Drazin-Theta Inverse for Rectangular Matrices

Divya Purushothama Shenoy

***Abstract*—The concept of Drazin-Theta matrices are extended for rectangular matrices and defined W-weighted Drazin-Theta matrices. Several characterizations and the algebraic and geometrical interpretations of Drazin-Theta matrices are obtained here. The Drazin-Theta matrices are special cases of W-weighted Drazin-Theta matrices. All the characterizations obtained for W-weighted Drazin-Theta matrices are applicable to its dual, that is Theta-Drazin matrices. Further, the concept of W-weighted Drazin-Theta matrices are applied in solving certain types of linear system of equations.**

***Index Terms*—Drazin inverse, Moore-Penrose inverse, Drazin-Theta inverse, DMP inverse, s-g inverse, secondary transpose**

I. Introduction

THE notion of Moore-Penrose inverse, Drazin inverse and Core inverse are well known in literature. Recent attempts at defining new generalized inverses and extending the existing notion of inverses resulted in the development of Core EP inverse, DMP inverse, CMP inverse, Drazin-Star inverse, Drazin-Theta inverse etc. Extending the notion of Core inverse, Mallik et al. [1] defined DMP inverse of a square matrix using its Moore-Penrose inverse and Drazin inverse. Using the core part of a matrix A and the Moore Penrose inverse $A^{\dagger}$, Mehdipour et al. [2] defined CMP inverse. For more properties and representations of Core EP inverse one can refer [3], [4]. Motivated by the popularity of these matrices, recent developments in this area led to the introdution of Drazin-Star matrix, Drazin-Theta matrix and Outer-Theta matrix [5]–[7].

The Moore-Penrose inverse is a unique inverse defined for any rectangular matrix while Drazin inverse, DMP inverse, CMP inverse, Drazin-Star inverse and Drazin-Theta inverse are defined only for square matrices. Cline et al. [8] defined W-weighted Drazin inverse by extending the definition of Drazin inverse for rectangular matrices. For several characterizations, properties and representations of W-weighted drazin inverse the authors are referred to [9], [10], [11]. An integral representation of W-weighted Drazin inverse is given by Wei in [12]. An attempt to generalize DMP, CMP and Drazin-Star inverses to rectangular matrices resulted in [13]–[16]. In this article, Drazin-Theta matrices and Theta-Drazin matrices are extended to rectangular matrices which give rise to W-weighted Drazin-Theta matrix and its dual. Equivalent expressions of W-weighted Drazin-Theta matrix $A_{W-d,\theta}$ and its characterizations are obtained here. Also, an application of these matrices in solving linear system of equations are provided.

Before moving to the main section, let us discuss some notations and preliminary results that are necessary.

Manuscript received March 19, 2023; revised August 21, 2023.

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II. Preliminaries

An $m \times n$ matrix A of rank r defined over the set of all complex numbers are denoted by $A \in \mathbb{C}_r^{m\times n}$. Let $\mathcal{R}(A)$, $\mathcal{N}(A)$, $Ind(A)$, A^{θ} be the column space, null space, index and secondary conjugate transpose of the matrix A respectively. Whenever $Ind(A) = k$, k is the smallest nonnegative integer satisfying the condition $rank(A^k) = rank(A^{k+1})$.

Definition 1: [17] Let $A \in \mathbb{C}^{n\times n}$. The secondary transpose of A denoted by A^s, is defined as $A^s = (b_{ij})$ where $b_{ij} = a_{n-j+1,n-i+1}$ for $i, j = 1, 2, ...n$.

Based on this idea, Vijayakumar [18] defined secondary conjugate transpose and s-g inverse of matrices.

Definition 2: [18] Let $A \in \mathbb{C}^{n\times n}$. Then the secondary conjugate transpose of A denoted by A^{θ} and is defined as $A^{\theta} = \overline{A}^{s} = (c_{ij})$ where $c_{ij} = \overline{a}_{n-j+1,n-i+1}$.

