Existence of Solutions for p-Laplacian Caputo-Hadamard Fractional Hybrid-Sturm-Liouville-Langevin Integro-Differential Equations with Functional Boundary Value Conditions

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***Abstract*—In the present work a class of p-Laplacian fractional hybrid-Sturm-Liouville-Langevin integro-differential equations with functional boundary value conditions involving Caputo-Hadamard fractional derivative is studied. Using the hybrid fixed point theorem for three operators by Dhage, the existence result is obtained. Finally, an example is given to illustrate the main result.**

***Index Terms*—Fractional hybrid-Sturm-Liouville-Langevin equation, Caputo-Hadamard fractional derivative, Fractional integro-differential equation, Fixed point theorem.**

I. Introduction

OVER the past few years, the fractional calculus (which is an expansion of integer derivatives and integrals) has become an important area of investigation due to their wide application. It has been shown in several literature works that fractional differential equations (FDEs) can be applied better than ordinary differential equations to describe some practical problems in the field of natural sciences. In particular, the FEDs have been successfully used to model the complexity in mathematics, physics and societies, such as the fractional evolution equations, control theory, anomalous diffusion processes, relaxation phenomena in complex viscoelastic materials, finance and so on [1-4]. Therefore, the qualitative analysis of FDEs has gained significant popularity and importance, especially for the equations with practical application background, which are particularly valued by scholars, for instance, fractional hybrid equations (FHEs), fractional Langevin equations (FLEs), and fractional Sturm-Liouville equations (FSLEs). On the other hand, the development of differential equations and the study of new operators are closely related to inequalities. Therefore, many researchers today employ various inequalities to prove the existence and uniqueness (EU) of solutions for ordinary (partial) differential equations. Scientists have established excellent results on the existence of solutions (ES) to FDEs by various fixed point theorem.

The Langevin equation is a stochastic differential equation which describes the time evolution of a subset of degrees of degrees of freedom. Some of the recent fractional Langevin's problem is discussed in [5-7]. Based on the Banach contraction principle and Schaefer's fixed point theorem, the EU results for a coupled system of FLEs with p-Laplacian were obtained by the author [5]. Salem et al. [6] considered EU results for FLEs with three-point boundary conditions by applying the Banach contraction principle, Krasnoselskii's fixed point theorem, nonlinear alternative Leray-Schauder theorem and Leray-Schauder degree theorem.

Another equation that is very well known and that plays an important role in engineering and mathematics is the Sturm-Liouville equation. In [8], Rivero et al. proposed the fractional form of the Sturm-Liouville equation. Later, Liu et al. [9] considered the ES to the FSLEs with two-point boundary conditions by means of the nonlinear alternative Leray-Schauder theorem. Batiha et al. [10] studied the fractional Sturm-Liouville and Langevin equations. The EU are established via the Banach contraction principle, Krasnoselskii's fixed point theorem and Leray-Schauder alternative.

In addition, the study of fractional hybrid differential equations has always been a particularly interesting topic. Zhao et al. [11] first studied hybrid equations of nonlinear FDEs with initial condition by using Dhage fixed point theorem. Ahmad et al. [12] studied the ES by using the Dhage fixed point theorem. In [13], Gao et al. further considered the ES for a system of coupled hybrid fractional integro-differential equations.

Recently, some scholars are keen to work on the generalized fractional Sturm-Liouville equations, Langevin equations and hybrid equations, such as combining Sturm-Liouville and Langevin fractional differential equations (FSLLEs)[14–16]; combining hybrid and Langevin fractional differential equations(FHLEs)[17, 18]; combining hybrid, Sturm-Liouville and Langevin fractional differential equations(FHSLLEs)[19, 20]. For example, Kiataramkul et al. [14] studied the EU of solutions for the FSLLEs of Hadamard

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type, with anti-periodic boundary conditions of the form

$$\begin{cases} {}^H_C\mathcal{D}^{\varrho}_{a+}([w(\sigma){}^H_C\mathcal{D}^{\rho}_{a+}+\Upsilon(\sigma)]\kappa(\sigma)=Z(\sigma,\kappa(\sigma)), \\ \kappa(1)=-\kappa(T), \quad \mathcal{D}^{\rho}\kappa(1)=-\mathcal{D}^{\rho}\kappa(T), \end{cases}$$

where ${}^H_C\mathcal{D}^{\rho}_{a+}$ and ${}^H_C\mathcal{D}^{\varrho}_{a+}$ are the Caputo-Hadamard fractional derivative with $\rho,\varrho\in(0,1)$; $\sigma\in(1,T)$, $w,\Upsilon\in C([1,T],\mathbf{R})$, $Z\in C([1,T]\times\mathbf{R},\mathbf{R})$ and $|w(t)|\geq Q>0$. The authors obtained EU results by utilizing Banach fixed-point theorem, and Leray-Schauder nonlinear alternative.

In [19], Boutiara et al. considered the ES for the nonlinear FHSLLEs, with two-point boundary conditions of the form

$$\begin{cases} {}^C\mathcal{D}^{\rho,Y}(\mathcal{O}(\sigma))=Z(\sigma,\kappa(\mu(\sigma))),\ \sigma\in(a,b), \\ \kappa(a)=0,\ u(b){}^C\mathcal{D}^{\varrho,Y}\big(\frac{\kappa(\sigma)}{m(\sigma,\kappa(v(\sigma)))}\big)_{\sigma=b}+g(b)\kappa(b)=0, \end{cases}$$

where $\mathcal{O}(\sigma)=u(\sigma){}^C\mathcal{D}^{\varrho,Y}\big(\frac{\kappa(\sigma)}{m(\sigma,\kappa(v(\sigma)))}\big)+g(\sigma)\kappa(\sigma)$, ${}^C\mathcal{D}^{\rho,Y}$ and ${}^C\mathcal{D}^{\varrho,Y}$ are the Y-Caputo fractional derivative with $\rho,\varrho\in(0,1)$; $u\in C(J,\mathbf{R}\backslash\{0\})$, $g\in C(J,\mathbf{R})$, $m\in C(J\times\mathbf{R},\mathbf{R}\backslash\{0\})$, $Z\in C(J\times\mathbf{R},\mathbf{R})$, $\mu,\nu:J\to J$ are given functions. From that Darbo's fixed point theorem, the author obtained the existence results.

