A Simplified Approach for Correlation Coefficient Similarity Measure

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Abstract—**In 1991, Gerstenkorn and Manko introduced a correlation coefficient similarity measure for intuitionistic fuzzy sets, aiming to investigate statistical relationships between objectives. However, their proof of the upper bound for the correlation coefficient similarity measure is overly complex and can be simplified. In 1995, Bustince and Burillo developed a similar correlation coefficient similarity measure for interval-valued intuitionistic fuzzy sets, but their proof of the upper bound closely resembles that of Gerstenkorn and Manko. Both proofs utilized the Schwarz inequality, yet failed to fully harness its power as a tool. In this paper, we begin by simplifying the intricate computations presented in the works of Gerstenkorn and Manko (1991) and Bustince and Burillo (1995). Subsequently, we demonstrate that their proofs can be directly derived by applying the Schwarz inequality. For a related transit bus model, we derive a new approximated solution that is very closed to the approximated solution of Wang et al. (2023) with the relative error less than 5%. We also consider an inventory model to show a possible new direction to locate the formulated closed-form optimal solution.**

Index Terms—**Similarity measure, Intuitionistic fuzzy sets, Correlation coefficient, information energy**

I. INTRODUCTION

GERSTENKORN and Manko [1] discussed correlation coefficient similarity measures under intuitionistic fuzzy sets environment. Up to now, there are 109 papers that had cited Gerstenkorn and Manko [1] in their references. One of them is Bustince and Burillo [2] to extend correlation coefficient similarity measures for interval-valued intuitionistic fuzzy sets. Up to now, there are 162 papers had cited Bustince and Burillo [2] in their references. The high citation rates for Gerstenkorn and Manko [1] and Bustince and Burillo [2] indicated the importance of these two papers in theoretical evolution and practical application to real problems. However, there is a lengthy verification procedure in Gerstenkorn and Manko [1] and then the similar tedious verification occurred in Bustince and Burillo [2]. We have run a comprehensive study of those papers referred to Gerstenkorn and Manko [1] or Bustince and Burillo [2], and then we can classify them into the following four categories:

(a) Developed new similarity measures, for example, Zhang et al. [3], Meng et al. [4].

(b) Applied measures to practical problems, for example, Abdullah and Ismail [5], Arefi and Taheri [6].

(c) Constructed a new algorithm to use measures to solve a decision problem, for example, Robinson and Amirtharaj [7], Sonia et al. [8].

(d) Extended measures to a more general setting to seek a theoretic structure, for example, Hong [9], Papakostas et al. [10].

Consequently, no papers had discussed the tedious results of Gerstenkorn and Manko [1] or Bustince and Burillo [2]. The purpose of this paper is to provide a simple proof for their verification.

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II. REVIEW FOR THE LENGTHY RESULTS IN GERSTENKORN AND MANKO

If $X = \{x_1, x_2, \dots, x_N\}$ is the universe of discourse, for a fuzzy set, $B = \{\langle x, \mu_B(x)\rangle : x \in X\}$, where $\mu_B(x)$ is the degree of member-ship, then Dumitrescu [11] assumed the informational energy of the fuzzy set B as

$$\sum_{k=1}^{N}\left[\mu_B^2(x_k) + (1-\mu_B(x_k))^2\right]. \quad (2.1)$$

Gerstenkorn and Manko [1] defined T(A) as

$$T(B) = \sum_{k=1}^{N}\left[\mu_B^2(x_k) + v_B^2(x_k)\right], \quad (2.2)$$

for an intuitionistic fuzzy set,

$$B = \{\langle x, \mu_B(x), v_B(x)\rangle : x \in X\}, \quad (2.3)$$

where $\mu_B(x)$ is the degree of membership and $v_B(x)$ is the degree of non-membership, with $\mu_B(x) \geq 0$ and $v_B(x) \geq 0$, with the restriction $1 \geq v_B(x) + \mu_B(x)$.

Gerstenkorn and Manko [1] assumed the correlation for two intuitionistic fuzzy sets, A and B as

$$C(A,B) = \sum_{k=1}^{N}[v_B(x_k)v_A(x_k) + \mu_B(x_k)\mu_A(x_k)]. \quad (2.4)$$

The definition of equation (2.4) coincides with the correlation for two fuzzy sets when the two intuitionistic fuzzy sets degenerated to two fuzzy sets, mentioned by Dumitrescu [12] as

$$\sum_{k=1}^{N}\left[(1-\mu_B(x_k))(1-\mu_A(x_k)) + \mu_B(x_k)\mu_A(x_k)\right]. \quad (2.5)$$

Moreover, Gerstenkorn and Manko [1] defined the correlation coefficient for two intuitionistic fuzzy sets, A and B as follows,

$$k(A,B) = \frac{C(A,B)}{(T(A)T(B))^{1/2}}. \quad (2.6)$$

We recall the Theorem 1 of Gerstenkorn and Manko [1] in the following.

