Connectedness and Compactness in Soft L-topological Spaces

Sandhya S Pai, Baiju Thankachan*

Abstract—**A soft set is a mapping from a parameter to a universe's power set. Molodstov suggested soft sets as a method for simulating ambiguous situations. Sandhya and Baiju describe soft L-topological spaces over a soft lattice L with a definite set of parameters M, and they have also looked into the continuity of soft L-topological space mappings. Soft L-T_i-space $(i = 0; 1; 2; 2\frac{1}{2}; 3; 3\frac{1}{2}; 4; 5; 6)$ on a soft L-topological space was explored with some of their properties by the same authors. In this paper, we introduce the concept of connectedness and compactness in soft L-topological spaces. We also analyze how soft L-connected spaces behave when subjected to soft L-continuous mappings, soft L-boundary, and soft L-closure.**

Index Terms—**Soft L-set, soft L-topology, soft L-connected space, soft L-compact space.**

I. Introduction

Molodtsov [15] was the first to describe soft set theory in 1999. The approach to modeling, ambiguity and uncertainty is altogether novel. Molodtsov [15] demonstrated several applications of soft set theory in various fields. Maji et.al [13], [19] conducted a study on Molodtsov soft sets, wherein they established definitions that revolved around the equivalence of two soft sets, soft set inclusion and containment, the soft set complement, the null soft set, and the absolute soft set. They complemented these definitions with illustrative examples and explanations of fundamental properties. Additionally, various researchers have explored the algebraic aspects of set theory in managing uncertain situations [2], [3], [4], [10], [11], [28], [16], [6], [21], [1], [32], [14], [33], [34]. In 2010, Li F [12] expanded the definition of a soft set to include soft lattices and soft fuzzy sets. Shabir and Naz [29] introduced the concept of soft topological spaces in the year 2011 and studied some basic properties. Sandhya and Baiju [23] used the notion of a soft set initiated by Molodtsov [15] and extended this idea to the field of soft lattices and obtained the topological properties of soft lattices. In 2016, Cigdem Gunduz Aras, Ayse Sonmez, and Huseyin Cakalli [5] presented soft continuous mappings. They also looked at some of its properties. The notion of continuity for soft mappings was provided in 2012 by Hazra, Majumdar, and Samanta [7]; numerous other authors [9], [18] have also explored soft mappings. Furthermore, in 2015, Yang et al. [8] first looked at the concept of soft continuous mapping between two soft topological spaces. In 2015, Tantawy et al. [31] established and studied the separation axioms T_i $(i = 0; 1; 2; 3; 4; 5)$ on a soft L-topological space. They clarified that these axioms are soft topological properties under certain soft mapping. Soft separation axioms are also studied in [30]. Ramkumar et al. [20] researched soft Urysohn space.

One of the most important topological characteristics for distinguishing topological spaces is connectedness. The concept of compactness was introduced into topology to generalize the characteristics of closed and bounded subsets of Euclidean space. In 2012, Peyghan et al. [17] introduced and researched soft connected topological spaces. Zorlutuna et al. [34] initially introduced soft compactness in 2012.

Sandhya and Baiju [23] propose Soft Lattice topological spaces (Soft L-topological spaces or Soft L-space) over an initial universe X with a predefined set of parameters M. They elucidated key attributes of soft L-topological spaces while providing explanations for what constitute soft L-open and soft L-closed sets. Furthermore, they introduced a generalized concept, the soft L-closure of a soft lattice, as an extension of set closure. In the context of parameterized topologies within an initial universe, the role of parameters was highlighted. Each parameter was assigned its individual topological space, underscoring their significance in the overall framework. The authors [23] established that a soft L-topological space generates a parameterized family of topologies in the initial universe, although the reverse may not hold true. This suggests that constructing a soft L-topological space is not feasible if specific topologies are given for each parameter. Sandhya and Baiju [24] defined properties related to the continuity of soft L-continuous mappings, encompassing aspects such as injectivity, surjectivity, bijectivity, and the composition of soft L-mappings. These mappings maintained a stable set of parameters across the initial universe. Intriguing findings included the exploration of soft open and closed L-mappings, soft L-homeomorphism, and various other concepts. Additionally, they examined soft L-continuous mappings between two soft L-topological spaces, the Cartesian product of soft L-topological spaces [25], and Soft L-separation axioms i.e., soft L-T_i-spaces $(i = 0; 1; 2; 2\frac{1}{2}; 3; 3\frac{1}{2}; 4; 5; 6)$ for soft L-topological spaces [26], [27]. Also, the requirement for a soft L-topological space to be a soft L-Ti-spaces $(i = 0; 1; 2; 2\frac{1}{2}; 3; 3\frac{1}{2}; 4; 5; 6)$ was proved. Soft L-invariant properties such as soft L-hereditary and soft L-topological properties were investigated in [26] and [27].

The concept and characteristics of soft L-connectedness and compactness in soft L-topological spaces are discussed in this paper. We also examine the behavior of soft L-connected spaces when soft L-continuous mappings, soft L-boundary and soft L-closure are applied. To demonstrate that several key findings in general topology do not apply to soft L-topological spaces, such as every compact Hausdorff space

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does not have to be normal, we construct a specific soft L-topological space.

II. Preliminaries and basic definitions

If we assume the consistency of L, we will consistently regard L as a complete lattice in the context of this investigation.

