Constructing Cospectral Non-isomorphic Signed Bipartite Graphs

Shupeng Li, Juan Liu, Hong Yang and Hong-Jian Lai

***Abstract*—Let $S = (G, \sigma)$ be a signed graph, where σ is the sign function on the edges of the underlying graph G. It is widely recognized that the adjacency spectrum alone cannot uniquely determine a signed graph. Therefore, it is of great interest to identify whether there exist any cospectral, non-isomorphic signed graphs within a specific class of signed graphs. In a significant contribution, Godsil et al. demonstrated that two components of $G_1 \times G_2$, where both G_1 and G_2 are connected bipartite graphs, are cospectral if and only if at the minimum one of G_1 and G_2 is balanced. In this paper, we first generalize Godsil's result for two connected signed bipartite graphs S_1 and S_2. Furthermore, we will define partitioned tensor product of two signed bipartite graphs, which will enable us to generate multiple pairs of cospectral non-isomorphic signed bipartite graphs.**

***Index Terms*—Signed bipartite graph, adjacency matrix, GM-switching, partitioned tensor product.**

I. Introduction

A *signed graph* is an ordered pair $S = (G, \sigma)$ consisting of the underlying graph G with vertex set $V(S)$, edge set $E(S)$ and a mapping σ: $E(S) \to \{1, -1\}$, called the *signature*. Each edge in the graph is associated with a value that can be either positive or negative. For simplicity, we assume that the graph G is simple, without multiple edges or self-loops. The *adjacency matrix* $A(S) = (a_{ij})$ for a signed graph S is a symmetric matrix with elements limited to 0, 1, and -1, where $a_{ij} = \sigma(x_i x_j)$ when x_i and x_j are neighboring vertices, and $a_{ij} = 0$ otherwise. The *adjacency spectrum* of a signed graph S is the set of all eigenvalues of $A(S)$ including multiplicities. Two signed graphs are *cospectral* for the adjacency matrices if they have the same adjacency spectrum.

Assume that $S = (G, \sigma)$ is a signed graph with vertex set $V(S)$ and edge set $E(S)$. For a given vertex $v \in V(S)$, we define $d_S^+(v)$ as the count of positive edges that are incident to vertex v in S, and $d_S^-(v)$ as the count of negative edges that are incident to vertex v in S. Additionally, we introduce $d_S^{\pm}(v) = d_S^+(v) - d_S^-(v)$.

If $V(S)$ can be partitioned into two parts X and Y such that every edge of S has one end in X and the other end in Y, then S is called the *signed bipartite graph*. We say that X and Y as the *partite sets* of S. If $|X| = |Y|$, then S is *balanced*. Furthermore, if S is a signed bipartite graph and its adjacency matrix can be expressed as $\begin{bmatrix} \mathbf{0} & B \\ B^T & \mathbf{0} \end{bmatrix}$, where $\mathbf{0}$ denotes the zero matrix, then the matrix B is the *biadjacency matrix* of S.

Let A be a square symmetric matrix with elements limited to 0, 1, and -1 such that diagonal entries are zero. Suppose S_A denotes the signed graph whose adjacency matrix is given by A. *The Kronecker product* of matrices A of size $m \times n$ and B of size $p \times q$, denoted by $A \otimes B$, is defined as the block matrix of size $mp \times nq$ constructed by replacing each entry a_{ij} of A with the matrix product $a_{ij}B$. On the other hand, *the partitioned tensor product of two partitioned matrices* $M = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$ and $H = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, denoted as $M \underline{\otimes} H$, is defined as the block matrix $\begin{bmatrix} P \otimes A & Q \otimes B \\ R \otimes C & S \otimes D \end{bmatrix}$, where each block is obtained by taking the Kronecker product of the corresponding submatrices. These notions were introduced by Godsil et al. in [6].

Let $S_1 = (G_1, \sigma_1)$ and $S_2 = (G_2, \sigma_2)$ be two signed graphs. Their *direct product* is the signed graph $S_1 \times S_2$, whose vertex set is $V(S_1) \times V(S_2)$, whose edges are all pairs $(x_i, y_k)(x_j, y_\ell)$ with $x_i x_j \in E(S_1)$ and $y_k y_\ell \in E(S_2)$. The signature of the edge $(x_i, y_k)(x_j, y_\ell)$ in $S_1 \times S_2$ is defined as $\sigma((x_i, y_k)(x_j, y_\ell)) = \sigma_1(x_i x_j)\sigma_2(y_k y_\ell)$, where $x_i x_j \in E(S_1)$ and $y_k y_\ell \in E(S_2)$. The direct product construction can also be applied to signed bipartite graphs.