Analogous to Moore-Penrose inverse, Vijayakumar [18] introduced s-g inverse $A^{\dagger_s}$ for a square matrix and the s-g inverse is unique whenever it exists. For more characterizations and construction of s-g inverse one can refer [19].

Definition 3: The s-g inverse of a square matrix $A \in \mathbb{C}^{n\times n}$ is denoted by $A^{\dagger_s}$ and is defined as

$$\begin{array}{llll} (1) & AXA = A & (2) & XAX = X \\ (3) & (AX)^{\theta} = AX & (4) & (XA)^{\theta} = XA. \end{array}$$

where A^{θ} denotes the secondary conjugate transpose of A. This definition can be extended to a rectangular matrix without loss of generality. It can be noted that s-g inverse and Moore-Penrose inverse are two different inverses as given in [6]. Consider a matrix $A \in \mathbb{C}^{m\times n}$. It is assured that there exists a unique matrix $G \in \mathbb{C}^{n\times m}$, satisfying all 4 conditions given below:

$$\begin{array}{llll} (1) & AXA = A & (2) & XAX = X \\ (3) & (AX)^{*} = AX & (4) & (XA)^{*} = XA. \end{array}$$

Any matrix which satisfies condition (1) is called the generalized inverse of A whereas that which satisfies (2) is the outer inverse. The matrix that satisfies (1) and (2) is the reflexive generalized inverse of A.

The Moore-Penrose inverse $A^{\dagger}$, of A, is the matrix satisfying the conditions $(1) - (4)$. Here $*$ denotes the conjugate transpose of A.

Assume $m = n$ and consider the following additional conditions -

$$(5)\ AX = XA \ \ (6)\ A^{k+1}X = A^k \text{ for some } k \in \{1, 2, 3, ...\}$$

The matrix that satisfies conditions $(1), (2)$ and (5), is the group inverse of A. Similarly the unique matrix that satisfies $(2), (5)$ and (6) is the Drazin inverse A^d of A. The readers are referred to [20]–[22] for more properties of Moore-Penrose inverse and Drazin inverse.

Cline [8] extended the notion of Drazin inverse for rectangular matrices and defined W-weighted Drazin inverse. In the same article, it was shown that whenever A is a square matrix, both $A^{\dagger}$ and A^{d} are special cases of W-weighted Drazin inverse.

Definition 4: [8] Let $A \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$. The matrix $X = A(WA)_d^2$ is the unique solution to the equations

$$(AW)^k = (AW)^{k+1}XW, \text{for some positive integer k} \quad (1)$$
$$X = XWAWX \quad (2)$$
$$AWX = XWA \quad (3)$$

is called the W-weighted Drazin inverse of A, and is denoted as $A_{d,W}$.
The properties of W-weighted Drazin inverse plays an important role in extending the definition of Drazin-Theta matrices to rectangular matrices.

Definition 5: [6] Let $A \in \mathbb{C}^{n\times n}$, $Ind(A) = p$. Then the Drazin-Theta matrix of A is denoted by $A^{D,\theta}$ and is defined as $A^{D,\theta} = A^{d}AA^{\theta}$ provided $A^{\dagger_s}$ exists.
The Drazin-Theta matrix $G = A^{D}AA^{\theta}$ provides a unique solution to the set of equations

$$G(A^{\dagger_s})^{\theta}G = G, \quad A^pG = A^pA^{\theta}, \quad G(A^{\dagger_s})^{\theta} = A^dA$$

Whenever the index of matrix A is one, the Drazin-Theta matrix reduces to Group-Theta matrix. The dual of Drazin-Theta matrix is Theta Drazin matrix $A^{\theta,D} = A^{\theta}AA^{d}$. Both Drazin-Theta matrices and Theta-Drazin matrices are helpful in solving linear system of equations.
In this article, we are extending the notion of Drazin-Theta matrix, whenever it exists, to rectangular matrices. We have also derived the algebraic and geometrical characterizations of W-weighted Drazin-Theta inverse. An application to solve the system of linear equations using W-weighted Drazin-Theta matrix is obtained. Analogous to these results, the characterizations for Theta-Drazin matrix which is the dual of Drazin-Theta matrix follows.