We define a unary operation denoted as $\prime\!: L \longrightarrow L$ as a quasi complementation, provided it exhibits two key properties: first, it is an involution, which means that for all elements $\alpha \in L$, we have $\alpha^{''} = \alpha$; second, it reverses the ordering, meaning that if $\alpha \leq \beta$, then $\beta' \leq \alpha'$.

Definition 1: [15] "Let's consider that we have an initial universe set represented by X, and a set of parameters represented by M. We denote the power set of X as $\wp(X)$, and if we have a subset A contained within M, we can define a pair (F, A) as a soft set over X. In this context, the mapping F can be described as $F : A \rightarrow \wp(X)$.
In simpler terms, a soft set over X can be thought of as a collection of subsets of the universe X, and each subset is associated with a parameter from the set A. To be precise, for any element $a \in A$, the collection of approximate elements of the soft set (F, A) is denoted as $F(a)$."

Definition 2: [12] "Let's consider a triple $P = (f, X, L)$, where:
(i)L represents a complete lattice, (ii)$f : X \longrightarrow \wp(L)$ is a mapping, (iii)X denotes a universe set. For each element x in the set X, we can define f^L_M as a soft lattice over L if the image of $f(x)$ under f is a sublattice of L.

In such a context, we refer to the triple $P = (f, X, L)$ as the soft lattice denoted by f^L_M."

Definition 3: [23] Consider an initial universe set denoted as X and a non-empty set of parameters denoted as M.

Let T be defined as the collection of complete and uniquely complemented soft lattices over L. This collection satisfies the following conditions:

(i) If ϕ and L belong to T. (ii) If you take the arbitrary union of soft lattices from T, the result also belongs to T. (iii) If you take the finite intersection of soft lattices from T, the result is a member of T. In such a case, we refer to T as a soft lattice topology on L, and the 3-tuple (L, T, M) is termed a soft lattice topological space, alternatively known as a soft topological lattice space or a soft L-space, defined over L.

Example 1: [23] Let $L = \{0, 1, l_1, l_2, l_3\}$ be the lattice where l_1, l_2, l_3 represents the students of class 12, $m = \{m_1, m_2\}$ be the parameter in which m_1: brilliant and m_2: average.
Let us consider a collection $T = \{\phi, L, f^L_{1M}, f^L_{2M}, f^L_{3M}, f^L_{4M}\}$, where $f^L_{1M}, f^L_{2M}, f^L_{3M}, f^L_{4M}$ are soft lattices over L in which f_1, f_2, f_3, f_4 represents subjects like Statistics, Economics, Accountancy and English respectively. It is defined as follows,
$f_1(m_1) = \{l_2\}, f_1(m_2) = \{l_1\}$,
$f_2(m_1) = \{l_2, l_3\}, f_2(m_2) = \{l_1, l_2\}$,
$f_3(m_1) = \{l_1, l_2\}, f_3(m_2) = \{L\}$,
$f_4(m_1) = \{l_1\}, f_4(m_2) = \{l_1, l_3\}$
Therefore T is a soft lattice topology.
Hence (L, T, M) is a soft L-topological space.
Further, $T_{m_1} = \{\phi, L, \{l_1\}, \{l_2\}, \{l_2, l_3\}, \{l_1, l_2\}\}$
and $T_{m_2} = \{\phi, L, \{l_1\}, \{l_1, l_3\}, \{l_1, l_2\}\}$ are topologies on

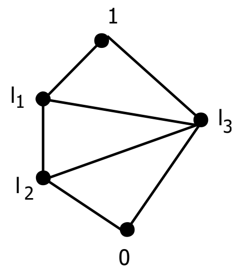

Fig. 1. Example for Complete Soft Lattice

L.
Thus, these collections depending on each parameter result in a soft topological lattice space (also known as a soft L-space) over L.

Definition 4: [23] "$(f^L_M)' = (f'^L_M)$ where $f' : P \longrightarrow \wp(L)$ is a mapping given by $f'(m) = L - f(m)$ for all $m \in M$ is called as the relative complement of a soft lattice f^L_M is denoted by $(f^L_M)'$"

Definition 5: [23] "(L, T, M) is a soft lattice topological space over L. Then soft L-open sets in L are members of T."

Definition 6: [23] "Consider a triple (L, T, M) constitutes as a soft lattice topological space defined over L. A soft lattice denoted as f^L_M over L is considered to be a soft L-closed set in L if its relative complement, denoted as $(f^L_M)'$, belongs to the set T."

Definition 7: [23] "Consider a lattice denoted as L and a set of parameters represented by M. Now, let's define T as the collection encompassing all possible soft lattices that can be constructed over L. In this context, we refer to T as the soft discrete lattice topology on L, and the triple (L, T, M) is termed a soft discrete lattice topological space defined over L."

Definition 8: [23] "Let's consider a lattice denoted as L and a set of parameters represented by M. Now, if we define a collection T as ϕ, L, we refer to T as the soft indiscrete lattice topology on L. In this context, the triple (L, T, M) is characterized as a soft indiscrete lattice topological space established over L."

Definition 9: [23] "Let's consider a soft lattice topological space denoted as (L, T, M) over L, and suppose we have a non-empty subset of L represented as Y. In this context, we can define a collection T_Y as follows: $T_Y = Y f^L_M \mid f^L_M \in T$. We refer to T_Y as the soft relative lattice topology on Y, and the triple (Y, T_Y, M) is termed a soft L-subspace of f^L_M. "

Definition 10: [24] "f^L_M is a soft lattice over L. The soft lattice f^L_M is called a soft lattice point, denoted by (l_m, M), for the element $m \in M$, if $f(m) = \{l\}$ and $f(m') = \phi$ for all $m' \in M - \{l\}$."

