A Newly Proposed Aggregation of Weighted Geometric Operator for Interval Valued Pentagonal Fuzzy Neutrosophic Set and its Application in Solving Multi-Attribute Decision Making Environment

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Abstract - **This paper proposes interval valued pentagonal fuzzy neutrosophic set in view with the combination of pentagonal fuzzy sets, single valued neutrosophic set and the interval valued neutrosophic set. Interval valued pentagonal fuzzy neutrosophic weighted geometric averaging operator is defined based on the operational laws, a theorem and some of its properties have been established by the proposed operator and also on the score and accuracy function. By assigning different weights to various features in accordance with each choice, the aggregation of the interval valued pentagonal fuzzy neutrosophic set is used on an example to validate the efficiency. Finally, a multi-attribute decision-making environment is examined using the proposed methodology's ranking order and collective ratings of each attribute's values for various alternatives.**

Index Terms- **Interval Valued Neutrosophic set, Neutrosophic set, Pentagonal fuzzy numbers, Score and Accuracy function.**

I. INTRODUCTION

By introducing a different dimension concerning membership functions ranging in the interval [0,1], Zadeh [13] introduced fuzzy sets and provided insight into interval valued fuzzy sets. The intuitionistic fuzzy set was discussed by Atanassov and Gargov [2, 3]. Alrefaei [1] dealt with operations on n-intuitionistic polygonal fuzzy numbers successfully. In order to deal with everyday ambiguities, inconsistent conditions, and complex circumstances, Smarandache [9] proposed the concept of neutrosophic set, which encompasses the degree of truthiness, indeterminacy, and falsity. The aggregation operator plays a vital role to in expressing the neutrosophic data of the alternatives in terms of a single number, which is validated in accordance with rating in multi-criteria decision-making problems. Aggregation of weighted arithmetic and weighted geometric operator for neutrosophic set was initiated by Lu et.al.[8]. Recently, Yen [12] explained trapezoidal fuzzy neutrosophic set and presented a method for trapezoidal fuzzy neutrosophic arithmetic and geometric weighted aggregation operators which is been used for decision making problems.

Chakraborty et. al. [4] studied the representation and properties of pentagonal fuzzy number. Interval valued pentagonal fuzzy numbers was initiated by Umamageswari et al. [10]. Literature survey reflects that this is the first time that weighted geometric aggregation operator for interval valued pentagonal fuzzy neutrosophic values has been studied which can be used effectively to deal with uncertain information.

Multi attribute decision making [MADM] based on single valued neutrosophic by Geng et. al. [6] and interval neutrosophic was initiated by Xu et. al. [11], which helps us to solve the problem with analytic approach. Comparison and analysis of MADM by ranking was well explained by Deng [5].

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II. PRELIMINARIES

A. Interval-valued pentagonal fuzzy neutrosophic

Definition 1. [7] "An interval-valued pentagonal fuzzy neutrosophic number (IVPFNN) $\check{N}$ is defined as an (IVPFNS) on X is represented by expressing $\check{N}(x) = [\check{N}^{\text{ɉ}}(x),\ \check{N}^{\text{ɥ}}(x)]$, where $\check{N}^{\text{ɉ}}$ and $\check{N}^{\text{ɥ}}$ are lower and upper pentagonal fuzzy neutrosophic sets on $\check{N}$ such that $\check{N}^{\text{ɉ}} \subseteq \check{N}^{\text{ɥ}}$.

$\check{N}^{\text{ɉ}} = \{[x, \text{Ŧ}^{\text{ɉ}}_{\check{N}}(x), \text{Ɨ}^{\text{ɉ}}_{\check{N}}(x), \text{Ƒ}^{\text{ɉ}}_{\check{N}}(x) : x \in X]\}$ where

$\text{Ŧ}^{\text{ɉ}}_{\check{N}}(x) \subset [0,1]$, $\text{Ɨ}^{\text{ɉ}}_{\check{N}}(x) \subset [0,1]$ and $\text{Ƒ}^{\text{ɉ}}_{\check{N}}(x) \subset [0,1]$ are lower pentagonal fuzzy neutrosophic numbers

$\text{Ŧ}^{\text{ɉ}}_{\check{N}}(x) = [\text{ŧ}^{\text{ɉ}1}_{\check{N}}(x), \text{ŧ}^{\text{ɉ}2}_{\check{N}}(x), \text{ŧ}^{\text{ɉ}3}_{\check{N}}(x), \text{ŧ}^{\text{ɉ}4}_{\check{N}}(x), \text{ŧ}^{\text{ɉ}5}_{\check{N}}(x)]: X \to [0,1]$,
$\text{Ɨ}^{\text{ɉ}}_{\check{N}}(x) = [\text{ɨ}^{\text{ɉ}1}_{\check{N}}(x), \text{ɨ}^{\text{ɉ}2}_{\check{N}}(x), \text{ɨ}^{\text{ɉ}3}_{\check{N}}(x), \text{ɨ}^{\text{ɉ}4}_{\check{N}}(x), \text{ɨ}^{\text{ɉ}5}_{\check{N}}(x)]: X \to [0,1]$, and
$\text{Ƒ}^{\text{ɉ}}_{\check{N}}(x) = [\text{ʄ}^{\text{ɉ}1}_{\check{N}}(x), \text{ʄ}^{\text{ɉ}2}_{\check{N}}(x), \text{ʄ}^{\text{ɉ}3}_{\check{N}}(x), \text{ʄ}^{\text{ɉ}4}_{\check{N}}(x), \text{ʄ}^{\text{ɉ}5}_{\check{N}}(x)]: X \to [0,1]$,
which satisfies the condition mentioned below: $0 \le \text{ŧ}^{\text{ɉ}5}_{\check{N}}(x) + \text{ɨ}^{\text{ɉ}5}_{\check{N}}(x) + \text{ʄ}^{\text{ɉ}5}_{\check{N}}(x) \le 3$.

For convenience of representation, we consider

$\text{Ŧ}^{\text{ɉ}}_{\check{N}}(x) = (\check{\text{ƍ}}, \check{\text{ƹ}}, \check{\text{ʂ}}, \check{\text{ƥ}}, \check{\text{ʒ}}): X \to [0,1]$,

$\text{Ɨ}^{\text{ɉ}}_{\check{N}}(x) = (\check{\text{ȣ}}, \check{\text{ʂ}}, \check{\text{ɕ}}, \check{\text{ɘ}}, \check{\text{ɣ}}): X \to [0,1]$ and

$\text{Ƒ}^{\text{ɉ}}_{\check{N}}(x) = (\check{\text{ɤ}}, \check{\text{ł}}, \check{\vartheta}, \check{\text{ƙ}}, \check{\text{ƺ}}): X \to [0,1]$.

Therefore,

$$\check{N}^{\text{ɉ}} = [(\check{\text{ƍ}}, \check{\text{ƹ}}, \check{\text{ʂ}}, \check{\text{ƥ}}, \check{\text{ʒ}}), (\check{\text{ȣ}}, \check{\text{ʂ}}, \check{\text{ɕ}}, \check{\text{ɘ}}, \check{\text{ɣ}}), (\check{\text{ɤ}}, \check{\text{ł}}, \check{\vartheta}, \check{\text{ƙ}}, \check{\text{ƺ}})]: X \to [0,1]$$"

Definition 2. [7] "Let $\check{N}^{l}$ be the lower pentagonal fuzzy neutrosophic number. Then $\mathrm{T}^{l}_{\check{N}}(x), \mathrm{I}^{l}_{\check{N}}(x)$ and $\mathrm{F}^{l}_{\check{N}}(x)$ is defined as follows:

$$\mathrm{T}^{l}_{\check{N}}(x)=\begin{cases}\left(\dfrac{x-\breve{g}}{\breve{\varepsilon}-\breve{g}}\right)\mathrm{T}^{l}_{\dot{N}} & \breve{g}\le x\le\breve{\varepsilon}\\ 1-\left(\dfrac{x-\breve{\varepsilon}}{\breve{s}-\breve{\varepsilon}}\right)\left(1-\mathrm{T}^{l}_{\dot{N}}\right) & \breve{\varepsilon}\le x\le\breve{s}\\ 1 & x=\breve{s}\\ 1-\left(\dfrac{\breve{p}-x}{\breve{p}-\breve{s}}\right)\left(1-\mathrm{T}^{l}_{\dot{N}}\right) & \breve{s}\le x\le\breve{p}\\ \left(\dfrac{\breve{z}-x}{\breve{z}-\breve{p}}\right)\mathrm{T}^{l}_{\dot{N}} & \breve{p}\le x\le\breve{z}\\ 0 & \text{otherwise}\end{cases}$$

$$\mathrm{I}^{l}_{\check{N}}(x)=\begin{cases}\left(\dfrac{\breve{\xi}-x}{\breve{\xi}-\breve{\delta}}\right)\mathrm{I}^{l}_{\dot{N}} & \breve{\delta}\le x\le\breve{\xi}\\ 1-\left(\dfrac{\breve{c}-x}{\breve{c}-\breve{\xi}}\right)\left(1-\mathrm{I}^{l}_{\dot{N}}\right) & \breve{\xi}\le x\le\breve{c}\\ 0 & x=\breve{c}\\ 1-\left(\dfrac{x-\breve{c}}{\breve{ə}-\breve{c}}\right)\left(1-\mathrm{I}^{l}_{\dot{N}}\right) & \breve{c}\le x\le\breve{ə}\\ \left(\dfrac{x-\breve{ə}}{\breve{o}-\breve{ə}}\right)\mathrm{I}^{l}_{\dot{N}} & \breve{ə}\le x\le\breve{o}\\ 1 & \text{otherwise}\end{cases}$$

$$\mathrm{F}^{l}_{\check{N}}(x)=\begin{cases}\left(\dfrac{\breve{t}-x}{\breve{t}-\breve{\tau}}\right)\mathrm{F}^{l}_{\dot{N}} & \breve{\tau}\le x\le\breve{t}\\ 1-\left(\dfrac{\breve{\vartheta}-x}{\breve{\vartheta}-\breve{t}}\right)\left(1-\mathrm{F}^{l}_{\dot{N}}\right) & \breve{t}\le x\le\breve{\vartheta}\\ 0 & x=\breve{\vartheta}\\ 1-\left(\dfrac{x-\breve{\vartheta}}{\breve{\kappa}-\breve{\vartheta}}\right)\left(1-\mathrm{F}^{l}_{\dot{N}}\right) & \breve{\vartheta}\le x\le\breve{\kappa}\\ \left(\dfrac{x-\breve{\kappa}}{\breve{\zeta}-\breve{\kappa}}\right)\mathrm{F}^{l}_{\dot{N}} & \breve{\kappa}\le x\le\breve{\zeta}\\ 1 & \text{otherwise}\end{cases}$$

$\check{N}^{u}=\left\{\left[x,\mathrm{T}^{u}_{\check{N}}(x),\mathrm{I}^{u}_{\check{N}}(x),\mathrm{F}^{u}_{\check{N}}(x): x\in X\right]\right\}$ where
$\mathrm{T}^{u}_{\check{N}}(x)\subset[0,1]$, $\mathrm{I}^{u}_{\check{N}}(x)\subset[0,1]$ and $\mathrm{F}^{u}_{\check{N}}(x)\subset[0,1]$
are upper pentagonal fuzzy neutrosophic numbers
$\mathrm{T}^{u}_{\dot{N}}(x)=\left[t^{u1}_{\dot{N}}(x),t^{u2}_{\dot{N}}(x),t^{u3}_{\dot{N}}(x),t^{u4}_{\dot{N}}(x),t^{u5}_{\dot{N}}(x)\right]: X\to[0,1]$,
$\mathrm{I}^{u}_{\dot{N}}(x)=\left[i^{u1}_{\dot{N}}(x),i^{u2}_{\dot{N}}(x),i^{u3}_{\dot{N}}(x),i^{u4}_{\dot{N}}(x),i^{u5}_{\dot{N}}(x)\right]: X\to[0,1]$,
and
$\mathrm{F}^{u}_{\dot{N}}(x)=\left[f^{u1}_{\dot{N}}(x),f^{u2}_{\dot{N}}(x),f^{u3}_{\dot{N}}(x),f^{u4}_{\dot{N}}(x),f^{u5}_{\dot{N}}(x)\right]: X\to[0,1]$
which satisfies the condition
$0\le t^{u5}_{\dot{N}}(x)+i^{u5}_{\dot{N}}(x)+f^{u5}_{\dot{N}}(x)\le 3$.
For convenience of representation, we consider

$\mathrm{T}^{u}_{\dot{N}}(x)=(\hat{g},\hat{\varepsilon},\hat{s},\hat{p},\hat{z}): X\to[0,1]$,
$\mathrm{I}^{u}_{\dot{N}}(x)=(\hat{\delta},\hat{\xi},\hat{c},\hat{ə},\hat{\gamma}): X\to[0,1]$ and
$\mathrm{F}^{u}_{\dot{N}}(x)=(\hat{\tau},\hat{t},\hat{\vartheta},\hat{\kappa},\hat{\zeta}): X\to[0,1]$.
Therefore

$$\dot{N}^{u}=\left\{\begin{matrix}\left[(\hat{g},\hat{\varepsilon},\hat{s},\hat{p},\hat{z}),(\hat{\delta},\hat{\xi},\hat{c},\hat{ə},\hat{\gamma}),(\hat{\tau},\hat{t},\hat{\vartheta},\hat{\kappa},\hat{\zeta})\right]:\\ X\to[0,1]\end{matrix}\right\}$$"

Definition 3. [7] "Let $\dot{N}^{u}$ be the upper pentagonal fuzzy neutrosophic number. Then $\mathrm{T}^{u}_{\dot{N}}(x), \mathrm{I}^{u}_{\dot{N}}(x)$ and $\mathrm{F}^{u}_{\dot{N}}(x)$ can be defined as follows:

$$\mathrm{T}^{u}_{\dot{N}}(x)=\begin{cases}\left(\dfrac{x-\hat{g}}{\hat{\varepsilon}-\hat{g}}\right)\mathrm{T}^{u}_{\dot{N}} & \hat{g}\le x\le\hat{\varepsilon}\\ 1-\left(\dfrac{x-\hat{\varepsilon}}{\hat{s}-\hat{\varepsilon}}\right)\left(1-\mathrm{T}^{u}_{\dot{N}}\right) & \hat{\varepsilon}\le x\le\hat{s}\\ 1 & x=\hat{s}\\ 1-\left(\dfrac{\hat{p}-x}{\hat{p}-\hat{s}}\right)\left(1-\mathrm{T}^{u}_{\dot{N}}\right) & \hat{s}\le x\le\hat{p}\\ \left(\dfrac{\hat{z}-x}{\hat{z}-\hat{p}}\right)\mathrm{T}^{u}_{\dot{N}} & \hat{p}\le x\le\hat{z}\\ 0 & \text{otherwise}\end{cases}$$

$$\mathrm{I}^{u}_{\dot{N}}(x)=\begin{cases}\left(\dfrac{\hat{\xi}-x}{\hat{\xi}-\hat{\delta}}\right)\mathrm{I}^{u}_{\dot{N}} & \hat{\delta}\le x\le\hat{\xi}\\ 1-\left(\dfrac{\hat{c}-x}{\hat{c}-\hat{\xi}}\right)\left(1-\mathrm{I}^{u}_{\dot{N}}\right) & \hat{\xi}\le x\le\hat{c}\\ 0 & x=\hat{c}\\ 1-\left(\dfrac{x-\hat{c}}{\hat{ə}-\hat{c}}\right)\left(1-\mathrm{I}^{u}_{\dot{N}}\right) & \hat{c}\le x\le\hat{ə}\\ \left(\dfrac{x-\hat{ə}}{\hat{\gamma}-\hat{ə}}\right)\mathrm{I}^{u}_{\dot{N}} & \hat{ə}\le x\le\hat{\gamma}\\ 1 & \text{otherwise}\end{cases}$$

$$\mathrm{F}^{u}_{\dot{N}}(x)=\begin{cases}\left(\dfrac{\hat{t}-x}{\hat{t}-\hat{\tau}}\right)\mathrm{F}^{u}_{\dot{N}} & \hat{\tau}\le x\le\hat{t}\\ 1-\left(\dfrac{\hat{\vartheta}-x}{\hat{\vartheta}-\hat{t}}\right)\left(1-\mathrm{F}^{u}_{\dot{N}}\right) & \hat{t}\le x\le\hat{\vartheta}\\ 0 & x=\hat{\vartheta}\\ 1-\left(\dfrac{x-\hat{\vartheta}}{\hat{\kappa}-\hat{\vartheta}}\right)\left(1-\mathrm{F}^{u}_{\dot{N}}\right) & \hat{\vartheta}\le x\le\hat{\kappa}\\ \left(\dfrac{x-\hat{\kappa}}{\hat{\zeta}-\hat{\kappa}}\right)\mathrm{F}^{u}_{\dot{N}} & \hat{\kappa}\le x\le\hat{\zeta}\\ 1 & \text{otherwise"}\end{cases}$$

An IVPFNN $\dot{\eta}$ is represented by

$$\dot{\eta}=\left\{\begin{matrix}\left\langle\left(\breve{g},\breve{\varepsilon},\breve{s},\breve{p},\breve{z}:\mathrm{T}^{l}_{\dot{\eta}}\right),\left(\hat{g},\hat{\varepsilon},\hat{s},\hat{p},\hat{z}:\mathrm{T}^{u}_{\dot{\eta}}\right)\right\rangle,\\ \left\langle\left(\breve{\delta},\breve{\xi},\breve{c},\breve{ə},\breve{\gamma}:\mathrm{I}^{l}_{\dot{\eta}}\right),\left(\hat{\delta},\hat{\xi},\hat{c},\hat{ə},\hat{\gamma}:\mathrm{I}^{u}_{\dot{\eta}}\right)\right\rangle,\\ \left\langle\left(\breve{\tau},\breve{t},\breve{\vartheta},\breve{\kappa},\breve{\zeta}:\mathrm{F}^{l}_{\dot{\eta}}\right),\left(\hat{\tau},\hat{t},\hat{\vartheta},\hat{\kappa},\hat{\zeta}:\mathrm{F}^{u}_{\dot{\eta}}\right)\right\rangle\end{matrix}\right\}$$

Definition 4. [6]: "Let $\dot{\eta}_1$ and $\dot{\eta}_2$ be two IVPFNNs,

$$\dot{\eta}_1=\left\{\begin{matrix}\left\langle\left[\left(\breve{g}_1,\breve{\varepsilon}_1,\breve{s}_1,\breve{p}_1,\breve{z}_1\right),\left(\hat{g}_1,\hat{\varepsilon}_1,\hat{s}_1,\hat{p}_1,\hat{z}_1\right)\right]:\mathrm{T}_{\dot{\eta}_1}\right\rangle\\ \left\langle\left[\left(\breve{\delta}_1,\breve{\xi}_1,\breve{c}_1,\breve{ə}_1,\breve{\gamma}_1\right),\left(\hat{\delta}_1,\hat{\xi}_1,\hat{c}_1,\hat{ə}_1,\hat{\gamma}_1\right)\right]:\mathrm{I}_{\dot{\eta}_1}\right\rangle\\ \left\langle\left[\left(\breve{\tau}_1,\breve{t}_1,\breve{\vartheta}_1,\breve{\kappa}_1,\breve{\zeta}_1\right),\left(\hat{\tau}_1,\hat{t}_1,\hat{\vartheta}_1,\hat{\kappa}_1,\hat{\zeta}_1\right)\right]:\mathrm{F}_{\dot{\eta}_1}\right\rangle\end{matrix}\right\}$$

$$\dot{\eta}_2=\left\{\begin{matrix}\left\langle\left[\left(\breve{g}_2,\breve{\varepsilon}_2,\breve{s}_2,\breve{p}_2,\breve{z}_2\right),\left(\hat{g}_2,\hat{\varepsilon}_2,\hat{s}_2,\hat{p}_2,\hat{z}_2\right)\right]:\mathrm{T}_{\dot{\eta}_2}\right\rangle\\ \left\langle\left[\left(\breve{\delta}_2,\breve{\xi}_2,\breve{c}_2,\breve{ə}_2,\breve{\gamma}_2\right),\left(\hat{\delta}_2,\hat{\xi}_2,\hat{c}_2,\hat{ə}_2,\hat{\gamma}_2\right)\right]:\mathrm{I}_{\dot{\eta}_2}\right\rangle\\ \left\langle\left[\left(\breve{\tau}_2,\breve{t}_2,\breve{\vartheta}_2,\breve{\kappa}_2,\breve{\zeta}_2\right),\left(\hat{\tau}_2,t_2,\hat{\vartheta}_2,\hat{\kappa}_2,\hat{\zeta}_2\right)\right]:\mathrm{F}_{\dot{\eta}_2}\right\rangle\end{matrix}\right\}$$