Theorem 1 of Gerstenkorn and Manko [1]. $0 \leq k(A, B) \leq 1$.

We will skip some repeated steps in their proof and then provide an outline for their proof for Theorem1.

They assumed that $\sum_{i=1}^{N} \mu_A^2(x_i) = a$, $\sum_{i=1}^{N} \mu_B^2(x_i) = b$, $\sum_{i=1}^{N} v_A^2(x_i) = c$, and $\sum_{i=1}^{N} v_B^2(x_i) = d$. They derived the following computation

$$k(A, B) = \frac{\sum_{i=1}^{N}\left(v_A(x_i)v_B(x_i)+\mu_A(x_i)\mu_B(x_i)\right)}{\left[\sum_{i=1}^{N}\left(\mu_B^2(x_i)+v_B^2(x_i)\right)\sum_{i=1}^{N}\left(\mu_A^2(x_i)+v_A^2(x_i)\right)\right]^{1/2}},$$

$$= \frac{\sum_{i=1}^{N} v_A(x_i)v_B(x_i)+\sum_{i=1}^{N}\mu_A(x_i)\mu_B(x_i)}{\left[\sum_{i=1}^{N}\left(\mu_B^2(x_i)+v_B^2(x_i)\right)\sum_{i=1}^{N}\left(\mu_A^2(x_i)+v_A^2(x_i)\right)\right]^{1/2}}. \quad (2.7)$$

They applied the Schwarz inequality to find that

$$\sum_{i=1}^{N} \mu_A(x_i)\mu_B(x_i) \leq \left[\sum_{i=1}^{N} \mu_A^2(x_i) \sum_{i=1}^{N} \mu_B^2(x_i)\right]^{1/2}, \quad (2.8)$$

and

$$\sum_{i=1}^{N} v_A(x_i)v_B(x_i) \leq \left[\sum_{i=1}^{N} v_A^2(x_i) \sum_{i=1}^{N} v_B^2(x_i)\right]^{1/2}, \quad (2.9)$$

They obtained that

$$k(A, B) \leq \frac{(cd)^{1/2}+(ab)^{1/2}}{\left((b+d)(a+c)\right)^{1/2}}. \quad (2.10)$$

They knew $C(A, B) \geq 0$ and $k(A, B) \geq 0$, so they considered $k^2(A, B)$. To verify $k(A, B) \leq 1$ is equivalent to show that

$$\left[\frac{(cd)^{1/2}+(ab)^{1/2}}{\left((b+d)(a+c)\right)^{1/2}}\right]^2 = ,$$

$$\frac{cd+2(cdab)^{1/2}+ab}{(b+d)(a+c)} \leq 1. \quad (2.11)$$

They computed

$$\frac{cd+2(cdab)^{1/2}+ab}{(b+d)(a+c)} - 1,$$

$$= \frac{cd+2(cdab)^{1/2}+ab-cd-bc-ad-ab}{(b+d)(a+c)},$$

$$= \frac{-\left[bc-2(cdab)^{1/2}+da\right]}{(b+d)(a+c)},$$

$$= \frac{-\left((cb)^{1/2}-(da)^{1/2}\right)^2}{(b+d)(a+c)} \leq 0. \quad (2.12)$$

Based on the inequality of equation (2.12), they showed that the inequality in equation (2.11) is valid such that $k^2(A, B) \leq 1$ and then $k(A, B) \leq 1$ is verified.

III. Review for the Tedious Results in Bustince and Burillo

When X is the universe of discourse, for an interval-valued intuitionistic fuzzy sets

$$A = \{\langle x, [M_{AL}(x), M_{AU}(x)], [N_{AL}(x), N_{AU}(x)]\rangle : x \in X\}, \quad (3.1)$$

Bustince and Burillo [2] assumed correlation for two interval-valued intuitionistic fuzzy sets A and B as

$$C_{IVIFS}(A, B) = \frac{1}{2}\sum_{k=1}^{N}[M_{AL}(x_i)M_{BL}(x_i)$$

$$+M_{AU}(x_i)M_{BU}(x_i)+N_{AL}(x_i)N_{BL}(x_i)$$

$$+N_{AU}(x_i)N_{BU}(x_i)], \quad (3.2)$$

and

$$E_{IVIFS}(A) = C_{IVIFS}(A, A). \quad (3.3)$$

Bustince and Burillo [2] defined the correlation coefficient for two interval-valued intuitionistic fuzzy sets, A and B as

$$K_{IVIFS}(A, B) = \frac{C_{IVIFS}(A,B)}{[K_{IVIFS}(A)K_{IVIFS}(B)]^{1/2}}. \quad (3.4)$$

We cite the critical part of Theorem 1, of Bustince and Burillo [2].

