Cubical Fuzzy Einstein Bonferroni Mean Geometric Aggregation Operators and Their Applications to Multiple Criteria Group Decision Making Problems

A. Thilagavathy, and S. Mohanaselvi, *Member, IAENG*

***Abstract*—Cubical fuzzy numbers offer an improved framework for assessing uncertain factors in decision-making. The accumulation of cubical fuzzy opinions is pivotal in every group decision-making process. This research paper focuses on creating innovative cubical fuzzy Bonferroni mean geometric aggregation operators and their weighted adaptations. We build these operators upon the principles of cubical fuzzy Einstein operations. The proposed operators can capture intricate relationships among evaluated cubical fuzzy factors. We formally demonstrate and validate the desirable properties of these introduced operators. In addressing the complexities of multiple criteria group decision-making, we present a Cubical Fuzzy Complex Proportional Assessment (CF-COPRAS) method as a practical solution. Determining the weights of criteria holds significant importance in any decision-making methodology. In this paper, we employ the inter-criteria correlation (CRITIC) method to establish objective criteria weights. To showcase the effectiveness of our developed method, we apply it to a real-world financial investment risk management scenario within a cubical fuzzy context. Subsequently, we conduct a sensitivity analysis and comparative evaluation to demonstrate the efficacy of our proposed approach.**

***Index Terms*—Cubical fuzzy set, Einstein operations, Bonferroni mean, Geometric aggregation operator, CRITIC, COPRAS, Financial investment, Risk management.**

I. Introduction

MULTIPLE criteria group decision-making (MCGDM) is instrumental in generating a ranking order of finite alternatives based on specific criteria, drawing upon insights from a group of experts. In today's complex real-world scenarios, MCGDM plays a pivotal role in various decision-making (DM) situations. However, in practice, experts may not consistently provide precise decisions due to the subjective nature of their thinking. To address this challenge, theories dealing with imprecise information prove valuable for decision experts in handling uncertain MCGDM situations. One such effective tool is fuzzy set (FS) theory, pioneered by Zadeh [1], which adeptly manages uncertain conditions and allows for the representation of imprecise data by assigning the degree of membership μ to the corresponding element.

In the realm of multiple criteria decision-making (MCDM) problems, researchers have developed various extensions of fuzzy sets (FS) and harnessed them effectively. These extensions address imprecise situations involving "yes" and "no" answers. Intuitionistic fuzzy set (IFS), introduced by Atanassov [2], extends the concept of FS by incorporating both the membership degree μ and the degree of non-membership ν to represent imprecise data. Pythagorean fuzzy set (PYFS), proposed by Yager [3], allows for assigning larger values to both μ and ν of imprecise data compared to IFS. Fermatean fuzzy set (FFS), developed by Senapati and Yager [4], further expands the representation arena by accommodating both membership degrees in a larger context than IFS and PYFS. Q-rung orthopair fuzzy set (Q-ROFS), introduced by Yager [5], surpasses IFS, PYFS, and FFS by encompassing all three as particular cases within its more extensive space. Thus IFs, PYFS, and FFS can be derived from Q-ROFS by setting Q to 1, 2, and 3, respectively.

In real-life scenarios, there are situations where answers can be categorized as "yes," "abstain," and "no." Handling such imprecise cases is effectively accomplished by picture Fuzzy Set (PFS), introduced by Cuong and Kreinovich [6], which includes the degree of neutral membership η in addition to μ and ν. Spherical fuzzy set (SFS), defined by G"undoğdu [7], proves to be more effective than PFS by assuming higher values for the three degrees of membership. However, the CF set (CFS), proposed by Khan et al. [8], stands out as highly influential, as it assigns larger values to all degrees of membership compared to PFS and SFS. The superiority of CFS over other extensions of FS is demonstrated in Table XXI.

Several researchers have significantly expanded the range of FS extensions to address various DM situations. The fusion of decision information represents a crucial initial step in any DM process, and aggregation operators play a pivotal role in this information fusion. Different researchers have developed numerous aggregation operators under various uncertain contexts to merge multiple decision matrices into a unified one. Liu and Chen [9] utilized IF Heronian mean aggregation operators based on Archimedean norms for aggregating multiple decision matrices of a group decision-making (GDM) problem. In a different context, Wei and Lu [10] employed power operators to tackle multiple attribute decision-making (MADM) problems in the PYF environment.

Manuscript received May 10, 2023; revised October 04, 2023.

A. Thilagavathy is a PhD student in the Department of Mathematics, Faculty of Engineering and Technology, SRM Institute of Science and Technology, Kattankulathur - 603203, Chengalpattu District, Tamil Nadu, India. (e-mail: ta1198@srmist.edu.in).

S. Mohanaselvi is an Assistant Professor in the Department of Mathematics, Faculty of Engineering and Technology, SRM Institute of Science and Technology, Kattankulathur - 603203, Chengalpattu District, Tamil Nadu, India. (Corresponding author. e-mail: mohanass@srmist.edu.in).

TABLE I
COMPARISON OF CFS WITH OTHER FS EXTENSIONS

Space of representation	Extension of FS	Deals with situations	Includes the membership degrees	Restricted to the condition	Remarks
2-Dimensional	IFS	"Yes" and "No"	μ and ν	$0 \leq \mu + \nu \leq 1$	
	PYFS	"Yes" and "No"	μ and ν	$0 \leq \mu^2 + \nu^2 \leq 1$	
	FFS	"Yes" and "No"	μ and ν	$0 \leq \mu^3 + \nu^3 \leq 1$	
	Q-ROFS	"Yes" and "No"	μ and ν	$0 \leq \mu^Q + \nu^Q \leq 1$	If $Q = 1$ then Q-ROFS $\rightarrow$ IFS If $Q = 2$ the Q-ROFS $\rightarrow$ PYFS If $Q = 3$ the Q-ROFS $\rightarrow$ FFS
3-Dimensional	PFS	"Yes", "abstain" and "No"	μ, η and ν	$0 \leq \mu + \eta + \nu \leq 1$	
	SFS	"Yes", "abstain" and "No"	μ, η and ν	$0 \leq \mu^2 + \eta^2 + \nu^2 \leq 1$	
	CFS	"Yes", "abstain" and "No"	μ, η and ν	$0 \leq \mu^3 + \eta^3 + \nu^3 \leq 1$	

Meanwhile, Aydemir and Yilmaz Gunduz [11] proposed FF Dombi aggregation operators for solving an MCDM problem. Wei [12] suggested PF hybrid averaging and geometric aggregation operators to tackle MADM problems. Wei [13] also introduced PF Hamacher aggregation operators to examine the MADM problem. Peng et al. [14] used exponential aggregation operators with a new score function to investigate a DM problem under the Q-ROF environment.

In specific application domains, researchers have explored the effectiveness of certain aggregation operators. For instance, Khan et al. [8] investigated an enterprise resource planning (ERP) system problem using CF weighted, ordered weighted, hybrid averaging, and geometric operators. Additionally, Jan et al. [15] established CF Hamacher averaging and geometric aggregation operators to examine cyclone disaster appraisal. Furthermore, Mahmood et al. [16] introduced T-spherical fuzzy (T-SF) geometric aggregation operators to address medical diagnostics problems using SFSs and T-SFSs. These advancements in aggregation operators offer valuable tools for handling diverse DM scenarios with ambiguous and uncertain information.

The literature shows that existing aggregation operators can combine decision information without considering the relationships between the aggregated arguments. However, there are certain aggregation functions that not only combine the arguments but also take into account their interrelationships. One such function is the Bonferroni mean (BOM) [17], which has applications in developing new aggregation operators for various uncertain environments. Researchers have explored different BOM-based aggregation function variants to address specific contexts. Xia et al. [18] introduced geometric BOM aggregation operators and applied them in the context of MCDM. Liu et al. [19] proposed IF BOM operators based on Dombi t-norm and t-conorm. Ate and Akay [20] provided PF BOM operators. Furthermore, Liu et al. [21] anticipated PF interactional BOM operators based on strict triangular norms. Farrokhizadeh et al. [22] defined SF BOM operators, and Yang and Pang [23] introduced T-SF BOM operators by utilizing interaction and Dombi operational laws.

The literature reveals that aggregation operators have been developed based on various T-norms and T-conorms. Among these, Einstein norms have garnered significant attention from researchers, developing diverse aggregation operators within different fuzzy set domains. Wang and Liu [24] were among the first to introduce IF geometric aggregation operators based on the Einstein operational laws. Subsequently, Rahman et al. [25] established the weighted variant of PYF Einstein geometric aggregation operators. Cao [26] contributed to the field by developing PF Einstein hybrid weighted aggregation operators. Khan et al. [27] introduced PF Einstein weighted and ordered weighted averaging operators.

Further expanding the applications, Rani and Mishra [28] presented FF Einstein aggregation operators for investigating the selection of electric vehicle charging stations using the MULTIMOORA method. Zulqarnain et al. [29] explored the utilization of Einstein geometric aggregation operators in solving MCDM problems under the Q-ROF soft set context. Munir et al. [30] proposed T-SF Einstein hybrid aggregation operators, while Zeng et al. [44] contributed T-SF aggregation operators based on the Einstein interactive operational laws. Li et al. [46] proposed the Einstein aggregation operator for generalized simplified neutrosophic numbers. Yang and Li [47] introduced the picture hesitant fuzzy normalized weighted BOM operator using Einstein operations.

Recently, numerous innovative MADM techniques have emerged and found applications in various DM scenarios with uncertainties. Karasan et al. [31] employed the PyF-AHP approach to select a landfill site in an MCGDM situation. Gündoğdu and Kahraman [32] utilized the SF-VIKOR method to address an MCGDM problem related to warehouse selection. In the context of solid waste management techniques, Garg [33] introduced novel aggregation operators under the IF environment, employing the MULTIMOORA method for assessment. Geng et al. [48] devised a technique based on TOPSIS and distance measures specifically for single-valued neutrosophic (SVN) linguistic sets. Gong et al. [34] developed an integrated BWM-TODIM method for solving Interval Type-2 fuzzy MCGDM problems. Xu et al. [50] expanded the classical TODIM method by incorporating cumulative prospect theory to cater to the requirements of SVN MADM. Yang et al. [35] devised the GDM method, utilizing a multiplicative consistent interval-valued IF preference relation. Furthermore, Chen et al. [36] proposed the TOPSIS method to investigate the MCDM problem in an IF environment. Ye et al. [49] constructed a new TOPSIS method to solve the MADM problem under a simplified SVN environment. Krishnakumar et al. [37] elaborated on the COPRAS approach for handling hesitant fuzzy DM situations. Lu et al. [38] introduced the

TABLE II
ADVANTAGE OF THE PROPOSED MCGDM APPROACH

Author	Operators	Operational laws involved	DM method used	Application used	Whether criteria weights are determined	Whether parameters involved	Whether criteria interrel-ationships considered
Khan et al. [8]	CFWG CFOWG CFHG	Algebraic	MCDM using score function	ERP system selection	No	No	No
Jan et al. [15]	CFHWG CFHOWG CFHHG	Hamacher	MCDM using score function	Cyclone disaster appraisal	No	No	No
Munir et al. [30]	T-SFEWG T-SFEOWG T-SFEHG	Einstein	MCDM using score function	Business selection for money investment	No	No	No
Zeng et al. [44]	T-SFEWIG T-SFEOWIG T-SFEHIG	Einstein Interaction	MCDM using score function	Selection of photovoltaic cell	No	No	No
Yang & Pang [23]	T-SFIGWA T-SFWIGBM T-SFWGDBM	Interaction Interaction Dombi	MCDM using score function	The furniture procurement	No	No Yes Yes	No Yes Yes
Proposed	CFEBOMWG CFEBOMOWG CFEBOMHG	Einstein	MCGDM using CRITIC based COPRAS	Finanacial investment risk management	Yes	Yes	Yes

PF COPRAS method for tackling the MCGDM problem of green supplier selection, highlighting its simplicity and ability to capture variations between alternatives. Kahraman et al. [39] introduced the SF CRITIC method for supplier selection. In contrast, Rong et al. [40] constructed an improved CRITIC method using Schweizer-Skalar operators to handle Q-ROF multiple attribute group decision-making (MAGDM) problems. Haktanr and Kahraman [41] combined the CRITIC and REGIME methods to address wearable health technology applications. Qi [42] proposed an integrated IF MAGDM method using GRA (Grey Relational Analysis) and CRITIC. Wan and Zhou [43] contributed a combined interval-valued Q-ROF CRITIC-WASPAS GDM method. Notably, the CRITIC method stands out for its ability to calculate the weights of aggregated arguments based on their correlations.

A. Contributions

Motivated by the above discussion, the significant contributions of this research work are as follows.

(1) Novel CF Einstein operational laws are introduced. Subsequently, these operations serve as the basis for defining a new CF Einstein BOM geometric operator. Moreover, the desirable properties of this operator are demonstrated and proven.
(2) Based on the initially defined operator, by including the weights of the aggregated CFNs, new CF Einstein BOM weighted, ordered weighted, and hybrid geometric aggregation operators are proposed. Furthermore, their fundamental properties are thoroughly examined and investigated.
(3) Utilizing the developed operators, an MCGDM method called the CF-CRITIC-based COPRAS method is formulated to address CF-MCGDM problems effectively.
(4) The weights of the aggregated CFNs are determined using the CRITIC method.
(5) We demonstrate the practicality of the proposed method by applying it to solve a real-life example of selecting the best enterprise for managing risk during financial investment.
(6) The influence of the parameters p and q on the decision results is explored through sensitivity analysis.
(7) We demonstrate the effectiveness of the proposed operators through a comparative study with existing aggregation operators. The advantages of the proposed approach over the existing ones are illustrated in Table XXII.

B. Structure

In Section 2, the fundamental concepts of CFS are presented. Section 3 begins by defining the CF Einstein operational laws. Subsequently, it introduces the definition of the BOM aggregation function. Additionally, this section introduces the CF Einstein BOM geometric aggregation operator and its weighted variants, namely the CF Einstein BOM weighted, ordered weighted, and hybrid geometric aggregation operators. The basic properties of these operators are thoroughly examined. Section 4 applies the proposed operators to establish a novel CF-MCGDM methodology named CF-CRITIC-based COPRAS methodology. Section 5 demonstrates the adaptability and compensatory nature of the proposed approach through a practical example of risk management during financial investment. It also includes a sensitivity analysis of the parameters involved in the operators and a comparative study of the decision results with those obtained using existing operators. Finally, Section 6 presents a concise conclusion of the research work.

II. PRELIMINARIES

This section presents some basics of CFS over the universal set $\ddot{\mathrm{U}}$.

Definition 2.1 *[8]*

A CFS, $\mathcal{L}$ on $\ddot{\mathbb{U}}$ is defined as $\mathcal{L} = \left\{\langle \alpha, \mu_{\mathcal{L}}(\alpha), \eta_{\mathcal{L}}(\alpha), \nu_{\mathcal{L}}(\alpha)\rangle \mid \alpha \in \ddot{\mathbb{U}}\right\}$ where the membership, neutral membership and non-membership degrees are defined by $\mu_{\mathcal{L}}, \eta_{\mathcal{L}}, \nu_{\mathcal{L}} : \ddot{\mathbb{U}} \to [0,1]$, respectively such that for all α, $0 \le (\mu_{\mathcal{L}}(\alpha))^3 + (\eta_{\mathcal{L}}(\alpha))^3 + (\nu_{\mathcal{L}}(\alpha))^3 \le 1$. For any $\mathcal{L}$ in $\ddot{\mathbb{U}}$, the degree of refusal for all α in $\mathcal{L}$ is defined by $\pi_{\mathcal{L}}(\alpha) = \sqrt[3]{1 - (\mu_{\mathcal{L}}(\alpha))^3 - (\eta_{\mathcal{L}}(\alpha))^3 - (\nu_{\mathcal{L}}(\alpha))^3}$. The cubical pair, $\langle \mu_{\mathcal{L}}, \eta_{\mathcal{L}}, \nu_{\mathcal{L}}\rangle$ is called a cubical fuzzy number (CFN).

Definition 2.2*[8]*

Let $\mathcal{L} = \left\{\langle \alpha, \mu_{\mathcal{L}}(\alpha), \eta_{\mathcal{L}}(\alpha), \nu_{\mathcal{L}}(\alpha)\rangle \mid \alpha \in \ddot{\mathbb{U}}\right\}$, $\mathcal{L}_1 = \left\{\langle \alpha, \mu_{\mathcal{L}_1}(\alpha), \eta_{\mathcal{L}_1}(\alpha), \nu_{\mathcal{L}_1}(\alpha)\rangle \mid \alpha \in \ddot{\mathbb{U}}\right\}$ and $\mathcal{L}_2 = \left\{\langle \alpha, \mu_{\mathcal{L}_2}(\alpha), \eta_{\mathcal{L}_2}(\alpha), \nu_{\mathcal{L}_2}(\alpha)\rangle \mid \alpha \in \ddot{\mathbb{U}}\right\}$, be any three CFSs, then their set operations are defined as

(i) $\mathcal{L}_1 \subseteq \mathcal{L}_2 \Leftrightarrow \mu_{\mathcal{L}_1}(\alpha) \le \mu_{\mathcal{L}_2}(\alpha), \eta_{\mathcal{L}_1}(\alpha) \le \eta_{\mathcal{L}_2}(\alpha)$ and $\nu_{\mathcal{L}_1}(\alpha) \ge \nu_{\mathcal{L}_2}(\alpha), \forall \alpha \in \ddot{\mathbb{U}}$;

(ii) $\mathcal{L}_1 \cup \mathcal{L}_2 = \left\{\left\langle \begin{array}{l} \alpha, \\ max\{\mu_{\mathcal{L}_1}(\alpha), \mu_{\mathcal{L}_2}(\alpha)\}, \\ min\{\eta_{\mathcal{L}_1}(\alpha), \eta_{\mathcal{L}_2}(\alpha)\}, \\ min\{\nu_{\mathcal{L}_1}(\alpha), \nu_{\mathcal{L}_2}(\alpha)\} \end{array}\right\rangle \mid \alpha \in \ddot{\mathbb{U}}\right\}$;

(iii) $\mathcal{L}_1 \cap \mathcal{L}_2 = \left\{\left\langle \begin{array}{l} \alpha, \\ min\{\mu_{\mathcal{L}_1}(\alpha), \mu_{\mathcal{L}_2}\}(\alpha), \\ min\{\eta_{\mathcal{L}_1}(\alpha), \eta_{\mathcal{L}_2}(\alpha)\}, \\ max\{\nu_{\mathcal{L}_1}(\alpha), \nu_{\mathcal{L}_2}(\alpha)\} \end{array}\right\rangle \mid \alpha \in \ddot{\mathbb{U}}\right\}$;

(iv) $\mathcal{L}^c = \left\{\langle \alpha, \nu_{\mathcal{L}}, \eta_{\mathcal{L}}, \mu_{\mathcal{L}}\rangle \mid \alpha \in \ddot{\mathbb{U}}\right\}$.

Definition 2.3*[8]*

Let $\mathcal{L} = \langle \mu_{\mathcal{L}}, \eta_{\mathcal{L}}, \nu_{\mathcal{L}}\rangle$ be a CFN, then the score function $\mathcal{S}$ of $\mathcal{L}$ is defined as

$$\mathcal{S}(\mathcal{L}) = \mu_{\mathcal{L}}^3 - \nu_{\mathcal{L}}^3 \in [-1, 1] \tag{1}$$

Definition 2.4*[8]*

Let $\mathcal{L} = \langle \mu_{\mathcal{L}}, \eta_{\mathcal{L}}, \nu_{\mathcal{L}}\rangle$ be a CFN, then the accuracy function $\mathcal{A}$ of $\mathcal{L}$ is defined as

$$\mathcal{A}(\mathcal{L}) = \mu_{\mathcal{L}}^3 + \nu_{\mathcal{L}}^3 \in [0, 1] \tag{2}$$

Definition 2.5*[8]*

Let $\mathcal{L}_1 = \langle \mu_{\mathcal{L}_1}, \eta_{\mathcal{L}_1}, \nu_{\mathcal{L}_1}\rangle$ and $\mathcal{L}_2 = \langle \mu_{\mathcal{L}_2}, \eta_{\mathcal{L}_2}, \nu_{\mathcal{L}_2}\rangle$ be any two CFNs and $\mathcal{S}(\mathcal{L}_j)$ and $\mathcal{A}(\mathcal{L}_j)$ for $(j = 1, 2)$ be their respective score and accuracy values, then we arrive at the following results:

(i) If $\mathcal{S}(\mathcal{L}_1) > \mathcal{S}(\mathcal{L}_2)$, then $\mathcal{L}_1 \succ \mathcal{L}_2$;
(ii) $\mathcal{S}(\mathcal{L}_1) < \mathcal{S}(\mathcal{L}_2)$, then $\mathcal{L}_1 \prec \mathcal{L}_2$;
(iii) $\mathcal{S}(\mathcal{L}_1) = \mathcal{S}(\mathcal{L}_2)$, then
  a) If $\mathcal{A}(\mathcal{L}_1) > \mathcal{A}(\mathcal{L}_2)$, then $\mathcal{L}_1 \succ \mathcal{L}_2$;
  b) $\mathcal{A}(\mathcal{L}_1) < \mathcal{A}(\mathcal{L}_2)$, then $\mathcal{L}_1 \prec \mathcal{L}_2$;
  c) $\mathcal{A}(\mathcal{L}_1) = \mathcal{A}(\mathcal{L}_2)$, then $\mathcal{L}_1 \sim \mathcal{L}_2$

III. Proposed Einstein BOM geometric aggregation operators for CFNs

In this section, we introduce novel Einstein operational laws of CFNs. Subsequently, we construct several CF-weighted, ordered weighted, and hybrid geometric aggregation operators, utilizing the newly proposed Einstein operational laws and employing the BOM aggregation function. Additionally, we explore the ideal properties of these operators to assess their performance and effectiveness.

A. Proposed CF Einstein operational laws

This section initially revisits the Einstein norms applicable to crisp numbers and presents novel CF Einstein operational laws. These laws encompass operations such as addition and multiplication between any two CFNs, scalar multiplication, and the exponentiation of a CFN.

Definition 3.1 *[30]*

The Einstein t-norm is defined as

$$TN_E(a, b) = \frac{ab}{1 + (1-a)(1-b)} \tag{3}$$

The Einstein t-conorm is defined as

$$TCN_E(a, b) = \frac{a + b}{1 + (ab)} \tag{4}$$

Next, we present the existing Einstein operations on any two CFNs in the following form:

Definition 3.2 *[52]*

Let $\mathcal{L}_1 = \langle \mu_{\mathcal{L}_1}, \eta_{\mathcal{L}_1}, \nu_{\mathcal{L}_1}\rangle$ and $\mathcal{L}_2 = \langle \mu_{\mathcal{L}_2}, \eta_{\mathcal{L}_2}, \nu_{\mathcal{L}_2}\rangle$ be two CFNs. Then their Einstein operations are defined as follows:

(i) $\mathcal{L}_1 \underset{E}{\bigotimes} \mathcal{L}_2 = \left\langle \begin{array}{l} \sqrt[3]{\frac{\mu_{\mathcal{L}_1}^3 \mu_{\mathcal{L}_2}^3}{1+(1-\mu_{\mathcal{L}_1}^3)(1-\mu_{\mathcal{L}_2}^3)}}, \sqrt[3]{\frac{\eta_{\mathcal{L}_1}^3 + \eta_{\mathcal{L}_2}^3}{1+\eta_{\mathcal{L}_1}^3 \eta_{\mathcal{L}_2}^3}}, \\ \sqrt[3]{\frac{\nu_{\mathcal{L}_1}^3 + \nu_{\mathcal{L}_2}^3}{1+\nu_{\mathcal{L}_1}^3 \nu_{\mathcal{L}_2}^3}} \end{array} \right\rangle$

(ii) $\mathcal{L}_1 \underset{E}{\bigoplus} \mathcal{L}_2 = \left\langle \begin{array}{l} \sqrt[3]{\frac{\mu_{\mathcal{L}_1}^3 + \mu_{\mathcal{L}_2}^3}{1+\mu_{\mathcal{L}_1}^3 \mu_{\mathcal{L}_2}^3}}, \sqrt[3]{\frac{\eta_{\mathcal{L}_1}^3 \eta_{\mathcal{L}_2}^3}{1+(1-\eta_{\mathcal{L}_1}^3)(1-\eta_{\mathcal{L}_2}^3)}}, \\ \sqrt[3]{\frac{\nu_{\mathcal{L}_1}^3 \nu_{\mathcal{L}_2}^3}{1+(1-\nu_{\mathcal{L}_1}^3)(1-\nu_{\mathcal{L}_2}^3)}} \end{array} \right\rangle$

(iii) $\lambda \mathcal{L}_j = \left\langle \begin{array}{l} \sqrt[3]{\frac{(1+\mu_{\mathcal{L}_j}^3)^\lambda - (1-\mu_{\mathcal{L}_j}^3)^\lambda}{(1+\mu_{\mathcal{L}_j}^3)^\lambda + (1-\mu_{\mathcal{L}_j}^3)^\lambda}}, \\ \sqrt[3]{\frac{2(\eta_{\mathcal{L}_j}^3)^\lambda}{(2-\eta_{\mathcal{L}_j}^3)^\lambda + (\eta_{\mathcal{L}_j}^3)^\lambda}}, \sqrt[3]{\frac{2(\nu_{\mathcal{L}_j}^3)^\lambda}{(2-\nu_{\mathcal{L}_j}^3)^\lambda + (\nu_{\mathcal{L}_j}^3)^\lambda}} \end{array} \right\rangle$
for $\lambda > 0$ and $j = 1, 2$.

(iv) $\mathcal{L}_j{}^\lambda$
$= \left\langle \begin{array}{l} \sqrt[3]{\frac{2(\mu_{\mathcal{L}_j}^3)^\lambda}{(2-\mu_{\mathcal{L}_j}^3)^\lambda + (\mu_{\mathcal{L}_j}^3)^\lambda}}, \sqrt[3]{\frac{(1+\eta_{\mathcal{L}_j}^3)^\lambda - (1-\eta_{\mathcal{L}_j}^3)^\lambda}{(1+\eta_{\mathcal{L}_j}^3)^\lambda + (1-\eta_{\mathcal{L}_j}^3)^\lambda}}, \\ \sqrt[3]{\frac{(1+\nu_{\mathcal{L}_j}^3)^\lambda - (1-\nu_{\mathcal{L}_j}^3)^\lambda}{(1+\nu_{\mathcal{L}_j}^3)^\lambda + (1-\nu_{\mathcal{L}_j}^3)^\lambda}} \end{array} \right\rangle$
for $\lambda > 0$ and $j = 1, 2$.

B. Bonferroni mean

The BOM is a powerful aggregation function that captures the interrelationships among the aggregated arguments. Here, we overview some BOMs defined in the existing literature.

Definition 3.3*[17]*

Let $x_j (j = 1, 2, \ldots, n)$ be a set of non-negative real numbers and $p, q \geq 0$. Then the (BOM) between x_j is mathematically calculated using the following formula:

$$BOM^{p,q}(x_1, x_2, \ldots, x_n) = \left[\frac{1}{r(r-1)}\left(\sum_{j,k=1;j\neq k}^{n} {x_j}^p {x_k}^q\right)\right]^{\frac{1}{p+q}} \tag{5}$$

Definition 3.4*[18]*

Let $x_j (j = 1, 2, \ldots, n)$ be a set of non-negative real numbers and $p, q \geq 0$. Then the geometric BOM $(GBOM)$ between x_j is mathematically calculated using the following formula:

$$GBOM^{p,q}(x_1, x_2, \ldots, x_n) = \frac{1}{p+q}\left[\left(\prod_{j,k=1;j\neq k}^{n}(px_j + qx_k)\right)^{\frac{1}{r(r-1)}}\right] \tag{6}$$

C. CF Einstein BOM geometric operators

In this section, we initiate the development of a new CF Einstein BOM geometric operator. Subsequently, we establish three additional CF Einstein BOM geometric aggregation operators: CF Einstein BOM weighted, CF Einstein BOM ordered weighted, and CF Einstein BOM hybrid geometric operators. These new operators consider the importance of the aggregated CFNs. Furthermore, we demonstrate the fundamental properties of these proposed operators to validate their applicability and usefulness.

1) ***CF Einstein BOM geometric operato****r:* Here, we define a new CF Einstein BOM geometric (CFEBOMG) operator.