Then the following operations on IVPFNNs are proposed as:

$$\dot{\eta}_1+\dot{\eta}_2=\left\{\begin{array}{c}\left[\begin{array}{l}(\breve{g}_1+\breve{g}_2-\breve{g}_1\breve{g}_2,\breve{\varepsilon}_1+\breve{\varepsilon}_2-\breve{\varepsilon}_1\breve{\varepsilon}_2,\breve{s}_1+\breve{s}_2-\breve{s}_1\breve{s}_2,\breve{p}_1+\breve{p}_2-\breve{p}_1\breve{p}_2,\breve{z}_1+\breve{z}_2-\breve{z}_1\breve{z}_2),\\(\hat{g}_1+\hat{g}_2-\hat{g}_1\hat{g}_2,\hat{\varepsilon}_1+\hat{\varepsilon}_2-\hat{\varepsilon}_1\hat{\varepsilon}_2,\hat{s}_1+\hat{s}_2-\hat{s}_1\hat{s}_2,\hat{p}_1+\hat{p}_2-\hat{p}_1\hat{p}_2,\hat{z}_1+\hat{z}_2-\hat{z}_1\hat{z}_2)\end{array}\right]\\ \left[(\breve{\delta}_1\breve{\delta}_2,\breve{\xi}_1\breve{\xi}_2,\breve{e}_1\breve{e}_2,\breve{\sigma}_1\breve{\sigma}_2,\breve{\gamma}_1\breve{\gamma}_2),(\hat{\delta}_1\hat{\delta}_2,\hat{\xi}_1\hat{\xi}_2,\hat{e}_1\hat{e}_2,\hat{\sigma}_1\hat{\sigma}_2,\hat{\gamma}_1\hat{\gamma}_2)\right],\\ \left[(\breve{\upsilon}_1\breve{\upsilon}_2,\breve{l}_1\breve{l}_2,\breve{\vartheta}_1\breve{\vartheta}_2,\breve{k}_1\breve{k}_2,\breve{\zeta}_1\breve{\zeta}_2),(\hat{\upsilon}_1\hat{\upsilon}_2,\hat{l}_1\hat{l}_2,\hat{\vartheta}_1\hat{\vartheta}_2,\hat{k}_1\hat{k}_2,\hat{\zeta}_1\hat{\zeta}_2)\right]\end{array}\right\}$$

$$\dot{\eta}_1\times\dot{\eta}_2=\left\{\begin{array}{c}\left[(\breve{g}_1\breve{g}_2,\breve{\varepsilon}_1\breve{\varepsilon}_2,\breve{s}_1\breve{s}_2,\breve{p}_1\breve{p}_2,\breve{z}_1\breve{z}_2),(\hat{g}_1\hat{g}_2,\hat{\varepsilon}_1\hat{\varepsilon}_2,\hat{s}_1\hat{s}_2,\hat{p}_1\hat{p}_2,\hat{z}_1\hat{z}_2)\right],\\ \left[\begin{array}{l}(\breve{\delta}_1+\breve{\delta}_2-\breve{\delta}_1\breve{\delta}_2,\breve{\xi}_1+\breve{\xi}_2-\breve{\xi}_1\breve{\xi}_2,\breve{e}_1+\breve{e}_2-\breve{e}_1\breve{e}_2,\breve{\sigma}_1+\breve{\sigma}_2-\breve{\sigma}_1\breve{\sigma}_2,\breve{\gamma}_1+\breve{\gamma}_2-\breve{\gamma}_1\breve{\gamma}_2),\\(\hat{\delta}_1+\hat{\delta}_2-\hat{\delta}_1\hat{\delta}_2,\hat{\xi}_1+\hat{\xi}_2-\hat{\xi}_1\hat{\xi}_2\hat{e}_1+\hat{e}_2-\hat{e}_1\hat{e}_2,\hat{\sigma}_1+\hat{\sigma}_2-\hat{\sigma}_1\hat{\sigma}_2,\hat{\gamma}_1+\hat{\gamma}_2-\hat{\gamma}_1\hat{\gamma}_2)\end{array}\right],\\ \left[\begin{array}{l}(\breve{\upsilon}_1+\breve{\upsilon}_2-\breve{\upsilon}_1\breve{\upsilon}_2,\breve{l}_1+\breve{l}_2-\breve{l}_1\breve{l}_2,\breve{\vartheta}_1+\breve{\vartheta}_2-\breve{\vartheta}_1\breve{\vartheta}_2,\breve{k}_1+\breve{k}_2-\breve{k}_1\breve{k}_2,\breve{\zeta}_1+\breve{\zeta}_2-\breve{\zeta}_1\breve{\zeta}_2),\\(\hat{\upsilon}_1+\hat{\upsilon}_2-\hat{\upsilon}_1\hat{\upsilon}_2,\hat{l}_1+\hat{l}_2-\hat{l}_1\hat{l}_2,\hat{\vartheta}_1+\hat{\vartheta}_2-\hat{\vartheta}_1\hat{\vartheta}_2,\hat{k}_1+\hat{k}_2-\hat{k}_1\hat{k}_2,\hat{\zeta}_1+\hat{\zeta}_2-\hat{\zeta}_1\hat{\zeta}_2)\end{array}\right]\end{array}\right\}$$

$$\upsilon\dot{\eta}_1=\left\{\begin{array}{c}\left[\begin{array}{l}(1-(1-\breve{g}_1)^{\upsilon},1-(1-\breve{\varepsilon}_1)^{\upsilon},1-(1-\breve{s}_1)^{\upsilon},1-(1-\breve{p}_1)^{\upsilon},1-(1-\breve{z}_1)^{\upsilon}),\\(1-(1-\hat{g}_1)^{\upsilon},1-(1-\hat{\varepsilon}_1)^{\upsilon},1-(1-\hat{s}_1)^{\upsilon},1-(1-\hat{p}_1)^{\upsilon},1-(1-\hat{z}_1)^{\upsilon})\end{array}\right],\\ \left[(\breve{\delta}_1^{\ \upsilon},\breve{\xi}_1^{\ \upsilon},\breve{e}_1^{\ \upsilon},\breve{\sigma}_1^{\ \upsilon},\breve{\gamma}_1^{\ \upsilon}),(\hat{\delta}_1^{\ \upsilon},\hat{\xi}_1^{\ \upsilon},\hat{e}_1^{\ \upsilon},\hat{\sigma}_1^{\ \upsilon},\hat{\gamma}_1^{\ \upsilon})\right],\\ \left[(\breve{\upsilon}_1^{\ \upsilon},\breve{l}_1^{\ \upsilon},\breve{\vartheta}_1^{\ \upsilon},\breve{k}_1^{\ \upsilon},\breve{\zeta}_1^{\ \upsilon}),(\hat{\upsilon}_1^{\ \upsilon},\hat{l}_1^{\ \upsilon},\hat{\vartheta}_1^{\ \upsilon},\hat{k}_1^{\ \upsilon},\hat{\zeta}_1^{\ \upsilon})\right]\end{array}\right\},\upsilon>0$$

$$\dot{\eta}_1^{\ \upsilon}=\left\{\begin{array}{c}\left[(\breve{g}_1^{\ \upsilon},\breve{\varepsilon}_1^{\ \upsilon},\breve{s}_1^{\ \upsilon},\breve{p}_1^{\ \upsilon},\breve{z}_1^{\ \upsilon}),(\hat{g}_1^{\ \upsilon},\hat{\varepsilon}_1^{\ \upsilon},\hat{s}_1^{\ \upsilon},\hat{p}_1^{\ \upsilon},\hat{z}_1^{\ \upsilon})\right],\\ \left[\begin{array}{l}(1-(1-\breve{\delta}_1)^{\upsilon},1-(1-\breve{\xi}_1)^{\upsilon},1-(1-\breve{e}_1)^{\upsilon},1-(1-\breve{\sigma}_1)^{\upsilon},1-(1-\breve{\gamma}_1)^{\upsilon}),\\(1-(1-\hat{\delta}_1)^{\upsilon},1-(1-\hat{\xi}_1)^{\upsilon},1-(1-\hat{e}_1)^{\upsilon},1-(1-\hat{\sigma}_1)^{\upsilon},1-(1-\hat{\gamma}_1)^{\upsilon})\end{array}\right],\\ \left[\begin{array}{l}(1-(1-\breve{\upsilon}_1)^{\upsilon},1-(1-\breve{l}_1)^{\upsilon},1-(1-\breve{\vartheta}_1)^{\upsilon},1-(1-\breve{k}_1)^{\upsilon},1-(1-\breve{\zeta}_1)^{\upsilon}),\\(1-(1-\hat{\upsilon}_1)^{\upsilon},1-(1-\hat{l}_1)^{\upsilon},1-(1-\hat{\vartheta}_1)^{\upsilon},1-(1-\hat{k}_1)^{\upsilon},1-(1-\hat{\zeta}_1)^{\upsilon})\end{array}\right]\end{array}\right\},\upsilon>0\text{”}$$

Definition 5. [7] “The score and accuracy function of IVPFNN based on the pentagonal neutrosophic numbers $\dot{\eta}$ are defined as follows

$$\mathcal{S}(\dot{\eta})=\frac{1}{6}\left[4+\frac{\breve{g}+\breve{\varepsilon}+\breve{s}+\breve{p}+\breve{z}}{5}+\frac{\hat{g}+\hat{\varepsilon}+\hat{s}+\hat{p}+\hat{z}}{5}-\frac{\breve{\delta}+\breve{\xi}+\breve{e}+\breve{\sigma}+\breve{\gamma}}{5}-\frac{\hat{\delta}+\hat{\xi}+\hat{e}+\hat{\sigma}+\hat{\gamma}}{5}-\frac{\breve{\upsilon}+\breve{l}+\breve{\vartheta}+\breve{k}+\breve{\zeta}}{5}-\frac{\hat{\upsilon}+\hat{l}+\hat{\vartheta}+\hat{k}+\hat{\zeta}}{5}\right]\qquad(1)$$

where $S(\dot{\eta})\in[0,1]$.

$\mathcal{S}(\dot{\eta})=0$, then

$$\dot{\eta}=\left\{\begin{array}{c}[(0,0,0,0,0),(0,0,0,0,0)],\\ [(1,1,1,1,1),(1,1,1,1,1)],\\ [(1,1,1,1,1),(1,1,1,1,1)]\end{array}\right\}$$

which is the smallest IVPFNN.

Larger value of $\mathcal{S}(\dot{\eta})$ implies higher IVPFNN $\dot{\eta}$.

If $\mathcal{S}(\dot{\eta})=1$, then $\dot{\eta}=\left\{\begin{array}{c}[(1,1,1,1,1),(1,1,1,1,1)],\\ [(0,0,0,0,0),(0,0,0,0,0)],\\ [(0,0,0,0,0),(0,0,0,0,0)]\end{array}\right\}$

which is the largest IVPFNN.

For convenience, it can also be written as

$$\dot{\eta}=\left\{\begin{array}{c}\langle[(\breve{g},\breve{\varepsilon},\breve{s},\breve{p},\breve{z}),(\hat{g},\hat{\varepsilon},\hat{s},\hat{p},\hat{z})]:T_{\dot{\eta}}\rangle\\ ,\langle[(\breve{\delta},\breve{\xi},\breve{e},\breve{\sigma},\breve{\gamma}),(\hat{\delta},\hat{\xi},\hat{e},\hat{\sigma},\hat{\gamma}):I_{\dot{\eta}}]\rangle,\\ \langle[(\breve{\upsilon},\breve{l},\breve{\vartheta},\breve{k},\breve{\zeta}),(\hat{\upsilon},\hat{l},\hat{\vartheta},\hat{k},\hat{\zeta})]:F_{\dot{\eta}}\rangle\end{array}\right\}$$

Accuracy function for IVPFNN $\dot{\eta}$ is given by

$$A_c(\dot{\eta})=\frac{1}{2}\left[\frac{\breve{g}+\breve{\varepsilon}+\breve{s}+\breve{p}+\breve{z}}{5}+\frac{\hat{g}+\hat{\varepsilon}+\hat{s}+\hat{p}+\hat{z}}{5}-\frac{\breve{\upsilon}+\breve{l}+\breve{\vartheta}+\breve{k}+\breve{\zeta}}{5}-\frac{\hat{\upsilon}+\hat{l}+\hat{\vartheta}+\hat{k}+\hat{\zeta}}{5}\right]$$ where

$A_c(\dot{\eta})\in[-1,1]$".

Definition 6. [7] “Let $\dot{\eta}_1$ and $\dot{\eta}_2$ be two IVPFNNs, $\mathcal{S}(\dot{\eta}_1),\mathcal{S}(\dot{\eta}_2)$ be the score functions and $A_c(\dot{\eta}_1),A_c(\dot{\eta}_2)$ be the accuracy functions

- If $\mathcal{S}(\dot{\eta}_1)>\mathcal{S}(\dot{\eta}_2)$, then $\dot{\eta}_1>\dot{\eta}_2$
- If $\mathcal{S}(\dot{\eta}_1)<\mathcal{S}(\dot{\eta}_2)$, then $\dot{\eta}_1<\dot{\eta}_2$
- If $\mathcal{S}(\dot{\eta}_1)=\mathcal{S}(\dot{\eta}_2)$ and if
  a) If $A_c(\dot{\eta})=A_c(\dot{\eta}_2)$, then $\dot{\eta}_1=\dot{\eta}_2$
  b) If $A_c(\dot{\eta}_1)>A_c(\dot{\eta}_2)$, then $\dot{\eta}_1>\dot{\eta}_2$
  c) If $A_c(\dot{\eta}_1)<A_c(\dot{\eta}_2)$, then $\dot{\eta}_1<\dot{\eta}_2$”.

III. INTERVAL-VALUED PENTAGONAL FUZZY NEUTROSOPHIC WEIGHTED GEOMETRIC AVERAGING OPERATOR

$$\text{Let,}\dot{\eta}_i=\left\{\begin{array}{c}\left[(\overleftarrow{t}_i,\overleftarrow{g}_i,\overleftarrow{b}_i,\overleftarrow{s}_i,\overleftarrow{z}_i),(\vec{t}_i,\vec{g}_i,\vec{b}_i,\vec{s}_i,\vec{z}_i)\right],\\ \left[(\overleftarrow{l}_i,\overleftarrow{h}_i,\overleftarrow{z}_i,\overleftarrow{n}_i,\overleftarrow{t}_i),(\vec{l}_i,\vec{h}_i,\vec{z}_i,\vec{n}_i,\vec{t}_i)\right],\\ \left[(\overleftarrow{\vartheta}_i,\overleftarrow{s}_i,\overleftarrow{\zeta}_i,\overleftarrow{v}_i,\overleftarrow{t}_i),(\vec{\vartheta}_i,\vec{s}_i,\vec{\zeta}_i,\vec{v}_i,\vec{t}_i)\right]\end{array}\right\}$$

where $i=1,2,\ldots,n$.

be the representation of IVPFNS is the set of real numbers, whereas IVPFNWG: $\varsigma^n\longrightarrow\varsigma$. The interval-valued pentagonal fuzzy neutrosophic weighted geometric (IVPFNWG) operator defined byIVPFNWG$(\dot{\eta}_1,\dot{\eta}_2,\ldots,\dot{\eta}_n)$

is defined as $\text{IVPFNWG}(\dot{\eta}_1,\dot{\eta}_2,\dots,\dot{\eta}_n)=\dot{\eta}_1^{w_1}\otimes\dot{\eta}_2^{w_1}\otimes\dots\otimes\dot{\eta}_n^{w_n}=\otimes_{i=1}^{n}\left(\dot{\eta}_i^{w_i}\right)$ where $w_i\in[0,1]$ is the exponential weight vector of $\dot{\eta}_i(i=1,2,\dots,n)$ such that $\sum_{i=1}^{n}w_i=1$.

Theorem 1. Let $\dot{\eta}_i=\left\{\begin{array}{l}\left[\left(\overleftarrow{t}_i,\overleftarrow{g}_i,\overleftarrow{b}_i,\overleftarrow{s}_i,\overleftarrow{z}_i\right),\left(\overrightarrow{t}_i,\overrightarrow{g}_i,\overrightarrow{b}_i,\overrightarrow{s}_i,\overrightarrow{z}_i\right)\right],\\ \left[\left(\overleftarrow{l}_i,\overleftarrow{\hbar}_i,\overleftarrow{\mathfrak{z}}_i,\overleftarrow{n}_i,\overleftarrow{\mathfrak{t}}_i\right),\left(\overrightarrow{l}_i,\overrightarrow{\hbar}_i,\overrightarrow{\mathfrak{z}}_i,\overrightarrow{n}_i,\overrightarrow{\mathfrak{t}}_i\right)\right],\\ \left[\left(\overleftarrow{\vartheta}_i,\overleftarrow{\varsigma}_i,\overleftarrow{\zeta}_i,\overleftarrow{\upsilon}_i,\overleftarrow{\tau}_i\right),\left(\overrightarrow{\vartheta}_i,\overrightarrow{\varsigma}_i,\overrightarrow{\zeta}_i,\overrightarrow{\upsilon}_i,\overrightarrow{\tau}_i\right)\right]\end{array}\right\}$

$i=1,2,\dots n$ be a collection of IVPFNNs in the set of real numbers. Then aggregated value obtained from IVPFNWG is also an IVPFNN, then we have

$$\text{IVPFNWG}(\dot{\eta}_1,\dot{\eta}_2,\dots,\dot{\eta}_n)=\dot{\eta}_1^{w_1}\otimes\dot{\eta}_2^{w_1}\otimes\dots\otimes\dot{\eta}_n^{w_n}=\otimes_{i=1}^{n}\left(\dot{\eta}_i^{w_i}\right)$$

where $w_i\in[0,1]$ is the weight vector of IVPFNN $\dot{\eta}_i$ $(i=1,2,\dots,n)$ such that $\sum_{i=1}^{n}w_i=1$.

Proof:
By mathematical induction we prove this theorem,

for $n=1$, it is trivial.

For $n=2$, $\otimes_{i=1}^{n}\left(\dot{\eta}_i\right)^{w_i}=\dot{\eta}_1^{w_1}\otimes\dot{\eta}_2^{w_2}$

$$=\left\{\begin{array}{l}\left[\left(\prod_{i=1}^{n}\overleftarrow{t}_i^{w_i},\prod_{i=1}^{n}\overleftarrow{g}_i^{w_i},\prod_{i=1}^{n}\overleftarrow{b}_i^{w_i},\prod_{i=1}^{n}\overleftarrow{s}_i^{w_i},\prod_{i=1}^{n}\overleftarrow{z}_i^{w_i}\right),\left(\prod_{i=1}^{n}\overrightarrow{t}_i^{w_i},\prod_{i=1}^{n}\overrightarrow{g}_i^{w_i},\prod_{i=1}^{n}\overrightarrow{b}_i^{w_i},\prod_{i=1}^{n}\overrightarrow{s}_i^{w_i},\prod_{i=1}^{n}\overrightarrow{z}_i^{w_i}\right)\right],\\ \left[\begin{array}{l}\left(1-\prod_{i=1}^{n}(1-\overleftarrow{l}_i)^{w_i},1-\prod_{i=1}^{n}(1-\overleftarrow{\hbar}_i)^{w_i},1-\prod_{i=1}^{n}(1-\overleftarrow{\mathfrak{z}}_i)^{w_i},1-\prod_{i=1}^{n}(1-\overleftarrow{n}_i)^{w_i},1-\prod_{i=1}^{n}(1-\overleftarrow{\mathfrak{t}}_i)^{w_i}\right),\\ \left(1-\prod_{i=1}^{n}(1-\overrightarrow{l}_i)^{w_i},1-\prod_{i=1}^{n}(1-\overrightarrow{\hbar}_i)^{w_i},1-\prod_{i=1}^{n}(1-\overrightarrow{\mathfrak{z}}_i)^{w_i},1-\prod_{i=1}^{n}(1-\overrightarrow{n}_i)^{w_i},1-\prod_{i=1}^{n}(1-\overrightarrow{\mathfrak{t}}_i)^{w_i}\right)\end{array}\right],\\ \left[\begin{array}{l}\left(1-\prod_{i=1}^{n}(1-\overleftarrow{\vartheta}_i)^{w_i},1-\prod_{i=1}^{n}(1-\overleftarrow{\varsigma}_i)^{w_i},1-\prod_{i=1}^{n}(1-\overleftarrow{\zeta}_i)^{w_i},1-\prod_{i=1}^{n}(1-\overleftarrow{\upsilon}_i)^{w_i},1-\prod_{i=1}^{n}(1-\overleftarrow{\tau}_i)^{w_i}\right),\\ \left(1-\prod_{i=1}^{n}(1-\overrightarrow{\vartheta}_i)^{w_i},1-\prod_{i=1}^{n}(1-\overrightarrow{\varsigma}_i)^{w_i},1-\prod_{i=1}^{n}(1-\overrightarrow{\zeta}_i)^{w_i},1-\prod_{i=1}^{n}(1-\overrightarrow{\upsilon}_i)^{w_i},1-\prod_{i=1}^{n}(1-\overrightarrow{\tau}_i)^{w_i}\right)\end{array}\right]\end{array}\right\} \qquad (2)$$

