# Properties of $n$- Independent Sets and $n$-Complete Sets of a Graph

Surekha Ravishankar Bhat, Ravishankar Bhat and Smitha Ganesh Bhat*

***Abstract*—The open neighborhood $N(w)$ of a vertex $w \in V$ is the set of all vertices that share an edge with $w$ in an undirected graph. In other words, $N(w)$ comprises all vertices that are directly connected to $w$ by an edge, excluding $w$ itself. The enclave of a vertex $w \in V$, denoted as $E(w)$, is the set of all vertices that are reachable from $w$ through a series of adjacent edges, including $w$ itself. In other words, $E(w)$ includes all vertices that can be reached from $w$ by following a path along the edges of the graph. The closed neighborhood of $w$ is known as enclave of $w$. We note that a vertex $w \in V$, $n$-covers an edge $x \in X$ if $x \in \langle N[v] \rangle$, the subgraph induced by the set $N[v]$. The $n$-covering number $\rho_n(H)$ introduced by Sampathkumar and Neeralagi [16] is the minimum number of vertices that $n$-cover all the edges of $H$. A set $S \subseteq V$ is said to be $n$-independent if every edge $x \in \langle S \rangle$ is $n$-covered by a vertex in $V - S$. On the other hand, $S$ is $n$-complete if, for every pair of nonadjacent vertices $u, v \in S$ there exists a vertex $w \in V - S$ such that $\{u, v, w\}$ is independent. The $n$-independence ($n$-complete) number $\alpha_N(H)(\omega_N(H))$ is the maximum order of $n$-independent ($n$-complete) set of $H$. In this paper, a Gallai's theorem type result $\rho_n(H) + \alpha_N(H) = p$ is proved. In addition to getting several bounds on $n$-independence number, we show that $\alpha_N(H) = \omega_N(\overline{H})$ and the chromatic number $\chi(H) \leq \alpha_N(H) + 1$.**

***Index Terms*—$n$-coverings, $n$-independence number, $n$-complete number, $n$-chromatic number and $n$-complete partition number.**

## I. Introduction

In cases where terminologies are not clearly defined, we can consult the references provided in [1], [21]. A graph $H$ in this context refers to a connected finite simple graph characterized by $p$ vertices and $q$ edges. The lower vertex covering number, denoted as $\beta(H)$, represents the minimum number of vertices needed to encompass all the edges within graph $H$. Conversely, the upper vertex covering number, $\Lambda(H)$, indicates the maximum count of vertices needed to cover all edges present in graph $H$. The upper independence number $\alpha(H)$ is the maximum number of vertices in any independent set of graph $H$, while the lower independence number $i(H)$ is the minimum number of vertices in any independent set of graph $H$ (see [8], [14]). The edge analogue of above parameters, edge covering number $\beta_1(H)$ and the matching number $\alpha_1(H)$ are similarly defined. The above parameters are related by classical theorem now known as Gallai's theorem, stated as for any graph $H$, $\alpha(H) + \beta(H) = i(H) + \Lambda(H) = \alpha_1(H) + \beta_1(H) = p$. A set $S$ is a dominating set of graph $H$ if, for every vertex $v$ in the graph $H$, either $v$ is in $S$ or there exists a vertex $w$ in $S$ such that there is an edge between $v$ and $w$ in the graph $H$. The lower domination number $\gamma(H)$ is the smallest number of vertices in any minimal dominating set of graph $H$, while the upper domination number $\Gamma(H)$ is the largest number of vertices in any minimal dominating set of graph $H$. The open neighborhood $N(w)$ of a vertex $w \in V$ is the set of all vertices that share an edge with $w$ in an undirected graph. In other words, $N(w)$ comprises all vertices that are directly connected to $w$ by an edge, excluding $w$ itself. The enclave of a vertex $w \in V$, denoted as $E(w)$, is the set of all vertices that are reachable from $w$ through a series of adjacent edges, including $w$ itself. In other words, $E(w)$ includes all vertices that can be reached from $w$ by following a path along the edges of the graph. The closed neighborhood of $w$ is also known as enclave of $w$. A set $S \subseteq V$ is called enclaveless (EL-set) (as defined by Slater [17]) if every vertex in $S$ has all of its neighbors within $S$. The upper enclaveless number $\Psi(H)$ is the maximum order of a maximal enclaveless set in graph $H$, while the lower enclaveless number $\psi(H)$ is the minimum order of a maximal enclaveless set in graph $H$. Furthermore, it is notable that the sum of the lower domination number and the upper enclaveless number, $\gamma(H) + \Psi(H)$, as well as the sum of the upper domination number and the lower enclaveless number, $\Gamma(H) + \psi(H)$, are both equal to number of vertices $p$. Similar Gallai-type results can also be found in other works, such as [2], [7], [12], [15], [18]. For an extensive exploration of domination parameters, please refer to the comprehensive investigations carried out in [6], [10]. Numerous compelling applications of these maximum and minimum problems arise in the fields of linear programming (LP) and integer programming. For a thorough discussion, please refer Chapter 1 in [9].