III. Results

Theorem 1: Let $B \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$ where $p = max\{Ind(BW),\ Ind(WB)\}$.
Consider

$$\begin{aligned} G(B^{\dagger_s})^{\theta}G = G(WB)^pG = (WB)^pB^{\theta} \\ G(B^{\dagger_s})^{\theta} = WB_{d,W}WB \end{aligned} \quad (4)$$

The system of equations (4) are consistent and $G = WB_{d,W}WBB^{\theta}$ is the unique solution.

Proof: Assume that $G = WB_{d,W}WBB^{\theta}$.
Then we have

$$\begin{aligned} G(B^{\dagger_s})^{\theta}G &= WB_{d,w}WBWB_{d,W}WBB^{\theta} \\ &= WB_{d,W}WBB^{\theta} = G. \\ G(B^{\dagger_s})^{\theta} &= WB_{d,W}WBB^{\theta}(B^{\dagger_s})^{\theta} \\ &= WB_{d,W}WB \end{aligned}$$

and

$$\begin{aligned} (WB)^pG &= (WB)^pWB_{d,W}WBB^{\theta} \\ &= (WB)^p(WB)^dWBB^{\theta} = (WB)^pB^{\theta} \end{aligned}$$

So $G = WB_{d,W}WBB^{\theta}$ satisfies the equations in (4).
Next, we will prove the uniqueness of G.
Assume there exists two $n \times m$ matrices G_1 and G_2 which satisfy the equation (4).

$$\begin{aligned} G_1 &= G_1(B^{\dagger_s})^{\theta}G_1 = WB_{d,W}WBB^{\theta}(B^{\dagger_s})^{\theta}G_1 \\ &= WB_{d,W}WBG_1 = (WB)^dWBG_1 \\ &= ((WB)^d)^p(WB)^pG_1 = ((WB)^d)^p(WB)^pB^{\theta} \\ &= ((WB)^d)^p(WB)^pG_2 = (WB)^dWBG_2 \\ &= WB_{d,W}WBG_2 = G_2(B^{\dagger_s})^{\theta}G_2 = G_2 \end{aligned}$$

Hence the uniqueness.

■

Definition 6: Let B and W are two complex square matrices of order $m \times n$ and $n \times m$ respectively with $p = max\{Ind(BW),\ Ind(WB)\}$. Then the W-weighted Drazin-Theta matrix of B is defined as $B_{W-d,\theta} = WB_{d,W}WBB^{\theta}$.

Remark 1: Let $B \in \mathbb{C}^{n\times n}$ and consider $W = I_{n\times n}$. Then we have the Drazin Theta matrix$B_{W-d,\theta} = B^dBB^{\theta} = B^{d,\theta}$. Here $Ind(B) = p$. Whenever $p = 1$ we get a particular case of W-weighted Drazin-Theta matrix of B, i.e. W-weighted Group-Theta matrix $B_{W-\sharp,\theta}$.
Observe that $B_{W-d,\theta}(B^{\dagger_s})^{\theta}B_{W-d,\theta} = B_{W-d,\theta}$.
Here is an example which shows that the Drazin-Theta outer inverse is different from Drazin Star outer inverse.