Theorem 1. $K_{IVIFS}(A, B) \leq 1$.

We cite their proof for $K_{IVIFS}(A, B) \leq 1$ in the following. From the Schwarz inequality, Bustince and Burillo [2] obtained

$$K_{IVIFS}(A, B) = \sum_{i=1}^{N}\{M_{AL}(x_i)\, M_{BL}(x_i) +,$$

$$N_{AL}(x_i)N_{BL}(x_i) + M_{AU}(x_i)M_{BU}(x_i) + N_{BU}(x_i)N_{AU}(x_i)\}$$

$$[M_{AL}^2(x_i) + M_{AU}^2(x_i) + N_{AL}^2(x_i) + N_{AU}^2(x_i)]^{1/2}$$

$$[M_{BL}^2(x_i) + M_{BU}^2(x_i) + N_{BL}^2(x_i) + N_{BU}^2(x_i)]^{1/2} \leq$$

$$\left\{\left[\sum_{i=1}^{N} M_{AL}^2(x_i) \sum_{i=1}^{N} M_{BL}^2(x_i)\right]^{1/2},\right.$$

$$+\left[\sum_{i=1}^{N} M_{AU}^2(x_i) \sum_{i=1}^{N} M_{BU}^2(x_i)\right]^{1/2},$$

$$+\left[\sum_{i=1}^{N} N_{AL}^2(x_i) \sum_{i=1}^{N} N_{BL}^2(x_i)\right]^{1/2},$$

$$\left.+\left[\sum_{i=1}^{N} N_{AU}^2(x_i) \sum_{i=1}^{N} N_{BU}^2(x_i)\right]^{1/2}\right\},$$

$$\left(\sum_{i=1}^{N}[M_{AL}^2(x_i) + M_{AU}^2(x_i) + N_{AL}^2(x_i) + N_{AU}^2(x_i)]\right)^{1/2},$$

$$\left(\sum_{i=1}^{N}[M_{BL}^2(x_i) + M_{BU}^2(x_i) + N_{BL}^2(x_i) + N_{BU}^2(x_i)]\right)^{1/2}. \quad (3.5)$$

Bustince and Burillo [2] adopted the following abbreviations:

$$\sum_{i=1}^{n} M_{AL}^2(x_i) = a, \quad (3.6)$$

$$\sum_{i=1}^{n} M_{BL}^2(x_i) = b, \quad (3.7)$$

$$\sum_{i=1}^{n} M_{AU}^2(x_i) = c, \quad (3.8)$$

$$\sum_{i=1}^{n} M_{BU}^2(x_i) = d, \quad (3.9)$$

$$\sum_{i=1}^{n} N_{AL}^2(x_i) = e, \quad (3.10)$$

$$\sum_{i=1}^{n} N_{BL}^2(x_i) = f, \quad (3.11)$$

$$\sum_{i=1}^{n} N_{AU}^2(x_i) = g, \quad (3.12)$$

and

$$\sum_{i=1}^{n} N_{BU}^2(x_i) = h, \quad (3.13)$$

to rewrite the inequality of equation (3.5) as

$$K_{IVIFS}(A, B) \le,$$

$$\frac{(hg)^{1/2}+(fe)^{1/2}+(dc)^{1/2}+(ba)^{1/2}}{\left((h+f+d+b)(g+e+c+a)\right)^{1/2}}. \quad (3.14)$$

Bustince and Burillo [2] already knew that

$$0 \le K_{IVIFS}(A, B), \quad (3.15)$$

such that they will try to prove that

$$K_{IVIFS}^2(A, B) \le$$

$$\left[\frac{(hg)^{1/2}+(fe)^{1/2}+(dc)^{1/2}+(ba)^{1/2}}{\left((h+f+d+b)(g+e+c+a)\right)^{1/2}}\right]^2,$$

$$= \frac{\left[(dc)^{1/2}+(ba)^{1/2}\right]^2+\left[(hg)^{1/2}+(fe)^{1/2}\right]^2}{(h+f+d+b)(g+e+c+a)},$$

$$+ \frac{2\left[(dc)^{1/2}+(ba)^{1/2}\right]\left[(hg)^{1/2}+(fe)^{1/2}\right]}{(h+f+d+b)(g+e+c+a)}. \quad (3.16)$$