Definition 11: [23] "Suppose we have a soft lattice topological space represented as (L, T, M) over L, and let f^L_M be a soft lattice defined over L. In this context, we can define the soft lattice closure of f^L_M and denote it as $\overline{f}M^L$. This closure is obtained by taking the intersection of all soft L-closed sets that contain fM^L."

Definition 12: [23] "Let's consider a soft lattice topological space denoted as (L, T, M) over L, and suppose we have a soft lattice f^L_M defined over L. With f^L_M, we can associate another soft lattice, denoted as $\overline{f}^L_M$, defined as follows: $\overline{f}(m) = \overline{f(m)}$. In this definition, $\overline{f(m)}$ represents the soft L-closure of $f(m)$ within T_M, for each parameter m in the set M."

Definition 13: [23] "Consider the soft lattice topological space (L, T, M) defined over L, and let g_M^L be a soft lattice over L. Additionally, suppose we have an element $x \in L$. We say that x qualifies as a soft L-interior point of g_M^L if there exists a soft L-open set f_M^L such that x belongs to f_M^L and f_M^L is a subset of g_M^L,denoted as $(f_M^L)^o$."

Definition 14: [23] "Let's consider the soft lattice topological space (L, T, M) defined over L, and suppose we have a soft lattice g_M^L over L. Additionally, let x be an element belonging to L. We say that g_M^L qualifies as a soft lattice neighborhood of x if there exists a soft L-open set f_M^L such that x is an element of f_M^L and f_M^L is a subset of g_M^L."

Definition 15: [24] "Take (L_1, T_1, M) and (L_2, T_2, M) as two soft lattice topological spaces. The mapping f_g is called a soft L-mapping from L_1 to L_2 denoted by $f_g\colon (L_1, T_1, M) \longrightarrow (L_2, T_2, M)$, where $f\colon L_1 \longrightarrow L_2$ and $g\colon M \longrightarrow M$ are two mappings. For each soft L-neighbourhood g_M^L of $(f(l)_M, M)$, if there exist a soft L-neighbourhood f_M^L of (l_m, M) such that $f_g(f_M^L \subset g_M^L)$, then f_g is a soft Lattice continuous mapping at (l_m, M).
If f_g is soft lattice continuous mapping for all (l_m, M), then f_g is considered as soft Lattice continuous mapping."

Definition 16: [24] "$(L_1, T_1, M), (L_2, T_2, M)$ be two soft lattice topological spaces, and $f_g\colon (L_1, T_1, M) \longrightarrow (L_2, T_2, M)$. Then
(a) If $f_g(f_M^L)$ of each soft L-open set f_M^L over L_1 is a soft L-open set in L_2, then f_g is known as a soft L-open mapping.
(b) If $f_g(h_M^L)$ of each soft L-closed set h_M^L over L_1 is a soft L-closed set in L_2, then f_g is called a soft L-closed mapping."

Definition 17: [24] "(L_1, T_1, M) and (L_2, T_2, M) as two soft lattice topological spaces, $f_g\colon (L_1, T_1, M) \longrightarrow (L_2, T_2, M)$ be a mapping. If f_g is a bijection, soft L-continuous and f_g^{-1} is a soft L-continuous mapping, then f_g is said to be soft L-homeomorphism from L_1 to L_2.
When a soft homeomorphism f_g exists between L_1 and L_2, we say that L_1 is soft L-homeomorphic to L_2."

Definition 18: [26] "(L, T, M) be a soft lattice topological spaces over L and $l_1, l_2 \in L$ such that $l_1 \neq l_2$. If there exist soft L-open sets f_M^L and g_M^L such that $l_1 \in f_M^L$ and $l_2 \notin f_M^L$ or $l_2 \in g_M^L$ and $l_1 \notin g_M^L$, then (L, T, M) is called a soft L-T_0-space."

Definition 19: [26] (L, T, M) be a soft lattice topological spaces over L and $l_1, l_2 \in L$ such that $l_1 \neq l_2$. If there exist soft L-open sets f_M^L and g_M^L such that $l_1 \in f_M^L$ and $l_2 \notin f_M^L$ and $l_2 \in g_M^L$ and $l_1 \notin g_M^L$, then (L, T, M) is called a soft L-T_1-space.

Definition 20: [26] "$l \in L$, therefore l_M^L denotes the soft L-set over L for which $l(m) = \{l\}$ for all $m \in M$."

Definition 21: [26] "(L, T, M) be a soft lattice topological space over L. and $l_1, l_2 \in L$ s.t. $l_1 \neq l_2$. If $\exists$ soft L-open sets f_M^L and g_M^L s.t. $l_1 \in f_M^L$ and $l_2 \in g_M^L$ and $f_M^L \cap g_M^L = \phi$, then (L, T, M) is said to be a soft L-T_2-space or soft L-Hausdorff space."

Definition 22: [26] (L, T, M) is a soft lattice topological spaces over L. Then (L, T, M) is a soft L-$T_{2\frac{1}{2}}$-space or soft L-Urysohn space if for $l_1, l_2 \in L$ such that $l_1 \neq l_2$, there exist two soft L-open sets f_M^L and g_M^L such that $l_1 \in f_M^L$ and $l_2 \in g_M^L$ and $\overline{f_M^L} \cap \overline{g_M^L} = \phi$.