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## II. $n$-Independent Sets

Another graph invariant of great interest is the neighborhood covering ($n$-covering) number, initially introduced and explored by Sampathkumar and Neeralagi [16]. Since its introduction, this concept has captivated numerous researchers, as evidenced by their works in [3], [11], [13], [19], [20]. We define the concept of a vertex $v \in V$ $n$-covers an edge $x \in X$ as follows: $x$ is an edge of the subgraph $\langle N[v] \rangle$, where $\langle N[v] \rangle$ represents the subgraph induced by the set of closed neighbors of vertex $v$. We refer to a set $S$ as an $n$-covering of graph $H$ if the vertices in $S$ collectively $n$-cover all the edges of $H$. In other words, every edge in $H$ is an element of the subgraph induced by the set of neighbors of at least one vertex in $S$. The lower $n$-covering number $\rho_n(H)$

is the smallest size among all minimal $n$-coverings of $H$, while the upper $n$-covering number $\rho_N(H)$ is the largest size among all minimal $n$-coverings of $H$. It is evident that the independence number $\alpha(H)$ is the complementary number to the covering number $\beta(H)$. This observation led us to inquire about the complementary number to the $n$-covering number. In response to this query, we introduced the concept of $n$-independent sets, where the vertices may be adjacent under specific conditions.

A set $D \subseteq V$ is said to be $n$-independent if every edge $x \in \langle D \rangle$ is $n$-covered by a vertex in $V-D$. The upper (lower) $n$-independence number $\alpha_N(H)(\alpha_n(H))$ is the maximum (minimum) order of a maximal $n$-independent set of $H$. For example, let the vertices of the cycle $C_5$ be labeled successively by $v_1, v_2, v_3, v_4, v_5$. Then $D = \{v_1, v_3\}$ is both maximum and minimum $n$-independent set and $\{v_1, v_3, v_4\}$ is both maximum and minimum $n$-covering. Therefore $\alpha_N(C_5) = \alpha_n(C_5) = 2 < 3 = \rho_n(C_5) = \rho_N(C_5)$. In fact, for any cycle $C_p$ with $p$ vertices $\alpha_N(C_p) = \alpha_n(C_p) = \left\lfloor \frac{p}{2} \right\rfloor$.

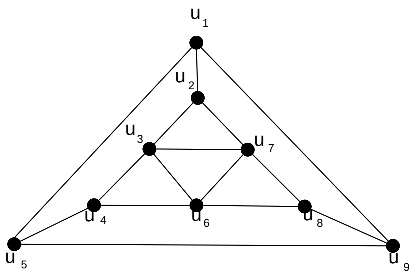


Fig. 1. A graph H with $\alpha(H) = 3 < 5 = \alpha_N(H) < 7 = \Psi(H)$ and $i(H) = 3 < 4 = \alpha_n(H) < 6 = \psi(H)$

Every independent set is a $n$-independent set of $H$ but not conversely. For any complete graph $K_p$, $\alpha_N(K_p) = p - 1$. As another example, for the graph $H$ in Fig. 1, $\alpha_N(H) = 5$ and $D = \{u_1, u_3, u_5, u_6, u_8\}$ is a maximum $n$-independent set of $H$ which is not independent. Observe that $V - D$ is a $\rho_n$-set of $H$. Again, $\alpha_n(H) = 4$ and $D_1 = \{u_2, u_3, u_4, u_8\}$ is a minimum $n$-independent set of $H$. Also $V - D_1$ is a $\rho_N$-set of $H$. As every subset of a $n$-independent set is also $n$-independent, the property of $n$-independence is a hereditary property while $n$-covering property is superhereditary. Some more properties of $n$-independent sets are revealed in the next result.