Differential equations containing p-Laplacian operators have extensive applications in heat conduction, engineering physics, and mathematics [21]. From these points of view, the p-Laplacian FDEs has been considered by many authors, see for example [22-24]. To the author's knowledge, there is currently no literature on the ES for fractional hybrid Sturm-Liouville-Langevin integro-differential equations with p-Laplacian operators.

Motivated by the remarkable developments indicated above, we want to examine the ES for p-Laplacian fractional hybrid-Sturm-Liouville-Langevin integro-differential equations with functional boundary value problems (BVPs) as follows:

$$\begin{cases} {}^H_C\mathcal{D}^{\rho}_{a+}\phi_p\big(\mathcal{H}(\sigma)\big)=Z(\sigma,\kappa(\mu(\sigma))),\ \sigma\in(a,b), \\ \kappa(a)=\theta(\kappa),\ \mathcal{H}(\sigma)_{\sigma=b}=0, \end{cases} \tag{1}$$

where ${}^H_C\mathcal{D}^{\rho}_{a+}$ and ${}^H_C\mathcal{D}^{\varrho}_{a+}$ are the Caputo-Hadamard fractional derivative with $\rho,\varrho\in(0,1)$; ${}^HI^{\gamma_i}_{a+}$ is the Hadamard fractional integral of order $\gamma_i>0$, $i=1,2\cdots l$, $\mathcal{H}(\sigma)=u(\sigma){}^H_C\mathcal{D}^{\varrho}_{a+}\Big(\frac{\kappa(\sigma)-\sum_{i=1}^{l}{}^HI^{\gamma_i}_{a+}h_i(\sigma,\kappa(\sigma))}{m(\sigma,\kappa(v(\sigma)))}\Big)+g(\sigma)\kappa(\sigma)$; $m\in C(J\times\mathbf{R},\mathbf{R}\backslash\{0\})$, $u\in C(J,\mathbf{R}\backslash\{0\})$, $g\in C(J,\mathbf{R})$ and $Z,h_i\in C(J\times\mathbf{R},\mathbf{R})$, $J=[a,b]$; $\mu,\nu:J\to J$ are given functions; $\phi_p(\varsigma)=|\varsigma|^{p-2}\varsigma$ represents the p-Laplacian operator, $(\phi_p)^{-1}=\phi_q$, $\frac{1}{p}+\frac{1}{q}=1$; the functional boundary value conditions in (1) is given function $\theta:C[J,\mathbf{R}]\to\mathbf{R}$.

The following summary summarizes the work's contribution based on that interpretation:

- A new problem consisting of p-Laplacian hybrid-Sturm-Liouville-Langevin integro-differential equations supplemented with functional BVPs is proposed.
- Sufficient criteria to ensure ES for the problem (1) are established and an example that illustrates and validates the theoretical contributions has been presented.
- The problem discussed in this paper are more general. By fixing different parameters in BVP (1) and obtain some new results in this paper. For the special case, if we consider the case when $p=2$, $h_i(\sigma,\kappa(\sigma))\equiv 0$, $\theta(\kappa)=0$, then the BVP (1) is reduced to the problem studied in [15]. By selecting $m(\sigma,\kappa(v(\sigma)))\equiv 1$, we obtain the results for a nonlinear p-Laplacian fractional Sturm-Liouville-Langevin integro-differential equations with functional boundary conditions; In case we choose $u(\sigma)\equiv 1$, the results correspond to nonlinear p-Laplacian fractional hybrid-Langevin equations equipped with functional boundary conditions.

The rest of the paper is organized as follows: Section 2 briefly introduces some basic concepts related to fractional calculus and fixed point theorems. The main results which are established by means of Dhage fixed point theorem in Section 3. An example is constructed for the illustration of the obtained results.

II. Preliminaries

This section, is assigned to recall some basic definitions of fractional calculus and some other related results to be used in section 3. We define a space $AC^n_\delta[a,b]$ as follow

$$AC^n_\delta[a,b]=\left\{\kappa:[a,b]\to R|\,\delta^{n-1}\kappa(\sigma)\in AC[a,b]\right\},$$

where $0<a<b<\infty$, $\kappa(\sigma)$ is a function and $AC[a,b]$ denote the space of all absolutely continuous real valued function on $[a,b]$ $\big(\delta=\sigma\frac{d}{d\sigma}\big)$.

Definition 1. (see [25]) The Hadamard fractional integral of the order ρ for a function $Z:J\to\mathbf{R}$ is outlined as

$${}^HI^{\rho}_{a+}Z(\sigma)=\frac{1}{\Gamma(\rho)}\int_a^{\sigma}\big(\ln\frac{\sigma}{\varphi}\big)^{\rho-1}Z(\varphi)\frac{d\varphi}{\varphi}.$$

Definition 2. (see [25]) Let $\rho>0$, $n=[\rho]+1$. The Hadamard fractional derivative of order ρ for a function $Z:J\to\mathbf{R}$ is outlined as

$${}^HD^{\rho}_{a+}Z(\sigma)=\frac{1}{\Gamma(n-\rho)}\big(\sigma\frac{d}{d\sigma}\big)^n\int_a^{\sigma}\big(\ln\frac{\sigma}{\varphi}\big)^{n-\rho-1}Z(\varphi)\frac{d\varphi}{\varphi}.$$

Definition 3. (see [26]) Let $\rho>0$, $n=[\rho]+1$. The Caputo-Hadamard fractional derivative of order ρ for a function $Z(\sigma)\in AC^n_\delta[a,b]$ is defined by

$$\begin{aligned} {}^H_C\mathcal{D}^{\rho}_{a+}Z(\sigma)&=({}^HI^{n-\rho}_{a+}\delta^nZ)(\sigma)\\ &\times\frac{1}{\Gamma(n-\rho)}\int_a^{\sigma}(\ln\frac{\sigma}{\varphi})^{n-\rho-1}\delta^nZ(\varphi)\frac{d\varphi}{\varphi}. \end{aligned}$$

