ partitioned conformally with $A_C[M_R(S_n^0)]$.

Consider,

$$[A_C[M_R(S_n^0)] - \lambda I]\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{vmatrix} (I-\lambda I)X+(J-I)Y \\ (J-I)X-\lambda IY \end{vmatrix}_{2n}. \quad (5)$$

Case 1. Let $X = X_j = e_1 - e_j$, $j = 2,3,\ldots,n$ and $Y = (1-\lambda)X_j$, where λ is any root of the equation,

$$\lambda^2-\lambda-1 = 0.$$

Using equation (5),

$$\begin{aligned} &(J-I)X_j - \lambda I(1-\lambda)X_j \\ &= -X_j - \lambda(1-\lambda)X_j \\ &= (\lambda^2-\lambda-1)X_j. \end{aligned}$$

So, $\lambda = \frac{1+\sqrt{5}}{2}$ and $\lambda = \frac{1-\sqrt{5}}{2}$ are the eigenvalues with multiplicity of at least $n-1$.

Case 2. Let $X = 1_n$ and $Y = \frac{(n-1)^2}{\lambda}1_n$, where λ is any root of the equation,

$$\lambda^2-\lambda-(n-1)^2 = 0.$$

Using equation (5),

$$\begin{aligned} (1-\lambda)I_n 1_n + (J-I)_n\frac{(n-1)^2}{\lambda}1_n &= (1-\lambda)1_n + \frac{(n-1)^2}{\lambda}1_n \\ &= \frac{\lambda^2-\lambda-(n-1)^2}{\lambda}1_n. \end{aligned}$$

So, $\lambda = \dfrac{1+\sqrt{1+4(n-1)^2}}{2}$ and $\lambda = \dfrac{1-\sqrt{1+4(n-1)^2}}{2}$ are the eigenvalues with multiplicity of at least 1.

Thus, the spectrum is given by,

$$\begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} & \lambda_1 & \lambda_2 \\ n-1 & n-1 & 1 & 1 \end{pmatrix}$$

where

$$\lambda_1 = \frac{1+\sqrt{1+4(n-1)^2}}{2},$$

$$\lambda_2 = \frac{1-\sqrt{1+4(n-1)^2}}{2}.$$

Therefore, $EA_C M_R(S_n^0) = \frac{1+\sqrt{5}}{2}(n-1)+\frac{1-\sqrt{5}}{2}(n-1)+\sqrt{1+4(n-1)^2}$. ■

Theorem IV.6. *A double star is denoted by $S(l,m)$. Let $V = \{u_i, v_j | i = 0,1,\ldots,l, j = 0,1,\ldots,m\}$ be the vertex set of the double star $S(l,m)$ with u_0 and v_0 as its centers. Then, characteristic polynomial of $S(l,m)$ is given by,*
$\phi\{M_R(S(l,m)\oplus S)\} = (-\lambda)^{l+m-2}(\lambda^4-2\lambda^3+\lambda^2(1-t^2-m+2m^2-m^3-l+4lm-3lm^2+2l^2-3l^2m-l^3)+\lambda(m-2m^2+m^3+l-4lm+3lm^2-2l^2+3l^2m+l^3)+lm-4lm^2+6lm^3-4lm^4+lm^5-4l^2m+12l^2m^2-12l^2m^3+4l^2m^4+6l^3m-12l^3m^2+6l^3m^3-4l^4m+4l^4m^2+l^5m.$

Proof: Let $S(l,m)$ be double star graph, then $C = \{u_0, v_0\}$. Then,

$$A_C M_R(S(l,m)) = \begin{bmatrix} J_2 & B_{2\times(l+m)} \\ B^T_{(l+m)\times 2} & 0_{(l+m)} \end{bmatrix}_{l+m+2}$$

is the minimum covering maximum reverse degree matrix of $S(l,m)$.

$$|A_C M_R(S(l,m)) - \lambda I| = \begin{vmatrix} (J-\lambda I)_2 B_{2\times(l+m)} \\ B^T_{(l+m)\times 2} - \lambda I_{(l+m)} \end{vmatrix}_{l+m+2} \quad (6)$$

where, B is given by

$$\begin{bmatrix} (l+m-1)J_{I\times l} & 0_{I\times m} \\ 0_{I\times l} & (l+m-1)J_{I\times M} \end{bmatrix}_{2\times(l+m)}$$