Definition 23: [26] (L, T, M) is a soft lattice topological spaces over L, g_M^L be a soft L-closed set in L and $l_1 \in L$ such that $l_2 \notin g_M^L$. If there exist soft L-open sets f_{1M}^L and f_{2M}^L such that $l_1 \in f_{1M}^L$, $g_M^L \subset f_{2M}^L$ and $f_{1M}^L \cap f_{2M}^L = \phi$, then (L, T, M) is called a soft L-regular space.

Definition 24: [26] (L, T, M) be a soft lattice topological spaces over L. Then (L, T, M) is said to be a soft L-T_3-space if it soft L-regular and soft L-T_1-space.

Definition 25: [26] (L, T, M) be a soft lattice topological spaces over L, then (L, T, M) is called a soft L-completely regular space if every soft L-closed subset f_M^L and any given soft L-point $l_M^L \notin f_M^L$, then there is a soft L-continuous function $f_g\colon (L, T, M) \longrightarrow (L, T, M)$ such that $f(l) = \phi$ and $f(f_M^L) = L$. Otherwise, we say l and f_M^L can be seperated by a soft L-continuous function.

Definition 26: [26] "A soft L-topological space, denoted as (L, T, M), is characterized as a soft L-$T_{3\frac{1}{2}}$-space if it possesses both the properties of being a soft L-completely regular space and a soft L-T_1-space."

Definition 27: [27] "(L, T, M) be a soft lattice topological space over L, f_M^L and g_M^L be a soft L-closed set s.t. $f_M^L \cap g_M^L = \phi$. If $\exists$ soft L-open sets f_{1M}^L and f_{2M}^L s.t. $f_M^L \subset f_{1M}^L$, $g_M^L \subset f_{2M}^L$ and $f_{1M}^L \cap f_{2M}^L = \phi$, then (L, T, M) is called a soft L-normal space."

Definition 28: [27] "Suppose we have a soft lattice topological space represented as (L, T, M) defined over L. In such a case, we classify (L, T, M) as a soft L-T_4-space provided that it satisfies both the conditions of being a soft L-normal space and a soft L-T_1-space."

Definition 29: [27] "(L, T, M) be a soft lattice topological space over L and f_M^L, g_M^L be two non-empty soft L-subset over L. Then we say A_M^L, B_M^L are two seperated soft L-sets if $A_M^L \cap \overline{B_M^L} = \phi$ and $\overline{A_M^L} \cap B_M^L = \phi$."

Definition 30: [27] "(L, T, M) be a soft lattice topological space over L. A soft L-topological space (L, T, M) is said to be soft L-completely normal space if for any two non-empty seperated soft L-sets A_M^L, B_M^L, $\exists$ $F_M^L, G_M^L \in T$ s.t. $A_M^L \subset F_M^L, B_M^L \subset G_M^L$ and $F_M^L \cap G_M^L = \phi$."

Definition 31: [27] "A soft L-topological space denoted as (L, T, M) is classified as a soft L-T_5-space when it exhibits the combined characteristics of being a soft L-completely normal space and a soft L-T_1-space."

Definition 32: [27] "(L, T, M) be a soft lattice topological space over L. A soft L-topological space (L, T, M) is said to be soft L-perfectly normal space if it is soft L-normal and every soft L-closed subset has countable intersection of soft L-open subsets."

Definition 33: [27] "(L, T, M) be a soft lattice topological space over L. A soft L-topological space (L, T, M) is called soft L-T_6-space when it is a soft L-perfectly T_4-space."

III. Soft L-connectness

Definition 34: Consider (L, T, M) as a soft lattice topological space over L. If $\exists$ no $f_M^L, g_M^L \in T - \{m\}$ s.t. $f_M^L \cap g_M^L = \phi$ and $f_M^L \cup g_M^L = L$, then (L, T, M) is called soft L-connected, otherwise (L, T, M) is called soft L-disconnected.

Definition 35: Let (L, T, M) be a soft lattice topological space over L. A soft L-topological space (L, T, M) is called soft L-connected if there does not have a separation of L.

Example 2: Let $L = \{l_1, l_2\}, M = \{m_1 = expensive, m_2 = cheap\}$ and $T = \{\phi, L, f^L_{1M}, f^L_{2M}, f^L_{3M}, f^L_{4M}, f^L_{5M}\}$ is a soft L-topological space over L, where $f^L_{1M}, f^L_{2M}, f^L_{3M}, f^L_{4M}, f^L_{5M}$ are soft lattices over L, defined as follows,
$f_1(m_1) = \{l_2\}, f_1(m_2) = \{l_1\}$,
$f_2(m_1) = \{l_2, l_3\}, f_2(m_2) = \{l_1, l_2\}$,
$f_3(m_1) = \{l_1, l_2\}, f_3(m_2) = L$,
$f_4(m_1) = \{l_1, l_2\}, f_4(m_2) = \{l_1, l_3\}$,
$f_5(m_1) = \{l_2\}, f_5(m_2) = \{l_1, l_2\}$.
Then T is a soft L-topology on L and (L, T, M) is a soft lattice topological space on L. Clearly, (L, T, M) is a soft L-connected space.

Definition 36: Let (L, T, M) be a soft lattice topological space over L and f^L_M be a soft L-subset over L. Then f^L_M is soft L-connected, if it is soft L-connected as a soft L-subspace.

Theorem 1: A soft L-topological space (L, T, M) is soft L- connected if and only if $\exists$ non-empty soft L-subset f^L_M that is both soft L-open and soft L-closed in (L, T, M).