Lemma 1. (see [26]) Let $\rho>0$, $Z\in AC^n_\delta[a,b]$ and $n=[\rho]+1$. Then

$$({}^HI^{\rho}_{a+}{}^H_C\mathcal{D}^{\rho}_{a+}Z)(\sigma)=Z(\sigma)-\sum_{l=0}^{n-1}\frac{\delta^lZ(a)}{l!}\big(\ln\frac{\sigma}{a}\big)^l.$$

Theorem 1. (see [27]) Let S be a nonempty, closed convex and bounded subset of a Banach algebra Ω and let $\mathcal{A},\mathcal{C}:\Omega\to\Omega$ and $\mathcal{B}:S\to\Omega$ be three operators satisfying:

(a) $\mathcal{A}$ and $\mathcal{C}$ are Lipschitzian with Lipschitz constants δ and ℓ, respectively;
(b) $\mathcal{B}$ is compact and continuous;
(c) $\kappa=\mathcal{A}\kappa\mathcal{B}\chi+\mathcal{C}\kappa\Rightarrow\kappa\in S$ for all $\chi\in S$;
(d) $\delta T+\ell<1$, $T=\|\mathcal{B}(S)\|$.

Then the operator equation $\kappa=\mathcal{A}\kappa\mathcal{B}\kappa+\mathcal{C}\kappa$ has a solution in S.

III. MAIN RESULTS

In this section, we prove the existence result for the BVPs (1) by using Theorem 1. Let $\Omega = C(J, \mathbf{R})$ be the space of continuous real-valued functions defined on $J = [a, b]$, we define a norm $\|\cdot\|$ in Ω by $\|\kappa\| = \sup_{\sigma\in J} |\kappa(\sigma)|$, $\forall\sigma \in J$.

Lemma 2. The BVPs (1) is equivalent to the following integral equation:

$$\begin{aligned}\kappa(\sigma) &= m(\sigma, \kappa(v(\sigma))) \\ &\times\Big[{}^H I^{\varrho}_{a+}\frac{1}{u(\sigma)}\phi_q\left(\mathcal{T}(\sigma)\right) - {}^H I^{\varrho}_{a+}\Big(\frac{g(\sigma)\kappa(\sigma)}{u(\sigma)}\Big) \\ &+ \frac{\theta(\kappa)}{m(a, \kappa(v(a)))}\Big] \\ &+ \sum\nolimits_{i=1}^{l} {}^H I^{\gamma_i}_{a+} h_i(\sigma, \kappa(\sigma)),\ \sigma \in J,\end{aligned} \tag{2}$$

where $\mathcal{T}(\sigma) = {}^H I^{\rho}_{a+} Z(\sigma, \kappa(\mu(\sigma))) - {}^H I^{\rho}_{a+} Z(\sigma, \kappa(\mu(\sigma)))_{\sigma=b}$.

Proof. Applying the operator ${}^H I^{\rho}_{a+}$ on both sides of (1) and then applying Lemma 1, we obtain

$$\phi_p(\mathcal{H}(\sigma)) = {}^H I^{\rho}_{a+} Z(\sigma, \kappa(\mu(\sigma))) + c_0,\ c_0 \in \mathbf{R}. \tag{3}$$

By using the boundary condition $\mathcal{H}(\sigma)_{\sigma=b} = 0$ in Eq.(3), we get $c_0 = -{}^H I^{\rho}_{a+} Z(\sigma, \kappa(\mu(\sigma)))_{\sigma=b}$. Thus we get

$$\begin{aligned}&{}^H_C\mathcal{D}^{\varrho}_{a+}\Big(\frac{\kappa(\sigma) - \sum_{i=1}^{l} {}^H I^{\gamma_i}_{a+} h_i(\sigma, \kappa(\sigma))}{m(\sigma, \kappa((\sigma)))}\Big) \\ &= \frac{\phi_q(\mathcal{T}(\sigma)) - g(\sigma)\kappa(\sigma)}{u(\sigma)}.\end{aligned} \tag{4}$$

Similarly, taking the ${}^H I^{\varrho}_{a+}$ to the both sides (4) and using Lemma 1, we have

$$\begin{aligned}&\frac{\kappa(\sigma) - \sum_{i=1}^{l} {}^H I^{\gamma_i}_{a+} h_i(\sigma, \kappa(\sigma))}{m(\sigma, \kappa(v(\sigma)))} \\ &= {}^H I^{\varrho}_{a+}\frac{1}{u(\sigma)}\phi_q\left(\mathcal{T}(\sigma)\right) - {}^H I^{\varrho}_{a+}\Big(\frac{g(\sigma)\kappa(\sigma)}{u(\sigma)}\Big) + c_1,\end{aligned} \tag{5}$$

where $c_1 \in \mathbf{R}$.

Applying the functional boundary conditions $\kappa(a) = \theta(\kappa)$ in (5). Then

$$\begin{aligned}&\frac{\kappa(a) - \sum_{i=1}^{l} {}^H I^{\gamma_i}_{a+} h_i(\sigma, \kappa(\sigma))_{\sigma=a}}{m(a, \kappa(v(a)))} \\ &= {}^H I^{\varrho}_{a+}\frac{1}{u(\sigma)}\phi_q\big(\mathcal{T}(\sigma)_{\sigma=a}\big) \\ &- {}^H I^{\varrho}_{a+}\Big(\frac{g(\sigma)\kappa(\sigma)}{u(\sigma)}\Big)_{\sigma=a} + c_1,\end{aligned}$$

it follows that,

$$c_1 = \frac{\theta(\kappa)}{m(a, \kappa(v(a)))}.$$

By substituting the value of c_1 in (5), we obtain the solution (2). This completes the proof.

In order to prove our main results, we list the following hypotheses.