Definition 3.5

Let $\mathcal{L}_j - \langle \mu_{\mathcal{L}_j}, \eta_{\mathcal{L}_j}, \nu_{\mathcal{L}_j} \rangle (j = 1, 2, \ldots, r)$ be a set of CFNs and $p, q \geq 0$. Then the CFEBOMG operator of dimension r is a mapping $CFEBOMG : \mathcal{L}^r \to \mathcal{L}$ such that

$$CFEBOMG^{p,q}(\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_r) = \frac{1}{p+q}\left[\bigotimes_{E\ j,k=1;j\neq k}^{r}\left(p\mathcal{L}_j \bigoplus_E q\mathcal{L}_k\right)\right]^{\frac{1}{r(r-1)}} \tag{7}$$

Theorem 3.1

The aggregated value obtained using CFEBOMG operator is also a CFN and can be expressed as follows:

$$CFEBOMG^{p,q}(\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_r) = \frac{1}{p+q}\left[\bigotimes_{E j,k=1;j\neq k}^{r}(p\mathcal{L}_j \oplus_E q\mathcal{L}_k)\right]^{\frac{1}{r(r-1)}}$$
$$= \left\langle \sqrt[3]{\frac{\phi^{\frac{1}{p+q}} - \varphi^{\frac{1}{p+q}}}{\phi^{\frac{1}{p+q}} + \varphi^{\frac{1}{p+q}}}}, \sqrt[3]{\frac{2\chi^{\frac{1}{p+q}}}{\psi^{\frac{1}{p+q}} + \chi^{\frac{1}{p+q}}}}, \sqrt[3]{\frac{2\rho^{\frac{1}{p+q}}}{\varrho^{\frac{1}{p+q}} + \rho^{\frac{1}{p+q}}}} \right\rangle \tag{8}$$

where

$$\phi = (2B_{jk} + 2C_{jk} - C_{jk})^{\frac{1}{r(r-1)}} + 3\,(A_{jk})^{\frac{1}{r(r-1)}}$$
$$\varphi = (2B_{jk} + 2C_{jk} - A_{jk})^{\frac{1}{r(r-1)}} - (A_{jk})^{\frac{1}{r(r-1)}}$$
$$\chi = (E_{jk} + F_{jk} + D_{jk})^{\frac{1}{r(r-1)}} - (E_{jk} + F_{jk} - D_{jk})^{\frac{1}{r(r-1)}}$$
$$\psi = (E_{jk} + F_{jk} + D_{jk})^{\frac{1}{r(r-1)}} + 3\,(E_{jk} + F_{jk} - D_{jk})^{\frac{1}{r(r-1)}}$$
$$\rho = (H_{jk} + I_{jk} + G_{jk})^{\frac{1}{r(r-1)}} - (H_{jk} + I_{jk} - G_{jk})^{\frac{1}{r(r-1)}}$$
$$\varrho = (H_{jk} + I_{jk} + G_{jk})^{\frac{1}{r(r-1)}} + 3\,(H_{jk} + I_{jk} - G_{jk})^{\frac{1}{r(r-1)}}$$

and

$$A_{jk} = \prod_{j,k=1;j\neq k}^{r} 2\left\{\begin{array}{l}[(1+\mu^3_{\mathcal{L}_j})^p(1+\mu^3_{\mathcal{L}_k})^q] \\ -[(1-\mu^3_{\mathcal{L}_j})^p(1-\mu^3_{\mathcal{L}_k})^q]\end{array}\right\}$$
$$B_{jk} = \prod_{j,k=1;j\neq k}^{r} 2\left\{\begin{array}{l}[(1+\mu^3_{\mathcal{L}_j})^p(1+\mu^3_{\mathcal{L}_k})^q] \\ +[(1-\mu^3_{\mathcal{L}_j})^p(1-\mu^3_{\mathcal{L}_k})^q]\end{array}\right\}$$
$$C_{jk} = \prod_{j,k=1;j\neq k}^{r} 4(1-\mu^3_{\mathcal{L}_j})^p(1-\mu^3_{\mathcal{L}_k})^q$$
$$D_{jk} = \sum_{j,k=1;j\neq k}^{r} 8(\eta^3_{\mathcal{L}_j})^p(\eta^3_{\mathcal{L}_k})^q\left\{\begin{array}{l}[(2-\eta^3_{\mathcal{L}_k})^p \\ (2-\eta^3_{\mathcal{L}_j})^q] \\ +[(\eta^3_{\mathcal{L}_k})^p(\eta^3_{\mathcal{L}_j})^q]\end{array}\right\}$$
$$E_{jk} = \prod_{j,k=1;j\neq k}^{r} 2\left\{\begin{array}{l}[(2-\eta^3_{\mathcal{L}_j})^p(2-\eta^3_{\mathcal{L}_k})^q] \\ +[(\eta^3_{\mathcal{L}_j})^p(\eta^3_{\mathcal{L}_k})^q]\end{array}\right\}$$
$$F_{jk} = \prod_{j,k=1;j\neq k}^{r} 2(\eta^3_{\mathcal{L}_j})^p(\eta^3_{\mathcal{L}_k})^q$$
$$G_{jk} = \sum_{j,k=1;j\neq k}^{r} 8(\nu^3_{\mathcal{L}_j})^p(\nu^3_{\mathcal{L}_k})^q\left\{\begin{array}{l}[(2-\nu^3_{\mathcal{L}_k})^p \\ (2-\nu^3_{\mathcal{L}_j})^q] \\ +[(\nu^3_{\mathcal{L}_k})^p(\nu^3_{\mathcal{L}_j})^q]\end{array}\right\}$$
$$H_{jk} = \prod_{j,k=1;j\neq k}^{r} 2\left\{\begin{array}{l}[(2-\nu^3_{\mathcal{L}_j})^p(2-\nu^3_{\mathcal{L}_k})^q] \\ +[(\nu^3_{\mathcal{L}_j})^p(\nu^3_{\mathcal{L}_k})^q]\end{array}\right\}$$
$$I_{jk} = \prod_{j,k=1;j\neq k}^{r} 2(\nu^3_{\mathcal{L}_j})^p(\nu^3_{\mathcal{L}_k})^q.$$

Proof. According to Definition III-A, we have

$$p\mathcal{L}_j = \left\langle \begin{array}{l}\sqrt[3]{\frac{(1+\mu^3_{\mathcal{L}_j})^p-(1-\mu^3_{\mathcal{L}_j})^p}{(1+\mu^3_{\mathcal{L}_j})^p+(1-\mu^3_{\mathcal{L}_j})^p}}, \\ \sqrt[3]{\frac{2(\eta^3_{\mathcal{L}_j})^p}{(2-\eta^3_{\mathcal{L}_j})^p+(\eta^3_{\mathcal{L}_j})^p}}, \sqrt[3]{\frac{2(\nu^3_{\mathcal{L}_j})^p}{(2-\nu^3_{\mathcal{L}_j})^p+(\nu^3_{\mathcal{L}_j})^p}}\end{array}\right\rangle$$

$$q\mathcal{L}_k = \left\langle \begin{array}{l}\sqrt[3]{\frac{(1+\mu^3_{\mathcal{L}_k})^q-(1-\mu^3_{\mathcal{L}_k})^q}{(1+\mu^3_{\mathcal{L}_k})^q+(1-\mu^3_{\mathcal{L}_k})^q}}, \\ \sqrt[3]{\frac{2(\eta^3_{\mathcal{L}_k})^q}{(2-\eta^3_{\mathcal{L}_k})^q+(\eta^3_{\mathcal{L}_k})^q}}, \sqrt[3]{\frac{2(\nu^3_{\mathcal{L}_k})^q}{(2-\nu^3_{\mathcal{L}_k})^q+(\nu^3_{\mathcal{L}_k})^q}}\end{array}\right\rangle$$

and

$$p\mathcal{L}_j \bigoplus_E q\mathcal{L}_k = \left\langle \begin{array}{l}\sqrt[3]{\frac{\left\{[(1+\mu^3_{\mathcal{L}_j})^p(1+\mu^3_{\mathcal{L}_k})^q]-[(1-\mu^3_{\mathcal{L}_j})^p(1-\mu^3_{\mathcal{L}_k})^q]\right\}}{\left\{[(1+\mu^3_{\mathcal{L}_j})^p(1+\mu^3_{\mathcal{L}_k})^q]+[(1-\mu^3_{\mathcal{L}_j})^p(1-\mu^3_{\mathcal{L}_k})^q]\right\}}}, \\ \sqrt[3]{\frac{4(\eta^3_{\mathcal{L}_j})^p(\eta^3_{\mathcal{L}_k})^q}{2\left\{[(2-\eta^3_{\mathcal{L}_j})^p(2-\eta^3_{\mathcal{L}_k})^q]+[(\eta^3_{\mathcal{L}_j})^p(\eta^3_{\mathcal{L}_k})^q]\right\}}}, \\ \sqrt[3]{\frac{4(\nu^3_{\mathcal{L}_j})^p(\nu^3_{\mathcal{L}_k})^q}{2\left\{[(2-\nu^3_{\mathcal{L}_j})^p(2-\nu^3_{\mathcal{L}_k})^q]+[(\nu^3_{\mathcal{L}_j})^p(\nu^3_{\mathcal{L}_k})^q]\right\}}}\end{array}\right\rangle$$

Therefore,

$$\bigotimes_{E\ j,k=1;j\neq k}^{r}\left(p\mathcal{L}_j \bigoplus_E q\mathcal{L}_k\right)$$

$$= \left\langle \begin{array}{l} \sqrt[3]{\dfrac{\prod_{j,k=1;j\neq k}^{r} 2\left\{ \begin{array}{l} [(1+\mu_{\mathcal{L}_j}^3)^p(1+\mu_{\mathcal{L}_k}^3)^q] \\ -[(1-\mu_{\mathcal{L}_j}^3)^p(1-\mu_{\mathcal{L}_k}^3)^q] \end{array} \right\}}{\prod_{j,k=1;j\neq k}^{r} 2\left\{ \begin{array}{l} [(1+\mu_{\mathcal{L}_j}^3)^p(1+\mu_{\mathcal{L}_k}^3)^q] \\ +[(1-\mu_{\mathcal{L}_j}^3)^p(1-\mu_{\mathcal{L}_k}^3)^q] \end{array} \right\} + \prod_{j,k=1;j\neq k}^{r} 4(1-\mu_{\mathcal{L}_j}^3)^p(1-\mu_{\mathcal{L}_k}^3)^q}}, \\ \sqrt[3]{\dfrac{\sum_{j,k=1;j\neq k}^{r} 8(\eta_{\mathcal{L}_j}^3)^p(\eta_{\mathcal{L}_k}^3)^q \left\{ \begin{array}{l} [(2-\eta_{\mathcal{L}_k}^3)^p \\ (2-\eta_{\mathcal{L}_j}^3)^q] \\ +[(\eta_{\mathcal{L}_k}^3)^p(\eta_{\mathcal{L}_j}^3)^q] \end{array} \right\}}{\prod_{j,k=1;j\neq k}^{r} 2\left\{ \begin{array}{l} [(2-\eta_{\mathcal{L}_j}^3)^p \\ (2-\eta_{\mathcal{L}_k}^3)^q] \\ +[(\eta_{\mathcal{L}_j}^3)^p(\eta_{\mathcal{L}_k}^3)^q] \end{array} \right\} + \prod_{j,k=1;j\neq k}^{r} 4(\eta_{\mathcal{L}_j}^3)^p(\eta_{\mathcal{L}_k}^3)^q}}, \\ \sqrt[3]{\dfrac{\sum_{j,k=1;j\neq k}^{r} 8(\nu_{\mathcal{L}_j}^3)^p(\nu_{\mathcal{L}_k}^3)^q \left\{ \begin{array}{l} [(2-\nu_{\mathcal{L}_k}^3)^p \\ (2-\nu_{\mathcal{L}_j}^3)^q] \\ +[(\nu_{\mathcal{L}_k}^3)^p(\nu_{\mathcal{L}_j}^3)^q] \end{array} \right\}}{\prod_{j,k=1;j\neq k}^{r} 2\left\{ \begin{array}{l} [(2-\nu_{\mathcal{L}_j}^3)^p \\ (2-\nu_{\mathcal{L}_k}^3)^q] \\ +[(\nu_{\mathcal{L}_j}^3)^p(\nu_{\mathcal{L}_k}^3)^q] \end{array} \right\} + \prod_{j,k=1;j\neq k}^{r} 4(\nu_{\mathcal{L}_j}^3)^p(\nu_{\mathcal{L}_k}^3)^q}} \end{array} \right\rangle$$

Let $A_{jk} = \prod_{j,k=1;j\neq k}^{r} 2\left\{ \begin{array}{l} [(1+\mu_{\mathcal{L}_j}^3)^p(1+\mu_{\mathcal{L}_k}^3)^q] \\ -[(1-\mu_{\mathcal{L}_j}^3)^p(1-\mu_{\mathcal{L}_k}^3)^q] \end{array} \right\}$

$B_{jk} = \prod_{j,k=1;j\neq k}^{r} 2\left\{ \begin{array}{l} [(1+\mu_{\mathcal{L}_j}^3)^p(1+\mu_{\mathcal{L}_k}^3)^q] \\ +[(1-\mu_{\mathcal{L}_j}^3)^p(1-\mu_{\mathcal{L}_k}^3)^q] \end{array} \right\}$

$C_{jk} = \prod_{j,k=1;j\neq k}^{r} 4(1-\mu_{\mathcal{L}_j}^3)^p(1-\mu_{\mathcal{L}_k}^3)^q$

$D_{jk} = \sum_{j,k=1;j\neq k}^{r} 8(\eta_{\mathcal{L}_j}^3)^p(\eta_{\mathcal{L}_k}^3)^q \left\{ \begin{array}{l} [(2-\eta_{\mathcal{L}_k}^3)^p \\ (2-\eta_{\mathcal{L}_j}^3)^q] \\ +[(\eta_{\mathcal{L}_k}^3)^p(\eta_{\mathcal{L}_j}^3)^q] \end{array} \right\}$

$E_{jk} = \prod_{j,k=1;j\neq k}^{r} 2\left\{ \begin{array}{l} [(2-\eta_{\mathcal{L}_j}^3)^p(2-\eta_{\mathcal{L}_k}^3)^q] \\ +[(\eta_{\mathcal{L}_j}^3)^p(\eta_{\mathcal{L}_k}^3)^q] \end{array} \right\}$

$F_{jk} = \prod_{j,k=1;j\neq k}^{r} 2(\eta_{\mathcal{L}_j}^3)^p(\eta_{\mathcal{L}_k}^3)^q$

$G_{jk} = \sum_{j,k=1;j\neq k}^{r} 8(\nu_{\mathcal{L}_j}^3)^p(\nu_{\mathcal{L}_k}^3)^q \left\{ \begin{array}{l} [(2-\nu_{\mathcal{L}_k}^3)^p \\ (2-\nu_{\mathcal{L}_j}^3)^q] \\ +[(\nu_{\mathcal{L}_k}^3)^p(\nu_{\mathcal{L}_j}^3)^q] \end{array} \right\}$

$H_{jk} = \prod_{j,k=1;j\neq k}^{r} 2\left\{ \begin{array}{l} [(2-\nu_{\mathcal{L}_j}^3)^p(2-\nu_{\mathcal{L}_k}^3)^q] \\ +[(\nu_{\mathcal{L}_j}^3)^p(\nu_{\mathcal{L}_k}^3)^q] \end{array} \right\}$

$I_{jk} = \prod_{j,k=1;j\neq k}^{r} 2(\nu_{\mathcal{L}_j}^3)^p(\nu_{\mathcal{L}_k}^3)^q.$

Then, $\bigotimes_{E\ j,k=1;j\neq k}^{r} \left(p\mathcal{L}_j \bigoplus_E q\mathcal{L}_k \right)$
$= \left\langle \sqrt[3]{\frac{A_{jk}}{B_{jk}+C_{jk}}}, \sqrt[3]{\frac{D_{jk}}{E_{jk}+F_{jk}}}, \sqrt[3]{\frac{G_{jk}}{H_{jk}+I_{jk}}} \right\rangle$

Thereafter, $\left[\bigotimes_{E\ j,k=1;j\neq k}^{r} \left(p\mathcal{L}_j \bigoplus_E q\mathcal{L}_k \right) \right]^{\frac{1}{r(r-1)}}$

$$= \left\langle \begin{array}{l} \sqrt[3]{\frac{2(A_{jk})^{\frac{1}{r(r-1)}}}{(2B_{jk}+2C_{jk}-A_{jk})^{\frac{1}{r(r-1)}}+(A_{jk})^{\frac{1}{r(r-1)}}}}, \\ \sqrt[3]{\frac{(E_{jk}+F_{jk}+D_{jk})^{\frac{1}{r(r-1)}}-(E_{jk}+F_{jk}-D_{jk})^{\frac{1}{r(r-1)}}}{(E_{jk}+F_{jk}+D_{jk})^{\frac{1}{r(r-1)}}+(E_{jk}+F_{jk}-D_{jk})^{\frac{1}{r(r-1)}}}}, \\ \sqrt[3]{\frac{(H_{jk}+I_{jk}+G_{jk})^{\frac{1}{r(r-1)}}-(H_{jk}+I_{jk}-G_{jk})^{\frac{1}{r(r-1)}}}{(H_{jk}+I_{jk}+G_{jk})^{\frac{1}{r(r-1)}}+(H_{jk}+I_{jk}-G_{jk})^{\frac{1}{r(r-1)}}}} \end{array} \right\rangle$$

And so, $\frac{1}{p+q}\left[\bigotimes_{E\ j,k=1;j\neq k}^{r} \left(p\mathcal{L}_j \bigoplus_E q\mathcal{L}_k \right) \right]^{\frac{1}{r(r-1)}}$

$$= \left\langle \begin{array}{l} \sqrt[3]{\dfrac{\left[\begin{array}{l} (2B_{jk}+2C_{jk}-A_{jk})^{\frac{1}{r(r-1)}} \\ +3\left(A_{jk}\right)^{\frac{1}{r(r-1)}} \end{array} \right]^{\frac{1}{p+q}} - \left[\begin{array}{l} (2B_{jk}+2C_{jk}-A_{jk})^{\frac{1}{r(r-1)}} \\ -\left(A_{jk}\right)^{\frac{1}{r(r-1)}} \end{array} \right]^{\frac{1}{p+q}}}{\left[\begin{array}{l} (2B_{jk}+2C_{jk}-A_{jk})^{\frac{1}{r(r-1)}} \\ +3\left(A_{jk}\right)^{\frac{1}{r(r-1)}} \end{array} \right]^{\frac{1}{p+q}} + \left[\begin{array}{l} (2B_{jk}+2C_{jk}-A_{jk})^{\frac{1}{r(r-1)}} \\ -\left(A_{jk}\right)^{\frac{1}{r(r-1)}} \end{array} \right]^{\frac{1}{p+q}}}}, \\ \sqrt[3]{\dfrac{2\left[\begin{array}{l} (E_{jk}+F_{jk}+D_{jk})^{\frac{1}{r(r-1)}} \\ -(E_{jk}+F_{jk}-D_{jk})^{\frac{1}{r(r-1)}} \end{array} \right]^{\frac{1}{p+q}}}{\left[\begin{array}{l} (E_{jk}+F_{jk}+D_{jk})^{\frac{1}{r(r-1)}} \\ +3\left(E_{jk}+F_{jk}-D_{jk}\right)^{\frac{1}{r(r-1)}} \end{array} \right]^{\frac{1}{p+q}} + \left[\begin{array}{l} (E_{jk}+F_{jk}+D_{jk})^{\frac{1}{r(r-1)}} \\ -(E_{jk}+F_{jk}-D_{jk})^{\frac{1}{r(r-1)}} \end{array} \right]^{\frac{1}{p+q}}}}, \\ \sqrt[3]{\dfrac{2\left[\begin{array}{l} (H_{jk}+I_{jk}+G_{jk})^{\frac{1}{r(r-1)}} \\ -(H_{jk}+I_{jk}-G_{jk})^{\frac{1}{r(r-1)}} \end{array} \right]^{\frac{1}{p+q}}}{\left[\begin{array}{l} (H_{jk}+I_{jk}+G_{jk})^{\frac{1}{r(r-1)}} \\ +3\left(H_{jk}+I_{jk}-G_{jk}\right)^{\frac{1}{r(r-1)}} \end{array} \right]^{\frac{1}{p+q}} + \left[\begin{array}{l} (H_{jk}+I_{jk}+G_{jk})^{\frac{1}{r(r-1)}} \\ -(H_{jk}+I_{jk}-G_{jk})^{\frac{1}{r(r-1)}} \end{array} \right]^{\frac{1}{p+q}}}} \end{array} \right\rangle$$

Let
$\phi = (2B_{jk}+2C_{jk}-C_{jk})^{\frac{1}{r(r-1)}} + 3\left(A_{jk}\right)^{\frac{1}{r(r-1)}}$
$\varphi = (2B_{jk}+2C_{jk}-A_{jk})^{\frac{1}{r(r-1)}} - \left(A_{jk}\right)^{\frac{1}{r(r-1)}}$
$\chi = (E_{jk}+F_{jk}+D_{jk})^{\frac{1}{r(r-1)}} - (E_{jk}+F_{jk}-D_{jk})^{\frac{1}{r(r-1)}}$
$\psi = (E_{jk}+F_{jk}+D_{jk})^{\frac{1}{r(r-1)}} + 3\left(E_{jk}+F_{jk}-D_{jk}\right)^{\frac{1}{r(r-1)}}$
$\rho = (H_{jk}+I_{jk}+G_{jk})^{\frac{1}{r(r-1)}} - (H_{jk}+I_{jk}-G_{jk})^{\frac{1}{r(r-1)}}$
$\varrho = (H_{jk}+I_{jk}+G_{jk})^{\frac{1}{r(r-1)}} + 3\left(H_{jk}+I_{jk}-G_{jk}\right)^{\frac{1}{r(r-1)}}$

Therefore,
$CFEBOMG^{p,q}(\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_r)$
$= \frac{1}{p+q}\left[\bigotimes_{E\ j,k=1;j\neq k}^{r} \left(p\mathcal{L}_j \bigoplus_E q\mathcal{L}_k \right) \right]^{\frac{1}{r(r-1)}}$
$= \left\langle \sqrt[3]{\frac{\phi^{\frac{1}{p+q}} - \varphi^{\frac{1}{p+q}}}{\phi^{\frac{1}{p+q}} + \varphi^{\frac{1}{p+q}}}}, \sqrt[3]{\frac{2\chi^{\frac{1}{p+q}}}{\psi^{\frac{1}{p+q}} + \chi^{\frac{1}{p+q}}}}, \sqrt[3]{\frac{2\rho^{\frac{1}{p+q}}}{\varrho^{\frac{1}{p+q}} + \rho^{\frac{1}{p+q}}}} \right\rangle$

where
$\phi = (2B_{jk}+2C_{jk}-C_{jk})^{\frac{1}{r(r-1)}} + 3\left(A_{jk}\right)^{\frac{1}{r(r-1)}}$
$\varphi = (2B_{jk}+2C_{jk}-A_{jk})^{\frac{1}{r(r-1)}} - \left(A_{jk}\right)^{\frac{1}{r(r-1)}}$
$\chi = (E_{jk}+F_{jk}+D_{jk})^{\frac{1}{r(r-1)}} - (E_{jk}+F_{jk}-D_{jk})^{\frac{1}{r(r-1)}}$
$\psi = (E_{jk}+F_{jk}+D_{jk})^{\frac{1}{r(r-1)}} + 3\left(E_{jk}+F_{jk}-D_{jk}\right)^{\frac{1}{r(r-1)}}$
$\rho = (H_{jk}+I_{jk}+G_{jk})^{\frac{1}{r(r-1)}} - (H_{jk}+I_{jk}-G_{jk})^{\frac{1}{r(r-1)}}$
$\varrho = (H_{jk}+I_{jk}+G_{jk})^{\frac{1}{r(r-1)}} + 3\left(H_{jk}+I_{jk}-G_{jk}\right)^{\frac{1}{r(r-1)}}$

and
$A_{jk} = \prod_{j,k=1;j\neq k}^{r} 2\left\{ \begin{array}{l} [(1+\mu_{\mathcal{L}_j}^3)^p(1+\mu_{\mathcal{L}_k}^3)^q] \\ -[(1-\mu_{\mathcal{L}_j}^3)^p(1-\mu_{\mathcal{L}_k}^3)^q] \end{array} \right\}$

$$B_{jk}=\prod_{j,k=1;j\neq k}^{r}2\left\{\begin{array}{l}[(1+\mu_{\mathcal{L}_j}^3)^p(1+\mu_{\mathcal{L}_k}^3)^q]\\+[(1-\mu_{\mathcal{L}_j}^3)^p(1-\mu_{\mathcal{L}_k}^3)^q]\end{array}\right\}$$
$$C_{jk}=\prod_{j,k=1;j\neq k}^{r}4(1-\mu_{\mathcal{L}_j}^3)^p(1-\mu_{\mathcal{L}_k}^3)^q$$
$$D_{jk}=\sum_{j,k=1;j\neq k}^{r}8(\eta_{\mathcal{L}_j}^3)^p(\eta_{\mathcal{L}_k}^3)^q\left\{\begin{array}{l}[(2-\eta_{\mathcal{L}_k}^3)^p\\(2-\eta_{\mathcal{L}_j}^3)^q]\\+[(\eta_{\mathcal{L}_k}^3)^p(\eta_{\mathcal{L}_j}^3)^q]\end{array}\right\}$$
$$E_{jk}=\prod_{j,k=1;j\neq k}^{r}2\left\{\begin{array}{l}[(2-\eta_{\mathcal{L}_j}^3)^p(2-\eta_{\mathcal{L}_k}^3)^q]\\+[(\eta_{\mathcal{L}_j}^3)^p(\eta_{\mathcal{L}_k}^3)^q]\end{array}\right\}$$
$$F_{jk}=\prod_{j,k=1;j\neq k}^{r}2(\eta_{\mathcal{L}_j}^3)^p(\eta_{\mathcal{L}_k}^3)^q$$
$$G_{jk}=\sum_{j,k=1;j\neq k}^{r}8(\nu_{\mathcal{L}_j}^3)^p(\nu_{\mathcal{L}_k}^3)^q\left\{\begin{array}{l}[(2-\nu_{\mathcal{L}_k}^3)^p\\(2-\nu_{\mathcal{L}_j}^3)^q]\\+[(\nu_{\mathcal{L}_k}^3)^p(\nu_{\mathcal{L}_j}^3)^q]\end{array}\right\}$$
$$H_{jk}=\prod_{j,k=1;j\neq k}^{r}2\left\{\begin{array}{l}[(2-\nu_{\mathcal{L}_j}^3)^p(2-\nu_{\mathcal{L}_k}^3)^q]\\+[(\nu_{\mathcal{L}_j}^3)^p(\nu_{\mathcal{L}_k}^3)^q]\end{array}\right\}$$
$$I_{jk}=\prod_{j,k=1;j\neq k}^{r}2(\nu_{\mathcal{L}_j}^3)^p(\nu_{\mathcal{L}_k}^3)^q.$$
which fulfills the proof of Theorem III-C1. ■

Example 3.1

Suppose $\mathcal{L}_1=\langle 0.83,0.53,0.63\rangle,\mathcal{L}_2=\langle 0.89,0.39,0.49\rangle,\mathcal{L}_3=\langle 0.82,0.62,0.22\rangle$ and $\mathcal{L}_4=\langle 0.6,0.2,0.8\rangle$ are four CFNs. If we take $p=1$ and $q=2$, then, CFEBOMG operator can be utilized to aggregate the four CFNs as follows:
$CFEBOMG^{1,2}(\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3,\mathcal{L}_4)$
$=\frac{1}{3}\left[\bigotimes_{E\,j,k=1;j\neq k}^{4}(1\mathcal{L}_j\bigoplus_E 2\mathcal{L}_k)\right]^{\frac{1}{12}}$
$=\left\langle\sqrt[3]{\frac{\phi^{\frac{1}{3}}-\varphi^{\frac{1}{3}}}{\phi^{\frac{1}{3}}+\varphi^{\frac{1}{3}}}},\sqrt[3]{\frac{2\chi^{\frac{1}{3}}}{\psi^{\frac{1}{3}}+\chi^{\frac{1}{3}}}},\sqrt[3]{\frac{2\rho^{\frac{1}{3}}}{\varrho^{\frac{1}{3}}+\rho^{\frac{1}{3}}}}\right\rangle$
$=\langle 0.8014,0.0602,0.0940\rangle$.