Proof: The proof is obvious. ■

Theorem 2: Let (L_1, T_1, m_1) and (L_2, T_2, m_2) be two soft lattice topological spaces and $f\colon L_1 \longrightarrow L_2$ and $g\colon m_1 \longrightarrow m_2$. Let $f_g\colon (L_1, T_1, m_1) \longrightarrow (L_2, T_2, m_2)$ be a soft L-continuous mapping and soft L-onto. If (L_1, T_1, m_1) is soft L-connected, then the soft L-image of (L_2, T_2, m_2) is also soft L-connected.

Proof: Let $f_g\colon (L_1, T_1, m_1) \longrightarrow (L_2, T_2, m_2)$ be soft L-continuous mapping and soft L-onto. Conversely, assume that (L_2, T_2, m_2) is soft L-disconnected and the pair $f^{L_2}_{1m_2}$ and $f^{L_2}_{2m_2}$ is a soft L-disconnections of (L_2, T_2, m_2). Since the mapping $f_g\colon (L_1, T_1, m_1) \longrightarrow (L_2, T_2, m_2)$ be a soft L-continuous, then $f^{-1}_g(f^{L_2}_{1m_2})$ and $f^{-1}_g(f^{L_2}_{2m_2})$ is a soft L-disconnected of (L_1, T_1, m_1) which is a contradiction. Hence (L_2, T_2, m_2) is also soft L-connected. ■

Definition 37: Let (L, T, M) be a soft lattice topological space over L, f^L_M be a soft L-subset of L and $l \in L$. Then l is termed as soft L-limit point of f^L_M if every soft L-neighbourhood of l soft L-intersects f^L_M at some point other than l itself. $(f^L_M)^l$ denotes the collection of all soft L-limit points of f^L_M.
In otherwords, if (L, T, M) be a soft lattice topological space, f^L_M be a soft L-subset of L and $l \in L$, then $l \in (f^L_M)^l$ iff $(g^L_M \cap (f^L_M \setminus \{l\})) \neq \phi$ for all soft L-open neighbourhoods g^L_M of l.

Remark 1: From definition 37, we say that l is a soft L-limit point of f^L_M iff $l \in \overline{((f^L_M) \setminus \{l\})}$.

Theorem 3: Let (L, T, M) be a soft lattice topological space over L and f^L_M be a soft L-subset of L. Then $f^L_M \cup (f^L_M)^l = \overline{f^L_M}$.

Proof: If $l \in (f^L_M \cup (f^L_M)^l)$, then $l \in f^L_M$ or $l \in (f^L_M)^l$.
If $l \in f^L_M$, then $l \in \overline{f^L_M}$.
If $l \in (f^L_M)^l$, then $(g^L_M \cap (f^L_M \setminus \{l\})) \neq \phi$ for all soft L-open neighbourhoods g^L_M of l and therefore $g^L_M \cap f^L_M \neq \phi$ for all soft L-open neighbourhoods g^L_M of l.
Hence $l \in \overline{f^L_M}$.
Conversely, if $l \in \overline{f^L_M}$, then $l \in f^L_M$ or $l \notin f^L_M$. If $l \in f^L_M$, it is trivial that $l \in (f^L_M \cup (f^L_M)^l)$. If $l \notin f^L_M$, then $(g^L_M \cap (f^L_M \setminus \{l\})) \neq \phi$ for all soft L-open neighbourhoods g^L_M of l.
So, $l \in (f^L_M)^l$ which implies $l \in (f^L_M \cup (f^L_M)^l)$.
Hence $f^L_M \cup (f^L_M)^l = \overline{f^L_M}$. ■

Theorem 4: Let (L, T, M) be a soft lattice topological space over L and f^L_M be a soft L-subset of L. Then f^L_M is soft L-closed if and only if $(f^L_M)^l \subseteq f^L_M$.

Proof: f^L_M is soft L-closed iff $f^L_M = \overline{f^L_M}$.
Also $f^L_M = \overline{f^L_M}$ iff $f^L_M = f^L_M \cup (f^L_M)^l$.
f^L_M is soft L-closed iff $f^L_M = f^L_M \cup (f^L_M)^l$.
i.e., f^L_M is soft L-closed iff $(f^L_M)^l \subseteq f^L_M$. ■

Theorem 5: Let (L, T, M) be a soft lattice topological space over L and f^L_M, g^L_M be two soft L-subsets of L. Then
(i) $f^L_M \subseteq g^L_M \Longrightarrow (f^L_M)^l \subseteq (g^L_M)^l$.
(ii) $(f^L_M \cap g^L_M)^l \subseteq (f^L_M)^l \cap (g^L_M)^l$.
(iii) $(f^L_M \cup g^L_M)^l = (f^L_M)^l \cup (g^L_M)^l$.
(iv) $\overline{((f^L_M)^l)^l} \subseteq (f^L_M)^l$.
(v) $\overline{(f^L_M)}^l = (f^L_M)^l$.