$$=\left\{\begin{array}{l}\left[\left(\prod_{i=1}^{2}\overleftarrow{t}_i^{w_i},\prod_{i=1}^{2}\overleftarrow{g}_i^{w_i},\prod_{i=1}^{2}\overleftarrow{b}_i^{w_i},\prod_{i=1}^{2}\overleftarrow{s}_i^{w_i},\prod_{i=1}^{2}\overleftarrow{z}_i^{w_i}\right),\left(\prod_{i=1}^{2}\overrightarrow{t}_i^{w_i},\prod_{i=1}^{2}\overrightarrow{g}_i^{w_i},\prod_{i=1}^{2}\overrightarrow{b}_i^{w_i},\prod_{i=1}^{2}\overrightarrow{s}_i^{w_i},\prod_{i=1}^{2}\overrightarrow{z}_i^{w_i}\right)\right],\\ \left[\begin{array}{l}\left(1-\prod_{i=1}^{2}(1-\overleftarrow{l}_i)^{w_i},1-\prod_{i=1}^{2}(1-\overleftarrow{\hbar}_i)^{w_i},1-\prod_{i=1}^{2}(1-\overleftarrow{\mathfrak{z}}_i)^{w_i},1-\prod_{i=1}^{2}(1-\overleftarrow{n}_i)^{w_i},1-\prod_{i=1}^{2}(1-\overleftarrow{\mathfrak{t}}_i)^{w_i}\right),\\ \left(1-\prod_{i=1}^{2}(1-\overrightarrow{l}_i)^{w_i},1-\prod_{i=1}^{2}(1-\overrightarrow{\hbar}_i)^{w_i},1-\prod_{i=1}^{2}(1-\overrightarrow{\mathfrak{z}}_i)^{w_i},1-\prod_{i=1}^{2}(1-\overrightarrow{n}_i)^{w_i},1-\prod_{i=1}^{2}(1-\overrightarrow{\mathfrak{t}}_i)^{w_i}\right)\end{array}\right],\\ \left[\begin{array}{l}\left(1-\prod_{i=1}^{2}(1-\overleftarrow{\vartheta}_i)^{w_i},1-\prod_{i=1}^{2}(1-\overleftarrow{\varsigma}_i)^{w_i},1-\prod_{i=1}^{2}(1-\overleftarrow{\zeta}_i)^{w_i},1-\prod_{i=1}^{2}(1-\overleftarrow{\upsilon}_i)^{w_i},1-\prod_{i=1}^{2}(1-\overleftarrow{\tau}_i)^{w_i}\right),\\ \left(1-\prod_{i=1}^{2}(1-\overrightarrow{\vartheta}_i)^{w_i},1-\prod_{i=1}^{2}(1-\overrightarrow{\varsigma}_i)^{w_i},1-\prod_{i=1}^{2}(1-\overrightarrow{\zeta}_i)^{w_i},1-\prod_{i=1}^{2}(1-\overrightarrow{\upsilon}_i)^{w_i},1-\prod_{i=1}^{2}(1-\overrightarrow{\tau}_i)^{w_i}\right)\end{array}\right]\end{array}\right\}$$

$$=\left\{\begin{array}{c}\left(\begin{array}{l}\left[\left(\overleftarrow{t}_1^{w_1},\overleftarrow{g}_1^{w_1},\overleftarrow{b}_1^{w_1},\overleftarrow{s}_1^{w_1},\overleftarrow{z}_1^{w_1}\right),\left(\overrightarrow{t}_1^{w_1},\overrightarrow{g}_1^{w_1},\overrightarrow{b}_1^{w_1},\overrightarrow{s}_1^{w_1},\overrightarrow{z}_1^{w_1}\right)\right],\\ \left[\begin{array}{l}\left(1-(1-\overleftarrow{l}_1)^{w_1},1-(1-\overleftarrow{\hbar}_1)^{w_1},1-(1-\overleftarrow{\mathfrak{z}}_1)^{w_1},1-(1-\overleftarrow{n}_1)^{w_1},1-(1-\overleftarrow{\mathfrak{t}}_1)^{w_1}\right),\\ \left(1-(1-\overrightarrow{l}_1)^{w_1},1-(1-\overrightarrow{\hbar}_1)^{w_1},1-(1-\overrightarrow{\mathfrak{z}}_1)^{w_1},1-(1-\overrightarrow{n}_1)^{w_1},1-(1-\overrightarrow{\mathfrak{t}}_1)^{w_1}\right)\end{array}\right],\\ \left[\begin{array}{l}\left(1-(1-\overleftarrow{\vartheta}_1)^{w_1},1-(1-\overleftarrow{\varsigma}_1)^{w_1},1-(1-\overleftarrow{\zeta}_1)^{w_1},1-(1-\overleftarrow{\upsilon}_1)^{w_1},1-(1-\overleftarrow{\tau}_1)^{w_1}\right),\\ \left(1-(1-\overrightarrow{\vartheta}_1)^{w_1},1-(1-\overrightarrow{\varsigma}_1)^{w_1},1-(1-\overrightarrow{\zeta}_1)^{w_1},1-(1-\overrightarrow{\upsilon}_1)^{w_1},1-(1-\overrightarrow{\tau}_1)^{w_1}\right)\end{array}\right]\end{array}\right)\\ \otimes\\ \left(\begin{array}{l}\left[\left(\overleftarrow{t}_2^{w_2},\overleftarrow{g}_2^{w_2},\overleftarrow{b}_2^{w_2},\overleftarrow{s}_2^{w_2},\overleftarrow{z}_2^{w_2}\right),\left(\overrightarrow{t}_2^{w_2},\overrightarrow{g}_2^{w_2},\overrightarrow{b}_2^{w_2},\overrightarrow{s}_2^{w_2},\overrightarrow{z}_2^{w_2}\right)\right],\\ \left[\begin{array}{l}\left(1-(1-\overleftarrow{l}_2)^{w_2},1-(1-\overleftarrow{\hbar}_2)^{w_2},1-(1-\overleftarrow{\mathfrak{z}}_2)^{w_2},1-(1-\overleftarrow{n}_2)^{w_2},1-(1-\overleftarrow{\mathfrak{t}}_2)^{w_2}\right),\\ \left(1-(1-\overrightarrow{l}_2)^{w_2},1-(1-\overrightarrow{\hbar}_2)^{w_2},1-(1-\overrightarrow{\mathfrak{z}}_2)^{w_2},1-(1-\overrightarrow{n}_2)^{w_2},1-(1-\overrightarrow{\mathfrak{t}}_2)^{w_2}\right)\end{array}\right],\\ \left[\begin{array}{l}\left(1-(1-\overleftarrow{\vartheta}_2)^{w_2},1-(1-\overleftarrow{\varsigma}_2)^{w_2},1-(1-\overleftarrow{\zeta}_2)^{w_2},1-(1-\overleftarrow{\upsilon}_2)^{w_2},1-(1-\overleftarrow{\tau}_2)^{w_2}\right),\\ \left(1-(1-\overrightarrow{\vartheta}_2)^{w_2},1-(1-\overrightarrow{\varsigma}_2)^{w_2},1-(1-\overrightarrow{\zeta}_2)^{w_2},1-(1-\overrightarrow{\upsilon}_2)^{w_2},1-(1-\overrightarrow{\tau}_2)^{w_2}\right)\end{array}\right]\end{array}\right)\end{array}\right\}$$

$$
= \left\{ \begin{array}{c}
\left[\begin{array}{c}
\left(\overleftarrow{t}_1^{\,w_1} \overleftarrow{t}_2^{\,w_2}, \overleftarrow{g}_1^{\,w_1} \overleftarrow{g}_2^{\,w_2}, \overleftarrow{b}_1^{\,w_1} \overleftarrow{b}_2^{\,w_2}, \overleftarrow{s}_1^{\,w_1} \overleftarrow{s}_2^{\,w_2}, \overleftarrow{z}_1^{\,w_1} \overleftarrow{z}_2^{\,w_2} \right), \\
\left(\overrightarrow{t}_1^{\,w_1} \overrightarrow{t}_2^{\,w_2}, \overrightarrow{g}_1^{\,w_1} \overrightarrow{g}_2^{\,w_2}, \overrightarrow{b}_1^{\,w_1} \overrightarrow{b}_2^{\,w_2}, \overrightarrow{s}_1^{\,w_1} \overrightarrow{s}_2^{\,w_2}, \overrightarrow{z}_1^{\,w_1} \overrightarrow{z}_2^{\,w_2} \right)
\end{array} \right], \\
\left(\begin{array}{c}
\left[\begin{array}{c}
\left(1 - (1 - \overleftarrow{l}_1)^{w_1} + 1 - (1 - \overleftarrow{l}_2)^{w_2} - \left(1 - (1 - \overleftarrow{l}_1)^{w_1} \right) \left(1 - (1 - \overleftarrow{l}_2)^{w_2} \right) \right), \\
\left(1 - (1 - \overleftarrow{h}_1)^{w_1} + 1 - (1 - \overleftarrow{h}_2)^{w_2} - \left(1 - (1 - \overleftarrow{h}_1)^{w_1} \right) \left(1 - (1 - \overleftarrow{h}_2)^{w_2} \right) \right), \\
\left(1 - (1 - \overleftarrow{z}_1)^{w_1} + 1 - (1 - \overleftarrow{z}_2)^{w_2} - (1 - (1 - \overleftarrow{z}_1)^{w_1})(1 - (1 - \overleftarrow{z}_2)^{w_2}) \right), \\
\left(1 - (1 - \overleftarrow{n}_1)^{w_1} + 1 - (1 - \overleftarrow{n}_2)^{w_2} - (1 - (1 - \overleftarrow{n}_1)^{w_1})(1 - (1 - \overleftarrow{n}_2)^{w_2}) \right), \\
\left(1 - (1 - \overleftarrow{f}_1)^{w_1} + 1 - (1 - \overleftarrow{f}_2)^{w_2} - \left(1 - (1 - \overleftarrow{f}_1)^{w_1} \right) \left(1 - (1 - \overleftarrow{f}_2)^{w_2} \right) \right)
\end{array} \right], \\
\left[\begin{array}{c}
\left(1 - (1 - \overrightarrow{l}_1)^{w_1} + 1 - (1 - \overrightarrow{l}_2)^{w_2} - \left(1 - (1 - \overrightarrow{l}_1)^{w_1} \right) \left(1 - (1 - \overrightarrow{l}_2)^{w_2} \right) \right), \\
\left(1 - (1 - \overrightarrow{h}_1)^{w_1} + 1 - (1 - \overrightarrow{h}_2)^{w_2} - \left(1 - (1 - \overrightarrow{h}_1)^{w_1} \right) \left(1 - (1 - \overrightarrow{h}_2)^{w_2} \right) \right) \\
\left(1 - (1 - \overrightarrow{z}_1)^{w_1} + 1 - (1 - \overrightarrow{z}_2)^{w_2} - (1 - (1 - \overrightarrow{z}_1)^{w_1})(1 - (1 - \overrightarrow{z}_2)^{w_2}) \right) \\
\left(1 - (1 - \overrightarrow{n}_1)^{w_1} + 1 - (1 - \overrightarrow{n}_2)^{w_2} - (1 - (1 - \overrightarrow{n}_1)^{w_1})(1 - (1 - \overrightarrow{n}_2)^{w_2}) \right) \\
\left(1 - (1 - \overrightarrow{f}_1)^{w_1} + 1 - (1 - \overrightarrow{f}_2)^{w_2} - \left(1 - (1 - \overrightarrow{f}_1)^{w_1} \right) \left(1 - (1 - \overrightarrow{f}_2)^{w_2} \right) \right)
\end{array} \right]
\end{array} \right), \\
\left(\begin{array}{c}
\left[\begin{array}{c}
\left(1 - (1 - \overleftarrow{\vartheta}_1)^{w_1} + 1 - (1 - \overleftarrow{\vartheta}_2)^{w_2} - \left(1 - (1 - \overleftarrow{\vartheta}_1)^{w_1} \right) \left(1 - (1 - \overleftarrow{\vartheta}_2)^{w_2} \right) \right), \\
\left(1 - (1 - \overleftarrow{s}_1)^{w_1} + 1 - (1 - \overleftarrow{s}_2)^{w_2} - (1 - (1 - \overleftarrow{s}_1)^{w_1})(1 - (1 - \overleftarrow{s}_2)^{w_2}) \right), \\
\left(1 - (1 - \overleftarrow{\zeta}_1)^{w_1} + 1 - (1 - \overleftarrow{\zeta}_2)^{w_2} - (1 - (1 - \overleftarrow{\zeta}_1)^{w_1})(1 - (1 - \overleftarrow{\zeta}_2)^{w_2}) \right), \\
\left(1 - (1 - \overleftarrow{\upsilon}_1)^{w_1} + 1 - (1 - \overleftarrow{\upsilon}_2)^{w_2} - (1 - (1 - \overleftarrow{\upsilon}_1)^{w_1})(1 - (1 - \overleftarrow{\upsilon}_2)^{w_2}) \right), \\
\left(1 - (1 - \overleftarrow{t}_1)^{w_1} + 1 - (1 - \overleftarrow{t}_2)^{w_2} - (1 - (1 - \overleftarrow{t}_1)^{w_1})(1 - (1 - \overleftarrow{t}_2)^{w_2}) \right)
\end{array} \right], \\
\left[\begin{array}{c}
\left(1 - (1 - \overrightarrow{\vartheta}_1)^{w_1} + 1 - (1 - \overrightarrow{\vartheta}_2)^{w_2} - \left(1 - (1 - \overrightarrow{\vartheta}_1)^{w_1} \right) \left(1 - (1 - \overrightarrow{\vartheta}_2)^{w_2} \right) \right), \\
\left(1 - (1 - \overrightarrow{s}_1)^{w_1} + 1 - (1 - \overrightarrow{s}_2)^{w_2} - (1 - (1 - \overrightarrow{s}_1)^{w_1})(1 - (1 - \overrightarrow{s}_2)^{w_2}) \right) \\
\left(1 - (1 - \overrightarrow{\zeta}_1)^{w_1} + 1 - (1 - \overrightarrow{\zeta}_2)^{w_2} - (1 - (1 - \overrightarrow{\zeta}_1)^{w_1})(1 - (1 - \overrightarrow{\zeta}_2)^{w_2}) \right) \\
\left(1 - (1 - \overrightarrow{\upsilon}_1)^{w_1} + 1 - (1 - \overrightarrow{\upsilon}_2)^{w_2} - (1 - (1 - \upsilon_1)^{w_1})(1 - (1 - \overrightarrow{\upsilon}_2)^{w_2}) \right) \\
\left(1 - (1 - \overrightarrow{t}_1)^{w_1} + 1 - (1 - \overrightarrow{t}_2)^{w_2} - (1 - (1 - \overrightarrow{t}_1)^{w_1})(1 - (1 - \overrightarrow{t}_2)^{w_2}) \right)
\end{array} \right]
\end{array} \right)
\end{array} \right\}
$$

$$
= \left\{ \begin{array}{c}
\left[\left(\prod_{i=1}^{2} \overleftarrow{t}_i^{\,w_i}, \prod_{i=1}^{2} \overleftarrow{g}_i^{\,w_i}, \prod_{i=1}^{2} \overleftarrow{b}_i^{\,w_i}, \prod_{i=1}^{2} \overleftarrow{s}_i^{\,w_i}, \prod_{i=1}^{2} \overleftarrow{z}_i^{\,w_i} \right), \left(\prod_{i=1}^{2} \overrightarrow{t}_i^{\,w_i}, \prod_{i=1}^{2} \overrightarrow{g}_i^{\,w_i}, \prod_{i=1}^{2} \overrightarrow{b}_i^{\,w_i}, \prod_{i=1}^{2} \overrightarrow{s}_i^{\,w_i}, \prod_{i=1}^{2} \overrightarrow{z}_i^{\,w_i} \right) \right], \\
\left[\begin{array}{c}
\left(1 - \prod_{i=1}^{2} (1 - \overleftarrow{l}_i)^{w_i}, 1 - \prod_{i=1}^{2} (1 - \overleftarrow{h}_i)^{w_i}, 1 - \prod_{i=1}^{2} (1 - \overleftarrow{z}_i)^{w_i}, 1 - \prod_{i=1}^{2} (1 - \overleftarrow{n}_i)^{w_i}, 1 - \prod_{i=1}^{2} (1 - \overleftarrow{f}_i)^{w_i} \right), \\
\left(1 - \prod_{i=1}^{2} (1 - \overrightarrow{l}_i)^{w_i}, 1 - \prod_{i=1}^{2} (1 - \overrightarrow{h}_i)^{w_i}, 1 - \prod_{i=1}^{2} (1 - \overrightarrow{z}_i)^{w_i}, 1 - \prod_{i=1}^{2} (1 - \overrightarrow{n}_i)^{w_i}, 1 - \prod_{i=1}^{2} (1 - \overrightarrow{f}_i)^{w_i} \right)
\end{array} \right], \\
\left[\begin{array}{c}
\left(1 - \prod_{i=1}^{2} (1 - \overleftarrow{\vartheta}_i)^{w_i}, 1 - \prod_{i=1}^{2} (1 - \overleftarrow{s}_i)^{w_i}, 1 - \prod_{i=1}^{2} (1 - \overleftarrow{\zeta}_i)^{w_i}, 1 - \prod_{i=1}^{2} (1 - \overleftarrow{\upsilon}_i)^{w_i}, 1 - \prod_{i=1}^{2} (1 - \overleftarrow{t}_i)^{w_i} \right), \\
\left(1 - \prod_{i=1}^{2} (1 - \overrightarrow{\vartheta}_i)^{w_i}, 1 - \prod_{i=1}^{2} (1 - \overrightarrow{s}_i)^{w_i}, 1 - \prod_{i=1}^{2} (1 - \overrightarrow{\zeta}_i)^{w_i}, 1 - \prod_{i=1}^{2} (1 - \overrightarrow{\upsilon}_i)^{w_i}, 1 - \prod_{i=1}^{2} (1 - \overrightarrow{t}_i)^{w_i} \right)
\end{array} \right]
\end{array} \right\}
$$

$$IVPFNWG(\dot{\eta}_1, \dot{\eta}_2, \dots, \dot{\eta}_k) = \dot{\eta}_1^{w_1} \otimes \dot{\eta}_2^{w_1} \otimes \dots \otimes \dot{\eta}_n^{w_k}$$

$$= \left\{ \begin{array}{l} \left[\left(\prod_{i=1}^{k} \overleftarrow{t}_i^{w_i}, \prod_{i=1}^{k} \overleftarrow{g}_i^{w_i}, \prod_{i=1}^{k} \overleftarrow{b}_i^{w_i}, \prod_{i=1}^{k} \overleftarrow{s}_i^{w_i}, \prod_{i=1}^{k} \overleftarrow{z}_i^{w_i} \right), \left(\prod_{i=1}^{k} \overrightarrow{t}_i^{w_i}, \prod_{i=1}^{k} \overrightarrow{g}_i^{w_i}, \prod_{i=1}^{k} \overrightarrow{b}_i^{w_i}, \prod_{i=1}^{k} \overrightarrow{s}_i^{w_i}, \prod_{i=1}^{k} \overrightarrow{z}_i^{w_i} \right) \right] \\ \left[\begin{array}{l} \left(1 - \prod_{i=1}^{k} (1 - \overleftarrow{l}_i)^{w_i}, 1 - \prod_{i=1}^{k} (1 - \overleftarrow{h}_i)^{w_i}, 1 - \prod_{i=1}^{k} (1 - \overleftarrow{z}_i)^{w_i}, 1 - \prod_{i=1}^{k} (1 - \overleftarrow{n}_i)^{w_i}, 1 - \prod_{i=1}^{k} (1 - \overleftarrow{t}_i)^{w_i} \right), \\ \left(1 - \prod_{i=1}^{k} (1 - \overrightarrow{l}_i)^{w_i}, 1 - \prod_{i=1}^{k} (1 - \overrightarrow{h}_i)^{w_i}, 1 - \prod_{i=1}^{k} (1 - \overrightarrow{z}_i)^{w_i}, 1 - \prod_{i=1}^{k} (1 - \overrightarrow{n}_i)^{w_i}, 1 - \prod_{i=1}^{k} (1 - \overrightarrow{t}_i)^{w_i} \right) \end{array} \right] \\ \left[\begin{array}{l} \left(1 - \prod_{i=1}^{k} (1 - \overleftarrow{\vartheta}_i)^{w_i}, 1 - \prod_{i=1}^{k} \left(1 - \overleftarrow{s}_i\right)^{w_i}, 1 - \prod_{i=1}^{k} \left(1 - \overleftarrow{\zeta}_i\right)^{w_i}, 1 - \prod_{i=1}^{k} (1 - \overleftarrow{\upsilon}_i)^{w_i}, 1 - \prod_{i=1}^{k} (1 - \overleftarrow{t}_i)^{w_i} \right), \\ \left(1 - \prod_{i=1}^{k} (1 - \overrightarrow{\vartheta}_i)^{w_i}, 1 - \prod_{i=1}^{k} \left(1 - \overrightarrow{s}_i\right)^{w_i}, 1 - \prod_{i=1}^{k} \left(1 - \overrightarrow{\zeta}_i\right)^{w_i}, 1 - \prod_{i=1}^{k} (1 - \overrightarrow{\upsilon}_i)^{w_i}, 1 - \prod_{i=1}^{k} (1 - \overrightarrow{t}_i)^{w_i} \right) \end{array} \right] \end{array} \right\}$$