(A_1) The functions $m : J\times\mathbf{R} \to \mathbf{R}\backslash\{0\}$ and $h_i : J\times\mathbf{R} \to \mathbf{R}$ are continuous and there exist positive functions ϕ and ψ_i, with bounds $\|\phi\|$ and $\|\psi_i\|$, such that

$$|m(\sigma, \kappa) - m(\sigma, \chi)| \le \phi(\sigma)\,|\kappa - \chi|\,, \tag{6}$$

and

$$|h_i(\sigma, \kappa) - h_i(\sigma, \chi)| \le \psi_i(\sigma)\,|\kappa - \chi|\,, \tag{7}$$

for any $\sigma \in J$, $\kappa, \chi \in \mathbf{R}$ and $i = 1, 2, \cdots l$.

(A_2) There exist a continuous function $Z : [a, b] \times \mathbf{R} \to \mathbf{R}$ and non-negative functions $d_1(\sigma), d_2(\sigma) \in J[a, b]$ such that

$$|Z(\sigma, \kappa)| \le d_1(\sigma) + d_2(\sigma)|\kappa|^{p-1}, \tag{8}$$

for any $\sigma \in J$ and $\kappa \in \mathbf{R}$.

(A_3) There exist a constant $h_0 > 0$, such that

$$|h_i(\sigma, \kappa)| \le h_0,\ (\sigma, \kappa) \in J \times \mathbf{R},\ i = 1, 2, \cdots l. \tag{9}$$

(A_4) There exist a constant $K > 0$, such that

$$\Big|\frac{\theta(\kappa)}{m(a, \kappa(v(a)))}\Big| \le K,\ \forall\kappa \in C(J, \mathbf{R}). \tag{10}$$

(A_5) There exist a number $r > 0$, such that

$$\begin{aligned}&\big(r\,\|\phi\| + m_0\big)\Big(\mathcal{V} + \frac{\bar{g}r(\ln b)^{\varrho}}{\Gamma(\varrho + 1)\bar{u}} + K\Big) \\ &+ \sum\nolimits_{i=1}^{l}\frac{(r\,\|\psi_i\| + h_0)(\ln\frac{b}{a})^{\gamma_i}}{\Gamma(\gamma_i + 1)} \le r.\end{aligned} \tag{11}$$

and

$$\|\phi\|\left[\mathcal{V} + \frac{\bar{g}r(\ln\frac{b}{a})^{\varrho}}{\Gamma(\varrho + 1)\bar{u}} + K\right]\sum\nolimits_{i=1}^{l}\frac{\|\psi_i\|\,(\ln\frac{b}{a})^{\gamma_i}}{\Gamma(\gamma_i + 1)} \le 1. \tag{12}$$

where $m_0 = \sup_{\sigma\in J} |m(\sigma, 0)|$, $\bar{u} = \min_{\sigma\in J} |u(\sigma)|$, $\bar{g} = \max_{\sigma\in J} |g(\sigma)|$, $h_0 = \sup_{\sigma\in J} |h_i(\sigma, 0)|$, $i = 1, 2, \cdots, l$, and

$$\mathcal{V} = \frac{2^{2q-2}(\ln b/a)^{\rho(q-1)+\varrho}(\|d_1\|^{q-1} + \|d_2\|^{q-1}r)}{\Gamma(\varrho + 1)\bar{u}(\Gamma(\rho + 1))^{q-1}}.$$

Theorem 2. Assume that the hypotheses (A_1)–(A_5) hold. Then the BVPs (1) has at least one solution defined on J.

Proof. we consider a subset S of Ω given by

$$S = \{\kappa \in E : \|\kappa\| \le r\},$$

where r satisfies inequality (11). Clearly S is closed, convex, and bounded subset of the Banach space Ω. By Lemma 2, the BVPs (1) is equivalent to the equation

$$\begin{aligned}\kappa(\sigma) &= m(\sigma, \kappa(v(\sigma)))\Big\{\frac{1}{\Gamma(\Psi)}\int_a^{\sigma}\Big(\ln\frac{\sigma}{\varphi}\Big)^{\varrho-1} \\ &\times\frac{1}{u(\varphi)}\Big[\phi_q\Big(\frac{1}{\Gamma(\Phi)}\int_a^{\varphi}\big(\ln\frac{\varphi}{\tau}\big)^{\rho-1}Z(\tau, \kappa(\mu(\tau)))\frac{d\tau}{\tau} \\ &- \frac{1}{\Gamma(\rho)}\int_a^{b}\big(\ln\frac{b}{\tau}\big)^{\rho-1}Z(\tau, \kappa(\mu(\tau)))\frac{d\tau}{\tau}\Big)\Big]\frac{d\varphi}{\varphi} \\ &- \frac{1}{\Gamma(\varrho)}\int_a^{\sigma}\big(\ln\frac{\sigma}{\varphi}\big)^{\varrho-1}\frac{g(\varphi)\kappa(\varphi)}{u(\varphi)}\frac{d\varphi}{\varphi} + \frac{\theta(\kappa)}{m(a, \kappa(v(a)))}\Big\} \\ &+ \sum\nolimits_{i=1}^{l} {}^H I^{\gamma_i}_{a+} h_i(\sigma, \kappa(\sigma)),\ \sigma \in J.\end{aligned} \tag{13}$$

Define three operators $\mathcal{A},\mathcal{C}:\Omega\to\Omega$ and $\mathcal{B}:S\to\Omega$ by

$$\mathcal{A}\kappa(\sigma)=m(\sigma,\kappa(v(\sigma))),\ \sigma\in J, \tag{14}$$

$$\begin{aligned}
B\kappa(\sigma)&=\frac{1}{\Gamma(\varrho)}\int_a^\sigma\big(\ln\frac{\sigma}{\varphi}\big)^{\varrho-1}\\
&\times\frac{1}{u(\varphi)}\Big[\phi_q\Big(\frac{1}{\Gamma(\Phi)}\int_a^\varphi\big(\ln\frac{\varphi}{\tau}\big)^{\rho-1}Z(\tau,\kappa(\mu(\tau)))\frac{d\tau}{\tau}\\
&-\frac{1}{\Gamma(\rho)}\int_a^b\big(\ln\frac{b}{\tau}\big)^{\rho-1}Z(\tau,\kappa(\mu(\tau)))\frac{d\tau}{\tau}\Big)\Big]\frac{d\varphi}{\varphi}\\
&-\frac{1}{\Gamma(\varrho)}\int_a^\sigma\big(\ln\frac{\sigma}{\varphi}\big)^{\varrho-1}\frac{g(\varphi)\kappa(\varphi)}{u(\varphi)}\frac{d\varphi}{\varphi}\\
&+\frac{\theta(\kappa)}{m(a,\kappa(v(a)))},\ \sigma\in J,
\end{aligned}$$

$$\mathcal{C}\kappa(\sigma)=\sum_{i=1}^{l}{}^{H}I_{a^+}^{\gamma_i}h_i(\sigma,\kappa(\sigma)),\ \sigma\in J. \tag{15}$$