If every signed graph that has the same spectrum as S is also isomorphic to S, then we say S is *determined by its spectrum*(DS). Otherwise, if there exist cospectral signed graphs that are not isomorphic to S, then we say that S has a cospectral mate or S is *not determined by its spectrum*(NDS). Godsil et al. [6] utilized the concept of partitioned tensor product to construct graphs that have same adjacency spectrum. Ji et al. [9] introduced a method of constructing cospectral bipartite graphs, which relies on adjacency and normalized Laplacian matrices and employs the unfolding technique. The notion of unfolding a bipartite graph initially introduced in [5] is further expanded upon by this construction method. This approach provided a more flexible and generalized framework for generating cospectral graphs. In recent research by Kannan et al. [10], bipartite graphs with the same eigenvalues for both adjacency and normalized Laplacian matrices were constructed using partitioned tensor product. For signed graphs, non-isomorphic Laplacian cospectral signed graphs were obtained by Ji et al. in [13] and used the operation of partial transpose. For more details, we refer to [2], [3], [4], [8], [9], [11], [12], [16].

The rest of the paper is organized as follows. In Section 2, we define the partitioned tensor product of two signed

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bipartite graphs and discuss the existence of cospectral non-isomorphic signed bipartite graphs. In Section 3, we use the GM-switching method to partitioned tensor product defined as Section 2, this enables us to construct additional examples of cospectral non-isomorphic signed bipartite graphs.

II. Cospectral signed bipartite graphs for partitioned tensor product

Next, we first discuss the presence of cospectral direct product for two signed graphs. Secondly, we define partitioned tensor product and give a sufficient and necessary condition of cospectral partitioned tensor product for two signed bipartite graphs. The following notations will be used in the rest of this paper.

Let $S_1 = (G_1, \sigma_1)$ and $S_2 = (G_2, \sigma_2)$ be two signed bipartite graphs with $V(S_1) = X \cup Y$ and $V(S_2) = U \cup W$, where $X = \{x_1, x_2, \ldots, x_m\}$, $Y = \{y_1, y_2, \ldots, y_n\}$, $U = \{u_1, u_2, \ldots, u_p\}$ and $W = \{w_1, w_2, \ldots, w_q\}$. Let $B = (b_{ij})_{m\times n}$ and $C = (c_{k\ell})_{p\times q}$ be the biadjacency matrices of S_1 and S_2, respectively, with

$$b_{ij} = \begin{cases} \sigma_1(x_iy_j), & \text{if } x_iy_j \in E(S_1), \\ 0, & \text{otherwise,} \end{cases}$$

and

$$c_{k\ell} = \begin{cases} \sigma_2(u_kw_\ell), & \text{if } u_kw_\ell \in E(S_2), \\ 0, & \text{otherwise.} \end{cases}$$

Then, S_1 and S_2 have adjacency matrices $A(S_1) = \begin{bmatrix} \mathbf{0} & B \\ B^T & \mathbf{0} \end{bmatrix}$ and $A(S_2) = \begin{bmatrix} \mathbf{0} & C \\ C^T & \mathbf{0} \end{bmatrix}$, respectively, relative to vertex orderings $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n$ and $u_1, u_2, \ldots, u_p, w_1, w_2, \ldots, w_q$, respectively. Define

$$A(S_1)^{\#} = \begin{bmatrix} \mathbf{0} & B^T \\ B & \mathbf{0} \end{bmatrix}$$

and

$$A(S_2)^{\#} = \begin{bmatrix} \mathbf{0} & C^T \\ C & \mathbf{0} \end{bmatrix}.$$

It is simple to verify that $S_1 \times S_2$ has adjacency matrix $A(S_1)\otimes A(S_2)$ relative to the ordering $(x_1, u_1), \ldots, (x_1, u_p)$, $(x_1, w_1), \ldots, (x_1, w_q), \ldots, (x_m, u_1), \ldots, (x_m, u_p), (x_m, w_1)$, $\ldots, (x_m, w_q), (y_1, u_1), \ldots, (y_1, u_p), (y_1, w_1), \ldots, (y_1, w_q), \ldots$ $, (y_n, u_1), \ldots, (y_n, u_p), (y_n, w_1), \ldots, (y_n, w_q)$ of its vertices.