Example 1: Consider a column matrix $A = \begin{pmatrix}1\\2\end{pmatrix} \in \mathbb{C}^{2\times 1}$ where $rank(A) = 1$.
Here $A^* = \begin{pmatrix}1 & 2\end{pmatrix}$ and $A^{\theta} = \begin{pmatrix}2 & 1\end{pmatrix}$
The Moore-Penrose inverse is $A^{\dagger} = \frac{1}{5}\begin{pmatrix}1 & 2\end{pmatrix}$
and the s-g inverse is $A^{\dagger_s} = \frac{1}{4}\begin{pmatrix}2 & 1\end{pmatrix}$
By Theorem 3 of [8], we obtain $A_{d,W} = \begin{pmatrix}1 & 2\end{pmatrix}$ where $W = \begin{pmatrix}1 & 0\end{pmatrix}$. Then the W-weighted Drazin-Theta matrix is $A_{W-d,\theta} = \begin{pmatrix}2 & 1\end{pmatrix}$ and the W-weighted Drazin-Star matrix is $A_{W-d,*} = \begin{pmatrix}1 & 2\end{pmatrix}$.
Infact, these are the Weighted Group-Theta matrix and Weighted Group-Star matrix respectively, since the index of A is 1.

Remark 2: Observe that the existance of W-weighted Drazin Star inverse is guaranteed. However, W-weighted Drazin-Theta of A exists only when $A^{\dagger_s}$ exists.

The following lemma characterizes the Drazin-Theta outer inverse as an outer inverse with prescribed column space and null space.

Lemma 1: Let $B \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$ with $p = max\{Ind(BW),\ Ind(WB)\}$. Then

(i) $(B^{\dagger_s})^{\theta}B_{W-d,\theta}$ is a projector onto $\mathcal{C}((B^{\dagger_s})^{\theta}WB_{d,W})$ along $\mathcal{N}(B_{d,W}B^{\theta})$
(ii) $B_{W-d,\theta}(B^{\dagger_s})^{\theta}$ is a projector onto $\mathcal{C}((WB)^p)$ along $\mathcal{N}((WB)^p)$.
(iii) $B_{W-d,\theta} = ((B^{\dagger_s})^{\theta})^{(2)}_{(\mathcal{C}((WB)^p),\mathcal{N}(B_{d,W}B^{\theta}))}$.

Proof: (i) Since $B_{W-d,\theta}$ is an outer inverse of $(B^{\dagger_s})^{\theta}$, $(B^{\dagger_s})^{\theta}B_{W-d,\theta}$ is a projector.
Also

$$(B^{\dagger_s})B_{W-d,\theta} = (B^{\dagger_s})^{\theta}WB_{d,W}WBB^{\theta}$$

and

$$(B^{\dagger_s})^{\theta}WB_{d,W} = (B^{\dagger_s})^{\theta}WB_{d,W}WBB^{\theta}(B^{\dagger_s})^{\theta}WB_{d,W}$$

implies that
$\mathcal{C}((B^{\dagger_s})^\theta B_{W-d,\theta}) = \mathcal{C}((B^{\dagger_s})^\theta WB_{d,W})$
Also,

$$\begin{aligned}\mathcal{N}((B^{\dagger_s})^\theta B_{W-d,\theta}) &= \mathcal{N}((B^{\dagger_s})^\theta WBWB_{d,W}A^\theta)\\ &\supseteq \mathcal{N}(B_{d,W}B^\theta)\end{aligned}$$

and

$$\begin{aligned}\mathcal{N}(B_{d,W}B^\theta) &= \mathcal{N}(B_{d,W}WB^{D,W}WBB^\theta)\\ &= \mathcal{N}(B_{d,W}B_{W-d,\theta})\\ &= \mathcal{N}(B_{d,W}WB_dD, WWBB^\theta(B^{\dagger_s})^\theta B_{W-d,\theta})\\ &\supseteq \mathcal{N}((B^{\dagger_s})^\theta B_{W-d}, B^\theta)\end{aligned}$$