The integral equations (13) can then be written as

$$\kappa(\sigma)=\mathcal{A}\kappa(\sigma)\mathcal{B}\kappa(\sigma)+\mathcal{C}\kappa(\sigma),\ \sigma\in J.$$

We now prove that the operators $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ satisfy the conditions of Theorem 1. To show this, we divided our prove into four steps.

Step 1. We first show that $\mathcal{A}$ and $\mathcal{C}$ are Lipschitzian on Ω. Let $\kappa,\chi\in\Omega$. Then by the condition (A_1), for $\sigma\in J$, we have

$$\begin{aligned}
&|\mathcal{A}\kappa(v(\sigma))-\mathcal{A}\chi(v(\sigma))|\\
&=|m(\sigma,\kappa(v(\sigma)))-m(\sigma,\chi(v(\sigma)))|\\
&\le\phi(\sigma)|\kappa(v(\sigma))-\chi(v(\sigma))|\\
&\le\|\phi\|\,\|\kappa-\chi\|,
\end{aligned}$$

taking the supremum over the interval J, we obtain

$$\|\mathcal{A}\kappa-\mathcal{A}\chi\|\le\|\phi\|\,\|\kappa-\chi\|,$$

for all $\kappa,\chi\in\Omega$. Thus, $\mathcal{A}$ is a Lipschitzian on Ω with Lipschitz constant $\|\phi\|$. Similarly, we have

$$\begin{aligned}
&|\mathcal{C}\kappa(\sigma)-\mathcal{C}\chi(\sigma)|\\
&=\Big|\sum_{i=1}^{l}{}^{H}I_{a^+}^{\gamma_i}h_i(\sigma,\kappa(\sigma))-\sum_{i=1}^{l}{}^{H}I_{a^+}^{\gamma_i}h_i(\sigma,\chi(\sigma))\Big|\\
&\le\sum_{i=1}^{l}\frac{1}{\Gamma(\gamma_i)}\int_a^\sigma\big(\ln\frac{\sigma}{\varphi}\big)^{\gamma_i-1}\psi_i(\varphi)|\kappa(\varphi)-\chi(\varphi)|\frac{d\varphi}{\varphi}\\
&\le\|\kappa-\chi\|\sum_{i=1}^{l}\frac{\|\psi_i\|}{\Gamma(\gamma_i+1)}\big(\ln\sigma-\ln\varphi\big)^{\gamma_i}\Big|_\sigma^a\\
&\le\|\kappa-\chi\|\sum_{i=1}^{l}\frac{(\ln\frac{b}{a})^{\gamma_i}\|\psi_i\|}{\Gamma(\gamma_i+1)},\ \sigma\in J,
\end{aligned}$$

which implies that,

$$\|\mathcal{C}\kappa-\mathcal{C}\chi\|\le\|\kappa-\chi\|\sum_{i=1}^{l}\frac{\big(\ln\frac{b}{a}\big)^{\gamma_i}\|\psi_i\|}{\Gamma(\gamma_i+1)},$$

for all $\kappa,\chi\in\Omega$. Thus, $\mathcal{C}$ is a Lipschitzian on Ω and constant $\ell=\sum_{i=1}^{l}\frac{(\ln\frac{b}{a})^{\gamma_i}\|\psi_i\|}{\Gamma(\gamma_i+1)}$.

Step 2. The operator $\mathcal{B}$ is completely continuous on S.

$$\begin{aligned}
\lim_{n\to\infty}B\kappa_n(\sigma)&=\lim_{n\to\infty}\Big\{\frac{1}{\Gamma(\varrho)}\int_a^\sigma(\ln\frac{\sigma}{\varphi})^{\varrho-1}\frac{1}{u(\varphi)}\\
&\times\Big[\phi_q\Big(\frac{1}{\Gamma(\rho)}\int_a^\varphi(\ln\frac{\varphi}{\tau})^{\rho-1}Z(\tau,\kappa_n(\mu(\tau)))\frac{d\tau}{\tau}\\
&-\frac{1}{\Gamma(\rho)}\int_a^b(\ln\frac{b}{\tau})^{\rho-1}Z(\tau,\kappa_n(\mu(\tau)))\frac{d\tau}{\tau}\Big)\Big]\frac{d\varphi}{\varphi}\\
&-\frac{1}{\Gamma(\varrho)}\int_a^\sigma(\ln\frac{\sigma}{\varphi})^{\varrho-1}\frac{g(\varphi)\kappa_n(\varphi)}{u(\varphi)}\frac{d\varphi}{\varphi}+\frac{\theta(\kappa)}{m(a,\kappa_n(v(a)))}\Big\}\\
&=\frac{1}{\Gamma(\varrho)}\int_a^\sigma\big(\ln\frac{\sigma}{\varphi}\big)^{\varrho-1}\frac{1}{u(\varphi)}\\
&\times\Big[\phi_q\Big(\frac{1}{\Gamma(\rho)}\int_a^\varphi\big(\ln\frac{\varphi}{\tau}\big)^{\rho-1}\lim_{n\to\infty}Z(\tau,\kappa_n(\mu(\tau)))\frac{d\tau}{\tau}\\
&-\frac{1}{\Gamma(\rho)}\int_a^b\big(\ln\frac{b}{\tau}\big)^{\rho-1}\lim_{n\to\infty}Z(\tau,\kappa_n(\mu(\tau)))\frac{d\tau}{\tau}\Big)\Big]\frac{d\varphi}{\varphi}\\
&-\frac{1}{\Gamma(\varrho)}\int_a^\sigma\big(\ln\frac{\sigma}{\varphi}\big)^{\rho-1}\lim_{n\to\infty}\frac{g(\varphi)\kappa_n(\varphi)}{u(\varphi)}\frac{d\varphi}{\varphi}\\
&+\lim_{n\to\infty}\frac{\theta(x)}{m(a,\kappa_n(v(a)))}=\mathcal{B}\kappa(\sigma),\ \sigma\in J
\end{aligned}$$