In [14], Zhang gave the next lemma.

Lemma 2.1: [14] Assume that $\lambda_1, \lambda_2, \ldots, \lambda_m$ are $A_{m\times m}$'s eigenvalues and $\mu_1, \mu_2, \ldots, \mu_n$ are $B_{n\times n}$'s eigenvalues. Then the eigenvalues of the Kronecker product $A \otimes B$ are $\lambda_i\mu_j$ for any i and j with $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

This lemma yields the following theorem.

Theorem 2.1: Assume that S_1, S_2, S_3 and S_4 are four signed graphs. If S_1 and S_3 are cospectral, S_2 and S_4 are cospectral, then signed graphs $S_1 \times S_2$ and $S_3 \times S_4$ are cospectral.

In order to construct cospectral non-isomorphic signed bipartite graphs, we will define a pair of signed bipartite graphs $\mathcal{T}_{S_1\underline{\times}S_2}$ as follows.

(i) The first signed bipartite graph of $\mathcal{T}_{S_1\underline{\times}S_2}$ has a vertex set $(X \times U) \cup (Y \times W)$, an edge set $\{(x_i, u_k)(y_j, w_\ell) | x_iy_j \in E(S_1), u_kw_\ell \in E(S_2)\}$ and a signature $\sigma((x_i, u_k)(y_j, w_\ell)) = \sigma_1(x_iy_j)\sigma_2(u_kw_\ell)$, where $x_iy_j \in E(S_1)$ and $u_kw_\ell \in E(S_2)$;

(ii) the second signed bipartite graph of $\mathcal{T}_{S_1\underline{\times}S_2}$ has a vertex set $(X \times W) \cup (Y \times U)$, an edge set $\{(x_i, w_\ell)(y_j, u_k) | x_iy_j \in E(S_1), u_kw_\ell \in E(S_2)\}$ and a signature $\sigma((x_i, w_\ell)(y_j, u_k)) = \sigma_1(x_iy_j)\sigma_2(u_kw_\ell)$, where $x_iy_j \in E(S_1)$ and $u_kw_\ell \in E(S_2)$.

By the definition of $\mathcal{T}_{S_1\underline{\times}S_2}$, the adjacency matrices of two signed bipartite graphs are $A(S_1)\underline{\otimes}A(S_2)$ relative to the ordering $(x_1, u_1), \ldots, (x_1, u_p), (x_2, u_1), \ldots, (x_2, u_p) \ldots,$ $(x_m, u_1), \ldots, (x_m, u_p), (y_1, w_1), \ldots, (y_1, w_q) \ldots, (y_n, w_1),$ $\ldots, (y_n, w_q)$ of its vertices and $A(S_1)\underline{\otimes}A(S_2)^{\#}$ relative to the ordering $(x_1, w_1), \ldots, (x_1, w_q), (x_2, w_1), \ldots, (x_2, w_q),$ $\ldots, (x_m, w_1), \ldots, (x_m, w_q), (y_1, u_1), \ldots, (y_1, u_p), (y_2, u_1),$ $\ldots, (y_2, u_p), \ldots, (y_n, u_1), \ldots, (y_nu_p)$ of its vertices, respectively. Therefore

$$\mathcal{T}_{S_1\underline{\times}S_2} = \{S_{A(S_1)\underline{\otimes}A(S_2)}, S_{A(S_1)\underline{\otimes}A(S_2)^{\#}}\}.$$

Moreover, if S_1 and S_2 are connected, then $S_1 \times S_2$ will consist of exactly two components: $S_{A(S_1)\underline{\otimes}A(S_2)}$ and $S_{A(S_1)\underline{\otimes}A(S_2)^{\#}}$.

We call $\mathcal{T}_{S_1\underline{\times}S_2}$ the *partitioned tensor product of two signed bipartite graphs* S_1 and S_2.

We start by recalling some properties of Kronecker product of matrices in the following.