For a collection of CFNs, we can easily prove the proposed CFEBOMG operator satisfy the idempotency, boundedness and monotonicity properties as follows:

Property 3.1

(Idempotency) Let $\mathcal{L}_j=\langle\mu_{\mathcal{L}_j},\eta_{\mathcal{L}_j},\nu_{\mathcal{L}_j}\rangle(j=1,2,\ldots,r)$ be a set of r CFNs. If $\mathcal{L}_j=\mathcal{L}=\langle\mu_{\mathcal{L}},\eta_{\mathcal{L}},\nu_{\mathcal{L}}\rangle$ for all j, then $CFEBOMG^{p,q}(\mathcal{L}_1,\mathcal{L}_2,\ldots,\mathcal{L}_r)=L$.
Proof. Since $\mathcal{L}_j=\mathcal{L}=\langle\mu_{\mathcal{L}},\eta_{\mathcal{L}},\nu_{\mathcal{L}}\rangle$ for all $j=1,2,\ldots,r$. Then,
$CFEBOMG^{p,q}(\mathcal{L}_1,\mathcal{L}_2,\ldots,\mathcal{L}_r)$
$$=\frac{1}{p+q}\left[\bigotimes_{E\;j,k=1;j\neq k}^{r}\left(p\mathcal{L}_j\bigoplus_E q\mathcal{L}_k\right)\right]^{\frac{1}{r(r-1)}}$$
$$=\frac{1}{p+q}\left[\bigotimes_{E\;j,k=1;j\neq k}^{r}\left(p\mathcal{L}\bigoplus_E q\mathcal{L}\right)\right]^{\frac{1}{r(r-1)}}$$
$$=\frac{1}{p+q}\left[\bigotimes_{E\;j,k=1;j\neq k}^{r}((p+q)\mathcal{L})\right]^{\frac{1}{r(r-1)}}$$
$$=\frac{1}{p+q}\left[((p+q)\mathcal{L})^{\frac{r(r-1)}{1}}\right]^{\frac{1}{r(r-1)}}$$
$=\mathcal{L}$
$\implies CFEBOMG^{p,q}(\mathcal{L}_1,\mathcal{L}_2,\ldots,\mathcal{L}_r)=\mathcal{L}$. ■

Property 3.2

(Monotonicity) For any two set of CFNs, $\mathcal{L}_j=\langle\mu_{\mathcal{L}_j},\eta_{\mathcal{L}_j},\nu_{\mathcal{L}_j}\rangle$ and $\mathcal{L}'_j=\langle\mu_{\mathcal{L}'_j},\eta_{\mathcal{L}'_j},\nu_{\mathcal{L}'_j}\rangle$ such that $\mathcal{L}_j\le\mathcal{L}'_j$ for all $j=1,2,\ldots,r$, we have $CFEBOMG^{p,q}(\mathcal{L}_1,\mathcal{L}_2,\ldots,\mathcal{L}_r)\le CFEBOMG^{p,q}(\mathcal{L}'_1,\mathcal{L}'_2,\ldots,\mathcal{L}'_r)$.

Proof. Suppose that $CFEBOMG^{p,q}(\mathcal{L}_1,\mathcal{L}_2,\ldots,\mathcal{L}_r)=\mathcal{L}=\langle\mu_{\mathcal{L}},\eta_{\mathcal{L}},\nu_{\mathcal{L}}\rangle$ and $CFEBOMG^{p,q}(\mathcal{L}'_1,\mathcal{L}'_2,\ldots,\mathcal{L}'_r)=\mathcal{L}'=\langle\mu'_{\mathcal{L}},\eta'_{\mathcal{L}},\nu'_L\rangle$. We know that $\mu_{\mathcal{L}_j}\le\mu_{\mathcal{L}'_j},\eta_{\mathcal{L}_j}\le\eta_{\mathcal{L}'_j}$ and $\nu_{\mathcal{L}_j}\ge\nu_{\mathcal{L}'_j}$ for $\mathcal{L}_j\le\mathcal{L}'_j$.
Let $\mu_{\mathcal{L}_j}\le\mu_{\mathcal{L}'_j}$ and $\mu_{\mathcal{L}_k}\le\mu_{\mathcal{L}'_k}$ for all $j=1,2,\ldots,r$; $k=j,j+1,\ldots,r$. Then, $\mu_{\mathcal{L}_j}\le\mu_{\mathcal{L}'_j}\implies\mu_{\mathcal{L}_j}^3\le\mu_{\mathcal{L}'_j}^3$ and $\mu_{\mathcal{L}_k}\le\mu_{\mathcal{L}'_k}\implies\mu_{\mathcal{L}_k}^3\le\mu_{\mathcal{L}'_k}^3$
Clearly,
$$A_{jk}=\prod_{j,k=1;j\neq k}^{r}2\left\{\begin{array}{l}[(1+\mu_{\mathcal{L}_j}^3)^p(1+\mu_{\mathcal{L}_k}^3)^q]\\-[(1-\mu_{\mathcal{L}_j}^3)^p(1-\mu_{\mathcal{L}_k}^3)^q]\end{array}\right\}$$
$$\le\prod_{j,k=1;j\neq k}^{r}2\left\{\begin{array}{l}[(1+\mu_{\mathcal{L}'_j}^3)^p(1+\mu_{\mathcal{L}'_k}^3)^q]\\-[(1-\mu_{\mathcal{L}'_j}^3)^p(1-\mu_{\mathcal{L}'_k}^3)^q]\end{array}\right\}=A'_{jk};$$
$$B_{jk}=\prod_{j,k=1;j\neq k}^{r}2\left\{\begin{array}{l}[(1+\mu_{\mathcal{L}_j}^3)^p(1+\mu_{\mathcal{L}_k}^3)^q]\\+[(1-\mu_{\mathcal{L}_j}^3)^p(1-\mu_{\mathcal{L}_k}^3)^q]\end{array}\right\}$$
$$\le\prod_{j,k=1;j\neq k}^{r}2\left\{\begin{array}{l}[(1+\mu_{\mathcal{L}'_j}^3)^p(1+\mu_{\mathcal{L}'_k}^3)^q]\\+[(1-\mu_{\mathcal{L}'_j}^3)^p(1-\mu_{\mathcal{L}'_k}^3)^q]\end{array}\right\}=B'_{jk};$$
$$C_{jk}=\prod_{j,k=1;j\neq k}^{r}4(1-\mu_{\mathcal{L}_j}^3)^p(1-\mu_{\mathcal{L}_k}^3)^q$$
$$\le\prod_{j,k=1;j\neq k}^{r}4(1-\mu_{\mathcal{L}'_j}^3)^p(1-\mu_{\mathcal{L}'_k}^3)^q=C'_{jk}$$
Let $\eta_{\mathcal{L}_j}\le\eta_{\mathcal{L}'_j}$ and $\eta_{\mathcal{L}_k}\le\eta_{\mathcal{L}'_k}$ for all $j=1,2,\ldots,r$; $k=j,j+1,\ldots,r$. Then, $\eta_{\mathcal{L}_j}\le\eta_{\mathcal{L}'_j}\implies\eta_{\mathcal{L}_j}^3\le\eta_{\mathcal{L}'_j}^3$ and $\eta_{\mathcal{L}_k}\le\eta_{\mathcal{L}'_k}\implies\eta_{\mathcal{L}_k}^3\le\eta_{\mathcal{L}'_k}^3$
Clearly,
$$D_{jk}=\sum_{j,k=1;j\neq k}^{r}8(\eta_{\mathcal{L}_j}^3)^p(\eta_{\mathcal{L}_k}^3)^q\left\{\begin{array}{l}[(2-\eta_{\mathcal{L}_k}^3)^p\\(2-\eta_{\mathcal{L}_j}^3)^q]\\+[(\eta_{\mathcal{L}_k}^3)^p(\eta_{\mathcal{L}_j}^3)^q]\end{array}\right\}$$
$$\le\sum_{j,k=1;j\neq k}^{r}8(\eta_{\mathcal{L}'_j}^3)^p(\eta_{\mathcal{L}'_k}^3)^q\left\{\begin{array}{l}[(2-\eta_{\mathcal{L}'_k}^3)^p\\(2-\eta_{\mathcal{L}'_j}^3)^q]\\+[(\eta_{\mathcal{L}'_k}^3)^p(\eta_{\mathcal{L}'_j}^3)^q]\end{array}\right\}=D'_{jk};$$
$$E_{jk}=\prod_{j,k=1;j\neq k}^{r}2\left\{\begin{array}{l}[(2-\eta_{\mathcal{L}_j}^3)^p(2-\eta_{\mathcal{L}_k}^3)^q]\\+[(\eta_{\mathcal{L}_j}^3)^p(\eta_{\mathcal{L}_k}^3)^q]\end{array}\right\}$$
$$\le\prod_{j,k=1;j\neq k}^{r}2\left\{\begin{array}{l}[(2-\eta_{\mathcal{L}'_j}^3)^p(2-\eta_{\mathcal{L}'_k}^3)^q]\\+[(\eta_{\mathcal{L}'_j}^3)^p(\eta_{\mathcal{L}'_k}^3)^q]\end{array}\right\}=E'_{jk};$$
$$F_{jk}=\prod_{j,k=1;j\neq k}^{r}2(\eta_{\mathcal{L}_j}^3)^p(\eta_{\mathcal{L}_k}^3)^q$$
$$\le\prod_{j,k=1;j\neq k}^{r}2(\eta_{\mathcal{L}'_j}^3)^p(\eta_{\mathcal{L}'_k}^3)^q=F'_{jk};$$
Let $\nu_{\mathcal{L}_j}\ge\nu_{\mathcal{L}'_j}$ and $\nu_{\mathcal{L}_k}\ge\nu_{\mathcal{L}'_k}$ for all $j=1,2,\ldots,r$; $k=j,j+1,\ldots,r$. Then, $\nu_{\mathcal{L}_j}\ge\nu_{\mathcal{L}'_j}\implies\nu_{\mathcal{L}_j}^3\ge\nu_{\mathcal{L}'_j}^3$ and $\nu_{\mathcal{L}_k}\ge\nu_{\mathcal{L}'_k}\implies\nu_{\mathcal{L}_k}^3\ge\nu_{\mathcal{L}'_k}^3$
Clearly,
$$G_{jk}=\sum_{j,k=1;j\neq k}^{r}8(\nu_{\mathcal{L}_j}^3)^p(\nu_{\mathcal{L}_k}^3)^q\left\{\begin{array}{l}[(2-\nu_{\mathcal{L}_k}^3)^p\\(2-\nu_{\mathcal{L}_j}^3)^q]\\+[(\nu_{\mathcal{L}_k}^3)^p(\nu_{\mathcal{L}_j}^3)^q]\end{array}\right\}$$
$$\ge\sum_{j,k=1;j\neq k}^{r}8(\nu_{\mathcal{L}'_j}^3)^p(\nu_{\mathcal{L}'_k}^3)^q\left\{\begin{array}{l}[(2-\nu_{\mathcal{L}'_k}^3)^p\\(2-\nu_{\mathcal{L}'_j}^3)^q]\\+[(\nu_{\mathcal{L}'_k}^3)^p(\nu_{\mathcal{L}'_j}^3)^q]\end{array}\right\}=G'_{jk};$$
$$H_{jk}=\prod_{j,k=1;j\neq k}^{r}2\left\{\begin{array}{l}[(2-\nu_{\mathcal{L}_j}^3)^p(2-\nu_{\mathcal{L}_k}^3)^q]\\+[(\nu_{\mathcal{L}_j}^3)^p(\nu_{\mathcal{L}_k}^3)^q]\end{array}\right\}$$

$$\geq \prod_{j,k=1;j\neq k}^{r} 2\left\{\begin{array}{l}[(2-\nu^3_{\mathcal{L}'_j})^p(2-\nu^3_{\mathcal{L}'_k})^q]\\+[(\nu^3_{\mathcal{L}'_j})^p(\nu^3_{\mathcal{L}'_k})^q]\end{array}\right\} = H'_{jk};$$

$$I_{jk} = \prod_{j,k=1;j\neq k}^{r} 2(\nu^3_{\mathcal{L}_j})^p(\nu^3_{\mathcal{L}_k})^q$$
$$\geq \prod_{j,k=1;j\neq k}^{r} 2(\nu^3_{\mathcal{L}'_j})^p(\nu^3_{\mathcal{L}'_k})^q = I'_{jk};$$

Thereafter,

$$\phi = (2B_{jk}+2C_{jk}-A_{jk})^{\frac{1}{r(r-1)}} + 3\,(A_{jk})^{\frac{1}{r(r-1)}}$$
$$\leq \left(2B'_{jk}+2C'_{jk}-A'_{jk}\right)^{\frac{1}{r(r-1)}} + 3\left(A'_{jk}\right)^{\frac{1}{r(r-1)}} = \phi';$$
$$\varphi = (2B_{jk}+2C_{jk}-A_{jk})^{\frac{1}{r(r-1)}} - (A_{jk})^{\frac{1}{r(r-1)}}$$
$$\leq \left(2B'_{jk}+2C'_{jk}-A'_{jk}\right)^{\frac{1}{r(r-1)}} - \left(A'_{jk}\right)^{\frac{1}{r(r-1)}} = \varphi';$$
$$\chi = (E_{jk}+F_{jk}+D_{jk})^{\frac{1}{r(r-1)}} - (E_{jk}+F_{jk}-D_{jk})^{\frac{1}{r(r-1)}}$$
$$\leq \left(E'_{jk}+F'_{jk}+D'_{jk}\right)^{\frac{1}{r(r-1)}} - \left(E'_{jk}+F'_{jk}-D'_{jk}\right)^{\frac{1}{r(r-1)}}$$
$$= \chi';$$
$$\psi = (E_{jk}+F_{jk}+D_{jk})^{\frac{1}{r(r-1)}} + 3\,(E_{jk}+F_{jk}-D_{jk})^{\frac{1}{r(r-1)}}$$
$$\leq \left(E'_{jk}+F'_{jk}+D'_{jk}\right)^{\frac{1}{r(r-1)}} + 3\left(E'_{jk}+F'_{jk}-D'_{jk}\right)^{\frac{1}{r(r-1)}}$$
$$= \psi';$$
$$\rho = (H_{jk}+I_{jk}+G_{jk})^{\frac{1}{r(r-1)}} - (H_{jk}+I_{jk}-G_{jk})^{\frac{1}{r(r-1)}}$$
$$\geq \left(H'_{jk}+I'_{jk}+G'_{jk}\right)^{\frac{1}{r(r-1)}} - \left(H'_{jk}+I'_{jk}-G'_{jk}\right)^{\frac{1}{r(r-1)}}$$
$$= \rho';$$
$$\varrho = (H_{jk}+I_{jk}+G_{jk})^{\frac{1}{r(r-1)}} + 3\,(H_{jk}+I_{jk}-G_{jk})^{\frac{1}{r(r-1)}}$$
$$\geq \left(H'_{jk}+I'_{jk}+G'_{jk}\right)^{\frac{1}{r(r-1)}} + 3\left(H'_{jk}+I'_{jk}-G'_{jk}\right)^{\frac{1}{r(r-1)}}$$
$$= \varrho'.$$

And so,

$$\sqrt[3]{\frac{\phi^{\frac{1}{p+q}}-\varphi^{\frac{1}{p+q}}}{\phi^{\frac{1}{p+q}}+\varphi^{\frac{1}{p+q}}}} \leq \sqrt[3]{\frac{\phi'^{\frac{1}{p+q}}-\varphi'^{\frac{1}{p+q}}}{\phi'^{\frac{1}{p+q}}+\varphi'^{\frac{1}{p+q}}}};$$
$$\sqrt[3]{\frac{2\chi^{\frac{1}{p+q}}}{\psi^{\frac{1}{p+q}}+\chi^{\frac{1}{p+q}}}} \leq \sqrt[3]{\frac{2\chi'^{\frac{1}{p+q}}}{\psi'^{\frac{1}{p+q}}+\chi'^{\frac{1}{p+q}}}};$$
$$\sqrt[3]{\frac{2\rho^{\frac{1}{p+q}}}{\varrho^{\frac{1}{p+q}}+\rho^{\frac{1}{p+q}}}} \geq \sqrt[3]{\frac{2\rho'^{\frac{1}{p+q}}}{\varrho'^{\frac{1}{p+q}}+\rho'^{\frac{1}{p+q}}}}.$$

Clearly

$$\left\langle \sqrt[3]{\frac{\phi^{\frac{1}{p+q}}-\varphi^{\frac{1}{p+q}}}{\phi^{\frac{1}{p+q}}+\varphi^{\frac{1}{p+q}}}}, \sqrt[3]{\frac{2\chi^{\frac{1}{p+q}}}{\psi^{\frac{1}{p+q}}+\chi^{\frac{1}{p+q}}}}, \sqrt[3]{\frac{2\rho^{\frac{1}{p+q}}}{\varrho^{\frac{1}{p+q}}+\rho^{\frac{1}{p+q}}}} \right\rangle$$
$$\leq \left\langle \sqrt[3]{\frac{\phi'^{\frac{1}{p+q}}-\varphi'^{\frac{1}{p+q}}}{\phi'^{\frac{1}{p+q}}+\varphi'^{\frac{1}{p+q}}}}, \sqrt[3]{\frac{2\chi'^{\frac{1}{p+q}}}{\psi'^{\frac{1}{p+q}}+\chi'^{\frac{1}{p+q}}}}, \sqrt[3]{\frac{2\rho'^{\frac{1}{p+q}}}{\varrho'^{\frac{1}{p+q}}+\rho'^{\frac{1}{p+q}}}} \right\rangle$$

$$\Longrightarrow \begin{array}{l}\frac{1}{p+q}\left[\bigotimes_{E\ j,k=1;j\neq k}^{r}\left(p\mathcal{L}_j \bigoplus_E q\mathcal{L}_k\right)\right]^{\frac{1}{r(r-1)}}\\ \leq \frac{1}{p+q}\left[\bigotimes_{E\ j,k=1;j\neq k}^{r}\left(p\mathcal{L}'_j \bigoplus_E q\mathcal{L}'_k\right)\right]^{\frac{1}{r(r-1)}}\end{array}$$

$$\Longrightarrow CFEBOMG^{p,q}(\mathcal{L}_1,\mathcal{L}_2,\ldots,\mathcal{L}_r)$$
$$\leq CFEBOMG^{p,q}(\mathcal{L}'_1,\mathcal{L}'_2,\ldots,\mathcal{L}'_n). \qquad \blacksquare$$

Property 3.3

(Boundedness) For a collection of CFNs $\mathcal{L}_j (j = 1,2,\ldots,r)$, if $\mathcal{L}^l = \min_j \mathcal{L}_j$ and $\mathcal{L}^u = \max_j \mathcal{L}_j$. Then, $\mathcal{L}^l \leq CFEBOMG^{p,q}(\mathcal{L}_1,\mathcal{L}_2,\ldots,\mathcal{L}_r) \leq \mathcal{L}^u$.

Proof. Clearly, $\mathcal{L}^l \leq \mathcal{L}_j$ and $\mathcal{L}^u \leq \mathcal{L}_j$ for each $j = 1,2,\ldots,r$ as $\mathcal{L}^l = \min_j \mathcal{L}_j = \langle \min \mu_{\mathcal{L}_j}, \min \eta_{\mathcal{L}_j}, \min \nu_{\mathcal{L}_j}\rangle$ and $\mathcal{L}^u = \max_j \mathcal{L}_j = \langle \max \mu_{\mathcal{L}_j}, \min \eta_{\mathcal{L}_j}, \max \nu_{\mathcal{L}_j}\rangle$.

Then, according to Property III-C1, we have

$$CFEBOMG^{p,q}(\mathcal{L}^l,\mathcal{L}^l,\ldots,\mathcal{L}^l) = \mathcal{L}^l$$
$$CFEBOMG^{p,q}(\mathcal{L}^u,\mathcal{L}^u,\ldots,\mathcal{L}^u) = \mathcal{L}^u$$

Thereafter, according to Property III-C1, we have

$$\mathcal{L}^l = CFEBOMG^{p,q}(\mathcal{L}^l,\mathcal{L}^l,\ldots,\mathcal{L}^l)$$
$$\leq CFEBOMG^{p,q}(\mathcal{L}_1,\mathcal{L}_2,\ldots,\mathcal{L}_r)$$
$$\leq CFEBOMG^{p,q}(\mathcal{L}^u,\mathcal{L}^u,\ldots,\mathcal{L}^u) = \mathcal{L}^u$$
$$\Longrightarrow \mathcal{L}^l \leq CFEBOMG^{p,q}(\mathcal{L}_1,\mathcal{L}_2,\ldots,\mathcal{L}_r) \leq \mathcal{L}^u. \qquad \blacksquare$$

The CFEBOMG operator defined in section III-C1 only includes the aggregated CFNs and their interrelationships but not their importance over each other. We should improve this shortcoming by considering the weight information of each aggregated CFN while defining new aggregation operators. A series of CF BOM geometric operators based on the Einstein operational laws, including the weight vector of the aggregated CFNs, are proposed in the following subsections.

2) CF Einstein BOM weighted geometric operator: Here, we define CF Einstein BOM weighted geometric (CFEBOMWG) operator including the weights of the aggregated CFNs.

Definition 3.6

Let $\mathcal{L}_j = \langle \mu_{\mathcal{L}_j}, \eta_{\mathcal{L}_j}, \nu_{\mathcal{L}_j}\rangle (j = 1,2,\ldots,r)$ be a set of CFNs and $p,q \geq 0$. Then the CFEBOMWG operator with the associated weight vector $\omega_j = (\omega_1,\omega_2,\ldots,\omega_r)^T$ satisfying $\sum_{j=1}^{r}\omega_j = 1$ is a mapping $CFEBOMWG_\omega : \mathcal{L}^r \rightarrow \mathcal{L}$ such that

$$CFEBOMWG_\omega{}^{p,q}(\mathcal{L}_1,\mathcal{L}_2,\ldots,\mathcal{L}_r)$$
$$= \frac{1}{p+q}\left[\bigotimes_{E\ j,k=1;j\neq k}^{r}\left(p(\mathcal{L}_j{}^{r\omega_j}) \bigoplus_E q(\mathcal{L}_k{}^{r\omega_k})\right)\right]^{\frac{1}{r(r-1)}} \tag{9}$$

Theorem 3.2

The aggregated value obtained using CFEBOMWG operator is also a CFN and can be expressed as follows:

$$CFEBOMWG_\omega{}^{p,q}(\mathcal{L}_1,\mathcal{L}_2,\ldots,\mathcal{L}_r)$$
$$= \frac{1}{p+q}\left[\bigotimes_{E\ j,k=1;j\neq k}^{r}\left(p(\mathcal{L}_j{}^{r\omega_j}) \bigoplus_E q(\mathcal{L}_k{}^{r\omega_k})\right)\right]^{\frac{1}{r(r-1)}}$$
$$= \left\langle \sqrt[3]{\frac{\phi^{\frac{1}{p+q}}-\varphi^{\frac{1}{p+q}}}{\phi^{\frac{1}{p+q}}+\varphi^{\frac{1}{p+q}}}}, \sqrt[3]{\frac{2\chi^{\frac{1}{p+q}}}{\psi^{\frac{1}{p+q}}+\chi^{\frac{1}{p+q}}}}, \sqrt[3]{\frac{2\rho^{\frac{1}{p+q}}}{\varrho^{\frac{1}{p+q}}+\rho^{\frac{1}{p+q}}}} \right\rangle \tag{10}$$

$$\phi = (2B_{jk}+2C_{jk}-C_{jk})^{\frac{1}{r(r-1)}} + 3\,(A_{jk})^{\frac{1}{r(r-1)}}$$
$$\varphi = (2B_{jk}+2C_{jk}-A_{jk})^{\frac{1}{r(r-1)}} - (A_{jk})^{\frac{1}{r(r-1)}}$$
$$\chi = (E_{jk}+F_{jk}+D_{jk})^{\frac{1}{r(r-1)}} - (E_{jk}+F_{jk}-D_{jk})^{\frac{1}{r(r-1)}}$$
$$\psi = (E_{jk}+F_{jk}+D_{jk})^{\frac{1}{r(r-1)}} + 3\,(E_{jk}+F_{jk}-D_{jk})^{\frac{1}{r(r-1)}}$$
$$\rho = (H_{jk}+I_{jk}+G_{jk})^{\frac{1}{r(r-1)}} - (H_{jk}+I_{jk}-G_{jk})^{\frac{1}{r(r-1)}}$$
$$\varrho = (H_{jk}+I_{jk}+G_{jk})^{\frac{1}{r(r-1)}} + 3\,(H_{jk}+I_{jk}-G_{jk})^{\frac{1}{r(r-1)}}$$

and

$$A_{jk} = \prod_{j,k=1;j\neq k}^{r} 2\left\{\begin{array}{l}\left[(2-\mu^3_{\mathcal{L}_j})^{r\omega_j}+3(\mu^3_{\mathcal{L}_j})^{r\omega_j}\right]^p\\ \left[(2-\mu^3_{\mathcal{L}_k})^{r\omega_k}+3(\mu^3_{\mathcal{L}_k})^{r\omega_k}\right]^q\\ -\left[(2-\mu^3_{\mathcal{L}_j})^{r\omega_j}-(\mu^3_{\mathcal{L}_j})^{r\omega_j}\right]^p\\ \left[(2-\mu^3_{\mathcal{L}_k})^{r\omega_k}-(\mu^3_{\mathcal{L}_k})^{r\omega_k}\right]^q\end{array}\right\}$$

$$B_{jk} = \prod_{j,k=1;j\neq k}^{r} 2\left\{\begin{array}{l} \left[(2-\mu^3_{\mathcal{L}_j})^{r\omega_j}+3(\mu^3_{\mathcal{L}_j})^{r\omega_j}\right]^p \\ \left[(2-\mu^3_{\mathcal{L}_k})^{r\omega_k}+3(\mu^3_{\mathcal{L}_k})^{r\omega_k}\right]^q \\ +\left[(2-\mu^3_{\mathcal{L}_j})^{r\omega_j}-(\mu^3_{\mathcal{L}_j})^{r\omega_j}\right]^p \\ \left[(2-\mu^3_{\mathcal{L}_k})^{r\omega_k}-(\mu^3_{\mathcal{L}_k})^{r\omega_k}\right]^q \end{array}\right\}$$

$$C_{jk} = \prod_{j,k=1;j\neq k}^{r} 2\left[\begin{array}{l} (2-\mu^3_{\mathcal{L}_j})^{r\omega_j} \\ -(\mu^3_{\mathcal{L}_j})^{r\omega_j} \end{array}\right]^p \left[\begin{array}{l} (2-\mu^3_{\mathcal{L}_k})^{r\omega_k} \\ -(\mu^3_{\mathcal{L}_k})^{r\omega_k} \end{array}\right]^q$$

$$D_{jk} = \sum_{j,k=1;j\neq k}^{r} 2\left[\begin{array}{l} (1+\eta^3_{\mathcal{L}_j})^{r\omega_j} \\ -(1-\eta^3_{\mathcal{L}_j})^{r\omega_j} \end{array}\right]^p \left[\begin{array}{l} (1+\eta^3_{\mathcal{L}_k})^{r\omega_k} \\ -(1-\eta^3_{\mathcal{L}_k})^{r\omega_k} \end{array}\right]^q \left\{\begin{array}{l} \left[(1+\eta^3_{\mathcal{L}_k})^{r\omega_k}+3(1-\eta^3_{\mathcal{L}_k})^{r\omega_k}\right]^p \\ \left[(1+\eta^3_{\mathcal{L}_j})^{r\omega_j}+3(1-\eta^3_{\mathcal{L}_j})^{r\omega_j}\right]^q \\ +\left[(1+\eta^3_{\mathcal{L}_k})^{r\omega_k}-(1-\eta^3_{\mathcal{L}_k})^{r\omega_k}\right]^p \\ \left[(1+\eta^3_{\mathcal{L}_j})^{r\omega_j}-(1-\eta^3_{\mathcal{L}_j})^{r\omega_j}\right]^q \end{array}\right\}$$

$$E_{jk} = \prod_{j,k=1;j\neq k}^{r} 2\left\{\begin{array}{l} \left[(1+\eta^3_{\mathcal{L}_j})^{r\omega_j}+3(1-\eta^3_{\mathcal{L}_j})^{r\omega_j}\right]^p \\ \left[(1+\eta^3_{\mathcal{L}_k})^{r\omega_k}+3(1-\eta^3_{\mathcal{L}_k})^{r\omega_k}\right]^q \\ +\left[(1+\eta^3_{\mathcal{L}_j})^{r\omega_j}-(1-\eta^3_{\mathcal{L}_j})^{r\omega_j}\right]^p \\ \left[(1+\eta^3_{\mathcal{L}_k})^{r\omega_k}-(1-\eta^3_{\mathcal{L}_k})^{r\omega_k}\right]^q \end{array}\right\}$$

$$F_{jk} = \prod_{j,k=1;j\neq k}^{r} 2\left[\begin{array}{l} (1+\eta^3_{\mathcal{L}_j})^{r\omega_j} \\ -(1-\eta^3_{\mathcal{L}_j})^{r\omega_j} \end{array}\right]^p \left[\begin{array}{l} (1+\eta^3_{\mathcal{L}_k})^{r\omega_k} \\ -(1-\eta^3_{\mathcal{L}_k})^{r\omega_k} \end{array}\right]^q$$

$$G_{jk} = \sum_{j,k=1;j\neq k}^{r} 2\left[\begin{array}{l} (1+\nu^3_{\mathcal{L}_j})^{r\omega_j} \\ -(1-\nu^3_{\mathcal{L}_j})^{r\omega_j} \end{array}\right]^p \left[\begin{array}{l} (1+\nu^3_{\mathcal{L}_k})^{r\omega_k} \\ -(1-\nu^3_{\mathcal{L}_k})^{r\omega_k} \end{array}\right]^q \left\{\begin{array}{l} \left[(1+\nu^3_{\mathcal{L}_k})^{r\omega_k}+3(1-\nu^3_{\mathcal{L}_k})^{r\omega_k}\right]^p \\ \left[(1+\nu^3_{\mathcal{L}_j})^{r\omega_j}+3(1-\nu^3_{\mathcal{L}_j})^{r\omega_j}\right]^q \\ +\left[(1+\nu^3_{\mathcal{L}_k})^{r\omega_k}-(1-\nu^3_{\mathcal{L}_k})^{r\omega_k}\right]^p \\ \left[(1+\nu^3_{\mathcal{L}_j})^{r\omega_j}-(1-\nu^3_{\mathcal{L}_j})^{r\omega_j}\right]^q \end{array}\right\}$$

$$H_{jk} = \prod_{j,k=1;j\neq k}^{r} 2\left\{\begin{array}{l} \left[(1+\nu^3_{\mathcal{L}_j})^{r\omega_j}+3(1-\nu^3_{\mathcal{L}_j})^{r\omega_j}\right]^p \\ \left[(1+\nu^3_{\mathcal{L}_k})^{r\omega_k}+3(1-\nu^3_{\mathcal{L}_k})^{r\omega_k}\right]^q \\ +\left[(1+\nu^3_{\mathcal{L}_j})^{r\omega_j}-(1-\nu^3_{\mathcal{L}_j})^{r\omega_j}\right]^p \\ \left[(1+\nu^3_{\mathcal{L}_k})^{r\omega_k}-(1-\nu^3_{\mathcal{L}_k})^{r\omega_k}\right]^q \end{array}\right\}$$

$$I_{jk} = \prod_{j,k=1;j\neq k}^{r} 2\left[\begin{array}{l} (1+\nu^3_{\mathcal{L}_j})^{r\omega_j} \\ -(1-\nu^3_{\mathcal{L}_j})^{r\omega_j} \end{array}\right]^p \left[\begin{array}{l} (1+\nu^3_{\mathcal{L}_k})^{r\omega_k} \\ -(1-\nu^3_{\mathcal{L}_k})^{r\omega_k} \end{array}\right]^q.$$