which yields $\mathcal{N}((B^{\dagger_s})^\theta B_{W-d,\theta}) = \mathcal{N}(B_{d,W}B^\theta)$.
(ii) Since $B_{W-d,\theta}(B^{\dagger_s})^\theta = (WB)^dWB$, we get
$\mathcal{C}(B_{W-d,\theta}(B^{\dagger_s})^\theta) = \mathcal{C}((WB)^p)$
and
$\mathcal{N}(B_{W-d,\theta}(B^{\dagger_s})^\theta) = \mathcal{N}((WB)^p)$.
(iii) From $\mathcal{C}(B_{W-d,\theta}) = \mathcal{C}((WB)^p)$ and
$\mathcal{N}(B_{W-d,\theta}) = \mathcal{N}((B^{\dagger_s})^\theta B_{W-d,\theta}) = \mathcal{N}(B_{d,W}B^\theta)$,
we have $B_{W-d,\theta} = ((B^{\dagger_s})^\theta)^{(2)}_{\mathcal{C}((WB)^p),\mathcal{N}(B_{d,W},B^\theta)}$. ■

IV. Characterizations of W-weighted Drazin-theta matrices

A geometrical and algebraic approach to charaterize W-weighted Drazin-Theta matrix is provided in this section.

Proposition 1: Consider two rectangular matrices $B \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$. Here $max\{Ind(BW), Ind(WB)\}= p$. The W-weighted Drazin-Theta matrix $G \in \mathbb{C}^{n\times m}$ of B satisfies the following matrix equations:

(i) $(WB)^pG = (WB)^pB^\theta$
(ii) $G(B^{\dagger_s})^\theta = WB_{d,W}WB$
(iii) $BG = BWB_{d,W}WBB^\theta$
(iv) $GB = WB_{d,W}WBB^\theta B$
(v) $(B^{\dagger_s})^\theta G = (B^{\dagger_s})^\theta WB_{d,W}WBB^\theta$
(vi) $WB_{d,W}WBG = G$
(vii) $WB_{d,W}WBGBB^{\dagger_s} = G$
(viii) $GBB^{\dagger_s} = G$
(ix) $(WB)^pG(B^{\dagger_s})^\theta = (WB)^p$
(x) $G(B^{\dagger_s})^\theta(WB)^p = (WB)^p$
(xi) $G(B^{\dagger_s})^\theta WBWB_{d,W}B^\theta = G$
(xii) $(B^{\dagger_s})^\theta WBWB_{d,W}G = (B^{\dagger_s})^\theta WB_{d,W}WBB^\theta$
(xiii) $G(B^{\dagger_s})^\theta WB_{d,W}WB = WB_{d,W}WB$.

Proof: The proofs follows directly from the definition of W-weighted Drazin-Theta matrix given in definition 6 and theorem 1. ■

The following theorem gives the equivalent conditions for a rectangular matrix to be a Drazin-Theta matrix.

Theorem 2: Let B and W be rectangular matrices of order $m \times n$ and $n \times m$ respectively, from the field of complex numbers. Let $p = max\{Ind(BW),\ Ind(WB)\}$. Then $G \in \mathbb{C}^{n\times m}$ is the W-weighted Drazin-Theta matrix of B if and only if any of the following statement is satisfied:

(i) $(WB)^pG = (WB)^pB^\theta$ and $WB_{d,W}WBG = G$.
(ii) $WB_{d,W}WBGBB^{\dagger_s} = G$ and
$(WB)^pG(B^{\dagger_s})^\theta = (WB)^p$.
(iii) $BG = BWB_{d,W}WBB^\theta$ and $WB_{d,W}WBG = G$.
(iv) $(B^{\dagger_s})^\theta G = (B^{\dagger_s})^\theta WB_{d,W}WBB^\theta$
and $WB_{d,W}WBG = G$.
(v) $G(B^{\dagger_s})^\theta = WB_{d,W}WB$ and $GBB^{\dagger_s} = G$.
(vi) $G(B^{\dagger_s})^\theta(WB)^p = (WB)^p$
and $G(B^{\dagger_s})^\theta WBWB_{d,W}B^\theta = G$.
(vii) $GB = WB_{d,W}WBB^\theta B$ and $GBB^{\dagger_s} = G$.
(viii) $WB_{d,W}WBG = G$ and
$(B^{\dagger_s})^\theta WBWB_{d,W}G = (B^{\dagger_s})^\theta WB_{d,W}WBB^\theta$.
(ix) $G(B^{\dagger_s})^\theta WBWB_{d,W}B^\theta = G$
and $G(B^{\dagger_s})^\theta WB_{d,W}WB = WB_{d,W}WB$.