Bustince and Burillo [2] subtracted one from both sides of the inequality (3.16) to imply that

$$K_{IVIFS}^2(A, B) - 1 \le \frac{-1}{(h+f+d+b)(g+e+c+a)},$$

$$\left\{\left((hc)^{1/2} - (gd)^{1/2}\right)^2 + \left((fc)^{1/2} - (ed)^{1/2}\right)^2,\right.$$

$$\left((gb)^{1/2} - (ha)^{1/2}\right)^2 + \left((eb)^{1/2} - (fa)^{1/2}\right)^2,$$

$$\left.\left((gf)^{1/2} - (he)^{1/2}\right)^2 + \left((cb)^{1/2} - (da)^{1/2}\right)^2\right\}. \quad (3.17)$$

Hence, Bustince and Burillo [2] obtained that $K_{IVIFS}^2(A, B) \le 1$ and then they finished their proof for $1 \ge K_{IVIFS}(A, B)$.

IV. Our Improvement for Both Papers

If we compare equations (2.12) and (3.14) to find out that the most important issue in their proof of Gerstenkorn and Manko [1] and Bustince and Burillo [2] is to verify that

$$\left(\sum_{k=1}^{J}(a_k b_k)^{1/2} / \left(\sum_{k=1}^{J} a_k \sum_{k=1}^{J} b_k\right)^{1/2}\right) \le 1. \quad (4.1)$$

The inequality in equation (4.1) can be directly derived from the Schwarz inequality, if we assume two vectors,

$$Y = \left(\sqrt{a_1}, \sqrt{a_2}, \dots, \sqrt{a_J}\right), \quad (4.2)$$

and

$$Z = \left(\sqrt{b_1}, \sqrt{b_2}, \dots, \sqrt{b_J}\right), \quad (4.3)$$

then the Schwarz inequality implies that the absolute value of the inner product (the dot product) of Y and Z is less than or equal to the multiplication of lengths of Y and of Z, that is,

$$\langle Y, Z\rangle = \sum_{k=1}^{J} \sqrt{a_k b_k},$$

$$\le \sqrt{\sum_{k=1}^{J}(\sqrt{a_k})^2} \sqrt{\sum_{k=1}^{J}\left(\sqrt{b_k}\right)^2},$$

$$= \sqrt{\sum_{k=1}^{J} a_k} \sqrt{\sum_{k=1}^{J} b_k}. \quad (4.4)$$

Hence, by Schwarz inequality, we can directly show the inequality of equation (4.1) is valid and then inequalities of equations (2.12) and (3.14) are also hold.

Consequently, the lengthy proof of equations (2.7-2.12) and tedious proof of equations (3.6-3.17) becomes unnecessary.

Moreover, if we assume two vectors

$$S = (s_1, \dots, s_{m1}, s_{1+m1}, \dots, s_{m2+m1},$$

$$s_{1+m(j-1)+\cdots+m1}, \dots, s_{mJ+m(J-1)+\cdots+m1}), \quad (4.5)$$

and

$$T = (t_1, \dots, t_{m1}, t_{1+m1}, \dots, t_{m2+m1},$$

$$t_{1+m(j-1)+\cdots+m1}, \dots, t_{mJ+m(J-1)+\cdots+m1}), \quad (4.6)$$

with a conventional expression, $m0 = 0$.

From the Schwarz inequality, researcher can directly derive that

$$\langle S, T\rangle \le \|S\| \, \|T\|, \quad (4.7)$$

that is,

$$\sum_{k=1}^{m1+m2+\cdots+mJ} s_k t_k \le,$$

$$\left(\sum_{k=1}^{m1+m2+\cdots+mJ} s_k^2 \sum_{k=1}^{m1+m2+\cdots+mJ} t_k^2\right)^{1/2}. \quad (4.8)$$

For Gerstenkorn and Manko [1], if we assume that

$$J = 2, \quad (4.9)$$

$$m1 = m2 = N, \quad (4.10)$$

for $k = 1,2, \dots, N$,

$$s_k = \mu_A(x_k), \quad (4.11)$$

$$s_{k+N} = v_A(x_k), \quad (4.12)$$

$$t_k = \mu_B(x_k), \tag{4.13}$$

and

$$t_{k+N} = v_B(x_k), \tag{4.14}$$

and then using equation (4.8), we derive that

$$\sum_{k=1}^{N} \mu_A(x_k)\mu_B(x_k) + \sum_{k=1}^{N} v_A(x_k)v_B(x_k),$$

$$\leq \left(\sum_{k=1}^{N} \mu_A^2(x_k) + \sum_{k=1}^{N} v_A^2(x_k)\right)^{1/2},$$

$$\left(\sum_{k=1}^{N} \mu_B^2(x_k) + \sum_{k=1}^{N} v_B^2(x_k)\right)^{1/2}. \tag{4.15}$$

From equation (4.15), we know that $1 \geq K(A, B)$ such that after we assume a new pair of vectors, and then Theorem 1 of Gerstenkorn and Manko [1] can be directly obtained.