$$= \left\{ \begin{array}{l} \left[\left(\prod_{i=1}^{k+1} \overleftarrow{t}_i^{w_i}, \prod_{i=1}^{k+1} \overleftarrow{g}_i^{w_i}, \prod_{i=1}^{k+1} \overleftarrow{b}_i^{w_i}, \prod_{i=1}^{k+1} \overleftarrow{s}_i^{w_i}, \prod_{i=1}^{k+1} \overleftarrow{z}_i^{w_i} \right), \left(\prod_{i=1}^{k+1} \overrightarrow{t}_i^{w_i}, \prod_{i=1}^{k+1} \overrightarrow{g}_i^{w_i}, \prod_{i=1}^{k+1} \overrightarrow{b}_i^{w_i}, \prod_{i=1}^{k+1} \overrightarrow{s}_i^{w_i}, \prod_{i=1}^{k+1} \overrightarrow{z}_i^{w_i} \right) \right], \\ \left[\begin{array}{l} \left(1 - \prod_{i=1}^{k+1} (1 - \overleftarrow{l}_i)^{w_i}, 1 - \prod_{i=1}^{k+1} (1 - \overleftarrow{h}_i)^{w_i}, 1 - \prod_{i=1}^{k+1} (1 - \overleftarrow{z}_i)^{w_i}, 1 - \prod_{i=1}^{k+1} (1 - \overleftarrow{n}_i)^{w_i}, 1 - \prod_{i=1}^{k+1} (1 - \overleftarrow{t}_i)^{w_i} \right), \\ \left(1 - \prod_{i=1}^{k+1} (1 - \overrightarrow{l}_i)^{w_i}, 1 - \prod_{i=1}^{k+1} (1 - \overrightarrow{h}_i)^{w_i}, 1 - \prod_{i=1}^{k+1} (1 - \overrightarrow{z}_i)^{w_i}, 1 - \prod_{i=1}^{k+1} (1 - \overrightarrow{n}_i)^{w_i}, 1 - \prod_{i=1}^{k+1} (1 - \overrightarrow{t}_i)^{w_i} \right) \end{array} \right], \\ \left[\begin{array}{l} \left(1 - \prod_{i=1}^{k+1} (1 - \overleftarrow{\vartheta}_i)^{w_i}, 1 - \prod_{i=1}^{k+1} \left(1 - \overleftarrow{s}_i\right)^{w_i}, 1 - \prod_{i=1}^{k+1} \left(1 - \overleftarrow{\zeta}_i\right)^{w_i}, 1 - \prod_{i=1}^{k+1} (1 - \overleftarrow{\upsilon}_i)^{w_i}, 1 - \prod_{i=1}^{k+1} (1 - \overleftarrow{t}_i)^{w_i} \right), \\ \left(1 - \prod_{i=1}^{k+1} (1 - \overrightarrow{\vartheta}_i)^{w_i}, 1 - \prod_{i=1}^{k+1} \left(1 - \overrightarrow{s}_i\right)^{w_i}, 1 - \prod_{i=1}^{k+1} \left(1 - \overrightarrow{\zeta}_i\right)^{w_i}, 1 - \prod_{i=1}^{k+1} (1 - \overrightarrow{\upsilon}_i)^{w_i}, 1 - \prod_{i=1}^{k+1} (1 - \overrightarrow{t}_i)^{w_i} \right) \end{array} \right] \end{array} \right\}$$

for $n = k + 1$, the following expression holds good.

$$IVPFNWG(\dot{\eta}_1, \dot{\eta}_2, \dots, \dot{\eta}_{k+1}) = \otimes_{i=1}^{k} (\dot{\eta}_i)^{w_i} \otimes (\dot{\eta}_{k+1})^{w_{k+1}}$$

Hence the theorem holds for all values of n.

By mathematical induction, the theorem has been incorporated based on the lower and upper intervals.

All the three membership functions $\dot{\eta}_i$ $(i = 1,2,\dots,n) \in [0,1]$, and the following relations are valid under the following conditions.

$0 \le \left(\prod_{i=1}^{n} \overleftarrow{z}_i^{w_i}\right) \le 1, 0 \le \left(\prod_{i=1}^{n} \overrightarrow{z}_i^{w_i}\right) \le 1$,

$$0 \le \left(1 - \prod_{i=1}^{n} (1 - \overleftarrow{t}_i)^{w_i} \right) \le 1,$$

$0 \le \left(1 - \prod_{i=1}^{n} (1 - \overrightarrow{t}_i)^{w_i}\right) \le 1$ and

$$0 \le \left(1 - \prod_{i=1}^{n} (1 - \overleftarrow{t}_i)^{w_i} \right) \le 1,$$

$$0 \le \left(1 - \prod_{i=1}^{n} (1 - \overrightarrow{t}_i)^{w_i} \right) \le 1$$

and it follows that $0 \le \left(\prod_{i=1}^{n} \overleftarrow{z}_i^{w_i} + 1 - \prod_{i=1}^{n} (1 - \overleftarrow{t}_i)^{w_i} + 1 - \prod_{i=1}^{n} (1 - \overleftarrow{t}_i)^{w_i}\right) \le 3$, and

$0 \le \prod_{i=1}^{n} \overrightarrow{z}_i^{w_i} + 1 - \prod_{i=1}^{n} (1 - \overrightarrow{t}_i)^{w_i} + 1 - \prod_{i=1}^{n} (1 - \overrightarrow{t}_i)^{w_i} \le 3$.

Hence the theorem is proved.

Property 1 (Idempotency):

If all $\dot{\eta}_i (i = 1,2,\dots,n)$ are equal, then for all i $IVPFNWG(\dot{\eta}_1, \dot{\eta}_2, \dots, \dot{\eta}_i) = \dot{\eta}$.

Proof: By (2), we have $IVPFNWG(\dot{\eta}_1, \dot{\eta}_2, \dots, \dot{\eta}_n) = IVPFNWG(\dot{\eta}, \dot{\eta}, \dots, \dot{\eta}) = \otimes_{i=1}^{n} (\dot{\eta}_i)^{w_i}$

$$
=\left\{\begin{array}{c}
\left[\left(\prod_{i=1}^{n}\overleftarrow{ţ}_i^{\,w_i},\prod_{i=1}^{n}\overleftarrow{ɠ}_i^{\,w_i},\prod_{i=1}^{n}\overleftarrow{ƃ}_i^{\,w_i},\prod_{i=1}^{n}\overleftarrow{ŝ}_i^{\,w_i},\prod_{i=1}^{n}\overleftarrow{ʒ}_i^{\,w_i}\right),\left(\prod_{i=1}^{n}\overrightarrow{ţ}_i^{\,w_i},\prod_{i=1}^{n}\overrightarrow{ɠ}_i^{\,w_i},\prod_{i=1}^{n}\overrightarrow{ƃ}_i^{\,w_i},\prod_{i=1}^{n}\overrightarrow{ŝ}_i^{\,w_i},\prod_{i=1}^{n}\overrightarrow{ʒ}_i^{\,w_i}\right)\right],\\
\left[\begin{array}{c}\left(1-\prod_{i=1}^{n}(1-\overleftarrow{ʅ}_i)^{w_i},1-\prod_{i=1}^{n}(1-\overleftarrow{ɧ}_i)^{w_i},1-\prod_{i=1}^{n}(1-\overleftarrow{ʓ}_i)^{w_i},1-\prod_{i=1}^{n}(1-\overleftarrow{ƞ}_i)^{w_i},1-\prod_{i=1}^{n}(1-\overleftarrow{ŧ}_i)^{w_i}\right),\\
\left(1-\prod_{i=1}^{n}(1-\overrightarrow{ʅ}_i)^{w_i},1-\prod_{i=1}^{n}(1-\overrightarrow{ɧ}_i)^{w_i},1-\prod_{i=1}^{n}(1-\overrightarrow{ʓ}_i)^{w_i},1-\prod_{i=1}^{n}(1-\overrightarrow{ƞ}_i)^{w_i},1-\prod_{i=1}^{n}(1-\overrightarrow{ŧ}_i)^{w_i}\right)\end{array}\right],\\
\left[\begin{array}{c}\left(1-\prod_{i=1}^{n}(1-\overleftarrow{\vartheta}_i)^{w_i},1-\prod_{i=1}^{n}(1-\overleftarrow{ş}_i)^{w_i},1-\prod_{i=1}^{n}(1-\overleftarrow{ƺ}_i)^{w_i},1-\prod_{i=1}^{n}(1-\overleftarrow{ʋ}_i)^{w_i},1-\prod_{i=1}^{n}(1-\overleftarrow{ŧ}_i)^{w_i}\right),\\
\left(1-\prod_{i=1}^{n}(1-\overrightarrow{\vartheta}_i)^{w_i},1-\prod_{i=1}^{n}(1-\overrightarrow{ş}_i)^{w_i},1-\prod_{i=1}^{n}(1-\overrightarrow{ƺ}_i)^{w_i},1-\prod_{i=1}^{n}(1-\overrightarrow{ʋ}_i)^{w_i},1-\prod_{i=1}^{n}(1-\overrightarrow{ŧ}_i)^{w_i}\right)\end{array}\right]
\end{array}\right\}
$$

$$
=\left\{\begin{array}{c}
\left[\left(\overleftarrow{ţ}^{\sum_{i=1}^{n}w_i},\overleftarrow{ɠ}^{\sum_{i=1}^{n}w_i},\overleftarrow{ƃ}^{\sum_{i=1}^{n}w_i},\overleftarrow{ŝ}^{\sum_{i=1}^{n}w_i},\overleftarrow{ʒ}^{\sum_{i=1}^{n}w_i}\right),\left(\overrightarrow{ţ}^{\sum_{i=1}^{n}w_i},\overrightarrow{ɠ}^{\sum_{i=1}^{n}w_i},\overrightarrow{ƃ}^{\sum_{i=1}^{n}w_i},\overrightarrow{ŝ}^{\sum_{i=1}^{n}w_i},\overrightarrow{ʒ}^{\sum_{i=1}^{n}w_i}\right)\right],\\
\left[\begin{array}{c}\left(1-(1-\overleftarrow{\vartheta})^{\sum_{i=1}^{n}w_i},1-(1-\overleftarrow{ş})^{\sum_{i=1}^{n}w_i},1-(1-\overleftarrow{ƺ})^{\sum_{i=1}^{n}w_i},1-(1-\overleftarrow{ʋ})^{\sum_{i=1}^{n}w_i},1-(1-\overleftarrow{ŧ})^{\sum_{i=1}^{n}w_i}\right),\\
\left(1-(1-\overrightarrow{\vartheta})^{\sum_{i=1}^{n}w_i},1-(1-\overrightarrow{ş})^{\sum_{i=1}^{n}w_i},1-(1-\overrightarrow{ƺ})^{\sum_{i=1}^{n}w_i},1-(1-\overrightarrow{ʋ})^{\sum_{i=1}^{n}w_i},1-(1-\overrightarrow{ŧ})^{\sum_{i=1}^{n}w_i}\right)\end{array}\right],\\
\left[\begin{array}{c}\left(1-(1-\overleftarrow{ʅ})^{\sum_{i=1}^{n}w_i},1-(1-\overleftarrow{ɧ})^{\sum_{i=1}^{n}w_i},1-(1-\overleftarrow{ʓ})^{\sum_{i=1}^{n}w_i},1-(1-\overleftarrow{ƞ})^{\sum_{i=1}^{n}w_i},1-(1-\overleftarrow{ŧ})^{\sum_{i=1}^{n}w_i}\right),\\
\left(1-(1-\overrightarrow{ʅ})^{\sum_{i=1}^{n}w_i},1-(1-\overrightarrow{ɧ})^{\sum_{i=1}^{n}w_i},1-(1-\overrightarrow{ʓ})^{\sum_{i=1}^{n}w_i},1-(1-\overrightarrow{ƞ})^{\sum_{i=1}^{n}w_i},1-(1-\overrightarrow{ŧ})^{\sum_{i=1}^{n}w_i}\right)\end{array}\right]
\end{array}\right\}
$$

$$
=\left\{\begin{array}{c}
[(\overleftarrow{ţ},\overleftarrow{ɠ},\overleftarrow{ƃ},\overleftarrow{ŝ},\overleftarrow{ʒ}),(\overrightarrow{ţ},\overrightarrow{ɠ},\overrightarrow{ƃ},\overrightarrow{ŝ},\overrightarrow{ʒ})],\\
[(\overleftarrow{ʅ},\overleftarrow{ɧ},\overleftarrow{ʓ},\overleftarrow{ƞ},\overleftarrow{ŧ}),(\overrightarrow{ʅ},\overrightarrow{ɧ},\overrightarrow{ʓ},\overrightarrow{ƞ},\overrightarrow{ŧ})],\\
[(\overleftarrow{\vartheta},\overleftarrow{ş},\overleftarrow{ƺ},\overleftarrow{ʋ},\overleftarrow{ŧ}),(\overrightarrow{\vartheta},\overrightarrow{ş},\overrightarrow{ƺ},\overrightarrow{ʋ},\overrightarrow{ŧ})]
\end{array}\right\}=\dot{ŋ}
$$

Hence the property is proved.

Property 2 (Boundedness):

$$
\dot{ŋ}_i=\left\{\begin{array}{c}
[(\overleftarrow{ţ}_i,\overleftarrow{ɠ}_i,\overleftarrow{ƃ}_i,\overleftarrow{ŝ}_i,\overleftarrow{ʒ}_i),(\overrightarrow{ţ}_i,\overrightarrow{ɠ}_i,\overrightarrow{ƃ}_i,\overrightarrow{ŝ}_i,\overrightarrow{ʒ}_i)],\\
[(\overleftarrow{ʅ}_i,\overleftarrow{ɧ}_i,\overleftarrow{ʓ}_i,\overleftarrow{ƞ}_i,\overleftarrow{ŧ}_i),(\overrightarrow{ʅ}_i,\overrightarrow{ɧ}_i,\overrightarrow{ʓ}_i,\overrightarrow{ƞ}_i,\overrightarrow{ŧ}_i)],\\
[(\overleftarrow{\vartheta}_i,\overleftarrow{ş}_i,\overleftarrow{ƺ}_i,\overleftarrow{ʋ}_i,\overleftarrow{ŧ}_i),(\overrightarrow{\vartheta}_i,\overrightarrow{ş}_i,\overrightarrow{ƺ}_i,\overrightarrow{ʋ}_i,\overrightarrow{ŧ}_i)]
\end{array}\right\},
$$

$i=1,2,\ldots n$ be a collection of IVPFNNs in the set of real numbers.

Assume

$$
\dot{ŋ}^{+}=\left\{\begin{array}{c}
\left[\begin{array}{c}\left(\max_i\overleftarrow{ţ}_i,\max_i\overleftarrow{ɠ}_i,\max_i\overleftarrow{ƃ}_i,\max_i\overleftarrow{ŝ}_i,\max_i\overleftarrow{ʒ}_i\right),\\ \left(\max_i\overrightarrow{ţ}_i,\max_i\overrightarrow{ɠ}_i,\max_i\overrightarrow{ƃ}_i,\max_i\overrightarrow{ŝ}_i,\max_i\overrightarrow{ʒ}_i\right)\end{array}\right],\\
\left[\begin{array}{c}\left(\min_i\overleftarrow{ʅ}_i,\min_i\overleftarrow{ɧ}_i,\min_i\overleftarrow{ʓ}_i,\min_i\overleftarrow{ƞ}_i,\min_i\overleftarrow{ŧ}_i\right),\\ \left(\min_i\overrightarrow{ʅ}_i,\min_i\overrightarrow{ɧ}_i,\min_i\overrightarrow{ʓ}_i,\min_i\overrightarrow{ƞ}_i,\min_i\overrightarrow{ŧ}_i\right)\end{array}\right],\\
\left[\begin{array}{c}\left(\min_i\overleftarrow{\vartheta}_i,\min_i\overleftarrow{ş}_i,\min_i\overleftarrow{ƺ}_i,\min_i\overleftarrow{ʋ}_i,\min_i\overleftarrow{ŧ}_i\right),\\ \left(\min_i\overrightarrow{\vartheta}_i,\min_i\overrightarrow{ş}_i,\min_i\overrightarrow{ƺ}_i,\min_i\overrightarrow{ʋ}_i,\min_i\overrightarrow{ŧ}_i\right)\end{array}\right]
\end{array}\right\}
$$

and

$$
\dot{ŋ}^{-}=\left\{\begin{array}{c}
\left[\begin{array}{c}\left(\min_i\overleftarrow{ţ}_i,\min_i\overleftarrow{ɠ}_i,\min_i\overleftarrow{ƃ}_i,\min_i\overleftarrow{ŝ}_i,\min_i\overleftarrow{ʒ}_i\right),\\ \left(\min_i\overrightarrow{ţ}_i,\min_i\overrightarrow{ɠ}_i,\min_i\overrightarrow{ƃ}_i,\min_i\overrightarrow{ŝ}_i,\min_i\overrightarrow{ʒ}_i\right)\end{array}\right],\\
\left[\begin{array}{c}\left(\max_i\overleftarrow{ʅ}_i,\max_i\overleftarrow{ɧ}_i,\max_i\overleftarrow{ʓ}_i,\max_i\overleftarrow{ƞ}_i,\max_i\overleftarrow{ŧ}_i\right),\\ \left(\max_i\overrightarrow{ʅ}_i,\max_i\overrightarrow{ɧ}_i,\max_i\overrightarrow{ʓ}_i,\max_i\overrightarrow{ƞ}_i,\max_i\overrightarrow{ŧ}_i\right)\end{array}\right],\\
\left[\begin{array}{c}\left(\max_i\overleftarrow{\vartheta}_i,\max_i\overleftarrow{ş}_i,\max_i\overleftarrow{ƺ}_i,\max_i\overleftarrow{ʋ}_i,\max_i\overleftarrow{ŧ}_i\right),\\ \left(\max_i\overrightarrow{\vartheta}_i,\max_i\overrightarrow{ş}_i,\max_i\overrightarrow{ƺ}_i,\max_i\overrightarrow{ʋ}_i,\max_i\overrightarrow{ŧ}_i\right)\end{array}\right]
\end{array}\right\}
$$

for all $i=1,2,\ldots n$.
Then $\dot{ŋ}^{-}\le \mathrm{IVPFNWG}(\dot{ŋ}_1,\dot{ŋ}_2,\ldots,\dot{ŋ}_n)\le\dot{ŋ}^{+}$.