**Theorem II.1.** *The following statements are equivalent.*
*S1: $D \subseteq V$ is a $n$-independent set of H*
*S2: For every $x \in \langle N[v] \rangle, v \in D$ there exists a vertex $w \in V - D$ such that $x \in \langle N[v] \rangle \cap \langle N[w] \rangle$.*
*S3:Any edge $uv = x \in \langle D \rangle$ is an edge of a triangle $\{u, v, w\}$ where $w \in V - D$.*
*S4:V-D is a $n$-covering of H*

*Proof:* To prove $S1 \Rightarrow S2$ : Let $D \subseteq V$ be a $n$-independent set of $H$ and $x \in \langle N[v] \rangle, v \in D$. Then the edge $x$ is of following two types.
Case (i): $x = vw, w \in V - D$: then it is evident that $x \in \langle N[w] \rangle$.
Case (ii): $x = uv$ where $u, v \in D$ : In this case, as $D$ is a $n$-independent set, there exists a $w \in V - D$ such that $x \in \langle N[w] \rangle$. Thus in any case $x \in \langle N[v] \rangle \cap \langle N[w] \rangle$.
To prove $S2 \Rightarrow S3$ : Suppose $S2$ holds. Consider an edge $uv = x \in \langle D \rangle$. Then both $u, v \in D$. Then as above, $x \in \langle N[w] \rangle$ where $w \in V - D$. Hence $\{u, v, w\}$ is a triangle in $H$.
To prove $S3 \Rightarrow S4$ : Suppose $S3$ holds. If $x = uv$ and $u, v \in V - D$ then $x$ is $n$- covered by $u \in V - D$. If $x = uv$, $u \in D$ and $v \in V - D$. Then $x$ is $n$-covered by $v \in V - D$. Finally, if $x = uv$, $u, v \in D$. Then from $S3$, there exists a $w \in V - D$ such that $\{u, v, w\}$ is a triangle in $H$. Hence $w$, $n$-covers $x$. Thus every edge of $H$ is $n$-covered by a vertex of $V - D$.
To prove $S4 \Rightarrow S1$ : Suppose $V - D$ is a $n$-covering of $H$. Assume that $D$ is not a $n$-independent set of $H$. Then there exists at least one $x \in \langle D \rangle$ which is not $n$-covered by any vertex in $V - D$. But then $V - D$ is not a $n$-covering of $H$ - a contradiction.

■

The corona of two graphs $H_1$ and $H_2$ is the graph $H = H_1 \cdot H_2$ formed from one copy of $H_1$ and $|V(H_1)|$ copies of $H_2$, where $i^{th}$ vertex of $H_1$ is adjacent to every vertex in the $i^{th}$ copy of $H_2$. From $S3$, every edge of $\langle D \rangle$ is in a triangle. Therefore, if $H$ is triangle free, every $n$-independent set is independent. But the converse is not true. For example, the three pendant vertices of the corona $C_3 \cdot K_1$ form a $n$-independent set in which no two vertices are adjacent, but the corona $C_3 \cdot K_1$ contains a triangle.

### *A. Gallai's theorem type results for $n$-independence number*

**Theorem II.2.** *For any graph H with p vertices*

$$\rho_n(H) + \alpha_N(H) = p \quad (1)$$

$$\rho_N(H) + \alpha_n(H) = p \quad (2)$$

*Proof:* Let $D$ be a minimum $n$-covering of $H$. Then by Theorem II.1, $V - D$ is a $n$-independent set of $H$. Hence $\alpha_N(H) \geq |V - D| = p - \rho_n$. To obtain the reverse inequality, we begin with a maximum $n$-independent set $S$ of $H$. Again by Theorem II.1, $V - S$ is a $n$-covering of $H$. Hence $\rho_n \leq |V - S| = p - \alpha_N(H)$. Then the desired result (1) follows. The proof of the result (2) is similar and we omit the proof. ■