Next, we will apply our approach for Bustince and Burillo [2].

For Bustince and Burillo [2], we assume that

$$J = 4, \tag{4.16}$$

$$m1 = m2 = m3 = m4 = N, \tag{4.17}$$

for $k = 1,2, \ldots, N$,

$$s_k = M_{AL}(x_k), \tag{4.18}$$

$$s_{k+N} = M_{AU}(x_k), \tag{4.19}$$

$$s_{k+2N} = N_{AL}(x_k), \tag{4.20}$$

$$s_{k+3N} = N_{AU}(x_k), \tag{4.21}$$

$$t_k = M_{BL}(x_k), \tag{4.22}$$

$$t_{k+N} = M_{BU}(x_k), \tag{4.23}$$

$$t_{k+2N} = N_{BL}(x_k), \tag{4.24}$$

and

$$t_{k+3N} = N_{BU}(x_k). \tag{4.25}$$

We apply equation (4.8), to derive that

$$\sum_{k=1}^{N} (N_{AL}(x_k)N_{BL}(x_k) + N_{AU}(x_k)N_{BU}(x_k),$$

$$M_{AL}(x_k)M_{BL}(x_k) + M_{AU}(x_k)M_{BU}(x_k)) \leq$$

$$\left(\sum_{k=1}^{N} M_{AL}^2(x_k) + M_{AU}^2(x_k)^{1/2},\right.$$

$$\left., +N_{AL}^2(x_k) + N_{AU}^2(x_k)\right)^{1/2}$$

$$\left(\sum_{k=1}^{N} M_{BL}^2(x_k) + M_{BU}^2(x_k),\right.$$

$$\left.+N_{BL}^2(x_k) + N_{BU}^2(x_k)\right)^{1/2}. \tag{4.26}$$

From equation (4.26), we directly show that $K_{IVIFS}(A, B) \leq 1$ to provide a direct proof for Theorem 1, of Bustince and Burillo [2].

V. MORE COMMENTS FOR GERSTENKORN AND MANKO

Gerstenkorn and Manko [1] mentioned that for two

$$A = \{\langle x, \mu_A(x), v_A(x)\rangle : x \in X\}, \tag{5.1}$$

and

$$B = \{\langle x, \mu_B(x), v_B(x)\rangle : x \in X\}, \tag{5.2}$$

the union operator $A \cup B$ is assumed as

$$A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x),$$

$$v_A(x) \vee v_B(x)\rangle : x \in X\}. \tag{5.3}$$

We recall Atanassov [13], then the union operator $A \cup B$ should be defined as follows,

$$A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x),,$$

$$v_A(x) \wedge v_B(x)\rangle : x \in X\}, \tag{5.4}$$

where

$$\mu_A(x) \vee \mu_B(x) = \max\{\mu_A(x), \mu_B(x)\}, \tag{5.5}$$

and

$$v_A(x) \wedge v_B(x) = \min\{v_A(x), v_B(x)\}. \tag{5.6}$$

We refer to Theorem 2 of Gerstenkorn and Manko [1] which claimed that $k(A, B) = 0$ if and only if for every x in the universe of discourse,

$$|\mu_A(x) - \mu_B(x)| = 1, \tag{5.7}$$

and

$$|v_A(x) - v_B(x)| = 1. \tag{5.8}$$

We agree that based on Equation (5.7), then two cases: (A) $\mu_A(x) = 1$, and $\mu_B(x) = 0$, or (B) $\mu_A(x) = 0$, and $\mu_B(x) = 1$, will happen.

However, for both cases (A) and (B), will imply that

$$\mu_A(x)\mu_B(x) = 0. \tag{5.9}$$

Similarly, from Equation (5.8), then two cases: (C) $v_A(x) = 1$, and $v_B(x) = 0$, or (D) $v_A(x) = 0$, and $v_B(x) = 1$, will occur.

However, for both cases (C) and (D), will imply that

$$v_A(x)v_B(x) = 0. \tag{5.10}$$

For every x in the universe of discourse, we know that Equations (5.9) and (5.10) are valid to yield that $k(A, B) = 0$ which is proved.