Proof. According to Definition III-A, we have

$$\mathcal{L}_j{}^{r\omega_j} = \left\langle \begin{array}{c} \sqrt[3]{\frac{2(\mu^3_{\mathcal{L}_j})^{r\omega_j}}{(2-\mu^3_{\mathcal{L}_j})^{r\omega_j}+(\mu^3_{\mathcal{L}_j})^{r\omega_j}}}, \\ \sqrt[3]{\frac{(1+\eta^3_{\mathcal{L}_j})^{r\omega_j}-(1-\eta^3_{\mathcal{L}_j})^{r\omega_j}}{(1+\eta^3_{\mathcal{L}_j})^{r\omega_j}+(1-\eta^3_{\mathcal{L}_j})^{r\omega_j}}}, \\ \sqrt[3]{\frac{(1+\nu^3_{\mathcal{L}_j})^{r\omega_j}-(1-\nu^3_{\mathcal{L}_j})^{r\omega_j}}{(1+\nu^3_{\mathcal{L}_j})^{r\omega_j}+(1-\nu^3_{\mathcal{L}_j})^{r\omega_j}}} \end{array} \right\rangle$$

$$\mathcal{L}_k{}^{r\omega_k} = \left\langle \begin{array}{c} \sqrt[3]{\frac{2(\mu^3_{\mathcal{L}_k})^{r\omega_k}}{(2-\mu^3_{\mathcal{L}_k})^{r\omega_k}+(\mu^3_{\mathcal{L}_k})^{r\omega_k}}}, \\ \sqrt[3]{\frac{(1+\eta^3_{\mathcal{L}_k})^{r\omega_k}-(1-\eta^3_{\mathcal{L}_k})^{r\omega_k}}{(1+\eta^3_{\mathcal{L}_k})^{r\omega_k}+(1-\eta^3_{\mathcal{L}_k})^{r\omega_k}}}, \\ \sqrt[3]{\frac{(1+\nu^3_{\mathcal{L}_k})^{r\omega_k}-(1-\nu^3_{\mathcal{L}_k})^{r\omega_k}}{(1+\nu^3_{\mathcal{L}_k})^{r\omega_k}+(1-\nu^3_{\mathcal{L}_k})^{r\omega_k}}} \end{array} \right\rangle$$

Now,

$$p(\mathcal{L}_j{}^{r\omega_j}) = \left\langle \begin{array}{c} \sqrt[3]{\frac{[(2-\mu^3_{\mathcal{L}_j})^{r\omega_j}+3(\mu^3_{\mathcal{L}_j})^{r\omega_j}]^p - [(2-\mu^3_{\mathcal{L}_j})^{r\omega_j}-(\mu^3_{\mathcal{L}_j})^{r\omega_j}]^p}{[(2-\mu^3_{\mathcal{L}_j})^{r\omega_j}+3(\mu^3_{\mathcal{L}_j})^{r\omega_j}]^p + [(2-\mu^3_{\mathcal{L}_j})^{r\omega_j}-(\mu^3_{\mathcal{L}_j})^{r\omega_j}]^p}}, \\ \sqrt[3]{\frac{2[(1+\eta^3_{\mathcal{L}_j})^{r\omega_j}-(1-\eta^3_{\mathcal{L}_j})^{r\omega_j}]^p}{[(1+\eta^3_{\mathcal{L}_j})^{r\omega_j}+3(1-\eta^3_{\mathcal{L}_j})^{r\omega_j}]^p + [(1+\eta^3_{\mathcal{L}_j})^{r\omega_j}-(1-\eta^3_{\mathcal{L}_j})^{r\omega_j}]^p}}, \\ \sqrt[3]{\frac{2[(1+\nu^3_{\mathcal{L}_j})^{r\omega_j}-(1-\nu^3_{\mathcal{L}_j})^{r\omega_j}]^p}{[(1+\nu^3_{\mathcal{L}_j})^{r\omega_j}+3(1-\nu^3_{\mathcal{L}_j})^{r\omega_j}]^p + [(1+\nu^3_{\mathcal{L}_j})^{r\omega_j}-(1-\nu^3_{\mathcal{L}_j})^{r\omega_j}]^p}} \end{array} \right\rangle$$

Silmilarly,

$$q(\mathcal{L}_k{}^{r\omega_k}) = \left\langle \begin{array}{c} \sqrt[3]{\frac{[(2-\mu^3_{\mathcal{L}_k})^{r\omega_k}+3(\mu^3_{\mathcal{L}_k})^{r\omega_k}]^q - [(2-\mu^3_{\mathcal{L}_k})^{r\omega_k}-(\mu^3_{\mathcal{L}_k})^{r\omega_k}]^q}{[(2-\mu^3_{\mathcal{L}_k})^{r\omega_k}+3(\mu^3_{\mathcal{L}_k})^{r\omega_k}]^q + [(2-\mu^3_{\mathcal{L}_k})^{r\omega_k}-(\mu^3_{\mathcal{L}_k})^{r\omega_k}]^q}}, \\ \sqrt[3]{\frac{2[(1+\eta^3_{\mathcal{L}_k})^{r\omega_k}-(1-\eta^3_{\mathcal{L}_k})^{r\omega_k}]^q}{[(1+\eta^3_{\mathcal{L}_k})^{r\omega_k}+3(1-\eta^3_{\mathcal{L}_k})^{r\omega_k}]^q + [(1+\eta^3_{\mathcal{L}_k})^{r\omega_k}-(1-\eta^3_{\mathcal{L}_k})^{r\omega_k}]^q}}, \\ \sqrt[3]{\frac{2[(1+\nu^3_{\mathcal{L}_k})^{r\omega_k}-(1-\nu^3_{\mathcal{L}_k})^{r\omega_k}]^q}{[(1+\nu^3_{\mathcal{L}_k})^{r\omega_k}+3(1-\nu^3_{\mathcal{L}_k})^{r\omega_k}]^q + [(1+\nu^3_{\mathcal{L}_k})^{r\omega_k}-(1-\nu^3_{\mathcal{L}_k})^{r\omega_k}]^q}} \end{array} \right\rangle$$

and

$$p(\mathcal{L}_j{}^{r\omega_j}) \bigoplus_E q(\mathcal{L}_k{}^{r\omega_k}) =$$

$$\left\langle \begin{array}{c} \sqrt[3]{\frac{\left[(2-\mu^3_{\mathcal{L}_j})^{r\omega_j}+3(\mu^3_{\mathcal{L}_j})^{r\omega_j}\right]^p \left[(2-\mu^3_{\mathcal{L}_k})^{r\omega_k}+3(\mu^3_{\mathcal{L}_k})^{r\omega_k}\right]^q - \left[(2-\mu^3_{\mathcal{L}_j})^{r\omega_j}-(\mu^3_{\mathcal{L}_j})^{r\omega_j}\right]^p \left[(2-\mu^3_{\mathcal{L}_k})^{r\omega_k}-(\mu^3_{\mathcal{L}_k})^{r\omega_k}\right]^q}{\left[(2-\mu^3_{\mathcal{L}_j})^{r\omega_j}+3(\mu^3_{\mathcal{L}_j})^{r\omega_j}\right]^p \left[(2-\mu^3_{\mathcal{L}_k})^{r\omega_k}+3(\mu^3_{\mathcal{L}_k})^{r\omega_k}\right]^q + \left[(2-\mu^3_{\mathcal{L}_j})^{r\omega_j}-(\mu^3_{\mathcal{L}_j})^{r\omega_j}\right]^p \left[(2-\mu^3_{\mathcal{L}_k})^{r\omega_k}-(\mu^3_{\mathcal{L}_k})^{r\omega_k}\right]^q}}, \\ \sqrt[3]{\frac{2\left[(1+\eta^3_{\mathcal{L}_j})^{r\omega_j}-(1-\eta^3_{\mathcal{L}_j})^{r\omega_j}\right]^p \left[(1+\eta^3_{\mathcal{L}_k})^{r\omega_k}-(1-\eta^3_{\mathcal{L}_k})^{r\omega_k}\right]^q}{\left[(1+\eta^3_{\mathcal{L}_j})^{r\omega_j}+3(1-\eta^3_{\mathcal{L}_j})^{r\omega_j}\right]^p \left[(1+\eta^3_{\mathcal{L}_k})^{r\omega_k}+3(1-\eta^3_{\mathcal{L}_k})^{r\omega_k}\right]^q + \left[(1+\eta^3_{\mathcal{L}_j})^{r\omega_j}-(1-\eta^3_{\mathcal{L}_j})^{r\omega_j}\right]^p \left[(1+\eta^3_{\mathcal{L}_k})^{r\omega_k}-(1-\eta^3_{\mathcal{L}_k})^{r\omega_k}\right]^q}}, \\ \sqrt[3]{\frac{2\left[(1+\nu^3_{\mathcal{L}_j})^{r\omega_j}-(1-\nu^3_{\mathcal{L}_j})^{r\omega_j}\right]^p \left[(1+\nu^3_{\mathcal{L}_k})^{r\omega_k}-(1-\nu^3_{\mathcal{L}_k})^{r\omega_k}\right]^q}{\left[(1+\nu^3_{\mathcal{L}_j})^{r\omega_j}+3(1-\nu^3_{\mathcal{L}_j})^{r\omega_j}\right]^p \left[(1+\nu^3_{\mathcal{L}_k})^{r\omega_k}+3(1-\nu^3_{\mathcal{L}_k})^{r\omega_k}\right]^q + \left[(1+\nu^3_{\mathcal{L}_j})^{r\omega_j}-(1-\nu^3_{\mathcal{L}_j})^{r\omega_j}\right]^p \left[(1+\nu^3_{\mathcal{L}_k})^{r\omega_k}-(1-\nu^3_{\mathcal{L}_k})^{r\omega_k}\right]^q}} \end{array} \right\rangle$$

Therefore,

$$\bigotimes_{E\ j,k=1;j\neq k}^{r}\left(p(\mathcal{L}_j{}^{r\omega_j})\bigoplus_{E} q(\mathcal{L}_k{}^{r\omega_k})\right)=$$

$$\left\langle \sqrt[3]{\frac{\prod\limits_{j,k=1;j\neq k}^{r}2\left\{\begin{array}{l}\left[(2-\mu^3_{\mathcal{L}_j})^{r\omega_j}+3(\mu^3_{\mathcal{L}_j})^{r\omega_j}\right]^p\\ \left[(2-\mu^3_{\mathcal{L}_k})^{r\omega_k}+3(\mu^3_{\mathcal{L}_k})^{r\omega_k}\right]^q\\ -\left[(2-\mu^3_{\mathcal{L}_j})^{r\omega_j}-(\mu^3_{\mathcal{L}_j})^{r\omega_j}\right]^p\\ \left[(2-\mu^3_{\mathcal{L}_k})^{r\omega_k}-(\mu^3_{\mathcal{L}_k})^{r\omega_k}\right]^q\end{array}\right\}}{\prod\limits_{j,k=1;j\neq k}^{r}2\left\{\begin{array}{l}\left[(2-\mu^3_{\mathcal{L}_j})^{r\omega_j}+3(\mu^3_{\mathcal{L}_j})^{r\omega_j}\right]^p\\ \left[(2-\mu^3_{\mathcal{L}_k})^{r\omega_k}+3(\mu^3_{\mathcal{L}_k})^{r\omega_k}\right]^q\\ +\left[(2-\mu^3_{\mathcal{L}_j})^{r\omega_j}-(\mu^3_{\mathcal{L}_j})^{r\omega_j}\right]^p\\ \left[(2-\mu^3_{\mathcal{L}_k})^{r\omega_k}-(\mu^3_{\mathcal{L}_k})^{r\omega_k}\right]^q\end{array}\right\}+\prod\limits_{j,k=1;j\neq k}^{r}2\left\{\begin{array}{l}\left[(2-\mu^3_{\mathcal{L}_j})^{r\omega_j}-(\mu^3_{\mathcal{L}_j})^{r\omega_j}\right]^p\\ \left[(2-\mu^3_{\mathcal{L}_k})^{r\omega_k}-(\mu^3_{\mathcal{L}_k})^{r\omega_k}\right]^q\end{array}\right\}}},\right.$$

$$\sqrt[3]{\frac{\begin{array}{l}\sum\limits_{j,k=1;j\neq k}^{r}2\left[(1+\eta^3_{\mathcal{L}_j})^{r\omega_j}-(1-\eta^3_{\mathcal{L}_j})^{r\omega_j}\right]^p\\ \left[(1+\eta^3_{\mathcal{L}_k})^{r\omega_k}-(1-\eta^3_{\mathcal{L}_k})^{r\omega_k}\right]^q\\ \left\{\begin{array}{l}\left[(1+\eta^3_{\mathcal{L}_k})^{r\omega_k}+3(1-\eta^3_{\mathcal{L}_k})^{r\omega_k}\right]^p\\ \left[(1+\eta^3_{\mathcal{L}_j})^{r\omega_j}+3(1-\eta^3_{\mathcal{L}_j})^{r\omega_j}\right]^q\\ +\left[(1+\eta^3_{\mathcal{L}_k})^{r\omega_k}-(1-\eta^3_{\mathcal{L}_k})^{r\omega_k}\right]^p\\ \left[(1+\eta^3_{\mathcal{L}_j})^{r\omega_j}-(1-\eta^3_{\mathcal{L}_j})^{r\omega_j}\right]^q\end{array}\right\}\end{array}}{\prod\limits_{j,k=1;j\neq k}^{r}2\left\{\begin{array}{l}\left[(1+\eta^3_{\mathcal{L}_j})^{r\omega_j}+3(1-\eta^3_{\mathcal{L}_j})^{r\omega_j}\right]^p\\ \left[(1+\eta^3_{\mathcal{L}_k})^{r\omega_k}+3(1-\eta^3_{\mathcal{L}_k})^{r\omega_k}\right]^q\\ +\left[(1+\eta^3_{\mathcal{L}_j})^{r\omega_j}-(1-\eta^3_{\mathcal{L}_j})^{r\omega_j}\right]^p\\ \left[(1+\eta^3_{\mathcal{L}_k})^{r\omega_k}-(1-\eta^3_{\mathcal{L}_k})^{r\omega_k}\right]^q\end{array}\right\}+\prod\limits_{j,k=1;j\neq k}^{r}2\left\{\begin{array}{l}\left[(1+\eta^3_{\mathcal{L}_j})^{r\omega_j}-(1-\eta^3_{\mathcal{L}_j})^{r\omega_j}\right]^p\\ \left[(1+\eta^3_{\mathcal{L}_k})^{r\omega_k}-(1-\eta^3_{\mathcal{L}_k})^{r\omega_k}\right]^q\end{array}\right\}}},$$

$$\left.\sqrt[3]{\frac{\begin{array}{l}\sum\limits_{j,k=1;j\neq k}^{r}2\left[(1+\nu^3_{\mathcal{L}_j})^{r\omega_j}-(1-\nu^3_{\mathcal{L}_j})^{r\omega_j}\right]^p\\ \left[(1+\nu^3_{\mathcal{L}_k})^{r\omega_k}-(1-\nu^3_{\mathcal{L}_k})^{r\omega_k}\right]^q\\ \left\{\begin{array}{l}\left[(1+\nu^3_{\mathcal{L}_k})^{r\omega_k}+3(1-\nu^3_{\mathcal{L}_k})^{r\omega_k}\right]^p\\ \left[(1+\nu^3_{\mathcal{L}_j})^{r\omega_j}+3(1-\nu^3_{\mathcal{L}_j})^{r\omega_j}\right]^q\\ +\left[(1+\nu^3_{\mathcal{L}_k})^{r\omega_k}-(1-\nu^3_{\mathcal{L}_k})^{r\omega_k}\right]^p\\ \left[(1+\nu^3_{\mathcal{L}_j})^{r\omega_j}-(1-\nu^3_{\mathcal{L}_j})^{r\omega_j}\right]^q\end{array}\right\}\end{array}}{\prod\limits_{j,k=1;j\neq k}^{r}2\left\{\begin{array}{l}\left[(1+\nu^3_{\mathcal{L}_j})^{r\omega_j}+3(1-\nu^3_{\mathcal{L}_j})^{r\omega_j}\right]^p\\ \left[(1+\nu^3_{\mathcal{L}_k})^{r\omega_k}+3(1-\nu^3_{\mathcal{L}_k})^{r\omega_k}\right]^q\\ +\left[(1+\nu^3_{\mathcal{L}_j})^{r\omega_j}-(1-\nu^3_{\mathcal{L}_j})^{r\omega_j}\right]^p\\ \left[(1+\nu^3_{\mathcal{L}_k})^{r\omega_k}-(1-\nu^3_{\mathcal{L}_k})^{r\omega_k}\right]^q\end{array}\right\}+\prod\limits_{j,k=1;j\neq k}^{r}2\left\{\begin{array}{l}\left[(1+\nu^3_{\mathcal{L}_j})^{r\omega_j}-(1-\nu^3_{\mathcal{L}_j})^{r\omega_j}\right]^p\\ \left[(1+\nu^3_{\mathcal{L}_k})^{r\omega_k}-(1-\nu^3_{\mathcal{L}_k})^{r\omega_k}\right]^q\end{array}\right\}}}\right\rangle$$

Let

$$A_{jk}=\prod_{j,k=1;j\neq k}^{r}2\left\{\begin{array}{l}\left[(2-\mu^3_{\mathcal{L}_j})^{r\omega_j}+3(\mu^3_{\mathcal{L}_j})^{r\omega_j}\right]^p\\ \left[(2-\mu^3_{\mathcal{L}_k})^{r\omega_k}+3(\mu^3_{\mathcal{L}_k})^{r\omega_k}\right]^q\\ -\left[(2-\mu^3_{\mathcal{L}_j})^{r\omega_j}-(\mu^3_{\mathcal{L}_j})^{r\omega_j}\right]^p\\ \left[(2-\mu^3_{\mathcal{L}_k})^{r\omega_k}-(\mu^3_{\mathcal{L}_k})^{r\omega_k}\right]^q\end{array}\right\}$$

$$B_{jk}=\prod_{j,k=1;j\neq k}^{r}2\left\{\begin{array}{l}\left[(2-\mu^3_{\mathcal{L}_j})^{r\omega_j}+3(\mu^3_{\mathcal{L}_j})^{r\omega_j}\right]^p\\ \left[(2-\mu^3_{\mathcal{L}_k})^{r\omega_k}+3(\mu^3_{\mathcal{L}_k})^{r\omega_k}\right]^q\\ +\left[(2-\mu^3_{\mathcal{L}_j})^{r\omega_j}-(\mu^3_{\mathcal{L}_j})^{r\omega_j}\right]^p\\ \left[(2-\mu^3_{\mathcal{L}_k})^{r\omega_k}-(\mu^3_{\mathcal{L}_k})^{r\omega_k}\right]^q\end{array}\right\}$$

$$C_{jk}=\prod_{j,k=1;j\neq k}^{r}2\left[\begin{array}{l}(2-\mu^3_{\mathcal{L}_j})^{r\omega_j}\\ -(\mu^3_{\mathcal{L}_j})^{r\omega_j}\end{array}\right]^p\left[\begin{array}{l}(2-\mu^3_{\mathcal{L}_k})^{r\omega_k}\\ -(\mu^3_{\mathcal{L}_k})^{r\omega_k}\end{array}\right]^q$$

$$D_{jk}=\sum_{j,k=1;j\neq k}^{r}2\left[\begin{array}{l}(1+\eta^3_{\mathcal{L}_j})^{r\omega_j}\\ -(1-\eta^3_{\mathcal{L}_j})^{r\omega_j}\end{array}\right]^p\left[\begin{array}{l}(1+\eta^3_{\mathcal{L}_k})^{r\omega_k}\\ -(1-\eta^3_{\mathcal{L}_k})^{r\omega_k}\end{array}\right]^q\left\{\begin{array}{l}\left[(1+\eta^3_{\mathcal{L}_k})^{r\omega_k}+3(1-\eta^3_{\mathcal{L}_k})^{r\omega_k}\right]^p\\ \left[(1+\eta^3_{\mathcal{L}_j})^{r\omega_j}+3(1-\eta^3_{\mathcal{L}_j})^{r\omega_j}\right]^q\\ +\left[(1+\eta^3_{\mathcal{L}_k})^{r\omega_k}-(1-\eta^3_{\mathcal{L}_k})^{r\omega_k}\right]^p\\ \left[(1+\eta^3_{\mathcal{L}_j})^{r\omega_j}-(1-\eta^3_{\mathcal{L}_j})^{r\omega_j}\right]^q\end{array}\right\}$$

$$E_{jk}=\prod_{j,k=1;j\neq k}^{r}2\left\{\begin{array}{l}\left[(1+\eta^3_{\mathcal{L}_j})^{r\omega_j}+3(1-\eta^3_{\mathcal{L}_j})^{r\omega_j}\right]^p\\ \left[(1+\eta^3_{\mathcal{L}_k})^{r\omega_k}+3(1-\eta^3_{\mathcal{L}_k})^{r\omega_k}\right]^q\\ +\left[(1+\eta^3_{\mathcal{L}_j})^{r\omega_j}-(1-\eta^3_{\mathcal{L}_j})^{r\omega_j}\right]^p\\ \left[(1+\eta^3_{\mathcal{L}_k})^{r\omega_k}-(1-\eta^3_{\mathcal{L}_k})^{r\omega_k}\right]^q\end{array}\right\}$$

$$F_{jk}=\prod_{j,k=1;j\neq k}^{r}2\left[\begin{array}{l}(1+\eta^3_{\mathcal{L}_j})^{r\omega_j}\\ -(1-\eta^3_{\mathcal{L}_j})^{r\omega_j}\end{array}\right]^p\left[\begin{array}{l}(1+\eta^3_{\mathcal{L}_k})^{r\omega_k}\\ -(1-\eta^3_{\mathcal{L}_k})^{r\omega_k}\end{array}\right]^q$$

$$G_{jk}=\sum_{j,k=1;j\neq k}^{r}2\left[\begin{array}{l}(1+\nu^3_{\mathcal{L}_j})^{r\omega_j}\\ -(1-\nu^3_{\mathcal{L}_j})^{r\omega_j}\end{array}\right]^p\left[\begin{array}{l}(1+\nu^3_{\mathcal{L}_k})^{r\omega_k}\\ -(1-\nu^3_{\mathcal{L}_k})^{r\omega_k}\end{array}\right]^q\left\{\begin{array}{l}\left[(1+\nu^3_{\mathcal{L}_k})^{r\omega_k}+3(1-\nu^3_{\mathcal{L}_k})^{r\omega_k}\right]^p\\ \left[(1+\nu^3_{\mathcal{L}_j})^{r\omega_j}+3(1-\nu^3_{\mathcal{L}_j})^{r\omega_j}\right]^q\\ +\left[(1+\nu^3_{\mathcal{L}_k})^{r\omega_k}-(1-\nu^3_{\mathcal{L}_k})^{r\omega_k}\right]^p\\ \left[(1+\nu^3_{\mathcal{L}_j})^{r\omega_j}-(1-\nu^3_{\mathcal{L}_j})^{r\omega_j}\right]^q\end{array}\right\}$$

$$H_{jk}=\prod_{j,k=1;j\neq k}^{r}2\left\{\begin{array}{l}\left[(1+\nu^3_{\mathcal{L}_j})^{r\omega_j}+3(1-\nu^3_{\mathcal{L}_j})^{r\omega_j}\right]^p\\ \left[(1+\nu^3_{\mathcal{L}_k})^{r\omega_k}+3(1-\nu^3_{\mathcal{L}_k})^{r\omega_k}\right]^q\\ +\left[(1+\nu^3_{\mathcal{L}_j})^{r\omega_j}-(1-\nu^3_{\mathcal{L}_j})^{r\omega_j}\right]^p\\ \left[(1+\nu^3_{\mathcal{L}_k})^{r\omega_k}-(1-\nu^3_{\mathcal{L}_k})^{r\omega_k}\right]^q\end{array}\right\}$$

$$I_{jk}=\prod_{j,k=1;j\neq k}^{r}2\left[\begin{array}{l}(1+\nu^3_{\mathcal{L}_j})^{r\omega_j}\\ -(1-\nu^3_{\mathcal{L}_j})^{r\omega_j}\end{array}\right]^p\left[\begin{array}{l}(1+\nu^3_{\mathcal{L}_k})^{r\omega_k}\\ -(1-\nu^3_{\mathcal{L}_k})^{r\omega_k}\end{array}\right]^q.$$

Then,

$$\bigotimes_{E\ j,k=1;j\neq k}^{r}\left(p(\mathcal{L}_j{}^{r\omega_j})\bigoplus_{E} q(\mathcal{L}_k{}^{r\omega_k})\right)$$

$$=\left\langle\sqrt[3]{\frac{A_{jk}}{B_{jk}+C_{jk}}},\sqrt[3]{\frac{D_{jk}}{E_{jk}+F_{jk}}},\sqrt[3]{\frac{G_{jk}}{H_{jk}+I_{jk}}}\right\rangle$$

Thereafter,

$$\left[\bigotimes_{E\ j,k=1;j\neq k}^{r}\left(p(\mathcal{L}_j{}^{r\omega_j})\bigoplus_{E} q(\mathcal{L}_k{}^{r\omega_k})\right)\right]^{\frac{1}{r(r-1)}}$$

$$=\left\langle\begin{array}{l}\sqrt[3]{\frac{(B_{jk}+C_{jk}+A_{jk})^{\frac{1}{r(r-1)}}-(B_{jk}+C_{jk}-A_{jk})^{\frac{1}{r(r-1)}}}{(B_{jk}+C_{jk}+A_{jk})^{\frac{1}{r(r-1)}}+(B_{jk}+C_{jk}-A_{jk})^{\frac{1}{r(r-1)}}}},\\ \sqrt[3]{\frac{2(D_{jk})^{\frac{1}{r(r-1)}}}{(2E_{jk}+2F_{jk}-D_{jk})^{\frac{1}{r(r-1)}}+(D_{jk})^{\frac{1}{r(r-1)}}}},\\ \sqrt[3]{\frac{2(G_{jk})^{\frac{1}{r(r-1)}}}{(2H_{jk}+2I_{jk}-G_{jk})^{\frac{1}{r(r-1)}}+(G_{jk})^{\frac{1}{r(r-1)}}}}\end{array}\right\rangle$$