**Proposition II.3.** *If a graph H has a $n$-covering which is also a $n$-independent set, then*

$$\rho_n(H) \leq \frac{p}{2}$$

*Proof:* Suppose $D$ be a $n$-covering which is also a n-independent set of $H$. Since $D$ is a $n$-covering of $H$, we have $\rho_n(H) \leq |D|$. Again, since $D$ is a $n$-independent set of $H$, from Theorem II.1, we have $V - D$ is a n-covering of $H$. Then $\rho_n(H) \leq |V - D| = p - |D|$. Adding these two inequalities we get, $2\rho_n(H) \leq p$ which yields the desired bound. ■

### *B. $n$-complete sets, $n$-complete partition and $n$-chromatic number*

The chromatic number $\chi(H)$ of a graph $H$ is the minimum number of colors required to ensure that no two adjacent vertices are assigned the same color in a proper coloring of the vertices of $H$. Any coloring of $H$ results in a partition of the vertex set into independent sets. Thus, the chromatic number $\chi(H)$ can also be defined as the smallest number of independent sets needed to partition the vertex set of graph $H$. A clique is a subset of vertices in $H$ where every vertex is directly connected to every other vertex in the subset, and it cannot be further extended while maintaining the same property. The upper clique number $\omega(H)$ is the largest size among all cliques in graph $H$, while the lower clique number $\vartheta(H)$ is the smallest size among all cliques in graph $H$. The partition number $\theta_0(H)$ is the minimum number of non-maximal cliques required to cover all the vertices of a graph which is introduced by Berge [1]. As independent sets and cliques exchange their characteristics under complementation, we arrive at the following relationship.

$$\chi(\overline{H}) = \theta_0(H); \alpha(\overline{H}) = \omega(H); i(\overline{H}) = \vartheta(H) \quad (3)$$

Motivated from these definitions, we are interested to see what are the properties of $n$-independent sets in complement of $H$. In view of this, we define the following graph parameters. A set $D \subseteq V$ is said to be $n$-complete if, for every pair of nonadjacent vertices $u, v \in D$ there exists a vertex $w \in V - D$ such that $\{u, v, w\}$ is independent. The upper (lower) $n$-complete number $\omega_N(H)(\vartheta_n(H))$ is the maximum (minimum) order of a maximal $n$-complete set of $H$.

We now introduce the concept of the *n-chromatic number*, denoted as $\chi_n(H)$, by considering the smallest number of independent sets into which the vertex set of $H$ can be partitioned. Conversely, we define the *n-complete partition number*, denoted as $\theta_n(H)$, as the minimum order of a partition of the vertex set into sets that are each $n$-complete.

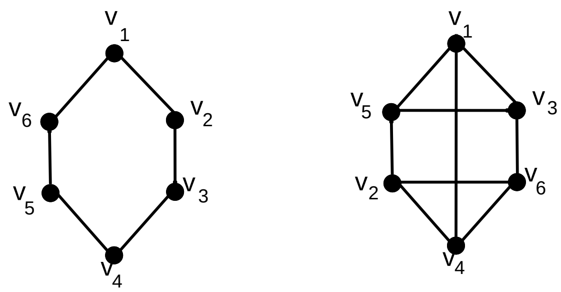


Fig. 2. Graphs $C_6$ and its complement.

We immediately note that every set $S$ which induce a complete subgraph is always $n$-complete. Therefore $\omega(H) \leq \omega_N(H)$. Observe that in Fig. 2, $D = \{v_1, v_2, v_6\}$ is a $\omega_N$-set of $C_6$. For the pair of nonadjacent vertices $v_2, v_6 \in D$ there exists a vertex $v_4 \in V - D$ such that the set $\{v_4, v_2, v_6\}$ is independent. Hence $\omega_N(C_6) = 3$. It is also true that $\{v_1, v_2, v_6\}$ is an $\alpha_N$-set of $\overline{C}_6$.
The partition $P = \{\{v_1, v_2, v_6\}, \{v_3, v_4, v_5\}\}$ is a $n$-complete partition of $C_6$. The same partition $P$ is a $n$-independent partition of $\overline{C}_6$. Hence $\theta_n(C_6) = \chi_n(\overline{C}_6) = 2$. From the above example it is interesting to see that the newly defined parameters also follow the results similar in (3).