On applying row operation $R_i \longrightarrow R_i - R_{i+1}$, $1 \le i \le l-1$, $1 \le j \le m-1$ and column operations $C_i \longrightarrow C_i + C_{i-1} + \ldots + C_1$, $1 \le i \le l$, $1 \le j \le m$, in equation (12) we get

$$(-\lambda)^{l+m-2}\begin{vmatrix} 1-\lambda & t & l(l+m-1) & 0 \\ t & 1-\lambda & 0 & m(l+m-1) \\ l+m-1 & 0 & -\lambda & 0 \\ 0 & l+m-1 & 0 & -\lambda \end{vmatrix} \quad (7)$$

where, $t = max\{C_{u_0}, C_{v_0}\}$

On further simplifying in equation (14), we get $\phi\{M_R(S(l,m) \oplus S)\} = (-\lambda)^{l+m-2}(\lambda^4 - 2\lambda^3 + \lambda^2(1 - t^2 - m + 2m^2 - m^3 - l + 4lm - 3lm^2 + 2l^2 - 3l^2m - l^3) + \lambda(m - 2m^2 + m^3 + l - 4lm + 3lm^2 - 2l^2 + 3l^2m + l^3) + lm - 4lm^2 + 6lm^3 - 4lm^4 + lm^5 - 4l^2m + 12l^2m^2 - 12l^2m^3 + 4l^2m^4 + 6l^3m - 12l^3m^2 + 6l^3m^3 - 4l^4m + 4l^4m^2 + l^5m.$ ∎

Theorem IV.7. *Let B_n^3 be triangular book graph and let $C = \{v_1, v_2\}$, where v_1v_2 is the base of n triangles, then $EA_CM_R(B_n^3) = \sqrt{1+2n^3}$.*

Proof: Let

$$A_CM_R(B_n^3) = \begin{bmatrix} 0_1 & nJ_{1\times 2} & 0_{1\times(n-1)} \\ nJ_{2\times 1} & J_2 & nJ_{2\times(n-1)} \\ 0(n-1)\times 1 & nJ_{(n-1)\times 2} & 0_{n-1} \end{bmatrix}_{n+2}$$

be the minimum covering maximum reverse degree matrix of B_n^3. The result is proved by showing $A_C[M_R(B_n^3)]W = \lambda Z$ for certain vector W and by making use of fact that the geometric multiplicity and algebraic multiplicity of each eigenvalue λ is same, as $A_C[M_R(B_n^3)]$ is real and symmetric.

Let $W = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$ be an eigenvector of order $n+2$ partitioned conformally with $A_C[M_R(B_n^3)]$. Consider,

$$\left[A_CM_R(B_n^3) - \lambda I\right]\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} -\lambda IX + nJY + 0Z \\ nJX + (J-\lambda I)Y + (nI)Z \\ 0X + (nJ)Y - \lambda IZ \end{bmatrix}. \quad (8)$$

Case 1. Let $X = X_j, j = 1,2,\ldots,n$, $Y = 0_2$ and $Z = 0_{n-1}$. Using equation (8), $[-\lambda I]X_j + [nJ]0_2$, then $\lambda = 0$ is an eigenvalue with multiplicity of at least n.

Case 2. Let $X = 0_1$ and $Y = 1_2$ and $Z = \frac{\lambda-2}{n^2}$ where λ is any root of the equation

$$\lambda^2 - 2\lambda - 2n^3 = 0.$$

From equation (8),

$$\begin{aligned}(nJ)Y - \lambda IZ &= nJ1_2 - \lambda I\frac{\lambda-2}{n^2} \\ &= \left\{2n - \lambda\frac{\lambda-2}{n^2}\right\}1_{n-1} \\ &= \frac{\lambda^2 - 2\lambda - 2n^3}{n^2}\end{aligned}$$

So, $\lambda = 1 + \sqrt{1+2n^3}$ and $\lambda = 1 - \sqrt{1+2n^3}$ are the eigenvalues with multiplicity of at least one.

Thus, spectrum of $A_CM_RB_n^3$ is $\begin{pmatrix} 0 & \lambda_1 & \lambda_2 \\ n & 1 & 1 \end{pmatrix}$, where $\lambda_1 = 1 + \sqrt{1+2n^3}$, $\lambda_2 = 1 + \sqrt{1+2n^3}$.

Therefore,

$$EA_CM_R(B_n^3) = \sqrt{1+2n^3}.$$

∎

Theorem IV.8. *Let $Amal(k, K_n)$ be the k times amalgamation of complete graph K_n. If $|C| = (n-1)(k-1)+1$, then $\phi_p\{A_CM_R(Amal(k,K_n))\} = (1-\lambda-C)^{k(n-3)}\{\lambda^2 + \lambda(3C-1-Cn) - C^2n + 2C^2\}^{k-1}\{-\lambda^3 + \lambda^2(Cn+2-3C) + \lambda(3C-1-2C^2-C^2k+C^2n-Cn+C^2kn) - C^2k - C^2n + 2C^2 + C^3kn\}$.*