And so,

$$\frac{1}{p+q}\left[\bigotimes_{E\ j,k=1;j\neq k}^{r}\left(p(\mathcal{L}_j{}^{r\omega_j})\bigoplus_{E} q(\mathcal{L}_k{}^{r\omega_k})\right)\right]^{\frac{1}{r(r-1)}}$$

$$=\left\langle \sqrt[3]{\frac{\left[\begin{array}{l}(2B_{jk}+2C_{jk}-A_{jk})^{\frac{1}{r(r-1)}}\\ +3\left(A_{jk}\right)^{\frac{1}{r(r-1)}}\end{array}\right]^{\frac{1}{p+q}}-\left[\begin{array}{l}(2B_{jk}+2C_{jk}-A_{jk})^{\frac{1}{r(r-1)}}\\ -\left(A_{jk}\right)^{\frac{1}{r(r-1)}}\end{array}\right]^{\frac{1}{p+q}}}{\left[\begin{array}{l}(2B_{jk}+2C_{jk}-A_{jk})^{\frac{1}{r(r-1)}}\\ +3\left(A_{jk}\right)^{\frac{1}{r(r-1)}}\end{array}\right]^{\frac{1}{p+q}}+\left[\begin{array}{l}(2B_{jk}+2C_{jk}-A_{jk})^{\frac{1}{r(r-1)}}\\ -\left(A_{jk}\right)^{\frac{1}{r(r-1)}}\end{array}\right]^{\frac{1}{p+q}}}}, \sqrt[3]{\frac{2\left[\begin{array}{l}(E_{jk}+F_{jk}+D_{jk})^{\frac{1}{r(r-1)}}\\ -(E_{jk}+F_{jk}-D_{jk})^{\frac{1}{r(r-1)}}\end{array}\right]^{\frac{1}{p+q}}}{\left[\begin{array}{l}(E_{jk}+F_{jk}+D_{jk})^{\frac{1}{r(r-1)}}\\ +3\left(E_{jk}+F_{jk}-D_{jk}\right)^{\frac{1}{r(r-1)}}\end{array}\right]^{\frac{1}{p+q}}+\left[\begin{array}{l}(E_{jk}+F_{jk}+D_{jk})^{\frac{1}{r(r-1)}}\\ -(E_{jk}+F_{jk}-D_{jk})^{\frac{1}{r(r-1)}}\end{array}\right]^{\frac{1}{p+q}}}}, \sqrt[3]{\frac{2\left[\begin{array}{l}(H_{jk}+I_{jk}+G_{jk})^{\frac{1}{r(r-1)}}\\ -(H_{jk}+I_{jk}-G_{jk})^{\frac{1}{r(r-1)}}\end{array}\right]^{\frac{1}{p+q}}}{\left[\begin{array}{l}(H_{jk}+I_{jk}+G_{jk})^{\frac{1}{r(r-1)}}\\ +3\left(H_{jk}+I_{jk}-G_{jk}\right)^{\frac{1}{r(r-1)}}\end{array}\right]^{\frac{1}{p+q}}+\left[\begin{array}{l}(H_{jk}+I_{jk}+G_{jk})^{\frac{1}{r(r-1)}}\\ -(H_{jk}+I_{jk}-G_{jk})^{\frac{1}{r(r-1)}}\end{array}\right]^{\frac{1}{p+q}}}} \right\rangle$$

Let

$\phi=(2B_{jk}+2C_{jk}-C_{jk})^{\frac{1}{r(r-1)}}+3\left(A_{jk}\right)^{\frac{1}{r(r-1)}}$

$\varphi=(2B_{jk}+2C_{jk}-A_{jk})^{\frac{1}{r(r-1)}}-\left(A_{jk}\right)^{\frac{1}{r(r-1)}}$

χ
$=(E_{jk}+F_{jk}+D_{jk})^{\frac{1}{r(r-1)}}-(E_{jk}+F_{jk}-D_{jk})^{\frac{1}{r(r-1)}}$

ψ
$=(E_{jk}+F_{jk}+D_{jk})^{\frac{1}{r(r-1)}}+3\left(E_{jk}+F_{jk}-D_{jk}\right)^{\frac{1}{r(r-1)}}$

ρ
$=(H_{jk}+I_{jk}+G_{jk})^{\frac{1}{r(r-1)}}-(H_{jk}+I_{jk}-G_{jk})^{\frac{1}{r(r-1)}}$

ϱ
$=(H_{jk}+I_{jk}+G_{jk})^{\frac{1}{r(r-1)}}+3\left(H_{jk}+I_{jk}-G_{jk}\right)^{\frac{1}{r(r-1)}}$

Therefore,

$CFEBOMWG^{p,q}(\mathcal{L}_1,\mathcal{L}_2,\ldots,\mathcal{L}_r)$

$$=\frac{1}{p+q}\left[\bigotimes_{E\ j,k=1;j\neq k}^{r}\left(p(\mathcal{L}_j{}^{r\omega_j})\bigoplus_{E} q(\mathcal{L}_k{}^{r\omega_k})\right)\right]^{\frac{1}{r(r-1)}}$$

$$=\left\langle\sqrt[3]{\frac{\phi^{\frac{1}{p+q}}-\varphi^{\frac{1}{p+q}}}{\phi^{\frac{1}{p+q}}+\varphi^{\frac{1}{p+q}}}},\sqrt[3]{\frac{2\chi^{\frac{1}{p+q}}}{\psi^{\frac{1}{p+q}}+\chi^{\frac{1}{p+q}}}},\sqrt[3]{\frac{2\rho^{\frac{1}{p+q}}}{\varrho^{\frac{1}{p+q}}+\rho^{\frac{1}{p+q}}}}\right\rangle$$

where

$\phi=(2B_{jk}+2C_{jk}-C_{jk})^{\frac{1}{r(r-1)}}+3\left(A_{jk}\right)^{\frac{1}{r(r-1)}}$

$\varphi=(2B_{jk}+2C_{jk}-A_{jk})^{\frac{1}{r(r-1)}}-\left(A_{jk}\right)^{\frac{1}{r(r-1)}}$

χ
$=(E_{jk}+F_{jk}+D_{jk})^{\frac{1}{r(r-1)}}-(E_{jk}+F_{jk}-D_{jk})^{\frac{1}{r(r-1)}}$

ψ
$=(E_{jk}+F_{jk}+D_{jk})^{\frac{1}{r(r-1)}}+3\left(E_{jk}+F_{jk}-D_{jk}\right)^{\frac{1}{r(r-1)}}$

ρ
$=(H_{jk}+I_{jk}+G_{jk})^{\frac{1}{r(r-1)}}-(H_{jk}+I_{jk}-G_{jk})^{\frac{1}{r(r-1)}}$

ϱ
$=(H_{jk}+I_{jk}+G_{jk})^{\frac{1}{r(r-1)}}+3\left(H_{jk}+I_{jk}-G_{jk}\right)^{\frac{1}{r(r-1)}}$

and

$$A_{jk}=\prod_{j,k=1;j\neq k}^{r}2\left\{\begin{array}{l}\left[(2-\mu^3_{\mathcal{L}_j})^{r\omega_j}+3(\mu^3_{\mathcal{L}_j})^{r\omega_j}\right]^p\\ \left[(2-\mu^3_{\mathcal{L}_k})^{r\omega_k}+3(\mu^3_{\mathcal{L}_k})^{r\omega_k}\right]^q\\ -\left[(2-\mu^3_{\mathcal{L}_j})^{r\omega_j}-(\mu^3_{\mathcal{L}_j})^{r\omega_j}\right]^p\\ \left[(2-\mu^3_{\mathcal{L}_k})^{r\omega_k}-(\mu^3_{\mathcal{L}_k})^{r\omega_k}\right]^q\end{array}\right\}$$

$$B_{jk}=\prod_{j,k=1;j\neq k}^{r}2\left\{\begin{array}{l}\left[(2-\mu^3_{\mathcal{L}_j})^{r\omega_j}+3(\mu^3_{\mathcal{L}_j})^{r\omega_j}\right]^p\\ \left[(2-\mu^3_{\mathcal{L}_k})^{r\omega_k}+3(\mu^3_{\mathcal{L}_k})^{r\omega_k}\right]^q\\ +\left[(2-\mu^3_{\mathcal{L}_j})^{r\omega_j}-(\mu^3_{\mathcal{L}_j})^{r\omega_j}\right]^p\\ \left[(2-\mu^3_{\mathcal{L}_k})^{r\omega_k}-(\mu^3_{\mathcal{L}_k})^{r\omega_k}\right]^q\end{array}\right\}$$

C_{jk}
$$=\prod_{j,k=1;j\neq k}^{r}2\left[\begin{array}{l}(2-\mu^3_{\mathcal{L}_j})^{r\omega_j}\\ -(\mu^3_{\mathcal{L}_j})^{r\omega_j}\end{array}\right]^p\left[\begin{array}{l}(2-\mu^3_{\mathcal{L}_k})^{r\omega_k}\\ -(\mu^3_{\mathcal{L}_k})^{r\omega_k}\end{array}\right]^q$$

D_{jk}
$$=\sum_{j,k=1;j\neq k}^{r}2\left[\begin{array}{l}(1+\eta^3_{\mathcal{L}_j})^{r\omega_j}\\ -(1-\eta^3_{\mathcal{L}_j})^{r\omega_j}\end{array}\right]^p\left[\begin{array}{l}(1+\eta^3_{\mathcal{L}_k})^{r\omega_k}\\ -(1-\eta^3_{\mathcal{L}_k})^{r\omega_k}\end{array}\right]^q$$
$$\left\{\begin{array}{l}\left[(1+\eta^3_{\mathcal{L}_k})^{r\omega_k}+3(1-\eta^3_{\mathcal{L}_k})^{r\omega_k}\right]^p\\ \left[(1+\eta^3_{\mathcal{L}_j})^{r\omega_j}+3(1-\eta^3_{\mathcal{L}_j})^{r\omega_j}\right]^q\\ +\left[(1+\eta^3_{\mathcal{L}_k})^{r\omega_k}-(1-\eta^3_{\mathcal{L}_k})^{r\omega_k}\right]^p\\ \left[(1+\eta^3_{\mathcal{L}_j})^{r\omega_j}-(1-\eta^3_{\mathcal{L}_j})^{r\omega_j}\right]^q\end{array}\right\}$$

E_{jk}
$$=\prod_{j,k=1;j\neq k}^{r}2\left\{\begin{array}{l}\left[(1+\eta^3_{\mathcal{L}_j})^{r\omega_j}+3(1-\eta^3_{\mathcal{L}_j})^{r\omega_j}\right]^p\\ \left[(1+\eta^3_{\mathcal{L}_k})^{r\omega_k}+3(1-\eta^3_{\mathcal{L}_k})^{r\omega_k}\right]^q\\ +\left[(1+\eta^3_{\mathcal{L}_j})^{r\omega_j}-(1-\eta^3_{\mathcal{L}_j})^{r\omega_j}\right]^p\\ \left[(1+\eta^3_{\mathcal{L}_k})^{r\omega_k}-(1-\eta^3_{\mathcal{L}_k})^{r\omega_k}\right]^q\end{array}\right\}$$

F_{jk}
$$=\prod_{j,k=1;j\neq k}^{r}2\left[\begin{array}{l}(1+\eta^3_{\mathcal{L}_j})^{r\omega_j}\\ -(1-\eta^3_{\mathcal{L}_j})^{r\omega_j}\end{array}\right]^p\left[\begin{array}{l}(1+\eta^3_{\mathcal{L}_k})^{r\omega_k}\\ -(1-\eta^3_{\mathcal{L}_k})^{r\omega_k}\end{array}\right]^q$$

G_{jk}
$$=\sum_{j,k=1;j\neq k}^{r}2\left[\begin{array}{l}(1+\nu^3_{\mathcal{L}_j})^{r\omega_j}\\ -(1-\nu^3_{\mathcal{L}_j})^{r\omega_j}\end{array}\right]^p\left[\begin{array}{l}(1+\nu^3_{\mathcal{L}_k})^{r\omega_k}\\ -(1-\nu^3_{\mathcal{L}_k})^{r\omega_k}\end{array}\right]^q$$
$$\left\{\begin{array}{l}\left[(1+\nu^3_{\mathcal{L}_k})^{r\omega_k}+3(1-\nu^3_{\mathcal{L}_k})^{r\omega_k}\right]^p\\ \left[(1+\nu^3_{\mathcal{L}_j})^{r\omega_j}+3(1-\nu^3_{\mathcal{L}_j})^{r\omega_j}\right]^q\\ +\left[(1+\nu^3_{\mathcal{L}_k})^{r\omega_k}-(1-\nu^3_{\mathcal{L}_k})^{r\omega_k}\right]^p\\ \left[(1+\nu^3_{\mathcal{L}_j})^{r\omega_j}-(1-\nu^3_{\mathcal{L}_j})^{r\omega_j}\right]^q\end{array}\right\}$$

H_{jk}
$$=\prod_{j,k=1;j\neq k}^{r}2\left\{\begin{array}{l}\left[(1+\nu^3_{\mathcal{L}_j})^{r\omega_j}+3(1-\nu^3_{\mathcal{L}_j})^{r\omega_j}\right]^p\\ \left[(1+\nu^3_{\mathcal{L}_k})^{r\omega_k}+3(1-\nu^3_{\mathcal{L}_k})^{r\omega_k}\right]^q\\ +\left[(1+\nu^3_{\mathcal{L}_j})^{r\omega_j}-(1-\nu^3_{\mathcal{L}_j})^{r\omega_j}\right]^p\\ \left[(1+\nu^3_{\mathcal{L}_k})^{r\omega_k}-(1-\nu^3_{\mathcal{L}_k})^{r\omega_k}\right]^q\end{array}\right\}$$

I_{jk}
$$=\prod_{j,k=1;j\neq k}^{r}2\left[\begin{array}{l}(1+\nu^3_{\mathcal{L}_j})^{r\omega_j}\\ -(1-\nu^3_{\mathcal{L}_j})^{r\omega_j}\end{array}\right]^p\left[\begin{array}{l}(1+\nu^3_{\mathcal{L}_k})^{r\omega_k}\\ -(1-\nu^3_{\mathcal{L}_k})^{r\omega_k}\end{array}\right]^q.$$

which fulfills the proof of Theorem III-C2. ■

Example 3.2

Suppose $\mathcal{L}_1=\langle 0.83,0.53,0.63\rangle,\mathcal{L}_2=\langle 0.89,0.39,0.49\rangle,\mathcal{L}_3=\langle 0.82,0.62,0.22\rangle$ and $\mathcal{L}_4=\langle 0.6,0.2,0.8\rangle$ are four CFNs, $\omega=(\omega_1,\omega_2,\omega_3,\omega_4)^T=(0.2,0.1,0.3,0.4)^T$ be the weight vector of $\mathcal{L}_j(j=1,2,3,4)$ and if we take $p=1$ and $q=2$, then, CFEBOMWG operator can be utilized to aggregate the four CFNs as follows:

$$CFEBOMWG_{\omega}{}^{1,2}(\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3,\mathcal{L}_4)$$
$$=\frac{1}{3}\left[\bigoplus_{E\ j,k=1;j\neq k}^{4}\left(1(\mathcal{L}_j{}^{4\omega_j})\bigotimes_E 2(\mathcal{L}_k{}^{4\omega_k})\right)\right]^{\frac{1}{12}}$$
$$=\left\langle\sqrt[3]{\frac{\phi^{\frac{1}{3}}-\varphi^{\frac{1}{3}}}{\phi^{\frac{1}{3}}+\varphi^{\frac{1}{3}}}},\sqrt[3]{\frac{2\chi^{\frac{1}{3}}}{\psi^{\frac{1}{3}}+\chi^{\frac{1}{3}}}},\sqrt[3]{\frac{2\rho^{\frac{1}{3}}}{\varrho^{\frac{1}{3}}+\rho^{\frac{1}{3}}}}\right\rangle$$
$$=\langle 0.6741,0,0\rangle$$

3) CF Einstein BOM ordered weighted geometric operator: Here, we define CF Einstein BOM ordered weighted geometric (CFEBOMOWG) operator that considers the weights of the ordered positions of the aggregated CFNs.

Definition 3.7

Let $\mathcal{L}_j-\langle\mu_{\mathcal{L}_j},\eta_{\mathcal{L}_j},\nu_{\mathcal{L}_j}\rangle(j=1,2,\ldots,r)$ be a set of CFNs and $p,q\geq 0$. Then the CFEBOMOWG operator with the associated weight vector $\omega_j=(\omega_1,\omega_2,\ldots,\omega_r)^T$ satisfying $\sum\limits_{j=1}^{r}\omega_j=1$ is a mapping $CFEBOMOWG_{\omega}:\mathcal{L}^r\rightarrow\mathcal{L}$ such that
$$CFEBOMOWG_{\omega}{}^{p,q}(\mathcal{L}_1,\mathcal{L}_2,\ldots,\mathcal{L}_r)$$
$$=\frac{1}{p+q}\left[\bigotimes_{E\ j,k=1;j\neq k}^{r}\left(\begin{array}{c}p(\mathcal{L}_{\sigma(j)}{}^{r\omega_j})\\ \bigoplus_E q(\mathcal{L}_{\sigma(k)}{}^{r\omega_k})\end{array}\right)\right]^{\frac{1}{r(r-1)}} \tag{11}$$
where $\sigma(j)$ is the permutation such that $\mathcal{L}_{\sigma(j-1)}\geq\mathcal{L}_{\sigma(j)}$ for any $j=1,2,\ldots,r$.

Theorem 3.3

The aggregated value obtained using CFEBOMOWG operator is also a CFN and can be expressed as follows:
$$CFEBOMOWG_{\omega}{}^{p,q}(\mathcal{L}_1,\mathcal{L}_2,\ldots,\mathcal{L}_r)$$
$$=\frac{1}{p+q}\left[\bigotimes_{E\ j,k=1;j\neq k}^{r}\left(\begin{array}{c}p(\mathcal{L}_{\sigma(j)}{}^{r\omega_j})\\ \bigoplus\limits_E q(\mathcal{L}_{\sigma(k)}{}^{r\omega_k})\end{array}\right)\right]^{\frac{1}{r(r-1)}}$$
$$=\left\langle\sqrt[3]{\frac{\phi^{\frac{1}{p+q}}-\varphi^{\frac{1}{p+q}}}{\phi^{\frac{1}{p+q}}+\varphi^{\frac{1}{p+q}}}},\sqrt[3]{\frac{2\chi^{\frac{1}{p+q}}}{\psi^{\frac{1}{p+q}}+\chi^{\frac{1}{p+q}}}},\sqrt[3]{\frac{2\rho^{\frac{1}{p+q}}}{\varrho^{\frac{1}{p+q}}+\rho^{\frac{1}{p+q}}}}\right\rangle \tag{12}$$
where
$$\phi=(2B_{jk}+2C_{jk}-C_{jk})^{\frac{1}{r(r-1)}}+3\,(A_{jk})^{\frac{1}{r(r-1)}}$$
$$\varphi=(2B_{jk}+2C_{jk}-A_{jk})^{\frac{1}{r(r-1)}}-(A_{jk})^{\frac{1}{r(r-1)}}$$
$$\chi=(E_{jk}+F_{jk}+D_{jk})^{\frac{1}{r(r-1)}}-(E_{jk}+F_{jk}-D_{jk})^{\frac{1}{r(r-1)}}$$
$$\psi=(E_{jk}+F_{jk}+D_{jk})^{\frac{1}{r(r-1)}}+3\,(E_{jk}+F_{jk}-D_{jk})^{\frac{1}{r(r-1)}}$$
$$\rho=(H_{jk}+I_{jk}+G_{jk})^{\frac{1}{r(r-1)}}-(H_{jk}+I_{jk}-G_{jk})^{\frac{1}{r(r-1)}}$$
$$\varrho=(H_{jk}+I_{jk}+G_{jk})^{\frac{1}{r(r-1)}}+3\,(H_{jk}+I_{jk}-G_{jk})^{\frac{1}{r(r-1)}}$$

and
$$A_{jk}=\prod_{j,k=1;j\neq k}^{r}2\left\{\begin{array}{c}\left[(2-\mu^3_{\mathcal{L}_{\sigma(j)}})^{r\omega_j}+3(\mu^3_{\mathcal{L}_{\sigma(j)}})^{r\omega_j}\right]^p\\ \left[(2-\mu^3_{\mathcal{L}_{\sigma(k)}})^{r\omega_k}+3(\mu^3_{\mathcal{L}_{\sigma(k)}})^{r\omega_k}\right]^q\\ -\left[(2-\mu^3_{\mathcal{L}_{\sigma(j)}})^{r\omega_j}-(\mu^3_{\mathcal{L}_{\sigma(j)}})^{r\omega_j}\right]^p\\ \left[(2-\mu^3_{\mathcal{L}_{\sigma(k)}})^{r\omega_k}-(\mu^3_{\mathcal{L}_{\sigma(k)}})^{r\omega_k}\right]^q\end{array}\right\}$$
$$B_{jk}=\prod_{j,k=1;j\neq k}^{r}2\left\{\begin{array}{c}\left[(2-\mu^3_{\mathcal{L}_{\sigma(j)}})^{r\omega_j}+3(\mu^3_{\mathcal{L}_{\sigma(j)}})^{r\omega_j}\right]^p\\ \left[(2-\mu^3_{\mathcal{L}_{\sigma(k)}})^{r\omega_k}+3(\mu^3_{\mathcal{L}_{\sigma(k)}})^{r\omega_k}\right]^q\\ +\left[(2-\mu^3_{\mathcal{L}_{\sigma(j)}})^{r\omega_j}-(\mu^3_{\mathcal{L}_{\sigma(j)}})^{r\omega_j}\right]^p\\ \left[(2-\mu^3_{\mathcal{L}_{\sigma(k)}})^{r\omega_k}-(\mu^3_{\mathcal{L}_{\sigma(k)}})^{r\omega_k}\right]^q\end{array}\right\}$$
$$C_{jk}=\prod_{j,k=1;j\neq k}^{r}2\left[\begin{array}{c}(2-\mu^3_{\mathcal{L}_{\sigma(j)}})^{r\omega_j}\\ -(\mu^3_{\mathcal{L}_{\sigma(j)}})^{r\omega_j}\end{array}\right]^p\left[\begin{array}{c}(2-\mu^3_{\mathcal{L}_{\sigma(k)}})^{r\omega_k}\\ -(\mu^3_{\mathcal{L}_{\sigma(k)}})^{r\omega_k}\end{array}\right]^q$$
$$D_{jk}=\sum_{j,k=1;j\neq k}^{r}2\left[\begin{array}{c}(1+\eta^3_{\mathcal{L}_{\sigma(j)}})^{r\omega_j}\\ -(1-\eta^3_{\mathcal{L}_{\sigma(j)}})^{r\omega_j}\end{array}\right]^p\left[\begin{array}{c}(1+\eta^3_{\mathcal{L}_{\sigma(k)}})^{r\omega_k}\\ -(1-\eta^3_{\mathcal{L}_{\sigma(k)}})^{r\omega_k}\end{array}\right]^q$$
$$\left\{\begin{array}{c}\left[(1+\eta^3_{\mathcal{L}_{\sigma(k)}})^{r\omega_k}+3(1-\eta^3_{\mathcal{L}_{\sigma(k)}})^{r\omega_k}\right]^p\\ \left[(1+\eta^3_{\mathcal{L}_{\sigma(j)}})^{r\omega_j}+3(1-\eta^3_{\mathcal{L}_{\sigma(j)}})^{r\omega_j}\right]^q\\ +\left[(1+\eta^3_{\mathcal{L}_{\sigma(k)}})^{r\omega_k}-(1-\eta^3_{\mathcal{L}_{\sigma(k)}})^{r\omega_k}\right]^p\\ \left[(1+\eta^3_{\mathcal{L}_{\sigma(j)}})^{r\omega_j}-(1-\eta^3_{\mathcal{L}_{\sigma(j)}})^{r\omega_j}\right]^q\end{array}\right\}$$
$$E_{jk}=\prod_{j,k=1;j\neq k}^{r}2\left\{\begin{array}{c}\left[(1+\eta^3_{\mathcal{L}_{\sigma(j)}})^{r\omega_j}+3(1-\eta^3_{\mathcal{L}_{\sigma(j)}})^{r\omega_j}\right]^p\\ \left[(1+\eta^3_{\mathcal{L}_{\sigma(k)}})^{r\omega_k}+3(1-\eta^3_{\mathcal{L}_{\sigma(k)}})^{r\omega_k}\right]^q\\ +\left[(1+\eta^3_{\mathcal{L}_{\sigma(j)}})^{r\omega_j}-(1-\eta^3_{\mathcal{L}_{\sigma(j)}})^{r\omega_j}\right]^p\\ \left[(1+\eta^3_{\mathcal{L}_{\sigma(k)}})^{r\omega_k}-(1-\eta^3_{\mathcal{L}_{\sigma(k)}})^{r\omega_k}\right]^q\end{array}\right\}$$
$$F_{jk}=\prod_{j,k=1;j\neq k}^{r}2\left[\begin{array}{c}(1+\eta^3_{\mathcal{L}_{\sigma(j)}})^{r\omega_j}\\ -(1-\eta^3_{\mathcal{L}_{\sigma(j)}})^{r\omega_j}\end{array}\right]^p\left[\begin{array}{c}(1+\eta^3_{\mathcal{L}_{\sigma(k)}})^{r\omega_k}\\ -(1-\eta^3_{\mathcal{L}_{\sigma(k)}})^{r\omega_k}\end{array}\right]^q$$
$$G_{jk}=\sum_{j,k=1;j\neq k}^{r}2\left[\begin{array}{c}(1+\nu^3_{\mathcal{L}_{\sigma(j)}})^{r\omega_j}\\ -(1-\nu^3_{\mathcal{L}_{\sigma(j)}})^{r\omega_j}\end{array}\right]^p\left[\begin{array}{c}(1+\nu^3_{\mathcal{L}_{\sigma(k)}})^{r\omega_k}\\ -(1-\nu^3_{\mathcal{L}_{\sigma(k)}})^{r\omega_k}\end{array}\right]^q$$
$$\left\{\begin{array}{c}\left[(1+\nu^3_{\mathcal{L}_{\sigma(k)}})^{r\omega_k}+3(1-\nu^3_{\mathcal{L}_{\sigma(k)}})^{r\omega_k}\right]^p\\ \left[(1+\nu^3_{\mathcal{L}_{\sigma(j)}})^{r\omega_j}+3(1-\nu^3_{\mathcal{L}_{\sigma(j)}})^{r\omega_j}\right]^q\\ +\left[(1+\nu^3_{\mathcal{L}_{\sigma(k)}})^{r\omega_k}-(1-\nu^3_{\mathcal{L}_{\sigma(k)}})^{r\omega_k}\right]^p\\ \left[(1+\nu^3_{\mathcal{L}_{\sigma(j)}})^{r\omega_j}-(1-\nu^3_{\mathcal{L}_{\sigma(j)}})^{r\omega_j}\right]^q\end{array}\right\}$$
$$H_{jk}=\prod_{j,k=1;j\neq k}^{r}2\left\{\begin{array}{c}\left[(1+\nu^3_{\mathcal{L}_{\sigma(j)}})^{r\omega_j}+3(1-\nu^3_{\mathcal{L}_{\sigma(j)}})^{r\omega_j}\right]^p\\ \left[(1+\nu^3_{\mathcal{L}_{\sigma(k)}})^{r\omega_k}+3(1-\nu^3_{\mathcal{L}_{\sigma(k)}})^{r\omega_k}\right]^q\\ +\left[(1+\nu^3_{\mathcal{L}_{\sigma(j)}})^{r\omega_j}-(1-\nu^3_{\mathcal{L}_{\sigma(j)}})^{r\omega_j}\right]^p\\ \left[(1+\nu^3_{\mathcal{L}_{\sigma(k)}})^{r\omega_k}-(1-\nu^3_{\mathcal{L}_{\sigma(k)}})^{r\omega_k}\right]^q\end{array}\right\}$$
$$I_{jk}=\prod_{j,k=1;j\neq k}^{r}2\left[\begin{array}{c}(1+\nu^3_{\mathcal{L}_{\sigma(j)}})^{r\omega_j}\\ -(1-\nu^3_{\mathcal{L}_{\sigma(j)}})^{r\omega_j}\end{array}\right]^p\left[\begin{array}{c}(1+\nu^3_{\mathcal{L}_{\sigma(k)}})^{r\omega_k}\\ -(1-\nu^3_{\mathcal{L}_{\sigma(k)}})^{r\omega_k}\end{array}\right]^q.$$

Proof. The proof is similar to Theorem III-C2. So we omit here. ■

Example 3.3

Suppose $\mathcal{L}_1=\langle 0.83,0.53,0.63\rangle,\mathcal{L}_2=\langle 0.89,0.39,0.49\rangle,\mathcal{L}_3=\langle 0.82,0.62,0.22\rangle$ and $\mathcal{L}_4=\langle 0.6,0.2,0.8\rangle$ are four CFNs, $\omega=(\omega_1,\omega_2,\omega_3,\omega_4)^T=(0.2,0.1,0.3,0.4)^T$ be the weight vector of $\mathcal{L}_j(j=1,2,3,4)$. Let us take $p=1$ and $q=2$. We need to permute the CFNs in order to aggregate them using (12). We first calculate the score values of $\mathcal{L}_j$ for

$j=1,2,3,4$ using (1). Thus, the obtained score values are $\mathbb{S}(\mathcal{L}_1)=0.32174, \mathbb{S}(\mathcal{L}_2)=0.58732, \mathbb{S}(\mathcal{L}_3)=0.54072$ and $\mathbb{S}(\mathcal{L}_4)=-0.296$. By Definition II, we have $\mathbb{S}(\mathcal{L}_2)>\mathbb{S}(\mathcal{L}_3)>\mathbb{S}(\mathcal{L}_1)>\mathbb{S}(\mathcal{L}_4)$ which implies that $\mathcal{L}_{\sigma(1)}=\langle 0.89,0.39,0.49\rangle, \mathcal{L}_{\sigma(2)}=\langle 0.82,0.62,0.22\rangle, \mathcal{L}_{\sigma(3)}=\langle 0.83,0.53,0.63\rangle$ and $\mathcal{L}_{\sigma(4)}=\langle 0.6,0.2,0.8\rangle$, then, CFEBOMOWG operator can be utilized to aggregate the four CFNs as follows:

$CFEBOMOWG_{\omega}{}^{1,2}(\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3,\mathcal{L}_4)$
$= \frac{1}{3}\left[\bigotimes_{E\ j,k=1;j\neq k}^{4}\left(1(\mathcal{L}_{\sigma(j)}{}^{4\omega_j})\bigoplus_E 2(\mathcal{L}_{\sigma(k)}{}^{4\omega_k})\right)\right]^{\frac{1}{12}}$
$= \left\langle \sqrt[3]{\frac{\phi^{\frac{1}{3}}-\varphi^{\frac{1}{3}}}{\phi^{\frac{1}{3}}+\varphi^{\frac{1}{3}}}}, \sqrt[3]{\frac{2\chi^{\frac{1}{3}}}{\psi^{\frac{1}{3}}+\chi^{\frac{1}{3}}}}, \sqrt[3]{\frac{2\rho^{\frac{1}{3}}}{\varrho^{\frac{1}{3}}+\rho^{\frac{1}{3}}}}\right\rangle$
$= \langle 0.6722, 0, 0\rangle$

4) CF Einstein BOM hybrid geometric operator: Here, we define CF Einstein BOM hybrid geometric (CFEBOMHG) operator in order to combine the different aspects of CFEBOMWG and CFEBOMOWG operators.