Proof: Let $G = WB_{d,W}WBB^\theta$. Now the conditions (i) to (ix) holds by Proposition (1). To prove the converse, it is enough to verify that every condition (i)-(ix) implies $G = WB_{d,W}WBB^\theta$.

(i) Assume that $WB_{d,W}WBG = G$ and $(WB)^pG = (WB)^pB^\theta$. Then
$G = WB_{d,W}WBG = ((WB)^d)^p(WB)^pG$
$= ((WB)^d)^p(WB)^pB^\theta = WB_{d,W}WBB^\theta$.
(ii) Suppose that $WB_{d,W}WBGBB^{\dagger_s} = G$
and $(WB)^pG(B^{\dagger_s})^\theta = (WB)^p$. Then
$G = WB_{d,W}WBGBB^{\dagger_s}$
$= ((WB)^d)^p(WB)^pG(B^{\dagger_s})^\theta B^\theta = WB_{d,W}WBB^\theta$.
(iii) Let $WB_{d,W}WBG = G$ and $BG = BWB_{d,W}WBB^\theta$. Then
$G = WB_{d,W}WBG = WB_{d,W}WBWB_{d,W}WBB^\theta$
$= WB_{d,W}WBB^\theta$.
(iv) Set $WB_{d,W}WBG = G$
and $(B^{\dagger_s})^\theta G = (B^{\dagger_s})^\theta WB_{d,W}WBB^\theta$. Then
$G = WB_{d,W}WBG = WB_{d,W}WBB^\theta(B^{\dagger_s})^\theta G$
$= WB_{d,W}WBB^\theta$.
(v) Assume that
$GBB^{\dagger_s} = G$ and $G(B^{\dagger_s})^\theta = WB_{d,W}WB$.
Then

$$G = GBB^{\dagger_s} = G(B^{\dagger_s})^\theta B^\theta = WB_{d,W}WBB^\theta.$$

The remaining part of the theorem can be proved in a similar manner. ■

A geometrical approach to characterize W-weighted Drazin-Theta matrix is given in the next theorem.

Theorem 3: Let $B \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$ where $p = max\{Ind(BW),\ Ind(WB)\}$. Then the W-weighted Drazin-Theta matrix $B_{W-d,\theta}$ is the unique matrix G satisfying the condition

$$(B^{\dagger_s})^\theta G = \mathcal{P}_{\mathcal{C}((B^{\dagger_s})^\theta WB_{d,W}),\mathcal{N}(B_{d,W}B^\theta)}, \mathcal{C}(G) \subseteq \mathcal{C}((WB)^p). \tag{5}$$

Proof: Let $G = B_{W-d,\theta}$. By lemma 1, it can be easily verified that $B_{W-d,\theta}$ is a solution of both conditions in (5). To prove the uniquness, assume that G_1 and G_2 satisfy the conditions in (5). Also $(B^{\dagger_s})^\theta(G_1 - G_2) = 0$. Now we have

$$\begin{aligned}\mathcal{C}(G_1 - G_2) &\subseteq \mathcal{N}((B^{\dagger_s})^\theta)\\ &= \mathcal{N}(B) \subseteq \mathcal{N}(WB_{d,W}WB) = \mathcal{N}((WB)^p).\end{aligned}$$

Since $\mathcal{C}(G_1 - G_2) \subseteq \mathcal{C}((WB)^p) \cap \mathcal{N}((WB)^p) = 0$.
Hence $G_1 = G_2$. ■

V. An application of W-weighted Drazin-Theta matrices

Here, we illustrate an application of W-weighted Drazin-Theta matrix in solving linear system of equations.