Therefore, the operator $\mathcal{B}$ is continuous in S. By using Aezelá-Ascoli Theorem, we get that $\mathcal{B}$ is a uniformly bounded and equicontinuous on S.

In fact, for any $\kappa(\sigma)\in S$, by using the conditions (A_2) and (A_4), we first prove the $\mathcal{B}$ is uniformly bounded on S.

$$\begin{aligned}
|\mathcal{B}\kappa(\sigma)|&\le\frac{1}{\Gamma(\varrho)}\int_a^\sigma\big(\ln\frac{\sigma}{\varphi}\big)^{\varrho-1}\\
&\times\frac{1}{|u(\varphi)|}\phi_q\Big(\frac{1}{\Gamma(\rho)}\int_a^\varphi(\ln\frac{\varphi}{\tau})^{\rho-1}|Z(\tau,\kappa(\mu(\tau)))|\frac{d\tau}{\tau}\\
&+\frac{1}{\Gamma(\rho)}\int_a^b\big(\ln\frac{b}{\tau}\big)^{\rho-1}|Z(\tau,\kappa(\mu(\tau)))|\frac{d\tau}{\tau}\Big)\frac{d\varphi}{\varphi}\\
&+\frac{1}{\Gamma(\varrho)}\int_a^\sigma\big(\ln\frac{\sigma}{\varphi}\big)^{\varrho-1}\frac{|g(\varphi)\kappa(\varphi)|}{|u(\varphi)|}\frac{d\varphi}{\varphi}\\
&+\frac{|\theta(\kappa)|}{|m(a,\kappa(v(a)))|}\\
&\le\frac{(\ln b/a)^\varrho}{\Gamma(\varrho+1)\bar{u}}\phi_q\Big(\frac{2(\ln b/a)^\rho}{\Gamma(\rho+1)}\big(\|d_1\|+\|d_2\|\,r^{p-1}\big)\Big)\\
&+\frac{\bar{g}r(\ln b/a)^\varrho}{\Gamma(\varrho+1)\bar{u}}+K,
\end{aligned} \tag{16}$$

by using the $(\kappa+\chi)^p\le2^p(\kappa^p+\chi^p)$, $\kappa,\chi,p>0$, then

$$\begin{aligned}
&\phi_q\Big[\frac{2(\ln b/a)^\rho}{\Gamma(\rho+1)}\big(\|d_1\|+\|d_2\|\,r^{p-1}\big)\Big]\\
&\le\frac{2^{2q-2}(\ln b/a)^{\rho(q-1)}}{(\Gamma(\rho+1))^{q-1}}\big(\|d_1\|^{q-1}+\|d_2\|^{q-1}r\big),
\end{aligned} \tag{17}$$

substitute (17) into (16),

$$\begin{aligned}
|\mathcal{B}\kappa(\sigma)|\le&\ \frac{2^{2q-2}(\ln b/a)^{\rho(q-1)+\varrho}\Big(\|d_1\|^{q-1}+\|d_2\|^{q-1}r\Big)}{\Gamma(\varrho+1)\bar{u}(\Gamma(\rho+1))^{q-1}}\\
&+\frac{\bar{g}r(\ln b/a)^\varrho}{\Gamma(\varrho+1)\bar{u}}+K,
\end{aligned}$$

which implies that $\mathcal{B}$ is uniformly bounded on S.

In the following part, for convenience in writing, put

$$y(\kappa)=\big(\ln\frac{\sigma_2}{\varphi}\big)^{\varrho-1}-\big(\ln\frac{\sigma_1}{\varphi}\big)^{\varrho-1}.$$

Secondly, we will prove that $\mathcal{B}: S \to \Omega$ is equicontinuous. Assume that $\sigma_1, \sigma_2 \in [a,b]$ and $\sigma_1 < \sigma_2$, for any $\kappa(\sigma) \in S$, then we have