Definition 3.8

Let $\mathcal{L}_j-\langle\mu_{\mathcal{L}_j},\eta_{\mathcal{L}_j},\nu_{\mathcal{L}_j}\rangle(j=1,2,\ldots,r)$ be a set of CFNs and $p,q\geq 0$. Then the CFEBOMHG operator with the associated weight vector $\omega_j=(\omega_1,\omega_2,\ldots,\omega_r)^T$ satisfying $\sum_{j=1}^{r}\omega_j=1$ is a mapping $CFEBOMHG_{\omega,w}:\mathcal{L}^r\to\mathcal{L}$ such that
$CFEBOMHG_{\omega,w}{}^{p,q}(\mathcal{L}_1,\mathcal{L}_2,\ldots,\mathcal{L}_r)$

$$= \frac{1}{p+q}\left[\bigotimes_{E\ j,k=1;j\neq k}^{r}\left(\begin{array}{c} p(\tilde{\mathcal{L}}_{\sigma(j)}{}^{r\omega_j}) \\ \bigoplus_E q(\tilde{\mathcal{L}}_{\sigma(k)}{}^{r\omega_k})\end{array}\right)\right]^{\frac{1}{r(r-1)}} \tag{13}$$

where $w=(w_1,w_2,\ldots w_r)^T$ is the weighting vector corresponding to CFNs $\mathcal{L}_j$ such that $\sum_{j=1}^{r}w_j=1$ and $\tilde{\mathcal{L}}_j=\mathcal{L}_j{}^{n*w}$ and $(\sigma(1),\sigma(2),\ldots,\sigma(3))$ is the permutation such that $\tilde{\mathcal{L}}_{\sigma(j-1)}\geq\tilde{\mathcal{L}}_{\sigma(j)}$ for all $j=1,2,\ldots,r$.

Theorem 3.4

The aggregated value obtained using CFEBOMHG operator is also a CFN and can be expressed as follows:
$CFEBOMHG_{\omega,w}{}^{p,q}(\mathcal{L}_1,\mathcal{L}_2,\ldots,\mathcal{L}_r)$

$$= \frac{1}{p+q}\left[\bigotimes_{E\ j,k=1;j\neq k}^{r}\left(\begin{array}{c} p(\tilde{\mathcal{L}}_{\sigma(j)}{}^{r\omega_j}) \\ \bigoplus_E q(\tilde{\mathcal{L}}_{\sigma(k)}{}^{r\omega_k})\end{array}\right)\right]^{\frac{1}{r(r-1)}}$$
$$= \left\langle \sqrt[3]{\frac{2\phi^{\frac{1}{p+q}}}{\varphi^{\frac{1}{p+q}}+\phi^{\frac{1}{p+q}}}}, \sqrt[3]{\frac{\chi^{\frac{1}{p+q}}-\psi^{\frac{1}{p+q}}}{\chi^{\frac{1}{p+q}}+\psi^{\frac{1}{p+q}}}}, \sqrt[3]{\frac{\rho^{\frac{1}{p+q}}-\varrho^{\frac{1}{p+q}}}{\rho^{\frac{1}{p+q}}+\varrho^{\frac{1}{p+q}}}}\right\rangle \tag{14}$$

where
$\phi=(2B_{jk}+2C_{jk}-C_{jk})^{\frac{1}{r(r-1)}}+3\,(A_{jk})^{\frac{1}{r(r-1)}}$
$\varphi=(2B_{jk}+2C_{jk}-A_{jk})^{\frac{1}{r(r-1)}}-(A_{jk})^{\frac{1}{r(r-1)}}$
χ
$=(E_{jk}+F_{jk}+D_{jk})^{\frac{1}{r(r-1)}}-(E_{jk}+F_{jk}-D_{jk})^{\frac{1}{r(r-1)}}$
ψ
$=(E_{jk}+F_{jk}+D_{jk})^{\frac{1}{r(r-1)}}+3\,(E_{jk}+F_{jk}-D_{jk})^{\frac{1}{r(r-1)}}$
ρ
$=(H_{jk}+I_{jk}+G_{jk})^{\frac{1}{r(r-1)}}-(H_{jk}+I_{jk}-G_{jk})^{\frac{1}{r(r-1)}}$
$\varrho=(H_{jk}+I_{jk}+G_{jk})^{\frac{1}{r(r-1)}}+3\,(H_{jk}+I_{jk}-G_{jk})^{\frac{1}{r(r-1)}}$
and

$$A_{jk}=\prod_{j,k=1;j\neq k}^{r}2\left\{\begin{array}{l}\left[(2-\mu^3_{\tilde{\mathcal{L}}_{\sigma(j)}})^{r\omega_j}+3(\mu^3_{\tilde{\mathcal{L}}_{\sigma(j)}})^{r\omega_j}\right]^p\\ \left[(2-\mu^3_{\tilde{\mathcal{L}}_{\sigma(k)}})^{r\omega_k}+3(\mu^3_{\tilde{\mathcal{L}}_{\sigma(k)}})^{r\omega_k}\right]^q\\ -\left[(2-\mu^3_{\tilde{\mathcal{L}}_{\sigma(j)}})^{r\omega_j}-(\mu^3_{\tilde{\mathcal{L}}_{\sigma(j)}})^{r\omega_j}\right]^p\\ \left[(2-\mu^3_{\tilde{\mathcal{L}}_{\sigma(k)}})^{r\omega_k}-(\mu^3_{\tilde{\mathcal{L}}_{\sigma(k)}})^{r\omega_k}\right]^q\end{array}\right\}$$

$$B_{jk}=\prod_{j,k=1;j\neq k}^{r}2\left\{\begin{array}{l}\left[(2-\mu^3_{\tilde{\mathcal{L}}_{\sigma(j)}})^{r\omega_j}+3(\mu^3_{\tilde{\mathcal{L}}_{\sigma(j)}})^{r\omega_j}\right]^p\\ \left[(2-\mu^3_{\tilde{\mathcal{L}}_{\sigma(k)}})^{r\omega_k}+3(\mu^3_{\tilde{\mathcal{L}}_{\sigma(k)}})^{r\omega_k}\right]^q\\ +\left[(2-\mu^3_{\tilde{\mathcal{L}}_{\sigma(j)}})^{r\omega_j}-(\mu^3_{\tilde{\mathcal{L}}_{\sigma(j)}})^{r\omega_j}\right]^p\\ \left[(2-\mu^3_{\tilde{\mathcal{L}}_{\sigma(k)}})^{r\omega_k}-(\mu^3_{\tilde{\mathcal{L}}_{\sigma(k)}})^{r\omega_k}\right]^q\end{array}\right\}$$

C_{jk}
$$=\prod_{j,k=1;j\neq k}^{r}2\left[\begin{array}{c}(2-\mu^3_{\tilde{\mathcal{L}}_{\sigma(j)}})^{r\omega_j}\\ -(\mu^3_{\tilde{\mathcal{L}}_{\sigma(j)}})^{r\omega_j}\end{array}\right]^p\left[\begin{array}{c}(2-\mu^3_{\tilde{\mathcal{L}}_{\sigma(k)}})^{r\omega_k}\\ -(\mu^3_{\tilde{\mathcal{L}}_{\sigma(k)}})^{r\omega_k}\end{array}\right]^q$$

D_{jk}
$$=\sum_{j,k=1;j\neq k}^{r}2\left[\begin{array}{c}(1+\eta^3_{\tilde{\mathcal{L}}_{\sigma(j)}})^{r\omega_j}\\ -(1-\eta^3_{\tilde{\mathcal{L}}_{\sigma(j)}})^{r\omega_j}\end{array}\right]^p\left[\begin{array}{c}(1+\eta^3_{\tilde{\mathcal{L}}_{\sigma(k)}})^{r\omega_k}\\ -(1-\eta^3_{\tilde{\mathcal{L}}_{\sigma(k)}})^{r\omega_k}\end{array}\right]^q$$
$$\left\{\begin{array}{l}\left[(1+\eta^3_{\tilde{\mathcal{L}}_{\sigma(k)}})^{r\omega_k}+3(1-\eta^3_{\tilde{\mathcal{L}}_{\sigma(k)}})^{r\omega_k}\right]^p\\ \left[(1+\eta^3_{\tilde{\mathcal{L}}_{\sigma(j)}})^{r\omega_j}+3(1-\eta^3_{\tilde{\mathcal{L}}_{\sigma(j)}})^{r\omega_j}\right]^q\\ +\left[(1+\eta^3_{\tilde{\mathcal{L}}_{\sigma(k)}})^{r\omega_k}-(1-\eta^3_{\tilde{\mathcal{L}}_{\sigma(k)}})^{r\omega_k}\right]^p\\ \left[(1+\eta^3_{\tilde{\mathcal{L}}_{\sigma(j)}})^{r\omega_j}-(1-\eta^3_{\tilde{\mathcal{L}}_{\sigma(j)}})^{r\omega_j}\right]^q\end{array}\right\}$$

E_{jk}
$$=\prod_{j,k=1;j\neq k}^{r}2\left\{\begin{array}{l}\left[(1+\eta^3_{\tilde{\mathcal{L}}_{\sigma(j)}})^{r\omega_j}+3(1-\eta^3_{\tilde{\mathcal{L}}_{\sigma(j)}})^{r\omega_j}\right]^p\\ \left[(1+\eta^3_{\tilde{\mathcal{L}}_{\sigma(k)}})^{r\omega_k}+3(1-\eta^3_{\tilde{\mathcal{L}}_{\sigma(k)}})^{r\omega_k}\right]^q\\ +\left[(1+\eta^3_{\tilde{\mathcal{L}}_{\sigma(j)}})^{r\omega_j}-(1-\eta^3_{\tilde{\mathcal{L}}_{\sigma(j)}})^{r\omega_j}\right]^p\\ \left[(1+\eta^3_{\tilde{\mathcal{L}}_{\sigma(k)}})^{r\omega_k}-(1-\eta^3_{\tilde{\mathcal{L}}_{\sigma(k)}})^{r\omega_k}\right]^q\end{array}\right\}$$

F_{jk}
$$=\prod_{j,k=1;j\neq k}^{r}2\left[\begin{array}{c}(1+\eta^3_{\tilde{\mathcal{L}}_{\sigma(j)}})^{r\omega_j}\\ -(1-\eta^3_{\tilde{\mathcal{L}}_{\sigma(j)}})^{r\omega_j}\end{array}\right]^p\left[\begin{array}{c}(1+\eta^3_{\tilde{\mathcal{L}}_{\sigma(k)}})^{r\omega_k}\\ -(1-\eta^3_{\tilde{\mathcal{L}}_{\sigma(k)}})^{r\omega_k}\end{array}\right]^q$$

G_{jk}
$$=\sum_{j,k=1;j\neq k}^{r}2\left[\begin{array}{c}(1+\nu^3_{\tilde{\mathcal{L}}_{\sigma(j)}})^{r\omega_j}\\ -(1-\nu^3_{\tilde{\mathcal{L}}_{\sigma(j)}})^{r\omega_j}\end{array}\right]^p\left[\begin{array}{c}(1+\nu^3_{\tilde{\mathcal{L}}_{\sigma(k)}})^{r\omega_k}\\ -(1-\nu^3_{\tilde{\mathcal{L}}_{\sigma(k)}})^{r\omega_k}\end{array}\right]^q$$
$$\left\{\begin{array}{l}\left[(1+\nu^3_{\tilde{\mathcal{L}}_{\sigma(k)}})^{r\omega_k}+3(1-\nu^3_{\tilde{\mathcal{L}}_{\sigma(k)}})^{r\omega_k}\right]^p\\ \left[(1+\nu^3_{\tilde{\mathcal{L}}_{\sigma(j)}})^{r\omega_j}+3(1-\nu^3_{\tilde{\mathcal{L}}_{\sigma(j)}})^{r\omega_j}\right]^q\\ +\left[(1+\nu^3_{\tilde{\mathcal{L}}_{\sigma(k)}})^{r\omega_k}-(1-\nu^3_{\tilde{\mathcal{L}}_{\sigma(k)}})^{r\omega_k}\right]^p\\ \left[(1+\nu^3_{\tilde{\mathcal{L}}_{\sigma(j)}})^{r\omega_j}-(1-\nu^3_{\tilde{\mathcal{L}}_{\sigma(j)}})^{r\omega_j}\right]^q\end{array}\right\}$$

H_{jk}
$$=\prod_{j,k=1;j\neq k}^{r}2\left\{\begin{array}{l}\left[(1+\nu^3_{\tilde{\mathcal{L}}_{\sigma(j)}})^{r\omega_j}+3(1-\nu^3_{\tilde{\mathcal{L}}_{\sigma(j)}})^{r\omega_j}\right]^p\\ \left[(1+\nu^3_{\tilde{\mathcal{L}}_{\sigma(k)}})^{r\omega_k}+3(1-\nu^3_{\tilde{\mathcal{L}}_{\sigma(k)}})^{r\omega_k}\right]^q\\ +\left[(1+\nu^3_{\tilde{\mathcal{L}}_{\sigma(j)}})^{r\omega_j}-(1-\nu^3_{\tilde{\mathcal{L}}_{\sigma(j)}})^{r\omega_j}\right]^p\\ \left[(1+\nu^3_{\tilde{\mathcal{L}}_{\sigma(k)}})^{r\omega_k}-(1-\nu^3_{\tilde{\mathcal{L}}_{\sigma(k)}})^{r\omega_k}\right]^q\end{array}\right\}$$

I_{jk}
$$=\prod_{j,k=1;j\neq k}^{r}2\left[\begin{array}{c}(1+\nu^3_{\tilde{\mathcal{L}}_{\sigma(j)}})^{r\omega_j}\\ -(1-\nu^3_{\tilde{\mathcal{L}}_{\sigma(j)}})^{r\omega_j}\end{array}\right]^p\left[\begin{array}{c}(1+\nu^3_{\tilde{\mathcal{L}}_{\sigma(k)}})^{r\omega_k}\\ -(1-\nu^3_{\tilde{\mathcal{L}}_{\sigma(k)}})^{r\omega_k}\end{array}\right]^q.$$

Proof. The proof is similar to Theorem III-C2. So we omit here. ■

Example 3.4

Suppose $\mathcal{L}_1 = \langle 0.83, 0.53, 0.63\rangle, \mathcal{L}_2 = \langle 0.89, 0.39, 0.49\rangle$ $\mathcal{L}_3 = \langle 0.82, 0.62, 0.22\rangle$ and $\mathcal{L}_4 = \langle 0.6, 0.2, 0.8\rangle$ are four CFNs, $\omega = (\omega_1, \omega_2, \omega_3, \omega_4)^T = (0.2, 0.1, 0.3, 0.4)^T$ be the weight vector associated with CFEBOMHG operator, the vector $w = (w_1, w_2, w_3, w_4)^T = (03, 0.2, 0.3, 0.2)^T$ denotes the weightage of $\mathcal{L}_j (j = 1, 2, 3, 4)$ and assume that $p = 1$ and $q = 2$. Now, $\tilde{\mathcal{L}}_j$ for $j = 1, 2, 3, 4$ calculated using Definition III-C4 are given as $\tilde{\mathcal{L}}_1 = \mathcal{L}_1^{4*0.3} = \langle 0.79, 0.56, 0.67\rangle, \tilde{\mathcal{L}}_2 = \mathcal{L}_2^{4*0.2} = \langle 0.91, 0.36, 0.46\rangle, \tilde{\mathcal{L}}_3 = \mathcal{L}_3^{4*0.3} = \langle 0.78, 0.66, 0.23\rangle$ and $\tilde{\mathcal{L}}_4 = \mathcal{L}_4^{4*0.2} = \langle 0.68, 0.19.0.75\rangle$. Further, the score values of $\tilde{\mathcal{L}}_j$ computed using Equation (1) are as follows $\mathbb{S}(\tilde{\mathcal{L}}_1) = 0.2027, \mathbb{S}(\tilde{\mathcal{L}}_2) = 0.6672, \mathbb{S}(\tilde{\mathcal{L}}_3) = 0.4649$ and $\mathbb{S}(\tilde{\mathcal{L}}_4) = -0.1120$. Then by Definition II, we have $\mathbb{S}(\tilde{\mathcal{L}}_2) > \mathbb{S}(\tilde{\mathcal{L}}_3) > \mathbb{S}(\tilde{\mathcal{L}}_1) > \mathbb{S}(\tilde{\mathcal{L}}_4)$ which implies that $\tilde{\mathcal{L}}_{\sigma(1)} = \langle 0.91, 0.36, 0.46\rangle, \tilde{\mathcal{L}}_{\sigma(2)} = \langle 0.78, 0.66, 0.23\rangle, \tilde{\mathcal{L}}_{\sigma(3)} = \langle 0.79, 0.56, 0.67\rangle$ and $\tilde{\mathcal{L}}_{\sigma(4)} = \langle 0.68, 0.19, 0.75\rangle$. Thereafter, CFEBOMHG operator can be utilized to aggregate the four CFNs as follows:

$$
\begin{aligned}
&CFEBOMHG_{\omega,w}{}^{1,2}(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4) \\
&= \frac{1}{3}\left[\bigotimes\nolimits^{4}_{E\ j,k=1;j\neq k}\left(1(\tilde{\mathcal{L}}_{\sigma(j)}{}^{4\omega_j}) \bigoplus_{E} 2(\tilde{\mathcal{L}}_{\sigma(k)}{}^{4\omega_k})\right)\right]^{\frac{1}{12}} \\
&= \left\langle \sqrt[3]{\frac{\phi^{\frac{1}{3}} - \varphi^{\frac{1}{3}}}{\phi^{\frac{1}{3}} + \varphi^{\frac{1}{3}}}}, \sqrt[3]{\frac{2\chi^{\frac{1}{3}}}{\psi^{\frac{1}{3}} + \chi^{\frac{1}{3}}}}, \sqrt[3]{\frac{2\rho^{\frac{1}{3}}}{\varrho^{\frac{1}{3}} + \rho^{\frac{1}{3}}}} \right\rangle \\
&= \langle 0.6807, 0, 0\rangle
\end{aligned}
$$

For a collection of CFNs, we can easily prove the proposed CFEBOMWG, CFEBOMOWG and CFEBOMHG operators satisfy the following idempotency, monotonicity and boundedness properties.

Property 3.4

(Idempotency) If $\mathcal{L}_j = \mathcal{L} = \langle \mu_{\mathcal{L}}, \eta_{\mathcal{L}}, \nu_{\mathcal{L}}\rangle$ for all $j = 1, 2, \ldots, r$, then

$$
\begin{aligned}
CFEBOMWG_{\omega}{}^{p,q}(\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_r) = \mathcal{L}; \\
CFEBOMWG_{\omega}{}^{p,q}(\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_r) = \mathcal{L}; \\
CFEBOMHG_{\omega,w}{}^{p,q}(\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_r) = \mathcal{L}.
\end{aligned}
$$

Property 3.5

(Monotonicity) For any two sets of CFNs, $\mathcal{L}_j = \langle \mu_{\mathcal{L}_j}, \eta_{\mathcal{L}_j}, \nu_{\mathcal{L}_j}\rangle$ and $\mathcal{L}'_j = \langle \mu_{\mathcal{L}'_j}, \eta_{\mathcal{L}'_j}, \nu_{\mathcal{L}'_j}\rangle$ such that $\mathcal{L}_j \leq \mathcal{L}'_j$ for all $j = 1, 2, \ldots, r$. We have

$$
\begin{aligned}
CFEBOMWG_{\omega}{}^{p,q}(\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_r) \leq \\
CFEBOMWG_{\omega}{}^{p,q}(\mathcal{L}'_1, \mathcal{L}'_2, \ldots, \mathcal{L}'_r); \\
CFEBOMOWG_{\omega}{}^{p,q}(\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_r) \leq \\
CFEBOMOWG_{\omega}{}^{p,q}(\mathcal{L}'_1, \mathcal{L}'_2, \ldots, \mathcal{L}'_r); \\
CFEBOMHG_{\omega,w}{}^{p,q}(\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_r) \leq \\
CFEBOMHG_{\omega,w}{}^{p,q}(\mathcal{L}'_1, \mathcal{L}'_2, \ldots, \mathcal{L}'_r).
\end{aligned}
$$

Property 3.6

(Boundedness) For a collection of CFNs, $\mathcal{L}_j$ for all $j = 1, 2, \ldots, r$, if $L^l = \min_j \mathcal{L}_j$ and $L^u = \max_j \mathcal{L}_j$. Then,

$$
\begin{aligned}
\mathcal{L}^l \leq CFEBOMWG_{\omega}{}^{p,q}(\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_r) \leq \mathcal{L}^u; \\
\mathcal{L}^l \leq CFEBOMOWG_{\omega}{}^{p,q}(\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_r) \leq \mathcal{L}^u; \\
\mathcal{L}^l \leq CFEBOMWG_{\omega,w}{}^{p,q}(\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_r) \leq \mathcal{L}^u.
\end{aligned}
$$

IV. CF-CRITIC-BASED COPRAS METHOD FOR SOLVING MCGDM PROBLEM

This section proposes a new CF-MCGDM approach using the CRITIC-based COPRAS method and the proposed operators. Assume that, there are a set of m alternatives $\mathfrak{A} = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m\}$ and r criteria $\mathfrak{C} = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_r\}$. Let $\omega = (\omega_1, \omega_2, \ldots, \omega_r)^T$ be the weight vector of criteria set $\mathfrak{C}$ such that $\omega_j \in [0, 1]$ and $\sum_{j=1}^{r} \omega_j = 1$. Let $\mathfrak{E} = \{\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_l\}$ be l number of experts with weight $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)^T$ such that each $\lambda_l \in [0, 1]$ and $\sum_{t=1}^{l} \lambda_t = 1$. The decision procedure is summarized as follows:

Step 1 The decision value of each decision expert $\mathcal{E}_t$ for $t = 1, 2, \ldots, l$ about each alternative $\mathcal{A}_i$ for $i = 1, 2, \ldots, m$ with respect to each criteria $\mathcal{C}_j$ for $j = 1, 2, \ldots, r$ is given by a CFN $\mathcal{L}^t_{ij}$. Each decision-maker accumulates the decision information in a matrix form called the CF decision matrix $\mathfrak{D}^t$ and represents it as follows:

$$
\mathfrak{D}^t = [\mathcal{L}^t_{ij}]_{m\times r} = \begin{array}{c} \\ \mathcal{A}_1 \\ \mathcal{A}_2 \\ \vdots \\ \mathcal{A}_m \end{array} \begin{array}{c} \begin{array}{cccc} \mathcal{C}_1 & \mathcal{C}_2 & \ldots & \mathcal{C}_r \end{array} \\ \begin{bmatrix} \mathcal{L}^t_{11} & \mathcal{L}^t_{12} & \cdots & \mathcal{L}^t_{1r} \\ \mathcal{L}^t_{21} & \mathcal{L}^t_{22} & \cdots & \mathcal{L}^t_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{L}^t_{m1} & \mathcal{L}^t_{m2} & \cdots & \mathcal{L}^t_{mr} \end{bmatrix} \end{array}
$$

where $\mathcal{L}^t_{ij} = \langle \mu_{\mathcal{L}^t_{ij}}, \eta_{\mathcal{L}^t_{ij}}, \nu_{\mathcal{L}^t_{ij}}\rangle$.

Step 2 Fuse the different decision matrices by aggregating the decision information of each expert using the CFEBOMWG operator to obtain the aggregated CF decision matrix $\mathfrak{D}$ given by:

$$
\mathfrak{D} = [\mathcal{L}_{ij}]_{m\times r} = \begin{array}{c} \\ \mathcal{A}_1 \\ \mathcal{A}_2 \\ \vdots \\ \mathcal{A}_m \end{array} \begin{array}{c} \begin{array}{cccc} \mathcal{C}_1 & \mathcal{C}_2 & \ldots & \mathcal{C}_r \end{array} \\ \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \cdots & \mathcal{L}_{1r} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \cdots & \mathcal{L}_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{L}_{m1} & \mathcal{L}_{m2} & \cdots & \mathcal{L}_{mr} \end{bmatrix} \end{array}
$$

where we obtain

$$
\mathcal{L}_{ij} = CFEBOMWG(\mathcal{L}^1_{ij}, \mathcal{L}^2_{ij}, \ldots, \mathcal{L}^l_{ij}) \quad (15)
$$

if we employ CFEBOMWG operator using Eq. (10).

Step 3 Utilize the CRITIC method developed by Diakoulaki et al. [45] to obtain the weight ω of the criteria set C. The computing procedure of the CRITIC method is presented below.

(i) Depending on the CF decision matrix obtained in Step 2, the correlation coefficient matrix $\Phi = [\Psi_{jk}]_{r\times r}$ is built by finding the correlation coefficients between the criteria $\mathcal{C}_j$ using Eq. (16).

$$
\Psi_{jk} = \frac{\sum_{i=1}^{m} (\mathcal{S}(\mathcal{L}_{ij}) - \mathcal{S}(\mathcal{L}_j))(\mathcal{S}(\mathcal{L}_{ik}) - \mathcal{S}(\mathcal{L}_k))}{\sqrt{\sum_{i=1}^{m} (\mathcal{S}(\mathcal{L}_{ij}) - \mathcal{S}(\mathcal{L}_j))^2}\sqrt{\sum_{i=1}^{m} (\mathcal{S}(\mathcal{L}_{ik}) - \mathcal{S}(\mathcal{L}_k))^2}}; \quad j, k = 1, 2, \ldots, r, \quad (16)
$$

where $\mathcal{S}(\mathcal{L}_j) = \frac{1}{m}\sum_{i=1}^{m} \mathcal{S}(\mathcal{L}_{ij})$ and $\mathcal{S}(\mathcal{L}_k) = \frac{1}{m}\sum_{i=1}^{m} \mathcal{S}(\mathcal{L}_{ik})$.