Proof: We infer that $\min_i(\overleftarrow{ʒ}_i)\le\overleftarrow{ʒ}_i\le\max_i(\overleftarrow{ʒ}_i)$,
$\min_i(\overleftarrow{ŧ}_i)\le\overleftarrow{ŧ}_i\le\max_i(\overleftarrow{ŧ}_i)$, $\min_i(\overleftarrow{ŧ}_i)\le\overleftarrow{ŧ}_i\le\max_i(\overleftarrow{ŧ}_i)$;
$\min_i(\overrightarrow{ʒ}_i)\le\overrightarrow{ʒ}_i\le\max_i(\overrightarrow{ʒ}_i)$, $\min_i(\overrightarrow{ŧ}_i)\le\overrightarrow{ŧ}_i\le\max_i(\overrightarrow{ŧ}_i)$,
$\min_i(\overrightarrow{ŧ}_i)\le\overrightarrow{ŧ}_i\le\max_i(\overrightarrow{ŧ}_i)$, for $i=1,2,\ldots,n$. (3)

Then,

$$
\prod_{i=1}^{n}\left(\min_i(\overleftarrow{ʒ}_i)\right)^{w_i}\le\prod_{i=1}^{n}(\overleftarrow{ʒ}_i)^{w_i}\le\prod_{i=1}^{n}\left(\max_i(\overleftarrow{ʒ}_i)\right)^{w_i}
$$

$$
\prod_{i=1}^{n}\left(\min_i(\overrightarrow{ʒ}_i)\right)^{w_i}\le\prod_{i=1}^{n}(\overrightarrow{ʒ}_i)^{w_i}\le\prod_{i=1}^{n}\left(\max_i(\overrightarrow{ʒ}_i)\right)^{w_i}
$$

$\left[\left(\min_i(\overleftarrow{ʒ}_i)\right)^{\sum_{i=1}^{n}w_i}\le\prod_{i=1}^{n}(\overleftarrow{ʒ}_i)^{w_i}\le\max_i(\overleftarrow{ʒ}_i)^{\sum_{i=1}^{n}w_i}\right]$,

$\min_i(\overleftarrow{ʒ}_i)\le\prod_{i=1}^{n}(\overleftarrow{ʒ}_i)^{w_i}\le\max_i(\overleftarrow{ʒ}_i)$ by (3), for $i=1,2,\ldots,n$.

$$\left[1-\prod_{i=1}^{n}\left(1-\min_{i}(\overleftarrow{t}_i)\right)^{w_i} \le 1-\prod_{i=1}^{n}(1-\overleftarrow{t}_i)^{w_i} \le 1-\prod_{i=1}^{n}\left(1-\max_{i}(\overleftarrow{t}_i)\right)^{w_i}\right]$$

$$\left[1-\left(1-\min_{i}(\overleftarrow{t}_i)\right)^{\sum_{i=1}^{n} w_i} \le 1-\prod_{i=1}^{n}(1-\overleftarrow{t}_i)^{w_i} \le 1-\left(1-\max_{i}(\overleftarrow{t}_i)\right)^{\sum_{i=1}^{n} w_i}\right]$$

$\min_{i}(\overleftarrow{t}_i) \le 1-\prod_{i=1}^{n}(1-\overleftarrow{t}_i)^{w_i} \le \max_{i}(\overleftarrow{t}_i)$ and

$$1-\prod_{i=1}^{n}\left(1-\min_{i}(\overleftarrow{b}_i)\right)^{w_i} \le 1-\prod_{i=1}^{n}(1-\overleftarrow{b}_i)^{w_i} \le 1-\prod_{i=1}^{n}\left(1-\max_{i}(\overleftarrow{b}_i)\right)^{w_i}$$

$$1-\left(1-\min_{i}(\overleftarrow{b}_i)\right)^{\sum_{i=1}^{n} w_i} \le 1-\prod_{i=1}^{n}(1-\overleftarrow{b}_i)^{w_i} \le 1-\left(1-\max_{i}(\overleftarrow{b}_i)\right)^{\sum_{i=1}^{n} w_i}$$

$$\min_{i}(\overleftarrow{b}_i) \le 1-\prod_{i=1}^{n}(1-\overleftarrow{b}_i)^{w_i} \le \max_{i}(\overleftarrow{b}_i)$$

In the same way,

$\min_{i}(\overrightarrow{3}_i) \le \prod_{i=1}^{n}(\overrightarrow{3}_i)^{w_i} \le \max_{i}(\overrightarrow{3}_i)$,
$\min_{i}(\overrightarrow{t}_i) \le 1-\prod_{i=1}^{n}(1-\overrightarrow{t}_i)^{w_i} \le \max_{i}(\overrightarrow{t}_i)$,

$\min_{i}(\overrightarrow{b}_i) \le 1-\prod_{i=1}^{n}(1-\overrightarrow{b}_i)^{w_i} \le \max_{i}(\overrightarrow{b}_i)$.

for $i = 1,2,\dots,n$.

Let $IVPFNWG(\eta_1,\eta_2,\dots,\eta_n) \le \eta$
By applying the aggregation operators with the weightage for $i = 1,2,\dots,n$.

$$IVPFNWG(\eta_1,\eta_2,\dots,\eta_n) \le \eta$$

where $\eta = \left\{\begin{array}{l}\langle[(\overleftarrow{t},\overleftarrow{g},\overleftarrow{b},\overleftarrow{s},\overleftarrow{3}),(\overrightarrow{t},\overrightarrow{g},\overrightarrow{b},\overrightarrow{s},\overrightarrow{3})]:T_{\eta}\rangle,\\ \langle[(\overleftarrow{l},\overleftarrow{h},\overleftarrow{z},\overleftarrow{n},\overleftarrow{t}),(\overrightarrow{l},\overrightarrow{h},\overrightarrow{z},\overrightarrow{n},\overrightarrow{t})]:I_{\eta}]\rangle,\\ \langle[(\overleftarrow{\vartheta},\overleftarrow{s},\overleftarrow{z},\overleftarrow{v},\overleftarrow{b}),(\overrightarrow{\vartheta},\overrightarrow{s},\overrightarrow{z},\overrightarrow{v},\overrightarrow{b})]:F_{\eta}\rangle\end{array}\right\}$

Similarly,

$\min_{i}(\overleftarrow{t}_i) \le \prod_{i=1}^{n}(\overleftarrow{t}_i)^{w_i} \le \max_{i}(\overleftarrow{t}_i)$,

$\min_{i}(\overleftarrow{g}_i) \le \prod_{i=1}^{n}(\overleftarrow{g}_i)^{w_i} \le \max_{i}(\overleftarrow{g}_i)$,

$\min_{i}(\overleftarrow{b}_i) \le \prod_{i=1}^{n}(\overleftarrow{b}_i)^{w_i} \le \max_{i}(\overleftarrow{b}_i)$,

$\min_{i}(\overleftarrow{s}_i) \le \prod_{i=1}^{n}(\overleftarrow{s}_i)^{w_i} \le \max_{i}(\overleftarrow{s}_i)$;

$\min_{i}(\overleftarrow{l}_i) \le 1-\prod_{i=1}^{n}(1-\overleftarrow{l}_i)^{w_i} \le \max_{i}(\overleftarrow{l}_i)$,
$\min_{i}(\overleftarrow{h}_i) \le 1-\prod_{i=1}^{n}(1-\overleftarrow{h}_i)^{w_i} \le \max_{i}(\overleftarrow{h}_i)$,

$\min_{i}(\overleftarrow{z}_i) \le 1-\prod_{i=1}^{n}(1-\overleftarrow{z}_i)^{w_i} \le \max_{i}(\overleftarrow{z}_i)$,
$\min_{i}(\overleftarrow{n}_i) \le 1-\prod_{i=1}^{n}(1-\overleftarrow{n}_i)^{w_i} \le \max_{i}(\overleftarrow{n}_i)$;

$\min_{i}(\overleftarrow{\vartheta}_i) \le 1-\prod_{i=1}^{n}(1-\overleftarrow{\vartheta}_i)^{w_i} \le \max_{i}(\overleftarrow{\vartheta}_i)$,

$$\min_{i}(\overleftarrow{s}_i) \le 1-\prod_{i=1}^{n}(1-\overleftarrow{s}_i)^{w_i} \le \max_{i}(\overleftarrow{s}_i)$$

$\min_{i}(\overleftarrow{z}_i) \le 1-\prod_{i=1}^{n}(1-\overleftarrow{z}_i)^{w_i} \le \max_{i}(\overleftarrow{z}_i)$,

$\min_{i}(\overleftarrow{v}_i) \le 1-\prod_{i=1}^{n}(1-\overleftarrow{v}_i)^{w_i} \le \max_{i}(\overleftarrow{v}_i)$.

And also,

$\min_{i}(\overrightarrow{t}_i) \le \prod_{i=1}^{n}(\overrightarrow{t}_i)^{w_i} \le \max_{i}(\overrightarrow{t}_i)$,
$\min_{i}(\overrightarrow{g}_i) \le \prod_{i=1}^{n}(\overrightarrow{g}_i)^{w_i} \le \max_{i}(\overrightarrow{g}_i)$,

$\min_{i}(\overrightarrow{b}_i) \le \prod_{i=1}^{n}(\overrightarrow{b}_i)^{w_i} \le \max_{i}(\overrightarrow{b}_i)$,
$\min_{i}(\overrightarrow{s}_i) \le \prod_{i=1}^{n}(\overrightarrow{s}_i)^{w_i} \le \max_{i}(\overrightarrow{s}_i)$;

$\min_{i}(\overrightarrow{l}_i) \le 1-\prod_{i=1}^{n}(1-\overrightarrow{l}_i)^{w_i} \le \max_{i}(\overrightarrow{l}_i)$,
$\min_{i}(\overrightarrow{h}_i) \le 1-\prod_{i=1}^{n}(1-\overrightarrow{h}_i)^{w_i} \le \max_{i}(\overrightarrow{h}_i)$,

$\min_{i}(\overrightarrow{z}_i) \le 1-\prod_{i=1}^{n}(1-\overrightarrow{z}_i)^{w_i} \le \max_{i}(\overrightarrow{z}_i)$,
$\min_{i}(\overrightarrow{n}_i) \le 1-\prod_{i=1}^{n}(1-\overrightarrow{n}_i)^{w_i} \le \max_{i}(\overrightarrow{n}_i)$;

$\min_{i}(\overrightarrow{\vartheta}_i) \le 1-\prod_{i=1}^{n}(1-\overrightarrow{\vartheta}_i)^{w_i} \le \max_{i}(\overrightarrow{\vartheta}_i)$,
$\min_{i}(\overrightarrow{s}_i) \le 1-\prod_{i=1}^{n}(1-\overrightarrow{s}_i)^{w_i} \le \max_{i}(\overrightarrow{s}_i)$,
$\min_{i}(\overrightarrow{z}_i) \le 1-\prod_{i=1}^{n}(1-\overrightarrow{z}_i)^{w_i} \le \max_{i}(\overrightarrow{z}_i)$,

$\min_{i}(\overrightarrow{v}_i) \le 1-\prod_{i=1}^{n}(1-\overrightarrow{v}_i)^{w_i} \le \max_{i}(\overrightarrow{v}_i)$.

The score function of $\dot{ŋ}$ is

$$S(\dot{ŋ})=\frac{1}{6}\Big[4+\frac{\overleftarrow{ţ}+\overleftarrow{ɠ}+\overleftarrow{ƀ}+\overleftarrow{ṡ}+\overleftarrow{ʒ}}{5}+\frac{\vec{ţ}+\vec{ɠ}+\vec{ƀ}+\vec{ṡ}+\vec{ʒ}}{5}-\frac{\overleftarrow{ļ}+\overleftarrow{ɧ}+\overleftarrow{ʓ}+\overleftarrow{ɲ}+\overleftarrow{ŧ}}{5}-\frac{\vec{ļ}+\vec{ɧ}+\vec{ʓ}+\vec{ɲ}+\vec{ŧ}}{5}-\frac{\overleftarrow{ϑ}+\overleftarrow{ş}+\overleftarrow{ʐ}+\overleftarrow{ʋ}+\overleftarrow{ƭ}}{5}-\frac{\vec{ϑ}+\vec{ş}+\vec{ʐ}+\vec{ʋ}+\vec{ƭ}}{5}\Big]$$

$$\leq\frac{1}{6}\left[\begin{array}{l}4+\dfrac{\max_i(\overleftarrow{ţ}_i)+\max_i(\overleftarrow{ɠ}_i)+\max_i(\overleftarrow{ƀ}_i)+\max_i(\overleftarrow{ṡ}_i)+\max_i(\overleftarrow{ʒ}_i)}{5}\\+\dfrac{\max_i(\vec{ţ}_i)+\max_i(\vec{ɠ}_i)+\max_i(\vec{ƀ}_i)+\max_i(\vec{ṡ}_i)+\max_i(\vec{ʒ}_i)}{5}\\-\dfrac{\min_i(\overleftarrow{ļ}_i)+\min_i(\overleftarrow{ɧ}_i)+\min_i(\overleftarrow{ʓ}_i)+\min_i(\overleftarrow{ɲ}_i)+\min_i(\overleftarrow{ŧ}_i)}{5}\\-\dfrac{\min_i(\vec{ļ}_i)+\min_i(\vec{ɧ}_i)+\min_i(\vec{ʓ}_i)+\min_i(\vec{ɲ}_i)+\min_i(\vec{ŧ}_i)}{5}\\-\dfrac{\min_i(\overleftarrow{ϑ}_i)+\min_i(\overleftarrow{ş}_i)+\min_i(\overleftarrow{ʐ}_i)+\min_i(\overleftarrow{ʋ}_i)+\min_i(\overleftarrow{ƭ}_i)}{5}\\-\dfrac{\min_i(\vec{ϑ}_i)+\min_i(\vec{ş}_i)+\min_i(\vec{ʐ}_i)+\min_i(\vec{ʋ}_i)+\min_i(\vec{ƭ}_i)}{5}\end{array}\right]=S(\dot{ŋ}^+)$$

In the same way,

$$S(\dot{ŋ})=\frac{1}{6}\Big[4+\frac{\overleftarrow{ţ}+\overleftarrow{ɠ}+\overleftarrow{ƀ}+\overleftarrow{ṡ}+\overleftarrow{ʒ}}{5}+\frac{\vec{ţ}+\vec{ɠ}+\vec{ƀ}+\vec{ṡ}+\vec{ʒ}}{5}-\frac{\overleftarrow{ļ}+\overleftarrow{ɧ}+\overleftarrow{ʓ}+\overleftarrow{ɲ}+\overleftarrow{ŧ}}{5}-\frac{\vec{ļ}+\vec{ɧ}+\vec{ʓ}+\vec{ɲ}+\vec{ŧ}}{5}-\frac{\overleftarrow{ϑ}+\overleftarrow{ş}+\overleftarrow{ʐ}+\overleftarrow{ʋ}+\overleftarrow{ƭ}}{5}-\frac{\vec{ϑ}+\vec{ş}+\vec{ʐ}+\vec{ʋ}+\vec{ƭ}}{5}\Big]$$

Here we discuss the different cases:

Case (i)
If $S(\dot{ŋ})<S(\dot{ŋ}^+)$ and $S(\dot{ŋ})>S(\dot{ŋ}^-)$ then,

$$\dot{ŋ}^-<IVPFNWG(\dot{ŋ}_1,\dot{ŋ}_2,\dots,\dot{ŋ}_n)<\dot{ŋ}^+.$$

Case (ii) If $S(\dot{ŋ})=S(\dot{ŋ}^+)$, we consider

$$S(\dot{ŋ})=\frac{1}{6}\left[\begin{array}{c}4+\dfrac{\overleftarrow{ţ}+\overleftarrow{ɠ}+\overleftarrow{ƀ}+\overleftarrow{ṡ}+\overleftarrow{ʒ}}{5}\\+\dfrac{\vec{ţ}+\vec{ɠ}+\vec{ƀ}+\vec{ṡ}+\vec{ʒ}}{5}-\dfrac{\overleftarrow{ļ}+\overleftarrow{ɧ}+\overleftarrow{ʓ}+\overleftarrow{ɲ}+\overleftarrow{ŧ}}{5}\\-\dfrac{\vec{ļ}+\vec{ɧ}+\vec{ʓ}+\vec{ɲ}+\vec{ŧ}}{5}-\dfrac{\overleftarrow{ϑ}+\overleftarrow{ş}+\overleftarrow{ʐ}+\overleftarrow{ʋ}+\overleftarrow{ƭ}}{5}\\-\dfrac{\vec{ϑ}+\vec{ş}+\vec{ʐ}+\vec{ʋ}+\vec{ƭ}}{5}\end{array}\right]$$

$$=\frac{1}{6}\left[\begin{array}{l}4+\frac{1}{5}\langle\max_i(\overleftarrow{ţ}_i)+\max_i(\overleftarrow{ɠ}_i)+\max_i(\overleftarrow{ƀ}_i)+\max_i(\overleftarrow{ṡ}_i)+\max_i(\overleftarrow{ʒ}_i)\rangle\\+\frac{1}{5}\langle\max_i(\vec{ţ}_i)+\max_i(\vec{ɠ}_i)+\max_i(\vec{ƀ}_i)+\max_i(\vec{ṡ}_i)+\max_i(\vec{ʒ}_i)\rangle\\-\frac{1}{5}\langle\min_i(\overleftarrow{ļ}_i)+\min_i(\overleftarrow{ɧ}_i)+\min_i(\overleftarrow{ʓ}_i)+\min_i(\overleftarrow{ɲ}_i)+\min_i(\overleftarrow{ŧ}_i)\rangle\\-\frac{1}{5}\langle\min_i(\vec{ļ}_i)+\min_i(\vec{ɧ}_i)+\min_i(\vec{ʓ}_i)+\min_i(\vec{ɲ}_i)+\min_i(\vec{ŧ}_i)\rangle\\-\frac{1}{5}\langle\min_i(\overleftarrow{ϑ}_i)+\min_i(\overleftarrow{ş}_i)+\min_i(\overleftarrow{ʐ}_i)+\min_i(\overleftarrow{ʋ}_i)+\min_i(\overleftarrow{ƭ}_i)\rangle\\-\frac{1}{5}\langle\min_i(\vec{ϑ}_i)+\min_i(\vec{ş}_i)+\min_i(\vec{ʐ}_i)+\min_i(\vec{ʋ}_i)+\min_i(\vec{ƭ}_i)\rangle\end{array}\right]$$

Then, it also follows

$$\frac{\overleftarrow{ţ}+\overleftarrow{ɠ}+\overleftarrow{ƀ}+\overleftarrow{ṡ}+\overleftarrow{ʒ}}{5}=\frac{\max_i(\overleftarrow{ţ}_i)+\max_i(\overleftarrow{ɠ}_i)+\max_i(\overleftarrow{ƀ}_i)+\max_i(\overleftarrow{ṡ}_i)+\max_i(\overleftarrow{ʒ}_i)}{5},$$

$$\frac{\vec{ţ}+\vec{ɠ}+\vec{ƀ}+\vec{ṡ}+\vec{ʒ}}{5}=\frac{\max_i(\vec{ţ}_i)+\max_i(\vec{ɠ}_i)+\max_i(\vec{ƀ}_i)+\max_i(\vec{ṡ}_i)+\max_i(\vec{ʒ}_i)}{5},$$

$$\frac{\overleftarrow{ļ}+\overleftarrow{ɧ}+\overleftarrow{ʓ}+\overleftarrow{ɲ}+\overleftarrow{ŧ}}{5}=\frac{\min_i(\overleftarrow{ļ}_i)+\min_i(\overleftarrow{ɧ}_i)+\min_i(\overleftarrow{ʓ}_i)+\min_i(\overleftarrow{ɲ}_i)+\min_i(\overleftarrow{ŧ}_i)}{5},$$

$$\frac{\vec{ļ}+\vec{ɧ}+\vec{ʓ}+\vec{ɲ}+\vec{ŧ}}{5}=\frac{\min_i(\vec{ļ}_i)+\min_i(\vec{ɧ}_i)+\min_i(\vec{ʓ}_i)+\min_i(\vec{ɲ}_i)+\min_i(\vec{ŧ}_i)}{5},$$

$$\frac{\overleftarrow{ϑ}+\overleftarrow{ş}+\overleftarrow{ʐ}+\overleftarrow{ʋ}+\overleftarrow{ƭ}}{5}=\frac{\min_i(\overleftarrow{ϑ}_i)+\min_i(\overleftarrow{ş}_i)+\min_i(\overleftarrow{ʐ}_i)+\min_i(\overleftarrow{ʋ}_i)+\min_i(\overleftarrow{ƭ}_i)}{5},\text{ and}$$

$$\frac{\vec{ϑ}+\vec{ş}+\vec{ʐ}+\vec{ʋ}+\vec{ƭ}}{5}=\frac{\min_i(\vec{ϑ}_i)+\min_i(\vec{ş}_i)+\min_i(\vec{ʐ}_i)+\min_i(\vec{ʋ}_i)+\min_i(\vec{ƭ}_i)}{5}.$$

The accuracy function

$$A_c(\dot{ŋ})=\frac{1}{2}\left[\begin{array}{l}\dfrac{\overleftarrow{ţ}+\overleftarrow{ɠ}+\overleftarrow{ƀ}+\overleftarrow{ṡ}+\overleftarrow{ʒ}}{5}+\dfrac{\vec{ţ}+\vec{ɠ}+\vec{ƀ}+\vec{ṡ}+\vec{ʒ}}{5}\\--\dfrac{\overleftarrow{ϑ}+\overleftarrow{ş}+\overleftarrow{ʐ}+\overleftarrow{ʋ}+\overleftarrow{ƭ}}{5}-\dfrac{\vec{ϑ}+\vec{ş}+\vec{ʐ}+\vec{ʋ}+\vec{ƭ}}{5}\end{array}\right]$$

$$=\frac{1}{2}\left[\begin{array}{l}\dfrac{\max_i(\overleftarrow{t}_i)+\max_i(\overleftarrow{g}_i)+\max_i(\overleftarrow{b}_i)+\max_i(\overleftarrow{s}_i)+\max_i(\overleftarrow{z}_i)}{5}\\+\dfrac{\max_i(\overrightarrow{t}_i)+\max_i(\overrightarrow{g}_i)+\max_i(\overrightarrow{b}_i)+\max_i(\overrightarrow{s}_i)+\max_i(\overrightarrow{z}_i)}{5}\\-\dfrac{\min_i(\overleftarrow{\vartheta}_i)+\min_i(\overleftarrow{s}_i)+\min_i(\overleftarrow{\zeta}_i)+\min_i(\overleftarrow{\upsilon}_i)+\min_i(\overleftarrow{t}_i)}{5}\\-\dfrac{\min_i(\overrightarrow{\vartheta}_i)+\min_i(\overrightarrow{s}_i)+\min_i(\overrightarrow{\zeta}_i)+\min_i(\overrightarrow{\upsilon}_i)+\min_i(\overrightarrow{t}_i)}{5}\end{array}\right]$$

$$=A_c(\dot{\eta}^+) \qquad (4)$$

which implies $IVPFNWG(\dot{\eta}_1,\dot{\eta}_2,\ldots,\dot{\eta}_n)\le\dot{\eta}^+$. In the same way,

$$A_c(\dot{\eta})=\frac{1}{2}\left[\begin{array}{l}\dfrac{\overleftarrow{t}+\overleftarrow{g}+\overleftarrow{b}+\overleftarrow{s}+\overleftarrow{z}}{5}+\dfrac{\overrightarrow{t}+\overrightarrow{g}+\overrightarrow{b}+\overrightarrow{s}+\overrightarrow{z}}{5}\\--\dfrac{\overleftarrow{\vartheta}+\overleftarrow{s}+\overleftarrow{\zeta}+\overleftarrow{\upsilon}+\overleftarrow{t}}{5}-\dfrac{\overrightarrow{\vartheta}+\overrightarrow{s}+\overrightarrow{\zeta}+\overrightarrow{\upsilon}+\overrightarrow{t}}{5}\end{array}\right]$$

$$=\frac{1}{2}\left[\begin{array}{l}\dfrac{\min_i(\overleftarrow{t}_i)+\min_i(\overleftarrow{g}_i)+\min_i(\overleftarrow{b}_i)+\min_i(\overleftarrow{s}_i)+\min_i(\overleftarrow{z}_i)}{5}\\+\dfrac{\min_i(\overrightarrow{t}_i)+\min_i(\overrightarrow{g}_i)+\min_i(\overrightarrow{b}_i)+\min_i(\overrightarrow{s}_i)+\min_i(\overrightarrow{z}_i)}{5}\\-\dfrac{\max_i(\overleftarrow{\vartheta}_i)+\max_i(\overleftarrow{s}_i)+\max_i(\overleftarrow{\zeta}_i)+\max_i(\overleftarrow{\upsilon}_i)+\max_i(\overleftarrow{t}_i)}{5}\\-\dfrac{\max_i(\overrightarrow{\vartheta}_i)+\max_i(\overrightarrow{s}_i)+\max_i(\overrightarrow{\zeta}_i)+\max_i(\overrightarrow{\upsilon}_i)+\max_i(\overrightarrow{t}_i)}{5}\end{array}\right]$$

$$=A_c(\dot{\eta}^-) \qquad (5)$$

which implies $IVPFNWG(\dot{\eta}_1,\dot{\eta}_2,\ldots,\dot{\eta}_n)\le\dot{\eta}^-$. From (4) and (5), we infer that $\dot{\eta}^-\le IVPFNWG(\dot{\eta}_1,\dot{\eta}_2,\ldots,\dot{\eta}_n)\le\dot{\eta}^+$.
Hence the proof is verified.