Proposition 2.1: [15] Let $M_1 = (m_{ij}^{(1)})_{p_1\times q_1}, M_2 = (m_{ij}^{(2)})_{p_2\times q_2}, H_1 = (h_{ij}^{(1)})_{s_1\times t_1}$ and $H_2 = (h_{ij}^{(2)})_{s_2\times t_2}$. Then each of the following holds:

(i) if both M_1 and H_1 are orthogonal matrices, then $M_1 \otimes H_1$ is also orthogonal matrix;

(ii) $(M_1 \otimes H_1)^T = M_1^T \otimes H_1^T$;

(iii) if $q_1 = p_2$ and $t_1 = s_2$, then $(M_1 \otimes H_1)(M_2 \otimes H_2) = (M_1M_2) \otimes (H_1H_2)$;

(iv) if $p_1 = p_2$ and $q_1 = q_2$, then $(M_1 + M_2) \otimes H = M_1 \otimes H + M_2 \otimes H$.

Next, we shall also develop a number of similar properties of partitioned tensor product for two partitioned matrices. All these will be applied in our arguments.

Proposition 2.2: Let $M_1 = (m_{ij}^{(1)})_{p_1\times q_1}, M_2 = (m_{ij}^{(2)})_{p_2\times q_2}, H_1 = (h_{ij}^{(1)})_{s_1\times t_1}$ and $H_2 = (h_{ij}^{(2)})_{s_2\times t_2}$. Then each of the following holds:

(i) $(M_1\underline{\otimes}H_1)^T = M_1^T\underline{\otimes}H_1^T$;

(ii) let $M_1 = \begin{bmatrix} P_1 & \mathbf{0} \\ \mathbf{0} & S_1 \end{bmatrix}$ and $H_1 = \begin{bmatrix} A_1 & \mathbf{0} \\ \mathbf{0} & D_1 \end{bmatrix}$, or $M_1 = \begin{bmatrix} \mathbf{0} & Q_1 \\ R_1 & \mathbf{0} \end{bmatrix}$ and $H_1 = \begin{bmatrix} \mathbf{0} & B_1 \\ C_1 & \mathbf{0} \end{bmatrix}$:

(ii-1) if $q_1 = p_2$ and $t_1 = s_2$, then $(M_1\underline{\otimes}H_1)(M_2\underline{\otimes}H_2) = (M_1M_2)\underline{\otimes}(H_1H_2)$;

(ii-2) if both M_1 and H_1 are orthogonal matrices, then $M_1\underline{\otimes}H_1$ is also orthogonal matrix.

Proof: For (i), let $M_1 = \begin{bmatrix} P_1 & Q_1 \\ R_1 & S_1 \end{bmatrix}$ and $H_1 = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}$. Then by Proposition 2.1 (ii),

$$\begin{aligned} (M_1\underline{\otimes}H_1)^T &= \begin{bmatrix} P_1^T \otimes A_1^T & R_1^T \otimes C_1^T \\ Q_1^T \otimes B_1^T & S_1^T \otimes D_1^T \end{bmatrix} \\ &= \begin{bmatrix} P_1^T & R_1^T \\ Q_1^T & S_1^T \end{bmatrix} \underline{\otimes} \begin{bmatrix} A_1^T & C_1^T \\ B_1^T & D_1^T \end{bmatrix} \\ &= M_1^T\underline{\otimes}H_1^T. \end{aligned}$$

For (ii), we assume that $M_1 = \begin{bmatrix} P_1 & \mathbf{0} \\ \mathbf{0} & S_1 \end{bmatrix}$ and $H_1 = \begin{bmatrix} A_1 & \mathbf{0} \\ \mathbf{0} & D_1 \end{bmatrix}$. If $q_1 = p_2$ and $t_1 = s_2$, we can define $M_2 = \begin{bmatrix} P_2 & Q_2 \\ R_2 & S_2 \end{bmatrix}$ and $H_2 = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}$ such that the number of columns of P_1 is equal to the number of rows of P_2, the number of columns of S_1 is equal to the number of rows of S_2, the number of columns of A_1 is equal to the number of rows of A_2 and the number of columns of D_1 is equal to the number of rows of D_2. Then, by Proposition 2.1 (iii),