$$\begin{aligned}
&|\mathcal{B}\kappa(\sigma_1) - \mathcal{B}\kappa(\sigma_2)| \\
&\leq \frac{|\theta(\kappa(\sigma_2)) - \theta(\kappa(\sigma_1))|}{m(a,\kappa(v(a)))} + \frac{1}{\Gamma(\varrho)} \int_a^{\sigma_1} \frac{y(\kappa)}{|u(\varphi)|} \phi_q \\
&\times \Big(\frac{1}{\Gamma(\rho)} \int_a^{\varphi} (\ln \frac{\varphi}{\tau})^{\rho-1} |Z(\tau, \kappa(\mu(\tau)))| \frac{d\tau}{\tau} \\
&+ \frac{1}{\Gamma(\rho)} \int_a^{b} \big(\ln \frac{b}{\tau}\big)^{\rho-1} |Z(\tau, \kappa(\mu(\tau)))| \frac{d\tau}{\tau} \Big) \frac{d\varphi}{\varphi} \\
&+ \frac{1}{\Gamma(\varrho)} \int_{\sigma_1}^{\sigma_2} \big(\ln \frac{\sigma_2}{\varphi}\big)^{\varrho-1} \frac{1}{|u(\varphi)|} \phi_q \\
&\times \Big(\frac{1}{\Gamma(\rho)} \int_a^{\varphi} \big(\ln \frac{\varphi}{\tau}\big)^{\rho-1} |Z(\tau, \kappa(\mu(\tau)))| \frac{d\tau}{\tau} \\
&+ \frac{1}{\Gamma(\rho)} \int_a^{b} \big(\ln \frac{b}{\tau}\big)^{\rho-1} |Z(\tau, \kappa(\mu(\tau)))| \frac{d\tau}{\tau} \Big) \frac{d\varphi}{\varphi} \\
&+ \frac{1}{\Gamma(\varrho)} \int_a^{\sigma_1} y(\kappa) \frac{|g(\varphi)\kappa(\varphi)|}{|u(\varphi)|} \frac{d\varphi}{\varphi} \\
&+ \frac{1}{\Gamma(\varrho)} \int_{\sigma_1}^{\sigma_2} \big(\ln \frac{\sigma_2}{\varphi}\big)^{\varrho-1} \frac{|g(\varphi)\kappa(\varphi)|}{|u(\varphi)|} \frac{d\varphi}{\varphi} \\
&\leq \frac{|\theta(\kappa(\sigma_2)) - \theta(\kappa(\sigma_1))|}{m(a,\kappa(v(a)))} \\
&+ \frac{1}{\bar{u}} \phi_q \left[\frac{(\ln b/a)^{\rho}}{\Gamma(\rho+1)} \left(\|d_1\| + \|d_2\| r^{p-1} \right) \right] \\
&\times \Big(\frac{1}{\Gamma(\varrho)} \int_a^{\sigma_1} y(\kappa) + \frac{1}{\Gamma(\varrho)} \int_{\sigma_1}^{\sigma_2} (\ln \frac{\sigma_2}{\varphi})^{\varrho-1} \Big) \frac{d\varphi}{\varphi} \\
&+ \frac{\bar{g}r}{\bar{u}} \Big(\frac{1}{\Gamma(\varrho)} \int_a^{\sigma_1} y(\kappa) + \frac{1}{\Gamma(\varrho)} \int_{\sigma_1}^{\sigma_2} (\ln \frac{\sigma_2}{\varphi})^{\varrho-1} \Big) \frac{d\varphi}{\varphi} \\
&\leq \frac{|\theta(\kappa(\sigma_2)) - \theta(\kappa(\sigma_1))|}{m(a,\kappa(v(a)))} + \frac{\bar{g}r}{\Gamma(\varrho+1)\bar{u}} \\
&\times \Big(\frac{2}{\varrho} (\ln \frac{\sigma_2}{\sigma_1})^{\varrho} + \frac{1}{\varrho} \big((\ln \frac{\sigma_1}{a})^{\varrho} - (\ln \frac{\sigma_2}{a})^{\varrho} \big) \Big) \\
&+ \frac{(\ln b/a)^{\rho(q-1)} (\|d_1\|^{q-1} + \|d_2\|^{q-1} r)}{\Gamma(\varrho+1)\bar{u}(\Gamma(\rho+1))^{q-1}} \\
&\times \Big(\frac{2}{\varrho} (\ln \frac{\sigma_2}{\sigma_1})^{\varrho} + \frac{1}{\varrho} \big((\ln \frac{\sigma_1}{a})^{\varrho} - (\ln \frac{\sigma_2}{a})^{\varrho} \big) \Big),
\end{aligned}$$

which is independent of $\kappa(\sigma) \in S$. As $\sigma_1 \to \sigma_2$, the right-hand side of the above inequality tends to zero. Hence, we obtain $\mathcal{B}$ is equicontinuous on S. Therefore, it follows from Aezel$\acute{a}$-Ascoli theorem that $\mathcal{B}$ is a completely continuous on S.

Step 3. We show that the condition (c) of Theorem 1 is satisfied. For any $\kappa \in \Omega$ and $\chi \in S$ be arbitrary elements such that $\kappa = \mathcal{A}\kappa\mathcal{B}\chi + \mathcal{C}\kappa$, by using the conditions (A_1)–(A_4), we have

$$\begin{aligned}
&|\kappa(\sigma)| \leq |A\kappa|\,|B\chi| + |C\kappa| \\
&\leq |m(\sigma, \kappa(v(\sigma))) - m(\sigma, 0) + m(\sigma, 0)| \\
&\times \left\{ \frac{1}{\Gamma(\varrho)} \int_a^{\sigma} (\ln \frac{\sigma}{\varphi})^{\varrho-1} \right. \\
&\times \frac{1}{|u(\varphi)|} \left[\phi_q \Big(\frac{1}{\Gamma(\rho)} \int_a^{\varphi} (\ln \frac{\varphi}{\tau})^{\rho-1} |Z(\tau, \chi(\mu(\tau)))| \frac{d\tau}{\tau} \right. \\
&+ \left. \frac{1}{\Gamma(\rho)} \int_a^{b} (\ln \frac{b}{\tau})^{\rho-1} |Z(\tau, \chi(\mu(\tau)))| \frac{d\tau}{\tau} \Big) \right] \frac{d\varphi}{\varphi} \\
&+ \frac{1}{\Gamma(\varrho)} \int_a^{\sigma} (\ln \frac{\sigma}{\varphi})^{\varrho-1} \frac{|g(\varphi)\kappa(\varphi)|}{|u(\varphi)|} \frac{d\varphi}{\varphi} \\
&+ \left. \frac{|\theta(\chi)|}{|m(a, \chi(v(a)))|} \right\} + \sum\nolimits_{i=1}^{l} \frac{1}{\Gamma(\gamma_i)} \\
&\times \int_a^{\sigma} \big(\ln \frac{\sigma}{\varphi}\big)^{\gamma_i-1} |h_i(\sigma, \kappa(\sigma)) - h_i(\sigma, 0) + h_i(\sigma, 0)| \frac{d\varphi}{\varphi} \\
&\leq (r\|\phi\| + m_0) \Big[\mathcal{V} + \frac{\bar{g}r(\ln b/a)^{\varrho}}{\Gamma(\varrho+1)\bar{u}} + K \Big] \\
&+ \sum\nolimits_{i=1}^{l} \frac{(r\|\psi_i\| + h_0)(\ln \frac{b}{a})^{\gamma_i}}{\Gamma(\gamma_i+1)} \leq r.
\end{aligned}$$

Therefore, $\kappa \in S$.