Proof: (i) Let $f^L_M \subseteq g^L_M$. Since $(f^L_M \setminus \{l\}) \subseteq (g^L_M \setminus \{l\})$, $\overline{(f^L_M \setminus \{l\})} \subseteq \overline{(g^L_M \setminus \{l\})}$, we get $(f^L_M)^l \subseteq (g^L_M)^l$.
(ii) We have $(f^L_M \cap g^L_M) \subseteq f^L_M$ and $(f^L_M \cap g^L_M) \subseteq g^L_M$. Then by (i), $(f^L_M \cap g^L_M)^l \subseteq (f^L_M)^l$ and $(f^L_M \cap g^L_M)^l \subseteq (g^L_M)^l$.
Hence $(f^L_M \cap g^L_M)^l \subseteq (f^L_M)^l \cap (g^L_M)^l$.
(iii) For all $l \in (f^L_M \cup g^L_M)^l \Longrightarrow l \in \overline{(f^L_M \cup g^L_M)^l \setminus \{l\}}$.
Therefore $\overline{(f^L_M \cup g^L_M)^l \setminus \{l\}} = \overline{(f^L_M \cup g^L_M) \cap \{l\}'}$
$= \overline{(f^L_M \cap \{l\}') \cup (g^L_M \cap \{l\}')}$
$= \overline{(f^L_M \cap \{l\}')} \cup \overline{(g^L_M \cap \{l\}')}$
$= \overline{(f^L_M \setminus \{l\})} \cup \overline{(g^L_M \setminus \{l\})}$ if and only if $l \in (f^L_M)^l \cup (g^L_M)^l$.
Hence $(f^L_M \cup g^L_M)^l = (f^L_M)^l \cup (g^L_M)^l$.
(iv) Let $l \notin (f^L_M)^l$. Then $l \notin \overline{(f^L_M \setminus \{l\})}$.
This implies that $\exists$ a soft L-open set g^L_M s.t. $l \in g^L_M$ and $(g^L_M \cap (f^L_M \setminus \{l\})) = \phi$.
Hence $l \notin ((f^L_M)^l)^l$.
Assume on the contrary that $l \in ((f^L_M)^l)^l$, then $l \in \overline{(f^L_M \setminus \{l\})}$.
Since $l \in g^L_M$, then $(g^L_M \cap (f^L_M \setminus \{l\})) \neq \phi$.
Therefore $\exists$ $m \neq l$ s.t. $m \in g^L_M \cap (f^L_M)^l$.
It follows that $m \in (g^L_M \setminus \{l\}) \cup (f^L_M \setminus \{m\})$.
Hence $(g^L_M \setminus \{l\}) \cup (f^L_M \setminus \{m\}) \neq \phi$ which is a contradiction to $(g^L_M \cap (f^L_M \setminus \{l\})) = \phi$.
This shows that $l \in ((f^L_M)^l)^l$ and hence $((f^L_M)^l)^l \subseteq (f^L_M)^l$.
(v) Using (ii), (iii) and (iv) and also by theorem 5.41, proof of (v) is obvious. ■

Theorem 6: Let (L, T, M) is a soft L-T_2-space and Y be the non-empty subset of L containing finite number of points, then Y is soft L-closed.

Proof: Let $Y = \{l_1\}$. Now we prove that Y is soft L-closed. If l_2 is a point of L different from l_1, then l_1 and l_2 have distinct soft L-neighbourhoods f^L_M and g^L_M respectively. Since f^L_M does not soft L-intersect $\{l_2\}$, point l_1 does not belong to the soft L-closure of $\{l_2\}$. As a result, the soft L-closure of $\{l_1\}$ is $\{l_1\}$ itself and hence it is soft L-closed. Since Y is arbitrary, it is true for all subsets of L containing a finite number of points, follows that Y is soft L-closed. ■

Definition 38: Let (L, T, M) be a soft lattice topological space over L. Then $\underline{f^L_M}$ denotes the soft L-boundary of soft lattice f^L_M over L and defined by $\underline{f^L_M} = \overline{(f^L_M)} \cap \overline{((f^L_M)')}$.

Theorem 7: A soft L-topological space (L, T, M) is soft L- connected if and only if every non-empty soft L-subspace has a non-empty soft L-boundary.

Proof: The proof is by contradiction. Assume that a non-empty soft L-subspace f^L_M of a soft L-connected space

(L,T,M) has empty soft L-boundary. Then f_M^L is soft L-open and $(\overline{f_M^L}) \cap (\overline{L\backslash(f_M^L)}) = \phi$. Let l be a soft L-limit point of f_M^L. Then $l \in (\overline{f_M^L})$ but $l \notin ((\overline{f_M^L})')$. Particularly, $l \notin (f_M^L)'$ and $l \in f_M^L$. So f_M^L is both soft L-open and closed. By theorem 4, (L,T,M) is soft L-disconnected. This contradiction shows that f_M^L has a non-empty soft L-boundary. Conversely, suppose that L is soft L-disconnected. Then by theorem 1, (L,T,M) has a soft L-subset f_M^L which is both soft L-open and soft L-closed. Then $(\overline{f_M^L}) = f_M^L$, $((\overline{f_M^L})') = (f_M^L)'$ and $(\overline{f_M^L}) \cap ((\overline{f_M^L})') = \phi$. So f_M^L has empty soft L-boundary, a contradiction, thereby (L,T,M) is soft L-connected. ■

Theorem 8: Let two soft lattices f_M^L and g_M^L be a soft L-disconnected in soft L-topological space (L,T,M) and $h_M^L \subseteq f_M^L$ or $h_M^L \subseteq g_M^L$.