$$\begin{aligned}
& (M_1 \underline{\otimes} H_1)(M_2 \underline{\otimes} H_2) \\
= & \begin{bmatrix} P_1 \otimes A_1 & \mathbf{0} \\ \mathbf{0} & S_1 \otimes D_1 \end{bmatrix} \begin{bmatrix} P_2 \otimes A_2 & Q_2 \otimes B_2 \\ R_2 \otimes C_2 & S_2 \otimes D_2 \end{bmatrix} \\
= & \begin{bmatrix} (P_1 \otimes A_1)(P_2 \otimes A_2) & (P_1 \otimes A_1)(Q_2 \otimes B_2) \\ (S_1 \otimes D_1)(R_2 \otimes C_2) & (S_1 \otimes D_1)(S_2 \otimes D_2) \end{bmatrix} \\
= & \begin{bmatrix} (P_1P_2) \otimes (A_1A_2) & (P_1Q_2) \otimes (A_1B_2) \\ (S_1R_2) \otimes (D_1C_2) & (S_1S_2) \otimes (D_1D_2) \end{bmatrix} \\
= & \begin{bmatrix} (P_1P_2) & (P_1Q_2) \\ (S_1R_2) & (S_1S_2) \end{bmatrix} \underline{\otimes} \begin{bmatrix} (A_1A_2) & (A_1B_2) \\ (D_1C_2) & (D_1D_2) \end{bmatrix} \\
= & (M_1M_2) \underline{\otimes} (H_1H_2).
\end{aligned}$$

Thus $(ii\text{-}1)$ holds.

If both M_1 and H_1 are orthogonal matrices, then by Proposition 2.2 (i) and $(ii\text{-}1)$,

$$\begin{aligned}
(M_1 \underline{\otimes} H_1)(M_1 \underline{\otimes} H_1)^T &= (M_1 \underline{\otimes} H_1)(M_1^T \underline{\otimes} H_1^T) \\
&= (M_1 M_1^T) \underline{\otimes} (H_1 H_1^T) \\
&= I_{p_1} \underline{\otimes} I_{s_1} \\
&= I_{p_1 s_1}.
\end{aligned}$$

So $M_1 \underline{\otimes} H_1$ is also an orthogonal matrix.

The same holds true when $M_1 = \begin{bmatrix} \mathbf{0} & Q_1 \\ R_1 & \mathbf{0} \end{bmatrix}$ and $H_1 = \begin{bmatrix} \mathbf{0} & B_1 \\ C_1 & \mathbf{0} \end{bmatrix}$, respectively. ■

Next, we will discuss the existence of cospectral non-isomorphic signed bipartite graphs in $\mathcal{T}_{S_1 \underline{\times} S_2}$. It is evident that S_1 and S_2 are cospectral if and only if the corresponding adjacency matrices exhibit orthogonal similarity.

Theorem 2.2: Assume that S_1 and S_2 are two signed bipartite graphs. Then two signed bipartite graphs of $\mathcal{T}_{S_1 \underline{\times} S_2}$ are cospectral if and only if S_1 or S_2 is balanced.

Proof: Assume first that S_1 or S_2 is balanced. We just need to show that the matrices $A(S_1) \underline{\otimes} A(S_2)$ and $A(S_1) \underline{\otimes} A(S_2)^{\#}$ are orthogonally similar, which implies that $S_{A(S_1) \underline{\otimes} A(S_2)}$ and $S_{A(S_1) \underline{\otimes} A(S_2)^{\#}}$ are cospectral. Therefore, two signed bipartite graphs in $\mathcal{T}_{S_1 \underline{\times} S_2}$ are cospectral.

If $m = n$, then it is possible to find two orthogonal matrices R_1 and R_2, which can lead to $R_1^T B R_2 = B^T$. We can define $R = \begin{bmatrix} \mathbf{0} & R_1 \\ R_2 & \mathbf{0} \end{bmatrix}$. Now, we have $R^T A(S_1) R = A(S_1)$. Let $F = \begin{bmatrix} \mathbf{0} & I_p \\ I_q & \mathbf{0} \end{bmatrix}$. It follows that $F^T A(S_2) F = A(S_2)^{\#}$. Clearly, both R and F are orthogonal matrices.