Theorem 4: Let $B \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$ with $max\{Ind(BW),\ Ind(WB)\} = k \leq t$. Let $u \in \mathbb{C}^m$. Consider

$$(WB)^\theta x = (WB)^t B^\theta u, \tag{6}$$

The solution of equation (6) is given by W-weighted Drazin-Theta matrix.
The general solution of (6) is

$$x = B_{W-d,\theta}u + (I - WB_{d,W}WB)y, \tag{7}$$

for arbitrary $y \in \mathbb{C}^n$.

Proof: It can be easily verified that $B_{W-d,\theta}$ is a solution of equation (6). Suppose $x = B_{W-d,\theta}b + (I - WB_{d,W}WB)y$ for arbitrary $y \in \mathbb{C}^n$. Then

$$\begin{aligned}(WB)^t x &= (WB)^t B_{W-d,\theta}u + (I - WB_{d,W}WB)y \\ &= (WB)^t B_{W-D,\theta}u = (WB)^t B^\theta u\end{aligned}$$

which implies x is a solution of equation (6).
Now let us assume that x is a solution of equation (6).

$$\begin{aligned}WB_{d,W}WBx &= (WB)^d WBx = ((WB)^d)^t (WB)^t x \\ &= ((WB)^d)^t (WB)^t B^\theta b \\ &= WB_{d,W}WBB^\theta b = B_{W-d,\theta}b\end{aligned}$$

Therefore

$$\begin{aligned}x &= B_{W-d,\theta}u + x - WA_{d,W}WBx \\ &= B_{W-d,\theta}u + (I - WB_{d,W}WB)x\end{aligned}$$

is a solution for equation (7).

■

VI. W-weighted Theta-Drazin Matrices

Analogous to W-weighted Drazin-Theta matrices, we can define its dual, the W-weighted Theta-Drazin matrix. All the proofs of the theorems defined here are also analogous to the theorems for $B_{W-d,\theta}$. We omit deriving those proofs here.

Theorem 5: Let $B \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$ with $p = max\{Ind(BW),\ Ind(WB)\}$. Then the system of equations

$$\begin{aligned}&G(B^{\dagger_s})^\theta G = G, \quad G(BW)^p = B^\theta (BW)^p \\ &(B^{\dagger_s})^\theta G = BWB_{d,W}W\end{aligned} \tag{8}$$

are consistent and have a unique solution given by $G = AB^\theta BWB_{d,W}W$.

Definition 7: Let $B \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$ with $p = max\{Ind(BW),\ Ind(WB)\}$. The W-weighted Theta-Drazin matrix of B is defined as $B_{\theta,W-d} = B^\theta BWB_{d,W}W$.

Lemma 2: Let $B \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$ with $p = max\{Ind(BW),\ Ind(WB)\}$. Then

(i) $(B^{\dagger_s})^\theta B_{\theta,W-d}$ is a projector onto $\mathcal{C}((BW)^p)$ along $\mathcal{N}((BW)^p)$.
(ii) $B_{\theta,W-d}(B^{\dagger_s})^\theta$ is a projector onto $\mathcal{C}(B^\theta B_{d,W})$ along $\mathcal{N}(B_{d,W}W(B^{\dagger_s})^\theta)$
(iii) $B_{\theta,W-d} = ((B^{\dagger_s})^\theta)^{(2)}_{\mathcal{C}(B^\theta B_{d,W}),\mathcal{N}((BW)^p)}$.