Proof: By contradiction, let h_M^L be neither contained in f_M^L nor in g_M^L. Then $(h_M^L \cap f_M^L)$, $(h_M^L \cap g_M^L)$ are both non-empty soft L-open subsets of h_M^L s.t. $(h_M^L \cap f_M^L) \cap (h_M^L \cap g_M^L) = \phi$ and $(h_M^L \cap f_M^L) \cup (h_M^L \cap g_M^L) = h_M^L$.
This implies that $(h_M^L \cap f_M^L)$ and $(h_M^L \cap g_M^L)$ is a soft L-disconnection of h_M^L. ■

Theorem 9: Let g_M^L be a soft L-connected subset of a soft L-topological space (L,T,M) and f_M^L be a soft L-subset of L s.t. $g_M^L \subseteq f_M^L \subseteq \overline{g_M^L}$. then f_M^L is soft L-connected.

Proof: It is sufficient to prove $\overline{g_M^L}$ is soft L-connected. By contradiction, let us assume that $\overline{g_M^L}$ is soft L-disconnected. Then $\exists$ a soft L-disconnection (h_M^L, k_M^L) of $\overline{g_M^L}$. That is $h_M^L \cap g_M^L$, $k_M^L \cap g_M^L$ are two soft L-open sets in g_M^L s.t. $(h_M^L \cap g_M^L) \cap (k_M^L \cap g_M^L) = (h_M^L \cap k_M^L) \cap (g_M^L) = \phi$ and $(h_M^L \cap g_M^L) \cup (k_M^L \cap g_M^L) = (h_M^L \cap k_M^L) \cup (g_M^L) = (g_M^L)$. This implies $(h_M^L \cap g_M^L, k_M^L \cap g_M^L)$ is a soft L-disconnection of g_M^L which is a contradiction.
Hence $\overline{g_M^L}$ is soft L-connected. ■

Corollary 1: If f_M^L is a soft L-connected and soft L-subspace of a soft L-topological space (L,T,M), then $(\overline{f_M^L})$ is soft L-connected.

IV. Soft L-compactness

Definition 39: A family $A = \{f_{a_M}^L\}_{a \in I}$ of a soft L-set is a cover of a soft L-set f_M^L if $f_M^L \subseteq \bigcup f_{a_M}^L$.
If each element of A is a soft L-open set, then it is a soft L-open cover. A subcover of A is a subfamily of A which is also a cover.

Definition 40: A soft L-topological space (L,T,M) is called soft L-compact if each soft L-open cover of L has a finite subcover.
Also let (L,T_1,M) and (L,T_2,M) be two soft L-topological spaces. If $T_1 \subseteq T_2$, then T_2 is finer than T_1. We say T_1 is soft L-comparable with T_2 if $T_1 \subseteq T_2$ or $T_2 \subseteq T_1$. Then we have the following.

Proposition 10: Consider (L,T_2,M) to be a soft L-compact space and $T_1 \subseteq T_2$. Then (L,T_1,M) is soft L-compact.

Proof: Let $\{f_{a_M}^L\}_{a \in I}$ be a soft L-open cover of L by soft L-open sets of (L,T_1,M). Since $T_1 \subseteq T_2$, then $\{f_{a_M}^L\}_{a \in I}$ is a soft L-open cover of L by soft L-open sets of (L,T_2,M). But (L,T_2,M) is soft L-compact. Therefore $L_M^L \subseteq \{f_{a_{1_M}}^L\} \cup \cdots \cup \{f_{a_{n_M}}^L\}$ for some $a_1, \cdots, a_n \in I$. Hence (L,T_1,M) is soft L-compact. ■

Proposition 11: Let f_M^L, g_M^L, h_M^L and i_M^L be soft L-sets in $SS(L)_M$. Then the following are demonstrated:
(i) $f_M^L \subseteq g_M^L$ if and only if $f_M^L \cap g_M^L = f_M^L$.
(ii) $f_M^L \subseteq g_M^L$ and $f_M^L \subseteq h_M^L$ if and only if $f_M^L \subseteq g_M^L \cap h_M^L$.
(iii) If $f_M^L \subseteq h_M^L$ and $g_M^L \subseteq i_M^L$, then $f_M^L \cup g_M^L \subseteq h_M^L \cup i_M^L$.
(iv) $f_M^L \cap (f_M^L)' = \phi_L$.
(v) $f_M^L \cap g_M^L = \phi_L$ if and only if $f_M^L \subseteq (g_M^L)'$.
(vi) $f_M^L \subseteq g_M^L$ if and only if $(g_M^L)' \subseteq (f_M^L)'$.

Proposition 12: Let f_M^L, g_M^L and h_M^L be soft L-sets and $\{f_{a_M}^L\}_{a \in I}$ be a family of soft L-sets in $SS(L)_M$. Then the following are demonstrated:
(i) $f_M^L \cap (f_M^L)' = \phi_L$.
(ii) $f_M^L \cup \phi_L = f_M^L$.
(iii) $f_M^L \cap (\cup_{a \in I}(f_{a_M}^L)) = \cup_{a \in I}(f_M^L \cap (f_{a_M}^L))$.
(iv) If $f_M^L \subseteq g_M^L$ and $g_M^L \cap h_M^L = \phi_L$, then $f_M^L \cap h_M^L = \phi_L$.
(v) $(\phi_L)' = L$.
(vi) $(L)' = \phi_L$.

Theorem 13: Let (Y,T_Y,M) be a soft L-subspace of soft L-topological space (L,T,M). Then (Y,T_Y,M) is soft L-compact if and only if every cover of Y by soft L-open sets in L has a finite subcover.