Property 3 (Containment):

If $\dot{\eta}_i^1=\left\{\begin{array}{l}\langle[(\overleftarrow{t}_i^1,\overleftarrow{g}_i^1,\overleftarrow{b}_i^1,\overleftarrow{s}_i^1,\overleftarrow{z}_i^1),(\overrightarrow{t}_i^1,\overrightarrow{g}_i^1,\overrightarrow{b}_i^1,\overrightarrow{s}_i^1,\overrightarrow{z}_i^1)]:T_{\dot{\eta}_i^1}\rangle\\\langle[(\overleftarrow{l}_i^1,\overleftarrow{h}_i^1,\overleftarrow{z}_i^1,\overleftarrow{n}_i^1,\overleftarrow{t}_i^1),(\overrightarrow{l}_i^1,\overrightarrow{h}_i^1,\overrightarrow{z}_i^1,\overrightarrow{n}_i^1,\overrightarrow{t}_i^1)]:I_{\dot{\eta}_i^1}\rangle\\\langle[(\overleftarrow{\vartheta}_i^1,\overleftarrow{s}_i^1,\overleftarrow{\zeta}_i^1,\overleftarrow{\upsilon}_i^1,\overleftarrow{t}_i^1),(\overrightarrow{\vartheta}_i^1,\overrightarrow{s}_i^1,\overrightarrow{\zeta}_i^1,\overrightarrow{\upsilon}_i^1,\overrightarrow{t}_i^1)]:F_{\dot{\eta}_i^1}\rangle\end{array}\right\}$ and

$$\dot{\eta}_i^2=\left\{\begin{array}{l}\langle[(\overleftarrow{t}_i^2,\overleftarrow{g}_i^2,\overleftarrow{b}_i^2,\overleftarrow{s}_i^2,\overleftarrow{z}_i^2),(\overrightarrow{t}_i^2,\overrightarrow{g}_i^2,\overrightarrow{b}_i^2,\overrightarrow{s}_i^2,\overrightarrow{z}_i^2,)]:T_{\dot{\eta}_i^2}\rangle\\\langle[(\overleftarrow{l}_i^2,\overleftarrow{h}_i^2,\overleftarrow{z}_i^2,\overleftarrow{n}_i^2,\overleftarrow{t}_i^2),(\overrightarrow{l}_i^2,\overrightarrow{h}_i^2,\overrightarrow{z}_i^2,\overrightarrow{n}_i^2,\overrightarrow{t}_i^2)]:I_{\dot{\eta}_i^2}\rangle\\\langle[(\overleftarrow{\vartheta}_i^2,\overleftarrow{s}_i^2,\overleftarrow{\zeta}_i^2,\overleftarrow{\upsilon}_i^2,\overleftarrow{t}_i^2),(\overrightarrow{\vartheta}_i^2,\overrightarrow{s}_i^2,\overrightarrow{\zeta}_i^2,\overrightarrow{\upsilon}_i^2,\overrightarrow{t}_i^2)]:F_{\dot{\eta}_i^2}\rangle\end{array}\right\}$$

$(i=1,2,\ldots,n)$be a collection of IVPFNNVs in the set of real numbers.

If $\overleftarrow{t}_i^1\le\overleftarrow{t}_i^2$, $\overleftarrow{g}_i^1\le\overleftarrow{g}_i^2$, $\overleftarrow{b}_i^1\le\overleftarrow{b}_i^2$, $\overleftarrow{s}_i^1\le\overleftarrow{s}_i^2$, $\overleftarrow{z}_i^1\le\overleftarrow{z}_i^2$; $\overrightarrow{t}_i^1\le\overrightarrow{t}_i^2$, $\overrightarrow{g}_i^1\le\overrightarrow{g}_i^2$, $\overrightarrow{b}_i^1\le\overrightarrow{b}_i^2$, $\overrightarrow{s}_i^1\le\overrightarrow{s}_i^2$, $\overrightarrow{z}_i^1\le\overrightarrow{z}_i^2$; $\overleftarrow{l}_i^1\ge\overleftarrow{l}_i^2$, $\overleftarrow{h}_i^1\ge\overleftarrow{h}_i^2$, $\overleftarrow{z}_i^1\ge\overleftarrow{z}_i^2$, $\overleftarrow{n}_i^1\ge\overleftarrow{n}_i^2$, $\overleftarrow{t}_i^1\ge\overleftarrow{t}_i^2$; $\overrightarrow{l}_i^1\ge\overrightarrow{l}_i^2$, $\overrightarrow{h}_i^1\ge\overrightarrow{h}_i^2$, $\overrightarrow{z}_i^1\ge\overrightarrow{z}_i^2$, $\overrightarrow{n}_i^1\ge\overrightarrow{n}_i^2$, $\overrightarrow{t}_i^1\ge\overrightarrow{t}_i^2$; $\overleftarrow{\vartheta}_i^1\ge\overleftarrow{\vartheta}_i^2$, $\overleftarrow{s}_i^1\ge\overleftarrow{s}_i^2$, $\overleftarrow{\zeta}_i^1\ge\overleftarrow{\zeta}_i^2$, $\overleftarrow{\upsilon}_i^1\ge\overleftarrow{\upsilon}_i^2$, $\overleftarrow{t}_i^1\ge\overleftarrow{t}_i^2$; $\overrightarrow{\vartheta}_i^1\ge\overrightarrow{\vartheta}_i^2$, $\overrightarrow{s}_i^1\ge\overrightarrow{s}_i^2$, $\overrightarrow{\zeta}_i^1\ge\overrightarrow{\zeta}_i^2$, $\overrightarrow{\upsilon}_i^1\ge\overrightarrow{\upsilon}_i^2$, $\overrightarrow{t}_i^1\ge\overrightarrow{t}_i^2$. Then $\dot{\eta}_i^1\le\dot{\eta}_i^2$ for $i=1,2,\ldots,n$.

Proof: In order to prove this, we consider $\overleftarrow{z}_i^1,\overrightarrow{z}_i^1,\overleftarrow{t}_i^1,\overrightarrow{t}_i^1,\overleftarrow{t}_i^1,\overrightarrow{t}_i^1$ of $\dot{\eta}_i^1$ and $\overleftarrow{z}_i^2,\overrightarrow{z}_i^2,\overleftarrow{t}_i^2,\overrightarrow{t}_i^2,\overleftarrow{t}_i^2,\overrightarrow{t}_i^2$of $\dot{\eta}_i^2$.
We assume, $\overleftarrow{z}_i^1\le\overleftarrow{z}_i^2$, $\overleftarrow{t}_i^1\ge\overleftarrow{t}_i^2$, $\overleftarrow{t}_i^1\ge\overleftarrow{t}_i^2$; $\overrightarrow{z}_i^1\le\overrightarrow{z}_i^2$, $\overrightarrow{t}_i^1\ge\overrightarrow{t}_i^2$, $\overrightarrow{t}_i^1\ge\overrightarrow{t}_i^2$ for $\dot{\eta}_i^1\le\dot{\eta}_i^2$ for $i=1,2,\ldots,n$. Then,

$\overleftarrow{z}_i^{1\,w_i}\ge\overleftarrow{z}_i^{2\,w_i}$, $(1-\overleftarrow{t}_i^1)^{w_i}\ge(1-\overleftarrow{t}_i^2)^{w_i}$,
$(1-\overleftarrow{t}_i^1)^{w_i}\ge(1-\overleftarrow{t}_i^2)^{w_i}$;

$\overrightarrow{z}_i^{1\,w_i}\ge\overrightarrow{z}_i^{2\,w_i}$, $(1-\overrightarrow{t}_i^1)^{w_i}\ge(1-\overrightarrow{t}_i^2)^{w_i}$,
$(1-\overrightarrow{t}_i^1)^{w_i}\ge(1-\overrightarrow{t}_i^2)^{w_i}$.

$\prod_{i=1}^n\overleftarrow{z}_i^{1\,w_i}\ge\prod_{i=1}^n\overleftarrow{z}_i^{2\,w_i}$,
$1-\prod_{i=1}^n(1-\overleftarrow{t}_i^1)^{w_i}\le1-\prod_{i=1}^n(1-\overleftarrow{t}_i^2)^{w_i}$,
$1-\prod_{i=1}^n(1-\overleftarrow{t}_i^1)^{w_i}\le1-\prod_{i=1}^n(1-\overleftarrow{t}_i^1)^{w_i}$;

$\prod_{i=1}^n\overrightarrow{z}_i^{1\,w_i}\ge\prod_{i=1}^n\overrightarrow{z}_i^{2\,w_i}$,
$1-\prod_{i=1}^n(1-\overrightarrow{t}_i^1)^{w_i}\le1-\prod_{i=1}^n(1-\overrightarrow{t}_i^2)^{w_i}$,
$1-\prod_{i=1}^n(1-\overrightarrow{t}_i^1)^{w_i}\le1-\prod_{i=1}^n(1-\overrightarrow{t}_i^1)^{w_i}$.

In the same way,

$\prod_{i=1}^n\overleftarrow{t}_i^{1\,w_i}\ge\prod_{i=1}^n\overleftarrow{t}_i^{2\,w_i}$,
$1-\prod_{i=1}^n(1-\overleftarrow{l}_i^1)^{w_i}\le1-\prod_{i=1}^n(1-\overleftarrow{l}_i^2)^{w_i}$,
$1-\prod_{i=1}^n(1-\overleftarrow{\vartheta}_i^1)^{w_i}\le1-\prod_{i=1}^n(1-\overleftarrow{\vartheta}_i^1)^{w_i}$;
$\prod_{i=1}^n\overleftarrow{g}_i^{1\,w_i}\ge\prod_{i=1}^n\overleftarrow{g}_i^{2\,w_i}$,
$1-\prod_{i=1}^n(1-\overleftarrow{h}_i^1)^{w_i}\le1-\prod_{i=1}^n(1-\overleftarrow{h}_i^2)^{w_i}$,
$1-\prod_{i=1}^n(1-\overleftarrow{s}_i^1)^{w_i}\le1-\prod_{i=1}^n(1-\overleftarrow{s}_i^2)^{w_i}$

$\prod_{i=1}^n\overleftarrow{b}_i^{1\,w_i}\ge\prod_{i=1}^n\overleftarrow{b}_i^{2\,w_i}$,
$1-\prod_{i=1}^n(1-\overleftarrow{z}_i^1)^{w_i}\le1-\prod_{i=1}^n(1-\overleftarrow{z}_i^2)^{w_i}$,
$1-\prod_{i=1}^n(1-\overleftarrow{\zeta}_i^1)^{w_i}\le1-\prod_{i=1}^n(1-\overleftarrow{\zeta}_i^2)^{w_i}$

$\prod_{i=1}^{n}\overleftarrow{t}_i^{1\,w_i} \ge \prod_{i=1}^{n}\overleftarrow{t}_i^{2\,w_i}$,
$1-\prod_{i=1}^{n}\left(1-\overleftarrow{n}_i^{1}\right)^{w_i} \le 1-\prod_{i=1}^{n}\left(1-\overleftarrow{n}_i^{2}\right)^{w_i}$,

$$1-\prod_{i=1}^{n}\left(1-\overleftarrow{v}_i^{1}\right)^{w_i} \le 1-\prod_{i=1}^{n}\left(1-\overleftarrow{v}_i^{2}\right)^{w_i}$$

and
$\prod_{i=1}^{n}\overrightarrow{t}_i^{1\,w_i} \ge \prod_{i=1}^{n}\overrightarrow{t}_i^{2\,w_i}$,
$1-\prod_{i=1}^{n}\left(1-\overrightarrow{l}_i^{1}\right)^{w_i} \le 1-\prod_{i=1}^{n}\left(1-\overrightarrow{l}_i^{2}\right)^{w_i}$,
$1-\prod_{i=1}^{n}\left(1-\overrightarrow{\vartheta}_i^{1}\right)^{w_i} \le 1-\prod_{i=1}^{n}\left(1-\overrightarrow{\vartheta}_i^{2}\right)^{w_i}$;

$\prod_{i=1}^{n}\overrightarrow{g}_i^{1\,w_i} \ge \prod_{i=1}^{n}\overrightarrow{g}_i^{2\,w_i}$,
$1-\prod_{i=1}^{n}\left(1-\overrightarrow{h}_i^{1}\right)^{w_i} \le 1-\prod_{i=1}^{n}\left(1-\overrightarrow{h}_i^{2}\right)^{w_i}$,

$$1-\prod_{i=1}^{n}\left(1-\overrightarrow{s}_i^{1}\right)^{w_i} \le 1-\prod_{i=1}^{n}\left(1-\overrightarrow{s}_i^{2}\right)^{w_i}$$

$\prod_{i=1}^{n}\overrightarrow{b}_i^{1\,w_i} \ge \prod_{i=1}^{n}\overrightarrow{b}_i^{2\,w_i}$,
$1-\prod_{i=1}^{n}\left(1-\overrightarrow{z}_i^{1}\right)^{w_i} \le 1-\prod_{i=1}^{n}\left(1-\overrightarrow{z}_i^{2}\right)^{w_i}$,

$$1-\prod_{i=1}^{n}\left(1-\overrightarrow{\zeta}_i^{1}\right)^{w_i} \le 1-\prod_{i=1}^{n}\left(1-\overrightarrow{\zeta}_i^{2}\right)^{w_i}$$

$\prod_{i=1}^{n}\overrightarrow{s}_i^{1\,w_i} \ge \prod_{i=1}^{n}\overrightarrow{s}_i^{2\,w_i}$,
$1-\prod_{i=1}^{n}\left(1-\overrightarrow{n}_i^{1}\right)^{w_i} \le 1-\prod_{i=1}^{n}\left(1-\overrightarrow{n}_i^{2}\right)^{w_i}$,

$$1-\prod_{i=1}^{n}\left(1-\overrightarrow{v}_i^{1}\right)^{w_i} \le 1-\prod_{i=1}^{n}\left(1-\overrightarrow{v}_i^{2}\right)^{w_i}$$

Hence $\eta_i^1 \le \eta_i^2$ for $i = 1,2,\dots,n$.

Property 4 (Monotonicity):

$$\text{If}\,\eta_i^1=\left\{\begin{array}{l}\langle[(\overleftarrow{t}_i^1,\overleftarrow{g}_i^1,\overleftarrow{b}_i^1,\overleftarrow{s}_i^1,\overleftarrow{z}_i^1),(\overrightarrow{t}_i^1,\overrightarrow{g}_i^1,\overrightarrow{b}_i^1,\overrightarrow{s}_i^1,\overrightarrow{z}_i^1)]:T_{\eta_i^1}\rangle\\ \langle[(\overleftarrow{l}_i^1,\overleftarrow{h}_i^1,\overleftarrow{z}_i^1,\overleftarrow{n}_i^1,\overleftarrow{t}_i^1),(\overrightarrow{l}_i^1,\overrightarrow{h}_i^1,\overrightarrow{z}_i^1,\overrightarrow{n}_i^1,\overrightarrow{t}_i^1)]:I_{\eta_i^1}\rangle\\ \langle[(\overleftarrow{\vartheta}_i^1,\overleftarrow{s}_i^1,\overleftarrow{\zeta}_i^1,\overleftarrow{v}_i^1,\overleftarrow{t}_i^1),(\overrightarrow{\vartheta}_i^1,\overrightarrow{s}_i^1,\overrightarrow{\zeta}_i^1,\overrightarrow{v}_i^1,\overrightarrow{t}_i^1)]:F_{\eta_i^1}\rangle\end{array}\right\}\text{ and}$$

$$\eta_i^2=\left\{\begin{array}{l}\langle[(\overleftarrow{t}_i^2,\overleftarrow{g}_i^2,\overleftarrow{b}_i^2,\overleftarrow{s}_i^2,\overleftarrow{z}_i^2),(\overrightarrow{t}_i^2,\overrightarrow{g}_i^2,\overrightarrow{b}_i^2,\overrightarrow{s}_i^2,\overrightarrow{z}_i^2,)]:T_{\eta_i^2}\rangle\\ \langle[(\overleftarrow{l}_i^2,\overleftarrow{h}_i^2,\overleftarrow{z}_i^2,\overleftarrow{n}_i^2,\overleftarrow{t}_i^2),(\overrightarrow{l}_i^2,\overrightarrow{h}_i^2,\overrightarrow{z}_i^2,\overrightarrow{n}_i^2,\overrightarrow{t}_i^2)]:I_{\eta_i^2}\rangle\\ \langle[(\overleftarrow{\vartheta}_i^2,\overleftarrow{s}_i^2,\overleftarrow{\zeta}_i^2,\overleftarrow{v}_i^2,\overleftarrow{t}_i^2),(\overrightarrow{\vartheta}_i^2,\overrightarrow{s}_i^2,\overrightarrow{\zeta}_i^2,\overrightarrow{v}_i^2,\overrightarrow{t}_i^2)]:F_{\eta_i^2}\rangle\end{array}\right\}$$

$(i = 1,2,\dots,n)$be a collection of IVPFNNVs in the set of real numbers.
If $\eta_i^1 \le \eta_i^2$ for $i = 1,2,\dots,n$ then $\mathrm{IVPFNWG}(\eta_1^1,\eta_2^1,\dots,\eta_n^1) \le \mathrm{IVPFNWG}(\eta_1^2,\eta_2^2,\dots,\eta_n^2)$.