**Proposition II.4.** *For any graph H,*

$$\omega_N(H) = \alpha_N(\overline{H})$$

$$\theta_n(H) = \chi_n(\overline{H})$$

*Proof:* Let $D \subseteq V$ be a $n$-complete set of $H$. Then for any pair of non adjacent vertices $u, v \in D$ there exists a vertex $w \in V - D$ such that $\{u, v, w\}$ is independent. This implies, for every edge $x = uv \in \langle D \rangle$ there exists a vertex $w \in V - D$ such that $uvw$ is a triangle in $\overline{H}$. Then by Theorem II.1, $D$ is a $n$-independent set of $\overline{H}$. Therefore $\omega_N(H) = \alpha_N(\overline{H})$.

Since every $n$-complete set is $n$-independent set in $\overline{H}$, we conclude that every $n$-complete partition of $V$ is also $n$-independent partition of $V$ in $\overline{H}$. Therefore $\theta_n(H) = \chi_n(\overline{H})$. ■

## III. Bounds on $n$-independence number

First we show that the new parameter fits best in between the known graph parameters.

**Proposition III.1.** *For any graph H*

$$\alpha(H) \leq \alpha_N(H) \leq \Psi(H) \quad (4)$$

$$i(H) \leq \alpha_n(H) \leq \psi(H) \quad (5)$$

*Proof:* Equations (4) and (5) follow from the fact that every independent set is a $n$-independent set and every $n$-independent set is an EL set. ■

One can observe that for the graph $H$ in Fig. 1, the inequalities (4) and (5) are strict and $\alpha(H) = 3 < 5 = \alpha_N(H) < 7 = \Psi(H)$. Also, $i(H) = 3 < 4 = \alpha_n(H) < 6 = \psi(H)$. We now show under what conditions equality hold in equations (4) and (5).

**Proposition III.2.** *For any graph H, $\alpha(H) = \alpha_N(H)$ if and only if there exists a maximum $n$-independent set D such that exactly one vertex of every triangle in H is in D.*

*Proof:* Let $\alpha(H) = \alpha_N(H)$. Then there exists a maximum $n$-independent set $D$ which is independent. If $H$ is triangle free then there is nothing to prove. So we assume that $H$ is not a triangle free graph. Let $\{u, v, w\}$ be any triangle in $H$. If $\mid D \cap \{u, v, w\} \mid \geq 2$ then $D$ cannot be independent and hence we assume that $\mid D \cap \{u, v, w\} \mid \leq 1$. Now we claim that $D \cap \{u, v, w\} \neq \phi$. If there exists a vertex $y \in D$ and a vertex $z \in \{u, v, w\}$ such that $D_1 = D - \{y\} \cup \{z\}$ is independent, then $D_1$ is the independent set with the required property. If this rearrangement is not possible, then $D \cup \{z\}$ is independent contradicting that $D$ is the maximum independent set of $H$. Thus $\mid D \cap \{u, v, w\} \mid = 1$ as desired. Conversely, suppose there exists a maximum $n$-independent set $D$ satisfying the condition stated in the proposition, then it is immediate that $D$ is a maximum independent set and hence $|D| = \alpha(H) = \alpha_N(H)$. ■

**Proposition III.3.** *For any graph H, $\alpha_N(H) = \Psi(H)$ if and only if there exists a maximum EL set D such that every edge of $\langle D \rangle$ is in a triangle $\{u, v, w\}$.*

*Proof:* Let $\alpha_N(H) = \Psi(H)$. Then there exists a maximum EL-set $D$ which is also a maximum $n$-independent set of $H$. Then from $S2$ of Theorem II.1 every edge of $\langle D \rangle$ is in a triangle $\{u, v, w\}$ and the result follows. Converse is straight forward as in Proposition III.2

■

**Corollary III.3.1.** *For any graph H,*
*(i) $i(H) = \alpha_n(H)$ if and only if there exists a maximal $n$-independent set D of minimum order such that exactly one*

*vertex of every triangle in H is in D.*
*(ii)* $\alpha_n(H) = \psi(H)$ *if and only if there exists a maximal EL set D of minimum order such that every edge of* $\langle D \rangle$ *is in a triangle* $\{u, v, w\}$.