On the other hand, we assume that $X = \{x\}$, and two intuitionistic fuzzy sets, A and B, with

$$A = \{\langle x, \mu_A(x) = 0.3, v_A(x) = 0\rangle : \{x\} = X\}, \tag{5.11}$$

and

$$B = \{\langle x, \mu_B(x) = 0, v_B(x) = 0.8\rangle : \{x\} = X\}, \tag{5.12}$$

then we derive that

$$c(A, B) = 0, \tag{5.13}$$

and then we obtain that

$$k(A, B) = 0, \tag{5.14}$$

However, we show that

$$|\mu_A(x) - \mu_B(x)| = 0.3 \neq 1, \quad (5.15)$$

and

$$|\nu_A(x) - \nu_B(x)| = 0.8 \neq 1. \quad (5.16)$$

Hence, Theorem 2 of Gerstenkorn and Manko [1] needs revisions.

In the following, we present a revised version for them, "If Equations (5.7) and (5.8) both are true, then Equation (5.14) is valid.".

We refer to Gerstenkorn and Manko [1] that claimed that $k(A, B) = 1$ if and only if for every x in the universe of discourse,

$$|\mu_A(x) - \mu_B(x)| = 0, \quad (5.17)$$

and

$$|\nu_A(x) - \nu_B(x)| = 0, \quad (5.18)$$

and then it will imply that $A = B$.

If $A = B$, we know that $c(A.A) = T(A)$, then $k(A, B) = 1$ is valid.

On the other hand, if $k(A, B) = 1$, based on the Cauchy-Schwarz inequality, then there is a positive number, denoted as s, that satisfies

$$A = sB. \quad (5.19)$$

There are infinite possible choice of s such that the implication of Gerstenkorn and Manko [1] to assert that $A = B$ which is false.

VI. A Related Transit Bus System

In this section, we will provide an alternative approximated solution for the bus transit model discussed by Wang et al. [14] and Jara-Diaz et al. [15].

The solution procedure of Wang et al. [14] was based on the following assumption: to find the optimal solution of

$$VCR(f_P, f_N) = A_p f_P + \frac{G_P}{f_P} + A_N f_N + \frac{G_N}{f_N} + \delta \frac{f_N}{f_P}, \quad (6.1)$$

denoted the optimal solutions as f_P^* and f_N^* that satisfy

$$f_P^* = (1+\varepsilon) f_P^\#, \quad (6.2)$$

and

$$f_N^* = (1-\varepsilon) f_N^\#, \quad (6.3)$$

where

$$f_P^\# = \sqrt{G_P / A_P}, \quad (6.4)$$

and

$$f_N^\# = \sqrt{G_N / A_N}, \quad (6.5)$$

are the solution for the traditional separated bus transit system for the peak period and normal period, respectively.

In Wang et al. [14], they implicitly assumed that the increasing rate of f_P from the initial point, $f_P^\#$, is equal to the decreasing rate of f_N from the initial point, $f_N^\#$.

We check that $870.679(1 + i) = 1083.23$, then $i = 0.2441$, the peak period frequency f_P increases 24.41%.

On the other hand, we find that $303.369(1 - j) = 182.99$, then $j = 0.3968$, the normal period frequency f_N decreases 39.68%.

The above computation points out that Wang et al. [14] assumed that the increasing percentage of the peak period frequency is closed to the decreasing percentage of the normal period frequency that did not supported by their results.
Therefore, we will provide an alternative approach to develop a new approximated formulated solutions for the peak frequency and normal frequency.

VII. Our Formulated Closed-form Solution

We consider approximated solution to treat

$$1+\varepsilon \approx 1-\varepsilon, \quad (7.1)$$

then we can simplify the previous quartic equation proposed by Wang et al. [14] to a quadratic equation,

$$2\delta\sqrt{\frac{ad}{bc}} - (2\varepsilon - \varepsilon^2)\sqrt{cd} - (2\varepsilon + \varepsilon^2)\sqrt{ab} = 0, \quad (7.2)$$

that can be further expressed as

$$\left(\sqrt{cd} - \sqrt{ab}\right)\varepsilon^2 - 2\left(\sqrt{cd} + \sqrt{ab}\right)\varepsilon + 2\delta\sqrt{\frac{ad}{bc}} = 0. \quad (7.3)$$