Proof: In order to prove this, we consider $\overleftarrow{z}_i^1,\overrightarrow{z}_i^1,\overleftarrow{t}_i^1,\overrightarrow{t}_i^1,\overleftarrow{\mathfrak{t}}_i^1,\overrightarrow{\mathfrak{t}}_i^1$ of η_i^1 and $\overleftarrow{z}_i^2,\overrightarrow{z}_i^2,\overleftarrow{t}_i^2,\overrightarrow{t}_i^2,\overleftarrow{\mathfrak{t}}_i^2,\overrightarrow{\mathfrak{t}}_i^2$ of η_i^2. We assume, $\overleftarrow{z}_i^1 \le \overleftarrow{z}_i^2, \overleftarrow{t}_i^1 \ge \overleftarrow{t}_i^2, \overleftarrow{\mathfrak{t}}_i^1 \ge \overleftarrow{\mathfrak{t}}_i^2; \overrightarrow{z}_i^1 \le \overrightarrow{z}_i^2, \overrightarrow{t}_i^1 \ge \overrightarrow{t}_i^2, \overrightarrow{\mathfrak{t}}_i^1 \ge \overrightarrow{\mathfrak{t}}_i^2$ for $\eta_i^1 \le \eta_i^2$ for $i = 1,2,\dots,n$. Then,
$\overleftarrow{z}_i^{1\,w_i} \ge \overleftarrow{z}_i^{2\,w_i}, \left(1-\overleftarrow{t}_i^1\right)^{w_i} \ge \left(1-\overleftarrow{t}_i^2\right)^{w_i}$,
$\left(1-\overleftarrow{\mathfrak{t}}_i^1\right)^{w_i} \ge \left(1-\overleftarrow{\mathfrak{t}}_i^2\right)^{w_i}$;

$\overrightarrow{z}_i^{1\,w_i} \ge \overrightarrow{z}_i^{2\,w_i}, \left(1-\overrightarrow{t}_i^1\right)^{w_i} \ge \left(1-\overrightarrow{t}_i^2\right)^{w_i}$,
$\left(1-\overrightarrow{\mathfrak{t}}_i^1\right)^{w_i} \ge \left(1-\overrightarrow{\mathfrak{t}}_i^2\right)^{w_i}$.

$\prod_{i=1}^{n}\overleftarrow{z}_i^{1\,w_i} \ge \prod_{i=1}^{n}\overleftarrow{z}_i^{2\,w_i}$,

$$1-\prod_{i=1}^{n}\left(1-\overleftarrow{t}_i^{1}\right)^{w_i} \le 1-\prod_{i=1}^{n}\left(1-\overleftarrow{t}_i^{2}\right)^{w_i},$$

$1-\prod_{i=1}^{n}\left(1-\overleftarrow{\mathfrak{t}}_i^{1}\right)^{w_i} \le 1-\prod_{i=1}^{n}\left(1-\overleftarrow{\mathfrak{t}}_i^{1}\right)^{w_i}$;
$\prod_{i=1}^{n}\overrightarrow{z}_i^{1\,w_i} \ge \prod_{i=1}^{n}\overrightarrow{z}_i^{2\,w_i}$,
$1-\prod_{i=1}^{n}\left(1-\overrightarrow{t}_i^{1}\right)^{w_i} \le 1-\prod_{i=1}^{n}\left(1-\overrightarrow{t}_i^{2}\right)^{w_i}$,
$1-\prod_{i=1}^{n}\left(1-\overrightarrow{\mathfrak{t}}_i^{1}\right)^{w_i} \le 1-\prod_{i=1}^{n}\left(1-\overrightarrow{\mathfrak{t}}_i^{1}\right)^{w_i}$.
In the same way
$\prod_{i=1}^{n}\overleftarrow{t}_i^{1\,w_i} \ge \prod_{i=1}^{n}\overleftarrow{t}_i^{2\,w_i}$,
$1-\prod_{i=1}^{n}\left(1-\overleftarrow{l}_i^{1}\right)^{w_i} \le 1-\prod_{i=1}^{n}\left(1-\overleftarrow{l}_i^{2}\right)^{w_i}$,
$1-\prod_{i=1}^{n}\left(1-\overleftarrow{\vartheta}_i^{1}\right)^{w_i} \le 1-\prod_{i=1}^{n}\left(1-\overleftarrow{\vartheta}_i^{1}\right)^{w_i}$;

$\prod_{i=1}^{n}\overleftarrow{g}_i^{1\,w_i} \ge \prod_{i=1}^{n}\overleftarrow{g}_i^{2\,w_i}$,
$1-\prod_{i=1}^{n}\left(1-\overleftarrow{h}_i^{1}\right)^{w_i} \le 1-\prod_{i=1}^{n}\left(1-\overleftarrow{h}_i^{2}\right)^{w_i}$,

$$1-\prod_{i=1}^{n}\left(1-\overleftarrow{s}_i^{1}\right)^{w_i} \le 1-\prod_{i=1}^{n}\left(1-\overleftarrow{s}_i^{2}\right)^{w_i}$$

$\prod_{i=1}^{n}\overleftarrow{b}_i^{1\,w_i} \ge \prod_{i=1}^{n}\overleftarrow{b}_i^{2\,w_i}$,
$1-\prod_{i=1}^{n}\left(1-\overleftarrow{z}_i^{1}\right)^{w_i} \le 1-\prod_{i=1}^{n}\left(1-\overleftarrow{z}_i^{2}\right)^{w_i}$,

$$1-\prod_{i=1}^{n}\left(1-\overleftarrow{\zeta}_i^{1}\right)^{w_i} \le 1-\prod_{i=1}^{n}\left(1-\overleftarrow{\zeta}_i^{2}\right)^{w_i}$$

$\prod_{i=1}^{n}\overleftarrow{t}_i^{1\,w_i} \ge \prod_{i=1}^{n}\overleftarrow{t}_i^{2\,w_i}$,
$1-\prod_{i=1}^{n}\left(1-\overleftarrow{n}_i^{1}\right)^{w_i} \le 1-\prod_{i=1}^{n}\left(1-\overleftarrow{n}_i^{2}\right)^{w_i}$,

$$1-\prod_{i=1}^{n}\left(1-\overleftarrow{v}_i^{1}\right)^{w_i} \le 1-\prod_{i=1}^{n}\left(1-\overleftarrow{v}_i^{2}\right)^{w_i}$$

and
$\prod_{i=1}^{n}\overrightarrow{t}_i^{1\,w_i} \ge \prod_{i=1}^{n}\overrightarrow{t}_i^{2\,w_i}$,
$1-\prod_{i=1}^{n}\left(1-\overrightarrow{l}_i^{1}\right)^{w_i} \le 1-\prod_{i=1}^{n}\left(1-\overrightarrow{l}_i^{2}\right)^{w_i}$,
$1-\prod_{i=1}^{n}\left(1-\overrightarrow{\vartheta}_i^{1}\right)^{w_i} \le 1-\prod_{i=1}^{n}\left(1-\overrightarrow{\vartheta}_i^{2}\right)^{w_i}$;

$\prod_{i=1}^{n}\overrightarrow{g}_i^{1\,w_i} \ge \prod_{i=1}^{n}\overrightarrow{g}_i^{2\,w_i}$,
$1-\prod_{i=1}^{n}\left(1-\overrightarrow{h}_i^{1}\right)^{w_i} \le 1-\prod_{i=1}^{n}\left(1-\overrightarrow{h}_i^{2}\right)^{w_i}$,

$$1-\prod_{i=1}^{n}\left(1-\overrightarrow{s}_i^{1}\right)^{w_i} \le 1-\prod_{i=1}^{n}\left(1-\overrightarrow{s}_i^{2}\right)^{w_i}$$

$\prod_{i=1}^{n}\overrightarrow{b}_i^{1\,w_i} \ge \prod_{i=1}^{n}\overrightarrow{b}_i^{2\,w_i}$,
$1-\prod_{i=1}^{n}\left(1-\overrightarrow{z}_i^{1}\right)^{w_i} \le 1-\prod_{i=1}^{n}\left(1-\overrightarrow{z}_i^{2}\right)^{w_i}$,

$$1-\prod_{i=1}^{n}\left(1-\overrightarrow{\zeta}_i^{1}\right)^{w_i} \le 1-\prod_{i=1}^{n}\left(1-\overrightarrow{\zeta}_i^{2}\right)^{w_i}$$

$\prod_{i=1}^{n}\overrightarrow{s}_i^{1\,w_i} \ge \prod_{i=1}^{n}\overrightarrow{s}_i^{2\,w_i}$,
$1-\prod_{i=1}^{n}\left(1-\overrightarrow{n}_i^{1}\right)^{w_i} \le 1-\prod_{i=1}^{n}\left(1-\overrightarrow{n}_i^{2}\right)^{w_i}$,

$$1-\prod_{i=1}^{n}\left(1-\overrightarrow{v}_i^{1}\right)^{w_i} \le 1-\prod_{i=1}^{n}\left(1-\overrightarrow{v}_i^{2}\right)^{w_i}$$

Consider,

$$\dot{ŋ}^1 = IVPFNWG(\dot{ŋ}_1^1, \dot{ŋ}_2^1, \ldots, \dot{ŋ}_n^1) = \begin{Bmatrix} \langle [(\overleftarrow{ƫ}^1, \overleftarrow{ɡ}^1, \overleftarrow{ƀ}^1, \overleftarrow{ƨ}^1, \overleftarrow{ʒ}^1), (\overrightarrow{ƫ}^1, \overrightarrow{ɡ}^1, \overrightarrow{ƀ}^1, \overrightarrow{ƨ}^1, \overrightarrow{ʒ}^1)] : Ŧ_{\dot{ŋ}^1} \rangle \\ \langle [(\overleftarrow{ɭ}^1, \overleftarrow{ɧ}^1, \overleftarrow{ʓ}^1, \overleftarrow{ƞ}^1, \overleftarrow{ẗ}^1), (\overrightarrow{ɭ}^1, \overrightarrow{ɧ}^1, \overrightarrow{ʓ}^1, \overrightarrow{ƞ}^1, \overrightarrow{ẗ}^1)] : Ɨ_{\dot{ŋ}^1} \rangle \\ \langle [(\overleftarrow{\vartheta}^1, \overleftarrow{ȿ}^1, \overleftarrow{\zeta}^1, \overleftarrow{\upsilon}^1, \overleftarrow{ƅ}^1), (\overrightarrow{\vartheta}^1, \overrightarrow{ȿ}^1, \overrightarrow{\zeta}^1, \overrightarrow{\upsilon}^1, \overrightarrow{ƅ}^1)] : Ƒ_{\dot{ŋ}^1} \rangle \end{Bmatrix}$$

and $\dot{ŋ}^2 = IVPFNWG(\dot{ŋ}_1^2, \dot{ŋ}_2^2, \ldots, \dot{ŋ}_n^2)$

$$= \begin{Bmatrix} \langle [(\overleftarrow{ƫ}^2, \overleftarrow{ɡ}^2, \overleftarrow{ƀ}^2, \overleftarrow{ƨ}^2, \overleftarrow{ʒ}^2), (\overrightarrow{ƫ}^2, \overrightarrow{ɡ}^2, \overrightarrow{ƀ}^2, \overrightarrow{ƨ}^2, \overrightarrow{ʒ}^2)] : Ŧ_{\dot{ŋ}^2} \rangle \\ \langle [(\overleftarrow{ɭ}^2, \overleftarrow{ɧ}^2, \overleftarrow{ʓ}^2, \overleftarrow{ƞ}^2, \overleftarrow{ẗ}^2), (\overrightarrow{ɭ}^2, \overrightarrow{ɧ}^2, \overrightarrow{ʓ}^2, \overrightarrow{ƞ}^2, \overrightarrow{ẗ}^2)] : Ɨ_{\dot{ŋ}^2} \rangle \\ \langle [(\overleftarrow{\vartheta}^2, \overleftarrow{ȿ}^2, \overleftarrow{\zeta}^2, \overleftarrow{\upsilon}^2, \overleftarrow{ƅ}^2), (\overrightarrow{\vartheta}^2, \overrightarrow{ȿ}^2, \overrightarrow{\zeta}^2, \overrightarrow{\upsilon}^2, \overrightarrow{ƅ}^2)] : Ƒ_{\dot{ŋ}^2} \rangle \end{Bmatrix}$$, were

$\overleftarrow{ƫ}^T = \prod_{i=1}^n (\overleftarrow{ƫ}_i^T)^{w_i}$, $\overleftarrow{ɡ}^T = \prod_{i=1}^n (\overleftarrow{ɡ}_i^T)^{w_i}$,
$\overleftarrow{ƀ}^T = \prod_{i=1}^n (\overleftarrow{ƀ}_i^T)^{w_i}$, $\overleftarrow{ƨ}^T = \prod_{i=1}^n (\overleftarrow{ƨ}_i^T)^{w_i}$,
$\overleftarrow{ʒ}^T = \prod_{i=1}^n (\overleftarrow{ʒ}_i^T)^{w_i}$;
$\overleftarrow{ɭ}^T = 1 - \prod_{i=1}^n (1 - \overleftarrow{ɭ}_i^T)^{w_i}$, $\overleftarrow{ɧ}^T = 1 - \prod_{i=1}^n (1 - \overleftarrow{ɧ}_i^T)^{w_i}$,
$\overleftarrow{ʓ}^T = 1 - \prod_{i=1}^n (1 - \overleftarrow{ʓ}_i^T)^{w_i}$,
$\overleftarrow{ƞ}^T = 1 - \prod_{i=1}^n (\overline{1 - ƞ_i^T})^{w_i}$, $\overleftarrow{ẗ}^T = 1 - \prod_{i=1}^n (1 - \overleftarrow{ẗ}_i^T)^{w_i}$ and
$\overleftarrow{\vartheta}^T = 1 - \prod_{i=1}^n (1 - \overleftarrow{\vartheta}_i^T)^{w_i}$, $\overleftarrow{ȿ}^T = 1 - \prod_{i=1}^n (1 - \overleftarrow{ȿ}_i^T)^{w_i}$,
$\overleftarrow{\zeta}^T = 1 - \prod_{i=1}^n (1 - \overleftarrow{\zeta}_i^T)^{w_i}$,
$\overleftarrow{\upsilon}^T = 1 - \prod_{i=1}^n (1 - \overleftarrow{\upsilon}_i^T)^{w_i}$, $\overleftarrow{ƅ}^T = 1 - \prod_{i=1}^n (1 - \overleftarrow{ƅ}_i^T)^{w_i}$.

Also

$\overrightarrow{ƫ}^T = \prod_{i=1}^n (\overrightarrow{ƫ}_i^T)^{w_i}$, $\overrightarrow{ɡ}^T = \prod_{i=1}^n (\overrightarrow{ɡ}_i^T)^{w_i}$,
$\overrightarrow{ƀ}^T = \prod_{i=1}^n (\overrightarrow{ƀ}_i^T)^{w_i}$, $\overrightarrow{ƨ}^T = \prod_{i=1}^n (\overrightarrow{ƨ}_i^T)^{w_i}$,
$\overrightarrow{ʒ}^T = \prod_{i=1}^n (\overrightarrow{ʒ}_i^T)^{w_i}$;
$\overrightarrow{ɭ}^T = 1 - \prod_{i=1}^n (1 - \overrightarrow{ɭ}_i^T)^{w_j}$, $\overrightarrow{ɧ}^T = 1 - \prod_{i=1}^n (1 - \overrightarrow{ɧ}_i^T)^{w_i}$,
$\overrightarrow{ʓ}^T = 1 - \prod_{i=1}^n (1 - \overrightarrow{ʓ}_i^T)^{w_i}$,

and

$$\overrightarrow{\vartheta}^T = 1 - \prod_{i=1}^n (1 - \overrightarrow{\vartheta}_i^T)^{w_j},$$

$$\overrightarrow{ȿ}^T = 1 - \prod_{i=1}^n (1 - \overrightarrow{ȿ}_i^T)^{w_i},$$

$$\overrightarrow{\zeta}^T = 1 - \prod_{i=1}^n (1 - \overrightarrow{\zeta}_i^T)^{w_i},$$

$$\overrightarrow{\upsilon}^T = 1 - \prod_{i=1}^n (1 - \overrightarrow{\upsilon}_i^T)^{w_i},$$

$$\overrightarrow{ƅ}^T = \prod_{i=1}^n (\overrightarrow{ƅ}_i^T)^{w_i} \text{for } T = 1,2.$$

The score function,

$$(\dot{ŋ}) = \frac{1}{6}\left[4 + \frac{\overleftarrow{ƫ}^1 + \overleftarrow{ɡ}^1 + \overleftarrow{ƀ}^1 + \overleftarrow{ƨ}^1 + \overleftarrow{ʒ}^1}{5} + \frac{\overrightarrow{ƫ}^1 + \overrightarrow{ɡ}^1 + \overrightarrow{ƀ}^1 + \overrightarrow{ƨ}^1 + \overrightarrow{ʒ}^1}{5} - \frac{\overleftarrow{ɭ}^1 + \overleftarrow{ɧ}^1 + \overleftarrow{ʓ}^1 + \overleftarrow{ƞ}^1 + \overleftarrow{ẗ}^1}{5} - \frac{\overrightarrow{ɭ}^1 + \overrightarrow{ɧ}^1 + \overrightarrow{ʓ}^1 + \overrightarrow{ƞ}^1 + \overrightarrow{ẗ}^1}{5} - \frac{\overleftarrow{\vartheta}^1 + \overleftarrow{ȿ}^1 + \overleftarrow{\zeta}^1 + \overleftarrow{\upsilon}^1 + \overleftarrow{ƅ}^1}{5} - \frac{\overrightarrow{\vartheta}^1 + \overrightarrow{ȿ}^1 + \overrightarrow{\zeta}^1 + \overrightarrow{\upsilon}^1 + \overrightarrow{ƅ}^1}{5} \right]$$

$$\leq \frac{1}{6}\left[4 + \frac{\overleftarrow{ƫ}^2 + \overleftarrow{ɡ}^2 + \overleftarrow{ƀ}^2 + \overleftarrow{ƨ}^2 + \overleftarrow{ʒ}^2}{5} + \frac{\overrightarrow{ƫ}^2 + \overrightarrow{ɡ}^2 + \overrightarrow{ƀ}^2 + \overrightarrow{ƨ}^2 + \overrightarrow{ʒ}^2}{5} - \frac{\overleftarrow{ɭ}^2 + \overleftarrow{ɧ}^2 + \overleftarrow{ʓ}^2 + \overleftarrow{ƞ}^2 + \overleftarrow{ẗ}^2}{5} - \frac{\overrightarrow{ɭ}^2 + \overrightarrow{ɧ}^2 + \overrightarrow{ʓ}^2 + \overrightarrow{ƞ}^2 + \overrightarrow{ẗ}^2}{5} - \frac{\overleftarrow{\vartheta}^2 + \overleftarrow{ȿ}^2 + \overleftarrow{\zeta}^2 + \overleftarrow{\upsilon}^2 + \overleftarrow{ƅ}^2}{5} - \frac{\overrightarrow{\vartheta}^2 + \overrightarrow{ȿ}^2 + \overrightarrow{\zeta}^2 + \overrightarrow{\upsilon}^2 + \overrightarrow{ƅ}^2}{5} \right]$$

Here we discuss the different cases:

Case (i):

If $S(\dot{ŋ}^1) < S(\dot{ŋ}^2)$, then

$$IVPFNWG(\dot{ŋ}_1^1, \dot{ŋ}_2^1, \ldots, \dot{ŋ}_n^1) < IVPFNWG(\dot{ŋ}_1^2, \dot{ŋ}_2^2, \ldots, \dot{ŋ}_n^2) \quad (6)$$

Case (ii):

If $S(\dot{ŋ}^1) = S(\dot{ŋ}^2)$,

then by score function, we consider

$$= \frac{1}{6}\left[4 + \frac{\overleftarrow{ƫ}^1 + \overleftarrow{ɡ}^1 + \overleftarrow{ƀ}^1 + \overleftarrow{ƨ}^1 + \overleftarrow{ʒ}^1}{5} + \frac{\overrightarrow{ƫ}^1 + \overrightarrow{ɡ}^1 + \overrightarrow{ƀ}^1 + \overrightarrow{ƨ}^1 + \overrightarrow{ʒ}^1}{5} - \frac{\overleftarrow{ɭ}^1 + \overleftarrow{ɧ}^1 + \overleftarrow{ʓ}^1 + \overleftarrow{ƞ}^1 + \overleftarrow{ẗ}^1}{5} - \frac{\overrightarrow{ɭ}^1 + \overrightarrow{ɧ}^1 + \overrightarrow{ʓ}^1 + \overrightarrow{ƞ}^1 + \overrightarrow{ẗ}^1}{5} - \frac{\overleftarrow{\vartheta}^1 + \overleftarrow{ȿ}^1 + \overleftarrow{\zeta}^1 + \overleftarrow{\upsilon}^1 + \overleftarrow{ƅ}^1}{5} - \frac{\overrightarrow{\vartheta}^1 + \overrightarrow{ȿ}^1 + \overrightarrow{\zeta}^1 + \overrightarrow{\upsilon}^1 + \overrightarrow{ƅ}^1}{5} \right]$$

$$=\frac{1}{6}\begin{bmatrix} 4+\frac{\overleftarrow{t}^2+\overleftarrow{g}^2+\overleftarrow{b}^2+\overleftarrow{s}^2+\overleftarrow{z}^2}{5} \\ +\frac{\overrightarrow{t}^2+\overrightarrow{g}^2+\overrightarrow{b}^2+\overrightarrow{s}^2+\overrightarrow{z}^2}{5} \\ -\frac{\overleftarrow{l}^2+\overleftarrow{h}^2+\overleftarrow{z}^2+\overleftarrow{n}^2+\overleftarrow{t}^2}{5} \\ -\frac{\overrightarrow{l}^2+\overrightarrow{h}^2+\overrightarrow{z}^2+\overrightarrow{n}^2+\overrightarrow{t}^2}{5} \\ -\frac{\overleftarrow{\vartheta}^2+\overleftarrow{s}^2+\overleftarrow{\zeta}^2+\overleftarrow{\upsilon}^2+\overleftarrow{b}^2}{5} \\ -\frac{\overrightarrow{\vartheta}^2+\overrightarrow{s}^2+\overrightarrow{\zeta}^2+\overrightarrow{\upsilon}^2+\overrightarrow{b}^2}{5} \end{bmatrix}$$

$$\mathrm{IVPFNWG}(\dot{\eta}_1^1,\dot{\eta}_2^1,\dots,\dot{\eta}_n^1)=\mathrm{IVPFNWG}(\dot{\eta}_1^2,\dot{\eta}_2^2,\dots,\dot{\eta}_n^2) \quad (7)$$