**Proposition III.4.** *For any graph H*

$$\Gamma(H) \leq \alpha_N(H)$$

*Proof:* Let $D$ be a $\Gamma$ set of $H$. If $D$ is a $n$-independent set of $H$ the result $\Gamma(H) \leq \alpha_N(H)$ is straight forward. Suppose $D$ is not a $n$-independent set of $H$. Then we construct a $n$-independent set of $H$ of order $|D|$ as follows. Since $D$ is not a $n$-independent set, there exists at least one edge $uv = x \in \langle D \rangle$ such that $x$ is not $n$-covered by any vertex $v \in V - D$. Let $x_1 = (u_1v_1), x_2 = (u_2v_2), \ldots, x_n = (u_nv_n) \in \langle D \rangle$ be the edges which are not $n$-covered by $V - D$. By minimality of $D$ there exists a $d_1 \in V - D$ such that $N(u_1) \cap D = \{d_1\}$. Then necessarily, $D_1 = D \cup \{d_1\} - \{u_1\}$ $n$-covers all the edges in $\langle N(u_1) \rangle$. If $D_1$ is a $n$-independent set of $H$, we stop and $\Gamma(H) = |D| = |D_1| \leq \alpha_N(H)$. Otherwise, we repeat the above process and suppose, at the $k^{th}$ stage ($k < n$) we get $|D| = |D_k|$ a $n$-independent set of $H$. Thus $\Gamma(H) =| D |=| D_k |\leq \alpha_N(H)$. ■

Cockayne and Mynhardt [4], [5] completely characterized the most popular inequality chain $\gamma(H) \leq i(H) \leq \alpha(H) \leq \Gamma(H)$. We strengthen this inequality chain by appending two more parameters to the upper end. It is interesting to see that six parameters fall in line, extending the inequality chain.

**Corollary III.4.1.** *For any graph H*

$$\gamma(H) \leq i(H) \leq \alpha(H) \leq \Gamma(H) \leq \alpha_N(H) \leq \Psi(H) \quad (6)$$

$$\gamma(H) \leq \rho_n(H) \leq \psi(H) \leq \beta(H) \leq \Lambda(H) \leq \Psi(H) \quad (7)$$

$$\gamma(\overline{H}) \leq \vartheta(H) \leq \omega(H) \leq \Gamma(\overline{H}) \leq \omega_N(H) \leq \Psi(\overline{H}) \quad (8)$$

*Proof:* It is already known that $\gamma(H) \leq i(H) \leq \alpha(H) \leq \Gamma(H)$ (See [5]). Then equation (6) is a consequence of inequalitiy (4) and Proposition III.4. Now, using Theorem III.1 and corresponding Gallai's type results, equation (6) may be written as $p - \Psi(H) \leq p - \Lambda(H) \leq p - \beta(H) \leq p - \psi(H) \leq p - \rho_n(H) \leq p - \gamma(H)$. Then equation (7) follows on simplification. Complementing the equation (6) using Proposition II.4, we get $\gamma(\overline{H}) \leq i(\overline{H}) = \vartheta(H) \leq \alpha(\overline{H}) = \omega(H) \leq \Gamma(\overline{H}) \leq \alpha_N(\overline{H}) = \omega_N(H) \leq \Psi(\overline{H})$ which is the desired equation (8). ■

A lower bound for $\alpha_n(H)$ in terms of minimum degree $\delta(H)$ is obtained in our next result. In what follows by $V_\delta$ we mean the set $\{v \in V \mid deg(v) = \delta\}$. Similarly, $V_\Delta$ is defined.