We obtain two formulated approximated solutions: if $\sqrt{cd} \neq \sqrt{ab}$,

$$\varepsilon_1 = \frac{1}{\sqrt{cd} - \sqrt{ab}}\left[\left(\sqrt{cd} + \sqrt{ab}\right) + \sqrt{\Psi}\right], \quad (7.4)$$

and

$$\varepsilon_2 = \frac{1}{\sqrt{cd} - \sqrt{ab}}\left[\left(\sqrt{cd} + \sqrt{ab}\right) - \sqrt{\Psi}\right], \quad (7.5)$$

where Ψ is an abbreviation to simplify the expressions, with

$$\Psi = \left(\sqrt{cd} + \sqrt{ab}\right)^2 - 2\delta\left(\sqrt{cd} - \sqrt{ab}\right)\sqrt{\frac{ad}{bc}}. \quad (7.6)$$

On the other hand, if $\sqrt{cd} = \sqrt{ab}$, then

$$\varepsilon = \delta \frac{\sqrt{ad}}{\left(\sqrt{ab} + \sqrt{cd}\right)\sqrt{bc}}. \quad (7.7)$$

We recall that a quadratic equation has a pair of closed-form formulated solutions as we mention in Equations (7.4) and (7.5).

We adopt the same numerical example as Wang et al. [14] and Jara-Diaz et al. [15] to use the following parameters: $a = 21.48$, $b = 16283611$, $c = 25.74$, and $d = 2368925$. Based on Equations (7.4) and (7.5), we derive that $\varepsilon_1 = 0.573$, and $\varepsilon_2 = -5.441$.

Based on $\varepsilon = 0.573$, then we obtain that the frequency for the peak period,

$$f_P = 1369.57854, \tag{7.8}$$

and the frequency for the normal period,

$$f_N = 129.538618. \tag{7.9}$$

We plug our results of Equations (7.8) and (7.9) into Equation (6.1) to derive the total cost for the bus transit system,

$$\mathrm{VCR}(f_P, f_N) = 67540.693. \tag{7.10}$$

We recall that in Wang et al. [14], they obtained that

$$f_N^{\#} = 198.1647. \tag{7.11}$$

and

$$f_P^{\#} = 1172.6195, \tag{7.12}$$

and then their approximated total cost is obtained as follows,

$$\mathrm{VCR}\left(f_P^{\#}, f_N^{\#}\right) = 64367.89. \tag{7.13}$$

We compute the relative error to show that

$$\frac{67540.693}{64367.89} = 1.04929. \tag{7.14}$$

Based on Equation (7.14), we know that our formulated approximated solutions are very closed to the formulated approximated solutions proposed by Wang et al. [14] such that the relative error is within the 5%.

VIII. A Related problem

In the next section, we will construct a new algebraic solution approach. First, we recall that Grubbström and Erdem [16] published a paper to solve inventory problem by an algebraic method of square root. This is the first paper to solve inventory models without referring to calculus. The purpose is to introduce inventory model to those students and practitioners, who are not familiar differential analysis. Up to now, there are more than one hundred papers that have cited Grubbström and Erdem [16] in their references. Cárdenas-Barrón [17] extended to the inventory model proposed by Grubbström and Erdem [16] with shortage. Ronald et al. [18] pointed out that Grubbström and Erdem [16] and Cárdenas-Barrón [17] knew the final solution in advanced such that they can construct their algebraic solution procedure. Ronald et al. [18] provided a complicate method to develop a two-step algebraic process to solve these inventory systems.

IX. Our proposed new algebraic procedure

We recall the algebraic process of Ronald et al. [18] that tried to minimize

$$f(x, y), \tag{9.1}$$

for $0 < x$ and $0 < y$, by locating a necessary condition of the optimal solution and then converting the two-variable problem into an one-variable problem and then they can solve the one-variable optimal problem.
Following this research trend, Chang et al. [19] constructed their new algebraic solution methods. Consequently, there are three solution approaches.

The first approach: in Ronald et al. [18], partition the domain from a infinite square to a collection of rays such that Ronald et al. [18] first solve the minimum problem along the ray $y = ax$ to convert the minimum problem from a two-variable problem into a one-variable problem of x,

$$f(x, y) = f(x, ax), \tag{9.2}$$

to solve the optimal solution, denote it as $x^*(a)$ that satisfies

$$f\left(x^*(a), ax^*(a)\right) = \min\{f(x, ax): 0 < x < \infty\}, \tag{9.3}$$

when "a" is temporarily treated as a constant, to consider $f(x, ax)$ only in variable x. In the next step, Ronald et al. [18] tried to find the optimal solution for "a" under the domain $0 < a < \infty$ as follows,

$$F(a) = f\left(x^*(a), ax^*(a)\right). \tag{9.4}$$

During a private conversation, one author of Ronald et al. [18] admitted that their solution approach was motivated by Montgomery et al. [20].