Proof: Let

$$A_CM_R(Amal(k,K_n))$$

$$= \begin{bmatrix} J_1 & CJ_{1\times n-1} & CJ_{1\times n-1} & \cdots & CJ_{1\times n-1} \\ CJ_{1\times n-1} & B_{n-1} & 0_{n-1} & \cdots & 0_{n-1} \\ CJ_{1\times n-1} & 0_{n-1} & B_{n-1} & \cdots & 0_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ CJ_{1\times n-1} & 0_{n-1} & 0_{n-1} & \cdots & B_{n-1} \end{bmatrix}_{k+1} \quad (9)$$

be the minimum covering maximum reverse degree matrix of $Amal(k,K_n)$, where $C = (n-1)(k-1)+1$ and

$$B = \begin{bmatrix} 0 & C & C & \cdots & C & C \\ C & 1 & C & \cdots & C & C \\ C & C & 1 & \cdots & C & C \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ C & C & C & \cdots & 1 & C \\ C & C & C & \cdots & C & 1 \end{bmatrix}_{n-1}.$$

$$|A_CM_R(Amal(k,K_n)) - \lambda I| =$$

$$\begin{vmatrix} 1-\lambda & CJ_{1\times n-1} & CJ_{1\times n-1} & \cdots & CJ_{1\times n-1} \\ CJ_{1\times n-1} & B_{n-1}-\lambda I & 0_{n-1} & \cdots & 0_{n-1} \\ CJ_{1\times n-1} & 0_{n-1} & B_{n-1}-\lambda I & \cdots & 0_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ CJ_{1\times n-1} & 0_{n-1} & 0_{n-1} & \cdots & B_{n-1}-\lambda I \end{vmatrix}_{k+1} \quad (10)$$

On replacing R_i by $R_i - R_{i+1}$ for $i = 2,\ldots,k+1$ and replacing C_i by $C_i + C_{i-1} + \cdots + C_2$ for $i = k+1, k, \ldots, 3, 2$ in (10), a new determinant say $det(D)$ is obtained.

$$det(D) = |(B-\lambda I)_{n-1})|^{(k-1)}\begin{vmatrix} 1-\lambda & CkJ \\ CJ & B-\lambda I \end{vmatrix}_n \quad (11)$$

Consider,

$$|(B-\lambda I)_{n-1})| = \begin{vmatrix} -\lambda & C & C & \cdots & C \\ C & 1-\lambda & C & \cdots & C \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C & C & C & \cdots & 1-\lambda \end{vmatrix}_{n-1} \quad (12)$$

On replacing R_i by $R_i - R_{i+1}$ for $i = 2,\ldots,n-1$ and replacing C_i by $C_i + C_{i-1} + \cdots + C_2 + C_1$ for $i = 2,\ldots,n-1$ in (12), we have

$$\{(1-\lambda-C)^{n-3}(\lambda^2 + \lambda(3C-1-Cn) - C^2n + 2C^2)\}^{k-1} \quad (13)$$

Next,

$$\begin{vmatrix} 1-\lambda & CkJ \\ CJ & B-\lambda I \end{vmatrix}_n = \begin{vmatrix} 1-\lambda & Ck & Ck & \cdots & Ck \\ C & -\lambda & C & \cdots & C \\ C & C & 1-\lambda & \cdots & C \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C & C & C & \cdots & 1-\lambda \end{vmatrix}_n \quad (14)$$

On replacing R_i by $R_i - R_{i+1}$ for $i = 2, \ldots, n$ and replacing C_i by $C_i + C_{i-1} + \cdots + C_2$, $i = 2, \ldots, n$ in (14), we have

$$\{(1-\lambda-C)^{n-3}\}\{\lambda^3 - \lambda^2(Cn+2-3C) - \lambda(3C-1-2C^2 -C^2k + C^2n - Cn + C^2kn) + C^2k + C^2n - 2C^2 - C^3kn\}. \tag{15}$$

Substituting (13) and (15) in (11), we obtain $\phi_p\{A_C M_R(Amal(k, K_n)\} = (1-\lambda-C)^{k(n-3)}\{\lambda^2 + \lambda(3C-1-Cn) - C^2n + 2C^2)\}^{k-1}\{-\lambda^3 + \lambda^2(Cn+2-3C) + \lambda(3C-1-2C^2 - C^2k + C^2n - Cn + C^2kn) - C^2k - C^2n + 2C^2 + C^3kn\}$. ■

V. Conclusion

Graph energies found unexpected applications in such areas of science and engineering as crystallography, air transportation, comparison of protein sequences, construction of spacecrafts, etc. In this paper, we have introduced minimum covering maximum reverse degree energy and obtained some bounds for minimum covering maximum reverse degree energy graphs. Also, a generalized expression for minimum covering maximum reverse degree energy of complete graph, star, cocktail party graph, crown graph, complete bipartite graph, double star graph, triangular book graph and amalgamation of complete graph are also computed.

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