Proposition 2: Let $B \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$ with $p = max\{Ind(BW),\ Ind(WB)\}$. The W-weighted Drazin-Theta matrix $G \in \mathbb{C}^{n\times m}$ of B satisfies the following matrix equations:

(i) $G(BW)^p = B^\theta (BW)^p$
(ii) $(B^{\dagger_s})^\theta G = BWB_{d,W}W$
(iii) $BG = BB^\theta BWB_{d,W}W$
(iv) $GB = B^\theta BWB_{d,W}WB$
(v) $G(B^{\dagger_s})^\theta = B^\theta BWB_{d,W}W(B^{\dagger_s})^\theta$
(vi) $GBWB_{d,W}W = G$
(vii) $B^{\dagger_s}BGBWB_{d,W}W = G$
(viii) $B^{\dagger_s}BG = G$
(ix) $(B^{\dagger_s})^\theta G(BW)^p = (BW)^p$
(x) $(BW)^p(B^{\dagger_s})^\theta G = (BW)^p$
(xi) $B^\theta BWB_{d,W}W(B^{\dagger_s})^\theta G = G$
(xii) $GB_{d,W}WBW(B^{\dagger_s})^\theta = B^\theta BWB_{d,W}W(B^{\dagger_s})^\theta$
(xiii) $BWB_{d,W}W(B^{\dagger_s})^\theta G = BWB_{d,W}W$.

Theorem 6: Let B and W be rectangular matrices from the field of complex numbers of order $m \times n$ and $n \times m$ respectively. Let $p = max\{Ind(BW),\ Ind(WB)\}$. Consider the notation of items as in proposition 2. Then $G \in \mathbb{C}^{n\times m}$ is a W-weighted Theta-Drazin matrix of B if and only if any of the following conditions are satisfied.

(1) (i) and (vi)
(2) (vii) and (ix)
(3) (iv) and (vi)
(4) (v) and (vi)
(5) (ii) and $(viii)$
(6) (x) and (xi)
(7) (iii) and $(viii)$
(8) (vi) and (xii)
(9) (xi) and $(xiii)$

Theorem 7: Let $B \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$ with $p = max\{Ind(BW),\ Ind(WB)\}$. Then the W-weighted Theta-Drazin matrix $B_{\theta,W-d}$ is the unique matrix G satisfying the condition

$$(B^{\dagger_s})^\theta G = \mathcal{P}_{\mathcal{C}((BW)^p),\mathcal{N}((BW)^p)}, \quad \mathcal{C}(G) \subseteq \mathcal{C}(B^\theta B_{d,W}). \tag{9}$$

Here, we give a relation connecting W-weighted Drazin-Theta matrix and W-weighted Theta-Drazin matrix.

Theorem 8: Let $B \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$ with $p = max\{Ind(BW),\ Ind(WB)\}$. Then the following conditions hold:

(i) $B_{W-d,\theta} = B_{\theta,W-d}$ if and only if $\mathcal{C}((WB)^p) = \mathcal{C}(B^\theta B_{d,W})$ and $\mathcal{N}(B_{d,W}B^\theta) = \mathcal{N}((BW)^p)$.
(ii) $B_{W-d,\theta}(B^{\dagger_s})^\theta = (B^{\dagger_s})^\theta$ if and only if $(WB)^d WB = BW(BW)^d$.

Proof: Proof of (i) follows directly from lemma (1) and lemma (2).
(ii) Using the definition of W-weighted Drazin-Theta inverse and W-weighted Theta Drazin inverse, $B_{W-d,\theta}(B^{\dagger_s})^\theta = (B^{\dagger_s})^\theta$ if and only if $WB_{d,W}WB = BWB_{d,W}W$ which is same as $(WB)^d WB = BW(BW)^d$. ■

VII. Conclusion

Here, the Drazin-Theta inverses that are defined for square matrices are extended to rectangular matrices. Exploring different characterizations, determinantal representation and iterative method to calculate W-weighted Drazin-Theta inverse etc. can be considered for future research. Also, extending the results obtained in this article to different algebraic structures will open up new area of research.

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