Step 4. Finally, we show that the (d) of Theorem 1 is satisfied, that is, $\delta T + \ell < 1$.

$$\begin{aligned}
T = \|\mathcal{B}(S)\| &= \sup_{\kappa \in S} \{ \sup_{\sigma \in J} |\mathcal{B}\kappa(\sigma)| \} \\
&\leq \mathcal{V} + \frac{\bar{g}r(\ln b/a)^{\varrho}}{\Gamma(\varrho+1)\bar{u}} + K.
\end{aligned}$$

Consequently, we have

$$\|\phi\|\, T + \sum\nolimits_{i=1}^{l} \frac{\big(\ln \frac{b}{a}\big)^{\gamma_i} \|\psi_i\|}{\Gamma(\gamma_i+1)} < 1,$$

where $\delta = \|\phi\|$ and $\ell = \sum_{i=1}^{l} \frac{(\ln \frac{b}{a})^{\gamma_i} \|\psi_i\|}{\Gamma(\gamma_i+1)}$.

In consequence, the problem (1) has a solution on J. This completes the proof.

IV. Example

Example 4.1. Considering the following p-Laplacian hybrid-Strum-Liouville-Langevin integro-differential equation with functional boundary value conditions equations, in the framework of Caputo-Hadamard derivative

$$\begin{cases} {}^{H}_{C}\mathcal{D}^{1/2}_{1^+} \phi_p \mathcal{H}(\sigma) = \frac{\sin\sigma + \sigma\cos(\kappa(\sigma))}{5}, \ \sigma \in (1,2) \\ \kappa(1) = \frac{\delta}{4} \sin^2(\sum_{j=1}^{n-2} \kappa(\xi_j)), \ \mathcal{H}(\sigma)_{\sigma=2} = 0, \end{cases} \quad (18)$$

where

$$\begin{aligned}
Z(\sigma, \kappa(\sigma)) &= \frac{\sin\sigma + \sigma\cos(\kappa(\sigma))}{5}, \\
m(\sigma, \kappa(\sigma)) &= \frac{\sigma}{100}(|\kappa(\sigma)| + \sin\kappa(\sigma)) + \frac{\sigma}{10}, \\
h_i(\sigma, \kappa(\sigma)) &= \frac{|\kappa(\sigma)|}{(14+i+\sigma)(2+|\kappa(\sigma)|)}, \ i = 1,2,3,4,
\end{aligned}$$

and

$$\mathcal{H}(\sigma) = {}^{H}_{C}\mathcal{D}^{3/4}_{a^+} \Big(\frac{\kappa(\sigma) - \sum_{i=1}^{4} {}^{H}I^{i+1}_{1^+} h_i(\sigma, \kappa(\sigma))}{m(\sigma, \kappa(v(\sigma)))} \Big) + \frac{1}{10}\kappa(\sigma).$$

Clear, for $\kappa, \chi \in \mathbf{R}$, we have

$$\begin{aligned}
|m(\sigma,\kappa) - m(\sigma,\chi)| &\leq \frac{\sigma}{100} |\kappa - \chi|, \\
|h_i(\sigma,\kappa) - h_i(\sigma,\chi)| &\leq \frac{1}{14+i+\sigma} |\kappa - \chi|, \ i = 1,2,3,4.
\end{aligned}$$

Here, we take $\rho = \frac{1}{2}$, $\varrho = \frac{3}{4}$, $k = 4$, $q = 3$, $p = \frac{3}{2}$, $\gamma_1 = 2$, $\gamma_2 = 3$, $\gamma_3 = 4$, $\gamma_4 = 5$; $\mu(\sigma) = v(\sigma) = \sigma$, $\theta(\kappa) = \frac{\delta}{4} \sin^2\big(\sum_{j=1}^{n-2} \kappa(\xi_j)\big)$, $0 < \delta < 1$, $j = 1, 2, \cdots, n-2$, $\xi_j \in (0,1)$, $\xi_j < \xi_j + 1$ and $n \geq 3$.

Setting $\phi(\sigma) = \sigma/100$ and $\psi_i(\sigma) = 1/(14+i+\sigma)$, which give norms $\|\phi\| = 1/50$ and $\|\psi_i\| = 1/(14+i)$, $i = 1,2,3,4$.

Hence we have,

$$\frac{|\theta(\kappa)|}{|m(1,\theta(\kappa))|} \le \frac{5}{2},\ |h_i(\sigma,\kappa)| \le \frac{1}{15},$$
$$|Z(\sigma,\kappa)| = \Big|\frac{\sin\sigma + \sigma\cos(\kappa(\sigma))}{5}\Big| \le \frac{1+\sigma}{5}.$$

It follows that $u(\sigma) = 1$, $g(\sigma) = \frac{1}{10}$, $d_1(\sigma) = \frac{\sin\sigma}{5}$ and $d_2(\sigma) = \frac{\sigma\cos(x(\sigma))}{5}$, we get $\bar{u} = 1$, $\bar{g} = 1/10$, $\|d_1\| \le \frac{1}{5}$, $\|d_2\| \le \frac{2}{5}$, $m_0 = \sup_{\sigma\in J}|m(\sigma,0)| = 1/5$ and $h_0 = \sup_{\sigma\in J}|h_i(\sigma,0)| = 0$. Using these values, it follows by inequalities (11) and (12) that constant $0.633 < r < 89.230$. Hence, the problem (18) has at least one solution on $(1,2)$.

V. CONCLUSION

In the present paper, we studied a kind of p-Laplacian fractional hybrid-Sturm-Liouville-Langevin integro-differential equation with functional boundary value conditions, involving Caputo-Hadamard factional derivative. By using the Dhage fixed point theorem, the existence of solutions of the problem (1) are proved. The main results are well illustrated with the aid of an example. In future, we plan to study the existence and stability results for p-Laplacian hybrid Sturm-Liouville-Langevin integro-differential equation using the ψ-Hilfer fractional derivative.

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