**Proposition III.5.** *For any graph H with minimum degree* $\delta$, $\alpha_n(H) \geq \delta$. *Further, equality holds if and only if for every vertex* $v \in V_\delta, N(v)$ *is a* $\alpha_n$*-set of H.*

*Proof:* Let $D$ be any $\alpha_n$- set of $H$ and $v$ be vertex of minimum degree $\delta$. Then $N(v)$ is a $n$-independent set of $H$. Since $D$ is a minimum $n$-independent set, $N(v) \subseteq D$. Hence $\alpha_n(H) \geq |N(v)| = \delta$.

If for every vertex $v \in V_\delta$, $N(v)$ is a minimum $n$-independent set of $H$, then it is immediate that $\alpha_n(H) = |N(v)| = \delta$. Conversely, let $\alpha_n(H) = \delta$. Suppose the contrary that $N(v)$ is not a minimum $n$-independent set of $H$ for some $v \in V_\delta$. Then there exists at least one $v \in V - N(v)$ such that $N(v) \cup \{u\}$ is a $n$-independent set of $H$. Therefor $\alpha_n(H) \geq| N(v) \cup \{u\} |= \delta + 1$- a contradiction. ■

**Corollary III.5.1.** *For any graph H with maximum degree* $\Delta$, $\alpha_N(H) \geq \Delta$
*Further, equality holds if and only if for every vertex* $v \in V_\Delta, N(v)$ *is a maximum n-independent set of H.*

Any complete bipartite graph $K_{m,n}$ and complete graph $K_n$ attain the bounds in Proposition III.5 and Corollary III.5.1

The next result provides bounds for chromatic number in terms of $n$-independence number of $H$.

**Proposition III.6.** *For any graph H with chromatic number* $\chi(H)$,

$$\chi_n(H) \leq \chi(H) \leq \alpha_N(H) + 1 \quad (9)$$

$$\theta_n(H) \leq \theta_0(H) \leq \omega_N(H) + 1 \quad (10)$$

*Proof:* It is well known that $\chi(H) \leq 1 + \Delta$ (see [21]). From Corollary III.5.1, $\Delta \leq \alpha_N(H)$. Then we have $\chi(H) \leq 1 + \Delta \leq 1 + \alpha_N(H)$. Since any partition of vertex set in to independent sets is also a $n$-independent partition, we have $\chi_n(H) \leq \chi(H)$. Complementing the equation (9) and using Proposition II.4 we get equation (10). The bounds in the Proposition III.6 are sharp is evident from the fact that for any complete graph $K_p$, $\chi(K_p) = p = 1 + (p - 1) = 1 + \alpha_N(H)$. For any even cycle, $\chi_n(C_p) = \chi(C_p)$. ■

A graph $H$ is called a block graph if every block of $H$ is a clique of $H$. From the above theorem, we note that every block graph which is $k$-clique regular and triangle free graphs attain the bound in the theorem.

For any two vertices $u, v \in V$ the distance $d(u, v)$ is the length of shortest path between $u$ and $v$. The diameter $d(H) = max_{u,v \in V} d(u, v)$. Brigham *et al.* [3] proved that $\rho_n(H) \geq \frac{d(H)}{2}$. Therefore $\frac{d(H)}{2} \leq \rho_n(H) \leq \rho_N(H)$. This lower bound for $\rho_N(H)$ is improved by 1, in our next proposition.

**Proposition III.7.** *For any graph H with diameter d(H),*

$$\frac{d(H) + 2}{2} \leq \rho_N(H) \quad (11)$$

*Further, the bound is sharp.*

*Proof:* Let $S$ be a maximum $n$-covering of $H$. Consider an arbitrary path of length $d(H)$ and let $u$ and $v$ be the end vertices of the diametral path. This diametral path includes at most two edges from the induced subgraph $\langle N[w] \rangle$ for each $w \in S - \{u, v\}$. The vertices $u$ and $v$ contribute at most one edge each from $\langle N[u] \rangle$ and $\langle N[v] \rangle$. Hence $d(H) \leq 2\rho_N(H) - 2$ which yields the desired bound. It is not hard to see that any path $P_n$ on $n$ vertices attains the bound. ■

Using Theorem II.2 we get the following

**Corollary III.7.1.** *For any graph H*

$$\alpha_n(H) \leq \frac{2p - d(H) - 2}{2}$$

$$\alpha_N(H) \leq \frac{2p - d(H)}{2}$$

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