(ii) Calculate the standard deviation ς_j of each $\mathcal{C}_j$ for $j = 1, 2, \ldots, r.$ using Eq. (17).

$$\varsigma_j = \sqrt{\frac{1}{m-1}\sum_{i=1}^{m}\left(\mathcal{S}(\mathcal{L}_{ij}) - \mathcal{S}(\mathcal{L}_j)\right)^2} \quad (17)$$

where $\mathcal{S}(\mathcal{L}_j) = \frac{1}{m}\sum_{i=1}^{m}\mathcal{S}(\mathcal{L}_{ij})$.
Then calculate the index $\Im_j$ of each $\mathcal{C}_j$ using Eq. (18).

$$\mathcal{I}_j = \varsigma_j \sum_{k=1}^{r}(1 - \Psi_{jk}); j = 1, 2, \ldots, r. \quad (18)$$

(iii) Find the criteria weights ω_j using Eq. (19).

$$\omega_j = \frac{I_j}{\sum_{j=1}^{r} I_j}; j = 1, 2, \ldots, r. \quad (19)$$

where $\omega_j \in [0, 1]$ and $\sum_{j=1}^{r}\omega_j = 1$.

Step 4 Derive the aggregated value $\mathcal{L}_i^+$ of the benefit criteria using the proposed CFEBOMWG, CFEBOMOWG or CFEBOMHG operator.
If we implement CFEBOMWG operator using Eq. (10), then we obtain
$\mathcal{L}_i^+ = CFEBOMWG_{\omega}{}^{p,q}(\mathcal{L}_{i1}^+, \mathcal{L}_{i2}^+, \ldots, \mathcal{L}_{ir}^+)$

$$= \frac{1}{p+q}\left[\bigotimes_{E\ j,k=1;j\neq k}^{r}\begin{pmatrix} p((\mathcal{L}_{ij}^+)^{r\omega_j}) \\ \bigoplus_E \\ q((\mathcal{L}_{ik}^+)^{r\omega_k}) \end{pmatrix}\right]^{\frac{1}{r(r-1)}} \quad (20)$$

If we implement CFEBOMOWG operator using Eq. (12), then we obtain
$\mathcal{L}_i^+ = CFEBOMOWG_{\omega}{}^{p,q}(\mathcal{L}_{i1}^+, \mathcal{L}_{i2}^+, \ldots, \mathcal{L}_{ir}^+)$

$$= \frac{1}{p+q}\left[\bigotimes_{E\ j,k=1;j\neq k}^{r}\begin{pmatrix} p((\mathcal{L}_{\sigma(ij)}^+)^{r\omega_j}) \\ \bigoplus_E \\ q((\mathcal{L}_{\sigma(ik)}^+)^{r\omega_k}) \end{pmatrix}\right]^{\frac{1}{r(r-1)}} \quad (21)$$

If we implement CFEBOMHG operator using Eq. (14), then we obtain
$\mathcal{L}_i^+ = CFEBOMHG_{\omega,w}{}^{p,q}(\mathcal{L}_{i1}^+, \mathcal{L}_{i2}^+, \ldots, \mathcal{L}_{ir}^+)$

$$= \frac{1}{p+q}\left[\bigotimes_{E\ j,k=1;j\neq k}^{r}\begin{pmatrix} p((\tilde{\mathcal{L}}_{\sigma(ij)}^+)^{r\omega_j}) \\ \bigoplus_E \\ q((\tilde{\mathcal{L}}_{\sigma(ik)}^+)^{r\omega_k}) \end{pmatrix}\right]^{\frac{1}{r(r-1)}} \quad (22)$$

where $w = (w_1, w_2, \ldots w_r)^T$ is the weighting vector corresponding to CFNs $\mathcal{L}_{ij}(j = 1, 2\ldots, r)$ such that $\sum_{j=1}^{r} w_j = 1$ and $\tilde{\mathcal{L}}_{ij} = \mathcal{L}_{ij}{}^{r*\omega_j}$ for any $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, r$.

Step 5 Derive the aggregated value $\mathcal{L}_i^-$ of all the cost criteria using the proposed CFEBOMWG, CFEBOMOWG or CFEBOMHG operator.
If we implement CFEBOMWG operator using Eq. (10), then we obtain
$\mathcal{L}_i^- = CFEBOMWG_{\omega}{}^{p,q}(\mathcal{L}_{i1}^-, \mathcal{L}_{i2}^-, \ldots, \mathcal{L}_{ir}^-)$

$$= \frac{1}{p+q}\left[\bigotimes_{E\ j,k=1;j\neq k}^{r}\begin{pmatrix} p((\mathcal{L}_{ij}^-)^{r\omega_j}) \\ \bigoplus_E \\ q((\mathcal{L}_{ik}^-)^{r\omega_k}) \end{pmatrix}\right]^{\frac{1}{r(r-1)}} \quad (23)$$

If we implement CFEBOMOWG operator using Eq. (12), then we obtain
$\mathcal{L}_i^- = CFEBOMOWG_{\omega}{}^{p,q}(\mathcal{L}_{i1}^-, \mathcal{L}_{i2}^-, \ldots, \mathcal{L}_{in}^-)$

$$= \frac{1}{p+q}\left[\bigotimes_{E\ j,k=1;j\neq k}^{r}\begin{pmatrix} p((\mathcal{L}_{\sigma(ij)}^-)^{r\omega_j}) \\ \bigoplus_E \\ q((\mathcal{L}_{\sigma(ik)}^-)^{r\omega_k}) \end{pmatrix}\right]^{\frac{1}{r(r-1)}} \quad (24)$$

If we implement CFEBOMHG operator using Eq. (14), then we obtain
$\mathcal{L}_i^- = CFEBOMHG_{\omega,w}{}^{p,q}(\mathcal{L}_{i1}^-, \mathcal{L}_{i2}^-, \ldots, \mathcal{L}_{in}^-)$

$$= \frac{1}{p+q}\left[\bigotimes_{E\ j,k=1;j\neq k}^{r}\begin{pmatrix} p((\tilde{\mathcal{L}}_{\sigma(ij)}^-)^{r\omega_j}) \\ \bigoplus_E \\ q((\tilde{\mathcal{L}}_{\sigma(ik)}^-)^{r\omega_k}) \end{pmatrix}\right]^{\frac{1}{r(r-1)}} \quad (25)$$

where $w = (w_1, w_2, \ldots w_r)^T$ is the weighting vector corresponding to CFNs $\mathcal{L}_{ij}(j = 1, 2, \ldots, r)$ such that $\sum_{j=1}^{r} w_j = 1$ and $\tilde{\mathcal{L}}_{ij} = \mathcal{L}_{ij}{}^{r*\omega_j}$ for any $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, r$.

Step 6 Compute the relative significance value RS_i of each alternative $\mathcal{A}_i(i = 1, 2, \ldots, m)$ based on the score values of $\mathcal{L}_i^+$ and $\mathcal{L}_i^-$ using Eq. (27).

$$RS_i = S(\mathcal{L}_i^+) + \frac{S(\mathcal{L}_{min}^-)\sum_{i=1}^{m} S(\mathcal{L}_i^-)}{S(\mathcal{L}_i^-)\sum_{i=1}^{m}\frac{S(\mathcal{L}_{min}^-)}{S(\mathcal{L}_i^-)}} \quad (26)$$

In simplified form,

$$RS_i = S(\mathcal{L}_i^+) + \frac{\sum_{i=1}^{m} S(\mathcal{L}_i^-)}{S(\mathcal{L}_i^-)\sum_{i=1}^{m}\frac{1}{S(\mathcal{L}_i^-)}} \quad (27)$$

Step 7 Obtain the degree of utility DU_i of each alternative $\mathcal{A}_i(i = 1, 2, \ldots, m)$ using Eq. (28).

$$DU_i = \frac{RS_i}{RS_{max}} \quad (28)$$

where $RS_{max} = max\{RS_i\}$.

Step 8 The alternatives are ranked based on the descending order of their respective degrees of utility and the highest DU_i gives the most preferred alternative $\mathcal{A}_i$.

V. Numerical Illustration and Comparative Analysis

This section presents a numerical illustration of a real-life instance to establish the applicability of the proposed MCGDM approach. Furthermore, we conduct a sensitivity analysis of the parameters and a comparative examination to analyze the advantages of the presented approach.

In today's financial risk environment, consumers worry about rising financial risks contributing to the financial system's instability. Hence, this section presents a real-life example of examining a financial investment risk to ensure the benefits of investment to demonstrate the proposed approach. Consider the problem of selecting the most deserving company for financial investment of an enterprise by analyzing the desirability of five deserving finance investment companies $\mathcal{A}_i(i = 1, 2, 3, 4, 5)$. We invite three experts $\mathcal{E}_t(t = 1, 2, 3)$ from the enterprise's management sector, each with a weight of $\lambda = (0.23, 0.40, 0.37)^T$, to make decisions by analyzing the five companies based on the following eight criteria: $\mathcal{C}_1$: Operating cost, $\mathcal{C}_2$: Assets structure risk, $\mathcal{C}_3$: Operation and management risk, $\mathcal{C}_4$: Capability of management, $\mathcal{C}_5$: Capability of handling emergencies, $\mathcal{C}_6$: Capital turnover, $\mathcal{C}_7$: Market risk, and $\mathcal{C}_8$: Environmental risk. Here, the criteria $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_7$, and $\mathcal{C}_8$ are cost type whereas $\mathcal{C}_4, \mathcal{C}_5$, and $\mathcal{C}_6$ are benefit type. The weight of the criteria is assumed to be unknown. The experts provide decision information for the five companies concerning the defined criteria in terms of CFNs.

We use the MCGDM approach proposed in the section to expound on this financial investment risk analysis problem. Here, we executed the process with $p = 1$ and $q = 2$ while we discussed other values for p and q in the subsection.

Step 1 The decision information of each expert $\mathcal{E}_t$ is accumulated in the form of a CF decision matrices and are given in Table III, IV and V.

TABLE III
CF DECISION INFORMATION GIVEN BY THE EXPERT E_1

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$	$\mathcal{C}_6$	$\mathcal{C}_7$	$\mathcal{C}_8$
$\mathcal{A}_1$	$\langle 0.8, 0.6, 0.6 \rangle$	$\langle 0.7, 0.7, 0.5 \rangle$	$\langle 0.7, 0.6, 0.6 \rangle$	$\langle 0.7, 0.6, 0.5 \rangle$	$\langle 0.9, 0.3, 0.4 \rangle$	$\langle 0.8, 0.7, 0.1 \rangle$	$\langle 0.6, 0.5, 0.4 \rangle$	$\langle 0.5, 0.7, 0.6 \rangle$
$\mathcal{A}_2$	$\langle 0.7, 0.4, 0.6 \rangle$	$\langle 0.7, 0.5, 0.7 \rangle$	$\langle 0.7, 0.5, 0.7 \rangle$	$\langle 0.6, 0.4, 0.5 \rangle$	$\langle 0.5, 0.6, 0.7 \rangle$	$\langle 0.9, 0.4, 0.2 \rangle$	$\langle 0.8, 0.5, 0.6 \rangle$	$\langle 0.7, 0.4, 0.6 \rangle$
$\mathcal{A}_3$	$\langle 0.8, 0.5, 0.6 \rangle$	$\langle 0.8, 0.7, 0.6 \rangle$	$\langle 0.8, 0.6, 0.7 \rangle$	$\langle 0.4, 0.6, 0.7 \rangle$	$\langle 0.8, 0.7, 0.3 \rangle$	$\langle 0.2, 0.5, 0.8 \rangle$	$\langle 0.7, 0.4, 0.8 \rangle$	$\langle 0.6, 0.5, 0.6 \rangle$
$\mathcal{A}_4$	$\langle 0.5, 0.6, 0.8 \rangle$	$\langle 0.6, 0.6, 0.6 \rangle$	$\langle 0.7, 0.6, 0.5 \rangle$	$\langle 0.6, 0.7, 0.6 \rangle$	$\langle 0.6, 0.9, 0.3 \rangle$	$\langle 0.3, 0.4, 0.6 \rangle$	$\langle 0.3, 0.7, 0.8 \rangle$	$\langle 0.7, 0.4, 0.5 \rangle$
$\mathcal{A}_5$	$\langle 0.4, 0.7, 0.7 \rangle$	$\langle 0.6, 0.7, 0.6 \rangle$	$\langle 0.6, 0.7, 0.4 \rangle$	$\langle 0.6, 0.5, 0.5 \rangle$	$\langle 0.3, 0.4, 0.8 \rangle$	$\langle 0.7, 0.8, 0.4 \rangle$	$\langle 0.4, 0.6, 0.5 \rangle$	$\langle 0.3, 0.9, 0.1 \rangle$

TABLE IV
CF DECISION INFORMATION GIVEN BY THE EXPERT E_2

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$	$\mathcal{C}_6$	$\mathcal{C}_7$	$\mathcal{C}_8$
$\mathcal{A}_1$	$\langle 0.3, 0.6, 0.5 \rangle$	$\langle 0.6, 0.3, 0.8 \rangle$	$\langle 0.8, 0.3, 0.4 \rangle$	$\langle 0.8, 0.2, 0.3 \rangle$	$\langle 0.3, 0.5, 0.5 \rangle$	$\langle 0.5, 0.7, 0.8 \rangle$	$\langle 0.8, 0.5, 0.1 \rangle$	$\langle 0.2, 0.5, 0.6 \rangle$
$\mathcal{A}_2$	$\langle 0.5, 0.3, 0.7 \rangle$	$\langle 0.7, 0.4, 0.5 \rangle$	$\langle 0.4, 0.7, 0.5 \rangle$	$\langle 0.3, 0.2, 0.6 \rangle$	$\langle 0.6, 0.7, 0.4 \rangle$	$\langle 0.7, 0.5, 0.6 \rangle$	$\langle 0.6, 0.7, 0.2 \rangle$	$\langle 0.4, 0.8, 0.4 \rangle$
$\mathcal{A}_3$	$\langle 0.4, 0.7, 0.6 \rangle$	$\langle 0.3, 0.3, 0.8 \rangle$	$\langle 0.6, 0.5, 0.8 \rangle$	$\langle 0.5, 0.7, 0.6 \rangle$	$\langle 0.9, 0, 0.2 \rangle$	$\langle 0.7, 0.4, 0.5 \rangle$	$\langle 0.5, 0.4, 0.8 \rangle$	$\langle 0.5, 0.8, 0.3 \rangle$
$\mathcal{A}_4$	$\langle 0.8, 0.3, 0.2 \rangle$	$\langle 0.7, 0.6, 0.3 \rangle$	$\langle 0.7, 0.5, 0.4 \rangle$	$\langle 0.9, 0.1, 0 \rangle$	$\langle 0.3, 0.2, 0.9 \rangle$	$\langle 0.8, 0.3, 0.2 \rangle$	$\langle 0.6, 0.4, 0.3 \rangle$	$\langle 0.8, 0.7, 0.2 \rangle$
$\mathcal{A}_5$	$\langle 0.7, 0.5, 0.6 \rangle$	$\langle 0.4, 0.7, 0.5 \rangle$	$\langle 0.3, 0.4, 0.6 \rangle$	$\langle 0.8, 0.3, 0.7 \rangle$	$\langle 0.4, 0.2, 0.7 \rangle$	$\langle 0.1, 0.3, 0.7 \rangle$	$\langle 0.7, 0.7, 0.4 \rangle$	$\langle 0.8, 0.3, 0.2 \rangle$

TABLE V
CF DECISION INFORMATION GIVEN BY THE EXPERT E_3

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$	$\mathcal{C}_6$	$\mathcal{C}_7$	$\mathcal{C}_8$
$\mathcal{A}_1$	$\langle 0.1, 0.3, 0.7 \rangle$	$\langle 0.7, 0.4, 0.5 \rangle$	$\langle 0.8, 0.3, 0.4 \rangle$	$\langle 0.3, 0.5, 0.5 \rangle$	$\langle 0.3, 0.5, 0.5 \rangle$	$\langle 0.5, 0.7, 0.8 \rangle$	$\langle 0.8, 0.5, 0.1 \rangle$	$\langle 0.2, 0.5, 0.6 \rangle$
$\mathcal{A}_2$	$\langle 0.4, 0.7, 0.7 \rangle$	$\langle 0.8, 0.7, 0.1 \rangle$	$\langle 0.3, 0.4, 0.6 \rangle$	$\langle 0.9, 0, 0.2 \rangle$	$\langle 0.6, 0.7, 0.4 \rangle$	$\langle 0.7, 0.5, 0.6 \rangle$	$\langle 0.6, 0.7, 0.2 \rangle$	$\langle 0.4, 0.8, 0.4 \rangle$
$\mathcal{A}_3$	$\langle 0.6, 0.4, 0.3 \rangle$	$\langle 0.5, 0.6, 0.8 \rangle$	$\langle 0.7, 0.4, 0.5 \rangle$	$\langle 0.8, 0.3, 0.2 \rangle$	$\langle 0.9, 0, 0.2 \rangle$	$\langle 0.7, 0.4, 0.5 \rangle$	$\langle 0.5, 0.4, 0.8 \rangle$	$\langle 0.5, 0.8, 0.3 \rangle$
$\mathcal{A}_4$	$\langle 0.6, 0.7, 0.4 \rangle$	$\langle 0.8, 0.7, 0.5 \rangle$	$\langle 0.8, 0.6, 0.6 \rangle$	$\langle 0.5, 0.1, 0.8 \rangle$	$\langle 0.3, 0.2, 0.9 \rangle$	$\langle 0.8, 0.3, 0.2 \rangle$	$\langle 0.6, 0.4, 0.3 \rangle$	$\langle 0.8, 0.7, 0.2 \rangle$
$\mathcal{A}_5$	$\langle 0.4, 0.2, 0.8 \rangle$	$\langle 0.3, 0.4, 0.6 \rangle$	$\langle 0.7, 0.5, 0.7 \rangle$	$\langle 0.3, 0.8, 0.5 \rangle$	$\langle 0.4, 0.2, 0.7 \rangle$	$\langle 0.1, 0.3, 0.7 \rangle$	$\langle 0.7, 0.7, 0.4 \rangle$	$\langle 0.8, 0.3, 0.2 \rangle$

Step 2 The aggregated CF decision matrix $\mathfrak{D}$ obtained by fusing the experts decision information using CFEBOMWG operator is given in Table VI.

TABLE VI
AGGREGATED CF DECISION INFORMATION

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$	$\mathcal{C}_6$	$\mathcal{C}_7$	$\mathcal{C}_8$
$\mathcal{A}_1$	$\langle 0.48, 0.09, 0.12 \rangle$	$\langle 0.70, 0.08, 0.13 \rangle$	$\langle 0.79, 0.07, 0.08 \rangle$	$\langle 0.69, 0.08, 0.07 \rangle$	$\langle 0.66, 0.09, 0.09 \rangle$	$\langle 0.59, 0.12, 0.13 \rangle$	$\langle 0.77, 0.07, 0.04 \rangle$	$\langle 0.61, 0.13, 0.10 \rangle$
$\mathcal{A}_2$	$\langle 0.60, 0.09, 0.14 \rangle$	$\langle 0.76, 0.10, 0.09 \rangle$	$\langle 0.54, 0.10, 0.11 \rangle$	$\langle 0.71, 0.04, 0.08 \rangle$	$\langle 0.66, 0.13, 0.10 \rangle$	$\langle 0.70, 0.10, 0.09 \rangle$	$\langle 0.75, 0.13, 0.06 \rangle$	$\langle 0.67, 0.15, 0.09 \rangle$
$\mathcal{A}_3$	$\langle 0.65, 0.10, 0.09 \rangle$	$\langle 0.60, 0.10, 0.17 \rangle$	$\langle 0.72, 0.09, 0.14 \rangle$	$\langle 0.64, 0.11, 0.10 \rangle$	$\langle 0.88, 0.08, 0.04 \rangle$	$\langle 0.66, 0.07, 0.13 \rangle$	$\langle 0.56, 0.06, 0.18 \rangle$	$\langle 0.69, 0.11, 0.06 \rangle$
$\mathcal{A}_4$	$\langle 0.69, 0.10, 0.09 \rangle$	$\langle 0.74, 0.13, 0.08 \rangle$	$\langle 0.76, 0.10, 0.09 \rangle$	$\langle 0.73, 0.05, 0.12 \rangle$	$\langle 0.67, 0.14, 0.10 \rangle$	$\langle 0.71, 0.05, 0.06 \rangle$	$\langle 0.61, 0.10, 0.11 \rangle$	$\langle 0.69, 0.09, 0.08 \rangle$
$\mathcal{A}_5$	$\langle 0.59, 0.09, 0.15 \rangle$	$\langle 0.52, 0.12, 0.10 \rangle$	$\langle 0.64, 0.10, 0.12 \rangle$	$\langle 0.67, 0.11, 0.11 \rangle$	$\langle 0.52, 0.06, 0.13 \rangle$	$\langle 0.66, 0.11, 0.10 \rangle$	$\langle 0.57, 0.13, 0.08 \rangle$	$\langle 0.71, 0.13, 0.05 \rangle$

Step 3 Weights of the criteria are calculated using the CRITIC method given in section 2

(i) The correlation coefficients between the criteria are calculated using Eq. (16) and presented in Table VII.

TABLE VII
CORRELATION COEFFICIENTS BETWEEN CRITERIA

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$	$\mathcal{C}_6$	$\mathcal{C}_7$	$\mathcal{C}_8$
$\mathcal{C}_1$	1	0.09	0.91	0.19	0.36	0.81	-0.69	0.70
$\mathcal{C}_2$	0.09	1	-0.24	0.83	-0.11	0.31	0.65	-0.47
$\mathcal{C}_3$	0.91	-0.24	1	-0.08	0.28	0.77	-0.85	0.92
$\mathcal{C}_4$	0.19	0.83	-0.08	1	-0.53	0.50	0.41	0.19
$\mathcal{C}_5$	0.36	-0.11	0.28	-0.53	1	-0.04	-0.30	0.02
$\mathcal{C}_6$	0.81	0.37	0.77	0.50	-0.04	1	-0.33	0.64
$\mathcal{C}_7$	-0.69	0.66	-0.85	0.41	-0.30	-0.33	1	-0.86
$\mathcal{C}_8$	0.70	-0.47	0.92	-0.19	0.02	0.64	-0.86	1

(ii) The standard deviation of each criteria $\mathcal{C}_j(j = 1, 2, \ldots, 8)$ calculated using Eq. (17) are given in Table VIII and the criteria indices evaluated using Eq. (18) are given in Table IX.

TABLE VIII
STANDARD DEVIATION OF EACH CRITERIA

$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$	$\mathcal{C}_6$	$\mathcal{C}_7$	$\mathcal{C}_8$
0.0823	0.1276	0.1538	0.0491	0.2028	0.0607	-0.1367	0.0502

TABLE IX
CRITERIA INDICES

$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$	$\mathcal{C}_6$	$\mathcal{C}_7$	$\mathcal{C}_8$
0.3810	0.7504	0.8135	0.2875	1.4834	0.2597	1.2241	0.3134

(iii) The criteria weights obtained using Eq. (19) are given in Table X.

TABLE X
CRITERIA WEIGHTS

$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$	$\mathcal{C}_6$	$\mathcal{C}_7$	$\mathcal{C}_8$
0.0691	0.1361	0.1476	0.0522	0.2691	0.0471	0.2220	0.0568

Step 4 For each alternative A_i, the aggregated values $\mathcal{L}_i^+(i = 1,2,3,4,5)$ of the benefit criteria $\mathcal{C}_4, \mathcal{C}_5$ and $\mathcal{C}_6$ evaluated using Eqs (23), (24) and (25) are given in Table XI.

TABLE XI
AGGREGATED VALUE OF BENEFIT CRITERIA

		Operator		
Alternative	Aggregated value	CFEBOMWG	CFEBOMOWG	CFEBOMHG $w=(0.25, 0.05, 0.09)^T$
$\mathcal{A}_1$	$\mathcal{L}_1^+$	$\langle 0.7851, 0, 0\rangle$	$\langle 0.7352, 0, 0\rangle$	$\langle 0.8420, 0, 0\rangle$
$\mathcal{A}_2$	$\mathcal{L}_2^+$	$\langle 0.7931, 0, 0\rangle$	$\langle 0.7871, 0, 0\rangle$	$\langle 0.8497, 0, 0\rangle$
$\mathcal{A}_3$	$\mathcal{L}_3^+$	$\langle 0.7611, 0, 0\rangle$	$\langle 0.7672, 0, 0\rangle$	$\langle 0.8325, 0, 0\rangle$
$\mathcal{A}_4$	$\mathcal{L}_4^+$	$\langle 0.8006, 0, 0\rangle$	$\langle 0.7935, 0, 0\rangle$	$\langle 0.8646, 0, 0\rangle$
$\mathcal{A}_5$	$\mathcal{L}_5^+$	$\langle 0.7830, 0, 0\rangle$	$\langle 0.7806, 0, 0\rangle$	$\langle 0.8429, 0, 0\rangle$

Step 5 For each alternative $\mathcal{A}_i$, the aggregated value $\mathcal{L}_i^-(i = 1,2,3,4,5)$ of the cost criteria $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_7$ and $\mathcal{C}_8$ evaluated using Eqs (23), (24) and (25) are given in Table XII.

TABLE XII
AGGREGATED VALUE OF COST CRITERIA

		Operator		
Alternative	Aggregated value	CFEBOMWG	CFEBOMOWG	CFEBOMHG $w=(0.125, 0.15, 0.2, 0.11, 0.025)^T$
$\mathcal{A}_1$	$\mathcal{L}_1^-$	$\langle 0.8622, 0, 0\rangle$	$\langle 0.8622, 0, 0\rangle$	$\langle 0.9554, 0, 0\rangle$
$\mathcal{A}_2$	$\mathcal{L}_2^-$	$\langle 0.8745, 0, 0\rangle$	$\langle 0.8668, 0, 0\rangle$	$\langle 0.9516, 0, 0\rangle$
$\mathcal{A}_3$	$\mathcal{L}_3^-$	$\langle 0.8880, 0, 0\rangle$	$\langle 0.8740, 0, 0\rangle$	$\langle 0.9564, 0, 0\rangle$
$\mathcal{A}_4$	$\mathcal{L}_4^-$	$\langle 0.8729, 0, 0\rangle$	$\langle 0.8634, 0, 0\rangle$	$\langle 0.9523, 0, 0\rangle$
$\mathcal{A}_5$	$\mathcal{L}_5^-$	$\langle 0.8473, 0, 0\rangle$	$\langle 0.8812, 0, 0.0280\rangle$	$\langle 0.9550, 0, 0\rangle$

Step 6 The relative significance values RS_i of each alternative $\mathcal{A}_i(i = 1,2,3,4,5)$ calculated using Eq. (26) based on the score values of $\mathcal{L}_i^+$ and $\mathcal{L}_i^-$ are given in Table XIII.

TABLE XIII
RELATIVE SIGNIFICANCE VALUES OF EACH ALTERNATIVE

		Operator		
Alternative	Relative Significance	CFEBOMWG	CFEBOMOWG	CFEBOMHG
$\mathcal{A}_1$	$\mathbf{RS_1}$	1.1225	1.1753	1.4874
$\mathcal{A}_2$	$\mathbf{RS_2}$	1.1358	1.0867	1.4606
$\mathcal{A}_3$	$\mathbf{RS_3}$	1.2289	1.1379	1.5115
$\mathcal{A}_4$	$\mathbf{RS_4}$	1.1194	1.0688	1.4320
$\mathcal{A}_5$	$\mathbf{RS_5}$	1.0937	1.1307	1.4845

Step 7 The degree of utility DU_i of each $\mathcal{A}_i(i = 1,2,3,4,5)$ calculated using (28) are shown in Table XIV.

TABLE XIV
DEGREE OF UTILITY OF EACH ALTERNATIVE

		Operator		
Alternative	Degree of utility	CFEBOMWG	CFEBOMOWG	CFEBOMHG
$\mathcal{A}_1$	$\mathbf{DU_1}$	0.9134	1	0.9841
$\mathcal{A}_2$	$\mathbf{DU_2}$	0.9243	0.9246	0.9663
$\mathcal{A}_3$	$\mathbf{DU_3}$	1	0.9682	1
$\mathcal{A}_4$	$\mathbf{DU_4}$	0.9109	0.9034	0.99474
$\mathcal{A}_5$	$\mathbf{DU_5}$	0.8900	0.9621	0.9821

Step 8 The ranking of the alternatives $\mathcal{A}_i(i = 1,2,3,4,5)$ based on the descending order of their utility degrees is displayed in Table XV.