However, to strive to publish a new paper, in Ronald et al. [18], the authors did not contribute the original idea to Montgomery et al. [20].

The second approach proposed by Chang et al. [19]: find a relation $x = x(y)$, to express x in the variable of y and then plug $x = x(y)$ into the original problem to convert

$$f(x, y) = f(x(y), y) \equiv g(y). \tag{9.5}$$

Hence, one-variable optimal problem, $g(y)$, will be solved by algebraic procedure proposed by researchers.

The third approach mentioned in Chang et al. [19]: find a relation of $y = y(x)$, to express y in the variable of x, and then plug $y = y(x)$ into the original problem to convert

$$f(x, y) = f(x, y(x)) \equiv h(x), \tag{9.6}$$

and then they will solve the one-variable optimal problem of $h(x)$.

Chang et al. [19] claimed that they improved Ronald et al. [18] by an easily algebraic approach. Sphicas [21] extended this line of examination for inventory models with two shortage

costs: linear and fixed.
However, the solution process of Sphicas [21] is too intuitive such that unless researchers know the final optimal solution in advanced, it is too difficult to image the solution process of Sphicas [21].
Consequently, several papers tried to provide different solution procedures for this kind of inventory models. Motivated by our previous discussions, in the following, we will present a new algebraic solution procedure.

Our goal is to solve the following minimum problem proposed by Chang et al. [19]:

$$\min\ f(x)=\sqrt{(1+a)x^2+b}-x, \tag{9.7}$$

to find restrictions of a, b, and c such that f(x) has a minimum solution for the restricted domain, $0<x<\infty$, under the condition $0<f(x)$ for the restricted domain, $0<x<\infty$.

We assume that the minimum value occurs at

$$f(x^*)=ax^*+E, \tag{9.8}$$

where E is the lower bound for the values of f(x).

We suppose the following possible solution algorithm:
Step 1, show that there is a lower bound, E for $f(x)$.
Step 2, define a new function $g(x)=f(x)-E$, then to minimize $g(x)$.
Step 3, to create $\theta^2(g(x)-ax)^2$.

Remark. We can say that it is almost impossible to derive $\theta^2(f(x)-ax-E)^2$ naturally, if researchers did not know the final minimum value.

We point out that (i) $\theta^2(g(x)-ax)^2$ contained three items, and (ii) $\theta^2(f(x)-ax-E)^2$ contained six items. We can predict that many inventory models may be solved by this new approach.

Recently, Luo and Chou [22], Lan et al. [23], and Chiu et al. [24] presented several different algebraic methods to solve this kind of inventory systems.

X. Direction for Future Research

There are several recently published papers that are worthy to recall to help readers understand those important research trends. For example, there are four noteworthy papers: Pappalardo et al. [25] Patil et al. [26], Zhu et al. [27], and Tan et al. [28] which practitioners should examine for their research topics. Conversely, there are another four articles: Wichapa and Sodsoon [29], Zhang et al. [30], Raghu and Prameela [31], and Atatalab and Najafabadi [32] which researchers should consider for future study directions. In addition, Wang and Chen [33] identified questionable results in Aguaron and Moreno-Jimenez [34], and proposed revisions. Furthermore, Wang and Chen [33] offered additional comments on Aguaron and Moreno-Jimenez [34] and Yen [35]. Yang and Chen [36] examined the works of Çalışkan [37], Çalışkan [38], Osler [39], and Yen [35], and presented their own revisions. Yen [40] identified significant issues in Çalışkan [41], Wee et al. [42], Çalışkan [43] and Çalışkan [44] and provided improvements for them. Yen [35] conducted a study on inventory systems using algebraic processes and addressed an open question proposed by Chang et al. [45]. These papers hold significance in guiding researchers and practitioners in their respective fields.

XI. Conclusion

In conclusion, our study extends the verification procedure presented by Gerstenkorn and Manko [1] and Bustince and Burillo [2] to an abstract setting. By directly applying the Schwarz inequality, we provide a simplified proof that renders the cumbersome computations in their original proofs redundant. Our findings highlight the effectiveness and power of the Schwarz inequality as a valuable tool in correlation coefficient similarity measures.

We also derived an approximated closed solution for a transit bus system which is very closed to the approximated solution proposed by Wang et al. [14].

At last, not the least, we recalled an inventory system proposed by Grubbström and Erdem [16] that had been studied by many researchers to apply algebraic methods to solve the closed-form optimal solution without referring the calculus. We present a possible solution approach that will help practitioners find new solution procedures.

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