Proof: Let (Y,T_Y,M) be a soft L-compact and $\{f_{a_M}^L\}_{a \in I}$ is a soft L-open cover of Y by soft L-open sets in L. Then by the proposition 11 and 12, $\{Y f_{a_M}^L\}_{a \in I}$ is a soft L-open cover of Y. Therefore $Y \subseteq \{f_{\alpha_{1_M}}^L\} \cup \cdots \cup \{f_{\alpha_{n_M}}^L\}$ for some $\alpha_1, \cdots, \alpha_n \in J$.
Hence $\{Y f_{\alpha_{i_M}}^L\}_{i=1}^n$ is a subcover of Y and (Y,T_Y,M) is soft L-compact. ■

Theorem 14: A soft L-Hausdorff space's soft L-compact subspaces are all soft L-closed.

Proof: Let (Y,T_Y,M) be a soft L-compact subspace of soft L-Hausdorff space (L,T,M). Let $l_1 \in L_M^L - Y_M^L$. Then for all $m_1 \in (Y,T_Y,M)$, $l_1 \neq m_1$. Therefore, $\exists$ soft L-open sets $f_{l_{1_M}}^L$ and $f_{l_1 m_{1_M}}^L$ containing l_1 and m_1 respectively s.t. $f_{l_{1_M}}^L \cap f_{l_1 m_{1_M}}^L = \phi$. Obviously, $\{f_{l_1 m_{1_M}}^L\}_{y \in Y}$ is a cover of Y by soft L-open sets in L. By theorem 13, $Y \subseteq \{f_{l_1 m_{1_M}}^L\} \cup \cdots \cup \{f_{l_1 m_{n_M}}^L\}$ for some $m_1, \cdots, m_n \in Y$. Next, $l_1 \in f_{m_{1_M}}^L \cap \cdots \cap f_{m_{n_M}}^L = f_{l_{1_M}}^L$ and by proposition 12, $f_{l_{1_M}}^L \cap Y_M^L = \phi$.
Hence $l_1 \in f_{l_{1_M}}^L \subseteq L_M^L - Y_M^L$.
Therefore $L_M^L - Y_M^L = f_{l_{1_M}}^L$. So $L_M^L - Y_M^L = \bigcup_{l_1 \in L_M^L - Y_M^L} f_{l_{1_M}}^L$ is soft L-open. Hence Y_M^L is soft L-closed. ■

Theorem 15: Soft L-compactness exists in every soft L-closed subspace of a soft L-compact space.

Proof: Let (Y,T_Y,M) be a soft L-subspace of a soft L-compact space (L,T,M) s.t. Y_M^L is soft L-closed in L. Let $\{f_{a_M}^L\}_{a \in I}$ be a soft L-open cover of Y by soft L-open sets of L. Since $(Y_M^L)'$ is a soft L-open set in L, then by proposition 11 and 12, we show that $\{f_{a_M}^L\}_{a \in I} \cup \{(Y_M^L)'\}$ form a soft L-open cover of L. Therefore $L_M^L \subseteq \{f_{a_{1_M}}^L\} \cup \cdots \cup \{f_{a_{n_M}}^L\} \cup \{(Y_M^L)'\}$, for some $a_1, \cdots, a_n \in J$.
By proposition 11 and 12, $\{Y f_{a_{i_M}}^L\}_{i=1}^n$ is a subcover of Y. ■

Definition 41: Consider a soft lattice topological space represented as (L,T,M) over L. If we have a collection B_M^L that is a subset of T, and if every element within T can be expressed as a combination of elements belonging to B_M^L, then we refer to B_M^L as a soft L-basis for the soft L-topology. Each individual element within B_M^L is termed a

soft L-basis element
Next, we characterize soft L-compact spaces in terms of basis element in the following theorem.

Theorem 16: A soft L-topological space (L, T, M) is soft L-compact if and only if there is a soft L-basis B_M^L for T that has a finite subcover for every cover of L by elements of B_M^L.

Proof: Let (L, T, M) be a soft L-compact. Then T is a soft L-basis for T trivially. As a result, any L cover by the members of T has a finite subcover.
Conversely, let $\{f_{a_M}^L\}_{a \in I}$ be a soft L-open cover of L. After that, $\{f_{a_M}^L\}$ may be represented as a union of basis elements for each $a \in I$. They combine to produce a soft L-open cover of L, such as $\{g_{B_M}^L\}_{B \in J}$.
Therefore $L_M^L \subseteq \{g_{B_{1_M}}^L\} \cup \cdots \cup \{g_{B_{n_M}}^L\}$, for some $B_1, \cdots, B_n \in J$.
Let $\{g_{B_{iM}}^L\} \subseteq \{f_{a_{iM}}^L\}$ for each $i = 1, ..., n$.
This implies $\{f_{a_M}^L\}_{a \in I}$ is a finite subcover of L. Hence (L, T, M) is a soft L-compact. ■

V. Conclusion

A method for dealing with uncertainty called soft set theory has been extended in this work. By utilizing soft lattice in a soft L-topology setting, we broaden the concept of soft set. This study focuses on the soft L-connected space and soft L-compact space on soft L-topological spaces. We come to the conclusion that this work is just the start of a new framework and that we have investigated a number of novel ideas that will be useful in future theoretical studies. This work has future research in the field of noncompact covering properties, viz., paracompactness, metacompactness, subparacompactness, submetacompactness, Lindelofness, Para-Lindelofness, etc. in soft L-topological spaces. We also suggest future directions that one can extend this work towards new directions like Linear Diophantine Fuzzy sets and Spherical linear Diophantine Fuzzy sets.

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