TABLE XV
RANKING OF THE ALTERNATIVES

Operator	Ranking	Most preferred alternative
CFEBOMWG	$\mathcal{A}_3 > \mathcal{A}_2 > \mathcal{A}_1 > \mathcal{A}_4 > \mathcal{A}_5$	$\mathcal{A}_3$
CFEBOMOWG	$\mathcal{A}_1 > \mathcal{A}_3 > \mathcal{A}_5 > \mathcal{A}_2 > \mathcal{A}_4$	$\mathcal{A}_1$
CFEBOMHGM	$\mathcal{A}_3 > \mathcal{A}_1 > \mathcal{A}_5 > \mathcal{A}_2 > \mathcal{A}_4$	$\mathcal{A}_3$

Hence, the most preferred alternative is $\mathcal{A}_3$ for CFEBOMWG and CFEBOMHG operators and $\mathcal{A}_1$ for CFEBOMOWG operator.

A. Sensitivity analysis of parameters p and q on decision results

Parameters are significant in the DM process as they affect the decision results. Analysis of the parameters helps us to find which one is more crucial during the DM process. Hence, we perform a sensitivity analysis on the parameters p and q for the proposed approach to investigate their effect on the decision results. Table XVI summarizes the ranking order of the alternatives for the proposed operators based on the different values for the parameters p and q.

TABLE XVI
SENSITIVITY ANALYSIS OF THE PARAMETERS p AND q

Operator	Parameter	Ranking	Most preferred alternative
CFEBOMWG	$p=1, q=3$	$\mathcal{A}_3 > \mathcal{A}_5 > \mathcal{A}_4 > \mathcal{A}_2 > \mathcal{A}_1$	$\mathcal{A}_3$
	$p=2, q=3$	$\mathcal{A}_3 > \mathcal{A}_5 > \mathcal{A}_2 > \mathcal{A}_4 > \mathcal{A}_1$	$\mathcal{A}_3$
	$p=2, q=4$	$\mathcal{A}_3 > \mathcal{A}_5 > \mathcal{A}_2 > \mathcal{A}_4 > \mathcal{A}_1$	$\mathcal{A}_3$
	$p=2, q=2$	$\mathcal{A}_3 > \mathcal{A}_5 > \mathcal{A}_2 > \mathcal{A}_4 > \mathcal{A}_1$	$\mathcal{A}_3$
	$p=3, q=3$	$\mathcal{A}_3 > \mathcal{A}_2 > \mathcal{A}_4 > \mathcal{A}_5 > \mathcal{A}_1$	$\mathcal{A}_3$
	$p=3, q=2$	$\mathcal{A}_3 > \mathcal{A}_5 > \mathcal{A}_2 > \mathcal{A}_4 > \mathcal{A}_1$	$\mathcal{A}_3$
CFEBOMOWG	$p=1, q=3$	$\mathcal{A}_5 > \mathcal{A}_3 > \mathcal{A}_4 > \mathcal{A}_2 > \mathcal{A}_1$	$\mathcal{A}_5$
	$p=2, q=3$	$\mathcal{A}_5 > \mathcal{A}_3 > \mathcal{A}_2 > \mathcal{A}_4 > \mathcal{A}_1$	$\mathcal{A}_5$
	$p=2, q=4$	$\mathcal{A}_5 > \mathcal{A}_3 > \mathcal{A}_2 > \mathcal{A}_4 > \mathcal{A}_1$	$\mathcal{A}_5$
	$p=2, q=2$	$\mathcal{A}_5 > \mathcal{A}_3 > \mathcal{A}_4 > \mathcal{A}_2 > \mathcal{A}_1$	$\mathcal{A}_5$
	$p=3, q=3$	$\mathcal{A}_5 > \mathcal{A}_3 > \mathcal{A}_2 > \mathcal{A}_4 > \mathcal{A}_1$	$\mathcal{A}_5$
	$p=3, q=2$	$\mathcal{A}_3 > \mathcal{A}_5 > \mathcal{A}_2 > \mathcal{A}_4 > \mathcal{A}_1$	$\mathcal{A}_3$
CFEBOMHG	$p=1, q=3$	$\mathcal{A}_5 > \mathcal{A}_3 > \mathcal{A}_4 > \mathcal{A}_2 > \mathcal{A}_1$	$\mathcal{A}_5$
	$p=2, q=3$	$\mathcal{A}_5 > \mathcal{A}_3 > \mathcal{A}_2 > \mathcal{A}_4 > \mathcal{A}_1$	$\mathcal{A}_5$
	$p=2, q=4$	$\mathcal{A}_5 > \mathcal{A}_3 > \mathcal{A}_2 > \mathcal{A}_4 > \mathcal{A}_1$	$\mathcal{A}_5$
	$p=2, q=2$	$\mathcal{A}_5 > \mathcal{A}_3 > \mathcal{A}_2 > \mathcal{A}_4 > \mathcal{A}_1$	$\mathcal{A}_5$
	$p=3, q=3$	$\mathcal{A}_5 > \mathcal{A}_3 > \mathcal{A}_2 > \mathcal{A}_4 > \mathcal{A}_1$	$\mathcal{A}_5$
	$p=3, q=2$	$\mathcal{A}_5 > \mathcal{A}_3 > \mathcal{A}_2 > \mathcal{A}_4 > \mathcal{A}_1$	$\mathcal{A}_5$

From the analysis table, we observe that for different values of parameters, the most preferred alternative is $\mathcal{A}_3$ for CFEBOMWG operator and $\mathcal{A}_5$ for CFEBOMOWG and CFEBOMHG operators. Table XVI shows that the decision-maker's pessimism can influence our proposed operators.

Decision-makers can assign higher values to the parameters based on their preference for pessimism to demonstrate the reliability of ranking results. The data from Table XVI is visually depicted in Figures 1, 2, and 3.

From Figure 1, we can see that for a fixed $'p'$ and varying $'q'$ like $p = 2, q = 3$ and $p = 2, q = 4$, the optimal alternative is A_3 for CFEBOMWG operator and A_5 for the CFEBOMOWG and CFEBOMHG operators.

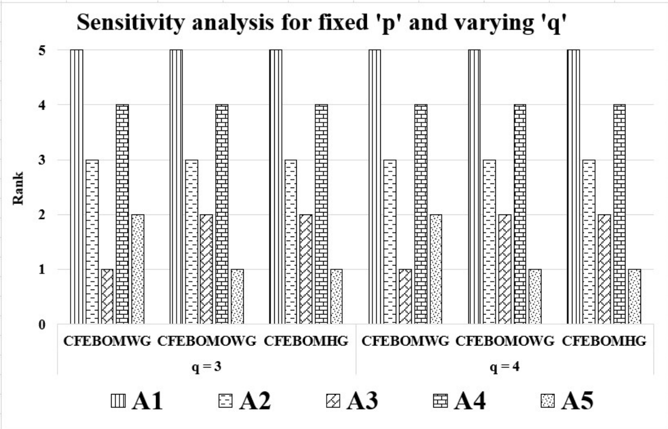

Fig. 1. Ranking of alternatives when p=2 and q=3,4 utilizing the proposed operators.

We can easily observe that Figure 2 is equivalent to Figure 1 with the same optimal results for a fixed $'q'$ and varying $'p'$ like $p = 1, q = 3$ and $p = 2, q = 3$.

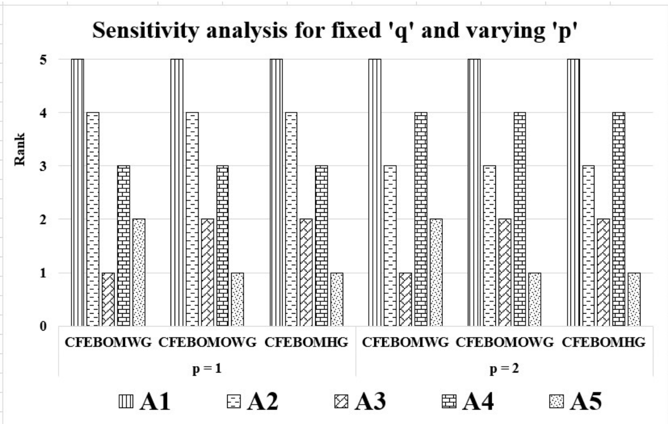

Fig. 2. Ranking of alternatives when p=1,2 and q=3 utilizing the proposed operators.

From Figure 3, we observe that for equal values of p and q, like $p = q = 2$ and $p = q = 3$, the ranking of alternatives remains unchanged for CFEBOMOWG and CFEBOMHG operators. In contrast, the optimal choice is A_3 for the CFEBOMWG operator.

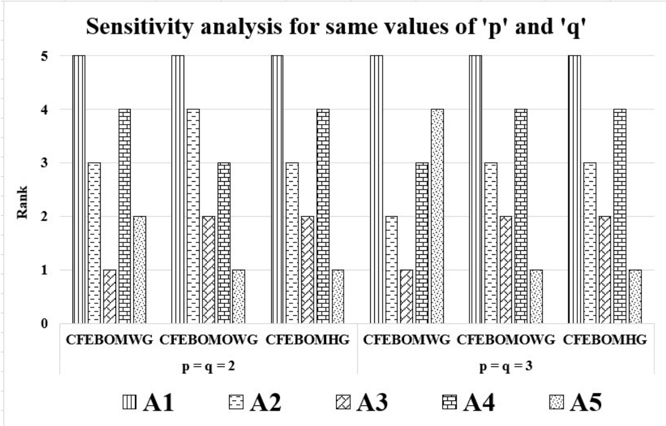

Fig. 3. Ranking of alternatives when p=q=2 and p=q=3 utilizing the proposed operators.

B. Comparative study

In this section, we conduct a detailed comparison study with the existing operators to verify the efficacy of the proposed aggregation operators.

Table XVII reveals the comparison results of the proposed CFEBOMWG and CFEBOMOWG operators with CF weighted geometric (CFWG) and CF weighted geometric (CFOWG) operators for the numerical example discussed in [8].

TABLE XVII
COMPARISON RESULT 1

Operator	Ranking result for the illustration discussed in [8]	Most preferred alternative
CFWG [8]	$A_1 > A_5 > A_3 > A_4 > A_2$	A_1
CFOWG [8]	$A_1 > A_2 > A_4 > A_3 > A_5$	A_1
Proposed CFEBOMWG (for $p = 1, q = 2$)	$A_1 > A_5 > A_2 > A_4 > A_3$	A_1
Proposed CFEBOMOWG (for $p = 1, q = 2$)	$A_1 > A_5 > A_2 > A_4 > A_3$	A_1

Table XVIII reveals the comparison results of the proposed CFEBOMHG operator with the CF Hamacher hybrid weighted geometric (CFHHWG) operator for the numerical example discussed in [15]. The CFHHWG operator becomes the CF Einstein hybrid weighted geometric (CFEHWG) operator when $\sigma = 2$.

TABLE XVIII
COMPARISON RESULT 2

Operator	Ranking result for the illustration discussed in [15]	Most preferred alternative
CFHHWG (for $\sigma = 2$) [15]	$A_1 > A_5 > A_4 > A_2 > A_3$	A_1
Proposed CFEBOMHG (for $p = 1, q = 2$)	$A_1 > A_4 > A_5 > A_2 > A_3$	A_1

Table XIX reveals the comparison results of the proposed CFEBOMHG operator with the T-SF Einstein hybrid geometric (T-SFEHG) operator for the numerical example discussed in [30].

TABLE XIX
COMPARISON RESULT 3

Operator	Ranking result for the illustration discussed in [30]	Most preferred alternative
T-SFEHG (for t=3)[30]	$b_1 > b_3 > b_2$	b_1
Proposed CFEBOMHG (for $p = 1, q = 2$)	$b_1 > b_3 > b_2$	b_1

Table XX reveals the comparison results of the proposed CFEBOMWG and CFEBOMOWG operators with T-SF interactive geometric weighted aggregation (T-SFIWGA), T-SF weighted interactive geometric BOM (T-SFWIGBM), and T-SF weighted geometric BOM (T-SFWGDBM) operators for the numerical example discussed in [23].

TABLE XX
COMPARISON RESULT 4

Operator	Ranking result for the illustration discussed in [23]	Most preferred alternative
T-SFIWGA (for t=3) [23]	$A_4 > A_3 > A_2 > A_5 > A_1$	A_4
T-SFWIGBM (for $p = 2, q = 2$) [23]	$A_4 > A_3 > A_2 > A_5 > A_1$	A_4
T-SFWGDBM (for $p = 2, q = 2$)[23]	$A_3 > A_4 > A_2 > A_5 > A_1$	A_3
Proposed CFEBOMWG (for $p = 2, q = 2$)	$A_3 > A_2 > A_4 > A_1 > A_5$	A_3
Proposed CFEBOMOWG (for $p = 2, q = 2$)	$A_2 > A_4 > A_3 > A_5 > A_1$	A_2

Table XXI reveals the comparison results of the proposed CFEBOMHG operator with the T-SF Einstein hybrid interactive geometric (T-SFEHIG) operator for the numerical example discussed in [44].

TABLE XXI
COMPARISON RESULT 5

Operator	Ranking result for the illustration discussed in [44]	Most preferred alternative
T-SFEHIG (for t=3) [44]	$d_3 > d_2 > d_1$	d_3
Proposed CFEBOMHG (for $p = 1, q = 2$)	$d_1 > d_3 > d_2$	d_1

Table XXII reveals the comparison results of the proposed CFEBOMWG operator with the Q-ROF weighted BOM (Q-ROFWBM) and the Q-ROF weighted geometric BOM (Q-ROFWGBM) operators for the numerical example discussed in [51].

TABLE XXII
COMPARISON RESULT 6

Operator	Ranking result for the illustration discussed in [51]	Most preferred alternative
Q-ROFWBM (for $Q = 3$ and $p = q = 1$) [51]	$A_2 > A_4 > A_5 > A_1 > A_3$	A_2
Q-ROFWGBM (for $Q = 3$ and $p = q = 1$) [51]	$A_2 > A_4 > A_3 > A_5 > A_1$	A_2
Proposed CFEBOMWG (for $p = q = 1$)	$A_2 > A_4 > A_5 > A_1 > A_3$	A_2

From Tables XVII,XVIII, and XIX, we observe that the most preferred alternative based on our proposed operators is identical (A_1) with CFWG, CFOWG, CFHHWG, and T-SFEHG operators. From Table XXI, we identify that the most preferred alternative is A_3 for the T-SFEHIG operator. In contrast, we obtain A_1 for our CFEBOMWG operator, and the overall ranking also changes due to the absence of parameters in the T-SFEHIG operator. The existing operators are not adequate to capture interrelationships among the aggregated CFNs. Meanwhile, our proposed operators successfully encounter this adequacy with the assistance of the BOM aggregation function. From Table XX, we find that the best alternative is A_4 for T-SFIWGA and T-SFWIGBM operators and A_3 for T-SFWGDBM and CFEBOMWG operators. This contradiction in the result is due to the absence of parameters in the T-SFWIGBM operator and the consumption of different operations like interactive, Dombi, and Einstein operations. Table XXII reveals that the optimal selection remains consistent for both the proposed and existing operators. However, the sequence changes for Q-ROFWGBM and the proposed CFEBOMWG operators due to distinct arithmetic operations. Additionally, the chosen CFS introduces an advantageous feature to Q-ROF by incorporating neutral membership.

VI. CONCLUSION

Aggregation operators serve as indispensable tools for any DM process. While the literature documents various aggregation operators, some cannot explore the interrelationships between the aggregated elements. However, the BOM operator successfully addresses this issue by considering the interrelationships among the aggregated factors using the parameters p and q. This article presents a set of novel CF BOM geometric aggregation operators, namely, CFEBMG, CFEBMWG, CFEBMOWG, and CFEBMHG operators, all derived from the proposed Einstein operational laws of CFNs. We introduce these operators to tackle the CF-MCGDM problem. The article thoroughly investigates the fundamental properties of these proposed operators.

Furthermore, we employ the proposed operators to develop a DM approach known as the CF-COPRAS approach. This approach effectively solves CF-MCGDM problems by determining the unknown weights of the criteria through an objective weighting method called the CRITIC method. We present a numerical illustration of a real-life scenario to demonstrate the applicability of the proposed approach. Specifically, the instance involves selecting the best financial investment company for an enterprise to minimize investment risks. The article concludes with a parameter analysis and comparative study, highlighting the effectiveness and superiority of the proposed approach. We plan to develop additional CF aggregation operators based on Dombi, Frank, Aczel-Alzina, Yager, and other operational laws. Additionally, we express interest in devising new CF-MGADM solving methods such as ELECTRE, PROMETHEE, VIKOR, and others.

REFERENCES

[1] L.A. Zadeh, "Fuzzy sets", *Information and Control*, vol. 8, no. 3, pp. 338-353, 1965.
[2] K.T. Atanassov, "Intuitionistic fuzzy sets", *Fuzzy Sets and Systems*, vol. 20, no. 1, pp. 87-96, 1986.
[3] R.R. Yager, "Pythagorean fuzzy subsets", *Joint IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS), IEEE*, pp. 57-61, 2013.
[4] T. Senapati and R.R. Yager, "Fermatean fuzzy sets", *Journal of Ambient Intelligence and Humanized Computing*, vol. 11, no. 2, pp. 663-674, 2020.
[5] R.R. Yager, "Generalized orthopair fuzzy set", *IEEE Transactions on Fuzzy Systems*, vol. 25, no. 5, pp. 1222-1230, 2016.
[6] B.C. Cuong and V. Kreinovich, "Picture fuzzy sets - a new concept for computational intelligence problems", *Third World Congress on Information and Communication Technologies*, pp. 1-6, 2013.
[7] F.K. Gündoğdu, "Principals of spherical fuzzy sets", *International Conference on Intelligent and Fuzzy Systems* , pp. 15-23, 2019.
[8] A. Khan, A.U. Jan, F. Amin, and A. Zeb, "Multiple attribute decision-making based on cubical fuzzy aggregation operators", *Granular Computing*, vol. 7, no. 2, pp. 393–410, 2022.
[9] P. Liu and S.M. Chen, "Group decision making based on Heronian aggregation operators of intuitionistic fuzzy numbers", *IEEE Transactions on Cybernetics*, vol. 47, no. 9, pp. 2514-2530, 2016.
[10] G. Wei and M. Lu, "Pythagorean fuzzy power aggregation operators in multiple attribute decision making", *International Journal of Intelligent Systems*, vol. 33, no. 1, pp. 169-186, 2017.
[11] S.B. Aydemir and S. Yilmaz Gunduz, "Fermatean fuzzy TOPSIS method with Dombi aggregation operators and its application in multi-criteria decision making", *Journal of Intelligent & Fuzzy Systems*, vol. 39, no. 1, pp. 851-869, 2020.
[12] G. Wei, "Picture fuzzy aggregation operators and their application to multiple attribute decision making", *Journal of Intelligent & Fuzzy Systems*, vol. 33, no. 2, pp. 713-724, 2017.
[13] G. Wei, "Picture fuzzy Hamacher aggregation operators and their application to multiple attribute decision making", *Fundamenta Informaticae*, vol. 157, no. 3, pp. 271-320, 2018.

[14] X. Peng, J. Dai, and H. Garg, "Exponential operation and aggregation operator for q-rung orthopair fuzzy set and their decision-making method with a new score function", *International Journal of Intelligent Systems*, vol. 33, no. 11, pp. 2255-2282, 2018.

[15] A.U. Jan, O. Barukab, A. Khan, Y.B. Jun, and S.A. Khan, "Cubical fuzzy Hamacher aggregation operators in multiple attribute decision-making problems", *Computational and Applied Mathematics*, vol. 42, no. 3, pp. 1-25, 2023.

[16] T. Mahmood, K. Ullah, Q. Khan, and N. Jan , "An approach toward decision-making and medical diagnosis problems using the concept of spherical fuzzy sets", *Neural Computing and Applications*, vol. 31, no. 11, pp. 7041-7053, 2019.

[17] C. Bonferroni, "Sulle medie multiple di potenze", *Bollettino dell'Unione Matematica Italiana*, vol. 5, no. 3-4, pp. 267-270, 1950.

[18] M. Xia, Z. Xu, and B. Zhu, "Geometric Bonferroni means with their application in multi-criteria decision making", *Knowledge-Based Systems*, vol. 40, pp. 88-100, 2013.

[19] P. Liu, J. Liu, and S.M. Chen, "Some intuitionistic fuzzy Dombi Bonferroni mean operators and their application to multi-attribute group decision making", *Journal of the Operational Research Society*, vol. 69, no. 1, pp. 1-24, 2018.

[20] F. Ate and D. Akay , "Some picture fuzzy Bonferroni mean operators with their application to multicriteria decision making", *International Journal of Intelligent Systems*, vol. 35, no. 4, pp. 625-649, 2020.

[21] L. Liu, X. Wu, and G. Che, "Picture fuzzy interactional Bonferroni mean operators via strict triangular norms and applications to multi-criteria decision making", *IEEE Transactions on Fuzzy Systems*, vol. 31, no. 8, pp. 2632-2644, 2023.

[22] E. Farrokhizadeh, S.A. Seyfi Shishavan, Y. Donyatalab, F.K. Gündoğdu, and C. Kahraman, "Spherical fuzzy Bonferroni mean aggregation operators and their applications to multiple-attribute decision making", *Decision Making with Spherical Fuzzy Sets*, vol. 392, pp. 111-134, 2021.

[23] W. Yang and Y. Pang, "T-spherical fuzzy Bonferroni mean operators and their application in multiple attribute decision making", *Mathematics*, vol. 10, no. 6, pp. 1-33, 2022.

[24] W. Wang and X. Liu, "Intuitionistic fuzzy geometric aggregation operators based on Einstein operations", *International Journal of Intelligent Systems*, vol. 26, no. 11, pp. 1049-1075, 2011.

[25] K. Rahman, S. Abdullah, R. Ahmed, and M. Ullah, "Pythagorean fuzzy Einstein weighted geometric aggregation operator and their application to multiple attribute group decision making", *Journal of Intelligent & Fuzzy Systems*, vol. 33, no. 1, pp. 635-647, 2017.

[26] G. Cao, "Picture fuzzy Einstein hybrid-weighted aggregation operator and its application to multi-criteria group decision making", *Computational Intelligence and Neuroscience*, vol. 2022, pp. 1-20, 2022.

[27] S. Khan, S. Abdullah, and S. Ashraf, "Picture fuzzy aggregation information based on Einstein operations and their application in decision making", *Mathematical Sciences*, vol. 13, no. 3, pp. 213–229, 2019.

[28] P. Rani and A.R. Mishra, "Fermatean fuzzy Einstein aggregation operators-based MULTIMOORA method for electric vehicle charging station selection", *Expert Systems with Applications*, vol. 182, pp. 1-23, 2021.

[29] R.M. Zulqarnain, R. Ali, J. Awrejcewicz, I. Siddique, F. Jarad, and A. Iampan, "Some Einstein geometric aggregation operators for q-rung orthopair fuzzy soft set with their application in MCDM", *IEEE Access*, vol. 10, pp. 88469-88494, 2022.

[30] M. Munir, H. Kalsoom, K. Ullah, T. Mahmood, and Y.M. Chu, "T-spherical fuzzy Einstein hybrid aggregation operators and their applications in multi-attribute decision making problems", *Symmetry*, vol. 12, no. 3, pp. 1-24, 2020.

[31] A. Karasan, E. Ilbahar, and C. Kahraman, "A novel pythagorean fuzzy AHP and its application to landfill site selection problem", *Soft Computing*, vol. 23, no. 21, pp. 10953-10968, 2019.

[32] F.K. Gündoğdu and C. Kahraman, "A novel VIKOR method using spherical fuzzy sets and its application to warehouse site selection", *Journal of Intelligent & Fuzzy Systems*, vol. 37, no. 1, pp. 1197-1211, 2019.

[33] H. Garg and D. Rani, "An efiicient intuitionistic MULTOMOORA approach based on novel aggregation operators for the assessment of solid waste management techniques", *Applied Intelligence*, vol. 52, no. 1, pp. 4330-4363, 2022.

[34] X. Gong, J. Zhang, and X. Zhao, "A BWM-TODIM based integrated decision framework for financial technology selection with interval type-2 fuzzy sets", *IAENG International Journal of Applied Mathematics*, vol. 52, no. 4, pp. 826-837, 2022.

[35] Z. Yang, L. Zhang, and C. Liang, "A method for group decision making with multiplicative consistent interval-valued intuitionistic fuzzy preference relation", *IAENG International Journal of Applied Mathematics*, vol. 51, no. 1, pp. 152-163, 2021.

[36] S.M. Chen, S.H. Cheng, and T.C. Lan, "Multicriteria decision making based on the TOPSIS method and similarity measures between intuitionistic fuzzy values", *Information Sciences*, vol. 367, pp. 279-295, 2016.

[37] R. Krishankumar, H. Garg, K. Arun, A. Saha, K.S. Ravichandran, and S. Kar, "An integrated decision-making COPRAS approach to probabilistic hesitant fuzzy set information", *Complex & Intelligent Systems*, vol. 7, no. 5, pp. 2281-2298, 2021.

[38] J. Lu, S. Zhang, J. Wu, and Y. Wei, "COPRAS method for multiple attribute group decision making under picture fuzzy environment and their application to green supplier selection", *Technological and Economic Development of Economy*, vol. 27, no. 2, pp. 369-385, 2021.

[39] C. Kahraman, S.C. Onar, and B. Öztayi, "A novel spherical fuzzy CRITIC method and its application to prioritization of supplier selection criteria", *Journal of Intelligent & Fuzzy Systems*, vol. 42, no. 1, pp. 29-36, 2022.

[40] Y. Rong, Q. Li, and Z. Pei, "A novel q-rung orthopair fuzzy multi-attribute group decision-making approach based on Schweizer-Sklar operations and improved COPRAS method", *Proceedings of the 4th International Conference on Computer Science and Application Engineering* , pp. 1-6, 2020.

[41] E. Haktanr and C. Kahraman, "A novel picture fuzzy CRITIC & REGIME methodology: wearable health technology application", *Engineering Applications of Artificial Intelligence*, vol. 113, pp. 1-14, 2022.

[42] Q.S. Qi, "GRA and CRITIC method for intuitionistic fuzzy multiattribute group decision making and application to development potentiality evaluation of cultural and creative garden", *Mathematical Problems in Engineering*, vol. 2021, pp. 1-9, 2021.

[43] B. Wan and S. Zhou, "A integrating CRITIC-WASPAS group decision making method under interval-valued q-rung orthogonal fuzzy enviroment", *arXiv:2201.01027*, pp. 1-47, 2022.

[44] S. Zeng, M. Munir, T. Mahmood, and M. Naeem , "Some t-spherical fuzzy Einstein interactive aggregation operators and their application to selection of photovoltaic cells", *Mathematical Problems in Engineering*, vol. 2020, pp. 1-16, 2020.

[45] D. Diakoulaki, G. Mavrotas, and L. Papayannakis, "Determining objective weights in multiple criteria problems: The critic method", *Computers & Operations Research*, vol. 22, no. 7, pp. 763-770, 1995.

[46] B. Li, J. Wang, L. Yang, and X. Li, "A novel generalized simplified neutrosophic number Einstein aggregation operator", *IAENG International Journal of Applied Mathematics*, vol. 48, no. 1, pp. 67-72, 2018.

[47] L. Yang and B. Li, "Picture hesitant fuzzy normalized weighted Bonferroni mean operator based on Einstein operations and its application", *Engineering Letters*, vol. 29, no. 1, pp. 35-42, 2021.

[48] J.J. Geng, H.Y. Wanhong, and D.S. Xu, "A method based on TOPSIS and distance measures for single-valued neutrosophic linguistic sets and its application", *IAENG International Journal of Applied Mathematics*, vol. 51, no. 3, pp. 538-545, 2021.

[49] W. Ye, J. Geng, X. Cui, and D. Xu, "A new method for multi-attribute decision-making based on single-valued neutrosophic sets", *Engineering Letters*, vol. 28, no. 4, pp. 1302-1309, 2020.

[50] D. Xu, X. Lu, and L. Peng, "An extended TODIM based on cumulative prospect theory for single-valued neutrosophic multi-attribute decision-making", *Engineering Letters*, vol. 30, no. 2, pp. 752-756, 2022.

[51] P. Liu and J. Liu, "Some q-rung orthopair fuzzy Bonferroni mean operators and their application to multi-attribute group decision making", *International Journal of Intelligent Systems*, vol. 33, no. 2, pp. 315-347, 2018.

[52] A. Thilagavathy and S. Mohanaselvi, "Cubical fuzzy Einstein Bonferrori mean averaging aggregation operators and their applications to multiple criteria group decision making problems", *Journal of Intelligent & Fuzzy Systems*, vol. Pre-press, no. Pre-press, pp. 1-27, 2023.