This implies $\dot{\eta}_h^1 \le \dot{\eta}_h^2$, $h=1,2,\dots,n$ i.e., for

$\overleftarrow{t}_h^1 \le \overleftarrow{t}_h^2$, $\overleftarrow{g}_h^1 \le \overleftarrow{g}_h^2$, $\overleftarrow{b}_h^1 \le \overleftarrow{b}_h^2$, $\overleftarrow{s}_h^1 \le \overleftarrow{s}_h^2$, $\overleftarrow{z}_h^1 \le \overleftarrow{z}_h^2$; $\overrightarrow{t}_h^1 \le \overrightarrow{t}_h^2$, $\overrightarrow{g}_h^1 \le \overrightarrow{g}_h^2$, $\overrightarrow{b}_h^1 \le \overrightarrow{b}_h^2$, $\overrightarrow{s}_h^1 \le \overrightarrow{s}_h^2$, $\overrightarrow{z}_h^1 \le \overrightarrow{z}_h^2$;

$\overleftarrow{l}_h^1 \ge \overleftarrow{l}_h^2$, $\overleftarrow{h}_h^1 \ge \overleftarrow{h}_h^2$, $\overleftarrow{z}_h^1 \ge \overleftarrow{z}_h^2$, $\overleftarrow{n}_h^1 \ge \overleftarrow{n}_h^2$, $\overleftarrow{t}_h^1 \ge \overleftarrow{t}_h^2$; $\overrightarrow{l}_h^1 \ge \overrightarrow{l}_h^2$, $\overrightarrow{h}_h^1 \ge \overrightarrow{h}_h^2$, $\overrightarrow{z}_h^1 \ge \overrightarrow{z}_h^2$, $\overrightarrow{n}_h^1 \ge \overrightarrow{n}_h^2$, $\overrightarrow{t}_h^1 \ge \overrightarrow{t}_h^2$; then by (6), (7) the following results follows:

$$\mathrm{IVPFNWG}(\dot{\eta}_1^1,\dot{\eta}_2^1,\dots,\dot{\eta}_n^1) \le \mathrm{IVPFNWG}(\dot{\eta}_1^2,\dot{\eta}_2^2,\dots,\dot{\eta}_n^2)$$

$\overleftarrow{\vartheta}_h^1 \ge \overleftarrow{\vartheta}_h^2$, $\overleftarrow{s}_h^1 \ge \overleftarrow{s}_h^2$, $\overleftarrow{\zeta}_h^1 \ge \overleftarrow{\zeta}_h^2$, $\overleftarrow{\upsilon}_h^1 \ge \overleftarrow{\upsilon}_h^2$, $\overleftarrow{b}_h^1 \ge \overleftarrow{b}_h^2$; $\overrightarrow{\vartheta}_h^1 \ge \overrightarrow{\vartheta}_h^2$, $\overrightarrow{s}_h^1 \ge \overrightarrow{s}_h^2$, $\overrightarrow{\zeta}_h^1 \ge \overrightarrow{\zeta}_h^2$, $\overrightarrow{\upsilon}_h^1 \ge \overrightarrow{\upsilon}_h^2$, $\overrightarrow{b}_h^1 \ge \overrightarrow{b}_h^2$.

If $\overleftarrow{t}^1=\overleftarrow{t}^2$, $\overleftarrow{g}^1=\overleftarrow{g}^2$, $\overleftarrow{b}^1=\overleftarrow{b}^2$, $\overleftarrow{s}^1=\overleftarrow{s}^2$, $\overleftarrow{z}^1=\overleftarrow{z}^2$,

$\overrightarrow{t}^1=\overrightarrow{t}^2$, $\overrightarrow{g}^1=\overrightarrow{g}^2$, $\overrightarrow{b}^1=\overrightarrow{b}^2$, $\overrightarrow{s}^1=\overrightarrow{s}^2$, $\overrightarrow{z}^1=\overrightarrow{z}^2$;
$\overleftarrow{l}^1=\overleftarrow{l}^2$, $\overleftarrow{h}^1=\overleftarrow{h}^2$, $\overleftarrow{z}^1=\overleftarrow{z}^2$, $\overleftarrow{n}^1=\overleftarrow{n}^2$, $\overleftarrow{t}^1=\overleftarrow{t}^2$,
$\overrightarrow{l}^1=\overrightarrow{l}^2$, $\overrightarrow{h}^1=\overrightarrow{h}^2$, $\overrightarrow{z}^1=\overrightarrow{z}^2$, $\overrightarrow{n}^1=\overrightarrow{n}^2$, $\overrightarrow{t}^1=\overrightarrow{t}^2$;
$\overleftarrow{\vartheta}^1=\overleftarrow{\vartheta}^2$, $\overleftarrow{s}^1=\overleftarrow{s}^2$, $\overleftarrow{\zeta}^1=\overleftarrow{\zeta}^2$, $\overleftarrow{\upsilon}^1=\overleftarrow{\upsilon}^2$, $\overleftarrow{b}^1=\overleftarrow{b}^2$,
$\overrightarrow{\vartheta}^1=\overrightarrow{\vartheta}^2$, $\overrightarrow{s}^1=\overrightarrow{s}^2$, $\overrightarrow{\zeta}^1=\overrightarrow{\zeta}^2$, $\overrightarrow{\upsilon}^1=\overrightarrow{\upsilon}^2$, $\overrightarrow{b}^1=\overrightarrow{b}^2$.

Then accuracy function is given by,

$$A_c(\dot{\eta}^1)=\frac{1}{2}\begin{bmatrix} \frac{\overleftarrow{t}^1+\overleftarrow{g}^1+\overleftarrow{b}^1+\overleftarrow{s}^1+\overleftarrow{z}^1}{5} \\ +\frac{\overrightarrow{t}^1+\overrightarrow{g}^1+\overrightarrow{b}^1+\overrightarrow{s}^1+\overrightarrow{z}^1}{5} \\ \frac{\overleftarrow{\vartheta}^1+\overleftarrow{s}^1+\overleftarrow{\zeta}^1+\overleftarrow{\upsilon}^1+\overleftarrow{b}^1}{5} \\ -\frac{\overrightarrow{\vartheta}^1+\overrightarrow{s}^1+\overrightarrow{\zeta}^1+\overrightarrow{\upsilon}^1+\overrightarrow{b}^1}{5} \end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix} \frac{\overleftarrow{t}^2+\overleftarrow{g}^2+\overleftarrow{b}^2+\overleftarrow{s}^2+\overleftarrow{z}^2}{5} \\ +\frac{\overrightarrow{t}^2+\overrightarrow{g}^2+\overrightarrow{b}^2+\overrightarrow{s}^2+\overrightarrow{z}^2}{5} \\ \frac{\overleftarrow{\vartheta}^2+\overleftarrow{s}^2+\overleftarrow{\zeta}^2+\overleftarrow{\upsilon}^2+\overleftarrow{b}^2}{5} \\ -\frac{\overrightarrow{\vartheta}^2+\overrightarrow{s}^2+\overrightarrow{\zeta}^2+\overrightarrow{\upsilon}^2+\overrightarrow{b}^2}{5} \end{bmatrix}=A_c(\dot{\eta}^2)$$

then by definition of $\dot{\eta}_1 \times \dot{\eta}_2$, we have

Thus, it completes the proof of the property.

Example. Let

$$\bar{A}_1=\langle\begin{bmatrix}(0.000,0.002,0.004,0.006,.008),\\(0.802,0.804,0.806,0.808,0.810)\end{bmatrix}, \begin{bmatrix}(0.212,0.214,0.216,0.218,0.220),\\(0.578,0.580,0.582,0.584,0.586\end{bmatrix}, \begin{bmatrix}(0.348,0.350,0.352,0.354,0.356),\\(0.702,0.704,0.706,0.708,0.710\end{bmatrix}\rangle,$$

$$\bar{A}_2=\langle\begin{bmatrix}(0.466,0.468,0.470,0.472,0.474),\\(0.684,0.686,0.688,0.690,0.700\end{bmatrix}, \begin{bmatrix}(0.110,0.112,0.114,0.116,0.118),\\(0.502,0.504,0.506,0.508,0.510\end{bmatrix} \begin{bmatrix}(0.380,0.382,0.384,0.386,0.388),\\(0.742,0.744,0.746,0.748,0.750)\end{bmatrix}\rangle$$

$$\bar{A}_3=\langle\begin{bmatrix}(0.222,0.224,0.226,0.228,0.230),\\(0.662,0.664,0.666,0.668,0.670)\end{bmatrix}, \begin{bmatrix}(0.532,0.534,0.536,0.538,0.540),\\(0.912,0.914,0.916,0.918,0.920\end{bmatrix} \begin{bmatrix}(0.460,0.462,0.464,0.466,0.468),\\(0.708,0.710,0.712,0.714,0.716)\end{bmatrix}\rangle$$

are three IVPFNVs. By using IVPFNWG Operator defined by (2) the aggregation is done for $\bar{A}_1, \bar{A}_2$ and $\bar{A}_3$ with the weight vector $w=(0.22,0.15,0.33,0.12,0.18)$ which is represented as

$$\bar{A}=\mathrm{IVFNWG}(\bar{A}_1,\bar{A}_2,\bar{A}_3)= w_1\bar{A}_1 \otimes w_2\bar{A}_2 \otimes w_3\bar{A}_3$$

$$=\langle\begin{bmatrix}(0,0.281,0.077,0.414,0.281),\\(0.800,0.860,0.720,0.888,0.840)\end{bmatrix}, \begin{bmatrix}(0.217,0.155,0.312,0.128,0.187),\\(0.584,0.453,0.738,0.388,0.524)\end{bmatrix}, \begin{bmatrix}(0.285,0.205,0.399,0.170,0.245),\\(0.506,0.436,0.718,0.371,0.503)\end{bmatrix}\rangle$$

IV. APPLICATION OF IVPFNWG OPERATOR TO SOLVE THE MULTI CRITERIA DECISION MAKING ENVIRONMENT

Let us assume an automobile company wants to recruit a public relations officer (PRO). A candidate selected for this position should be prioritized in many aspects without leaving any minor qualities, a person should have for this designation. After initial stages of scrutinizing, four candidates $P_i (i=1,2,3,4)$ were selected for the final evaluation with five criterions to decide $C_i (i=1,2,3,4,5)$ namely,

1. Communication skill - C_1
2. Experience - C_2
3. Confidence - C_3
4. Knowledge - C_4
5. Personality - C_5

The decision matrix is given in terms of number of candidates and the criterion with the weightage of $w=(0.16,0.12,0.22,0.24,0.26)^T$ which is expressed in the following table I

Steps involved in solving the above problem using IVPFNWG operator:

Step 1: Aggregation of the rating values is calculated using the INPFNWG operator.

Step 2: The Score value is calculated for all the alternatives.

Step 3: Ranking is allotted according to the highest calculated score value.

Step 4: We arrive at the conclusion based upon the ranking.

TABLE I
RATING VALUES IN TERMS OF IVPFNNs

	C1
A	[(0.000,0.002,0.004,0.006,0.008),(0.502,0.504,0.506,0.508,0.510)],[(0.202,0.204,0.206,0.208,0.210),(0.600,0.602,0.604,0.606,0.608)],[(0.300,0.302,0.304,0.306,0.308),(0.802,0.804,0.806,0.808,0.810)]
B	[(0.222,0.224,0.226,0.228,0.230),(0.568,0.570,0.572,0.574,0.576)],[(0.462,0.464,0.466,0.468,0.500),(0.702,0.704,0.706,0.708,0.710)], [(0.352,0.354,0.356,0.358,0.360),(0.752,0.754,0.756,0.758,0.760)]
C	[(0.212,0.214,0.216,0.218,0.220),(0.572,0.574,0.576,0.578,0.580)],[(0.432,0.434,0.436,0.438,0.440),(0.666,0.668,0.670,0.672,0.674)],[(0.612,0.614,0.616,0.618,0.620),(0.918,0.920,0.922,0.924,0.926)]
D	[(0.292,0.294,0.296,0.298,0.300),(0.564,0.566,0.568,0.570,0.572)],[(0.640,0.642,0.644,0.646,0.648),(0.864,0.866,0.868,0.870,0.872)],[(0.472,0.474,0.476,0.478,0.500),(0.808,0.810,0.812,0.814,0.816)]
	C2
A	[(0.332,0.334,0.336,0.338,0.340),(0.742,0.744,0.746,0.748,0.750)],[(0.436,0.438,0.440,0.442,0.444),(0.720,0.722,0.724,0.726,0.728)],[(0.078,0.080,0.082,0.084,0.086),(0.642,0.644,0.646,0.648,0.650)]
B	[(0.012,0.014,0.016,0.018,0.020),(0.332,0.334,0.336,0.338,0.340)],[(0.072,0.074,0.076,0.078,0.080),(0.552,0.554,0.556,0.558,0.560)],[(0.674,0.676,0.678,0.680,0.682),(0.844,0.846,0.848,0.850,0.852)]
C	[(0.368,0.370,0.372,0.374,0.376),(0.642,0.644,0.646,0.648,0.650)],[(0.074,0.076,0.078,0.080,0.082),(0.512,0.514,0.516,0.518,0.520)],[(0.502,0.504,0.506,0.508,0.510),(0.936,0.938,0.940,0.942,0.944)]
D	[(0.064,0.066,0.068,0.070,0.072),(0.452,0.454,0.456,0.458,0.460)],[(0.372,0.374,0.376,0.378,0.380),(0.772,0.774,0.776,0.778,0.780)],[(0.590,0.592,0.594,0.596,0.598),(0.882,0.884,0.886,0.888,0.890)]
	C3
A	[(0.098,0.100,0.102,0.104,0.106),(0.562,0.564,0.566,0.568,0.570)],[(0.362,0.364,0.366,0.368,0.370),(0.712,0.714,0.716,0.718,0.720)],[(0.702,0.704,0.706,0.708,710),(0.972,0.974,0.976,0.978,0.980)]
B	[(0.188,0.190,0.192,0.194,196),(0.444,0.446,0.448,0.450,0.452)],[(0.282,0.284,0.286,0.288,0.290),(0.668,0.670,0.672,0.674,0.676)],[(0.502,0.504,0.506,0.508,0.510),(0.882,0.884,0.886,0.888,0.890)]
C	[(0.042,0.044,0.046,0.048,0.050),(0.452,0.454,0.456,0.458,0.460)],[(0.312,0.314,0.316,0.318,0.320),(0.606,0.608,0.610,0.612,0.614)],[(0.552,0.554,0.556,0.558,0.560),(0.972,0.974,0.976,0.978,0.980)]
D	[(0.052,0.054,0.056,0.058,0.060),(0.222,0.224,0.226,0.228,0.230)],[(0.072,0.074,0.076,0.078,0.080),(0.512,0.514,0.516,0.518,0.520)],[(0.652,0.654,0.656,0.658,0.660),(0.882,0.884,0.886,0.888,0.890)]
	C4
A	[(0.032,0.034,0.036,0.038,0.040),(0.342,0.344,0.346,0.348,0.350)],[(0.056,0.058,0.060,0.062,0.064),(0.552,0.554,0.556,0.558,0.560)],[(0.432,0.434,0.436,0.438,0.440),(0.712,0.714,0.716,0.718,0.720)]
B	[(0.112,0.114,0.116,0.118,0.120),(0.412,0.414,0.416,0.418,0.420)],[(0.302,0.304,0.306,0.308,0.310),(0.612,0.614,0.616,0.618,620)],[(0.502,0.504,0.506,0.508,0.510),(0.910,0.912,0.914,0.916,0.918)]

C	[(0.008,0.010,0.012,0.014,0.016),(0.442,0.444,0.446,0.448,0.450)],[(0.202,0.204,0.206,0.208,0.210),(0.662,0.664,0.666,0.668,0.670)],[(0.500,0.502,0.504,0.506,0.508),(0.712,0.714,0.716,0.718,0.720)]
D	[(0.090,0.092,0.094,0.096,0.098),(0.440,0.442,0.444,0.446,0.448)],[(0.222,0.224,0.226,0.228,0.230),(0.542,0.544,0.546,0.548,0.550)],[(0.378,0.380,0.382,0.384,0.386),(0.846,0.848,0.850,0.852,0.854)]
	C5
A	[(0.070,0.072,0.074,0.076,0.078),(0.356,0.358,0.360,0.362,0.364)],[(0.264,0.266,0.268,0.270,0.272),(0.572,0.574,0.576,0.578,0.580)],[(0.622,0.624,0.626,0.628,0.630),(0.950,0.952,0.954,0.956,0.958)]
B	[(0.082,0.084,0.086,0.088,0.090),(0.442,0.444,0.446,0.448,0.450)],[(0.272,0.274,0.276,0.278,0.280),(0.682,0.684,0.686,0.688,0.690)],[(0.526,0.528,0.530,0.532,0.534),(0.942,0.944,0.946,0.948,0.950)]
C	[(0.172,0.174,0.176,0.178,0.180),(0.376,0.378,0.380,0.382,0.384)],[(0.402,0.404,0.406,0.408,0.410),(0.732,0.734,0.736,0.738,0.740)],[(0.522,0.524,0.526,0.528,0.530),(0.902,0.904,0.906,0.908,0.910)]
D	[(0.342,0.344,0.346,0.348,0.350),(0.552,0.554,0.556,0.558,0.560)],[(0.072,0.074,0.076,0.078,0.080),(0.652,0.654,0.656,0.658,0.660)],[(0.554,0.556,0.558,0.560,0.562),(0.972,0.974,0.976,0.978,0.980)

TABLE II
AGGREGATED RATING VALUES OF IVPFNNs

A	[(0.000,0.044,0.050,0.055,0.059),(0.450,0.452,0.454,0.456,0.458)],[(0.257,0.259,0.261,0.263,0.265),(0.627,0.629,0.631,0.633,0.635)],[(0.514,0.516,0.518,0.521,0.523),(0.894,0.897,0.901,0.904,0.908)]
B	[(0.099,0.102,0.105,0.108,0.111),(0.438,0.440,0.442,0.444,0.446)],[(0.295,0.297,0.300,0.301,0.310),(0.653,0.655,0.657,0.659,0.661)],[(0.513,0.515,0.517,0.519,0.521),(0.893,0.895,0.897,0.900,0.902)]
C	[(0.068,0.073,0.078,0.082,0.086),(0.464,0.466,0.468,0.470,0.472)],[(0.309,0.311,0.313,0.315,0.317),(0.657,0.659,0.661,0.663,0.665)],[(0.537,0.539,0.541,0.543,0.545),(0.911,0.914,0.917,0.919,0.922)]
D	[(0.131,0.133,0.136,0.139,0.141),(0.419,0.421,0.423,0.426,0.428)],[(0.271,0.273,0.275,0.277,0.279),(0.673,0.675,0.677,0.679,0.682)],[(0.535,0.537,0.539,0.541,0.546),(0.906,0.909,0.912,0.915,0.918)]

TABLE III
RANKING BASED ON SCORE VALUES

	Score Value	Rank
A	0.3641	1
B	0.3625	2
C	0.3523	4
D	0.3593	3

V. CONCLUSION

This work seeks to increase decision-making precision in multi-criteria decision-making (MCDM) situations, which frequently arise in complicated and uncertain situations in daily life. Fuzzy decision making, which involves ranking the preferences, is one of the best techniques for handling MCDM problems. We have outlined the notions of IVPFNS and established some of its operational laws for IVPFNNs. The goal of this work is to overcome the limits that were pointed out while managing the value of dishonesty, ambiguity, and falsehood in a constrained context. The score and accuracy functions in this study were used to construct the IVPFNWG operator, a theorem, and some of its properties. An MCDM problem has finally been resolved using the proposed operator to increase the accuracy level for each alternate being evaluated with different weights for different attributes. This method is suitable for all fields when the decision-making problem is uncertain. To increase the efficacy and level of accuracy in medical diagnostics, pattern recognition, ranking preferences, etc., this recommended strategy can be widely employed in various MCDM scenarios. The suggested score and accuracy function will serve as the foundation for future work because it makes it simple to answer other MCDM problems with higher accuracy using fewer computations.

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