Let $P = R \underline{\otimes} F = \begin{bmatrix} \mathbf{0} & R_1 \otimes I_p \\ R_2 \otimes I_q & \mathbf{0} \end{bmatrix}$. By Proposition 2.2 $(ii\text{-}2)$, P is an orthogonal matrix. Now,

$$\begin{aligned}
P^T(A(S_1) \underline{\otimes} A(S_2))P &= (R \underline{\otimes} F)^T (A(S_1) \underline{\otimes} A(S_2))(R \underline{\otimes} F) \\
&= (R^T A(S_1) R) \underline{\otimes} (F^T A(S_2) F) \\
&= A(S_1) \underline{\otimes} A(S_2)^{\#}.
\end{aligned}$$

Be aware that the second step uses Proposition 2.2 (i) and $(ii\text{-}1)$. Therefore, we can conclude that $S_{A(S_1) \underline{\otimes} A(S_2)}$ and $S_{A(S_1) \underline{\otimes} A(S_2)^{\#}}$ are cospectral.

If $p = q$, then it is possible to find two orthogonal matrices F_1 and F_2, which can lead to $F_1^T C F_2 = C^T$. Clearly, it is also possible to find a permutation matrix $R = \begin{bmatrix} I_m & \mathbf{0} \\ \mathbf{0} & I_n \end{bmatrix}$, which can lead to $R^T A(S_1) R = A(S_1)$. Let $F = \begin{bmatrix} F_1 & \mathbf{0} \\ \mathbf{0} & F_2 \end{bmatrix}$. We have, $F^T A(S_2) F = A(S_2)^{\#}$. It is evident that both R and F are orthogonal matrices.

Let $P = R \underline{\otimes} F = \begin{bmatrix} I_m \otimes F_1 & \mathbf{0} \\ \mathbf{0} & I_n \otimes F_2 \end{bmatrix}$. By Proposition 2.2 $(ii\text{-}2)$, P is an orthogonal matrix. Now,

$$\begin{aligned}
P^T(A(S_1) \underline{\otimes} A(S_2))P &= (R^T \underline{\otimes} F^T)(A(S_1) \underline{\otimes} A(S_2))(R \underline{\otimes} F) \\
&= (R^T A(S_1) R) \underline{\otimes} (F^T A(S_2) F) \\
&= A(S_1) \underline{\otimes} A(S_2)^{\#}.
\end{aligned}$$

It should be noted that in the second step, Proposition 2.2 (i) and $(ii\text{-}1)$ are utilized. As a result, we can conclude that $S_{A(S_1) \underline{\otimes} A(S_2)}$ and $S_{A(S_1) \underline{\otimes} A(S_2)^{\#}}$ are cospectral.

In contrast, suppose $S_{A(S_1) \underline{\otimes} A(S_2)}$ and $S_{A(S_1) \underline{\otimes} A(S_2)^{\#}}$ are cospectral. This implies that $mp + nq = mq + np$, which further simplifies to $(m - n)(p - q) = 0$. From this, we can conclude that $m = n$ or $p = q$. Therefore, we have proved the theorem. ■

Hammack et al. in [7] established a cancellation law for $(0, 1)$-matrices. This result can be further specialized to $(0, 1, -1)$-matrices and the following lemma may be proven using a similar approach as Lemma 3 in [7].

Lemma 2.2: Assume that A_1, A_2 and C are $(0, 1, -1)$-matrices for which $C \neq 0$, and A_1 is square and has at least one nonzero entry in each row. Suppose it is possible to find two permutation matrices Q_1 and R_1, which can lead to $Q_1(C \otimes A_1)R_1 = C \otimes A_2$. Then it is possible to find two permutation matrices Q_2 and R_2, which can lead to $Q_2 A_1 R_2 = A_2$. Also, if $Q_1(A_1 \otimes C)R_1 = A_2 \otimes C$, then it is also possible to find two permutation matrices Q_2 and R_2, which can lead to $Q_2 A_1 R_2 = A_2$.

Then we get the following theorem.

Theorem 2.3: Assume that S_1 and S_2 are connected signed bipartite graphs whose biadjacency matrices are B and C, respectively. Then two signed bipartite graphs of $\mathcal{T}_{S_1 \underline{\times} S_2}$ are isomorphic if and only if it is possible to find two permutation matrices R_1 and R_2, which can lead to $R_1^T B R_2 = B^T$ or $R_1^T C R_2 = C^T$.

Proof: Assuming the given condition, it is possible to find a permutation matrix R that can be expressed in one of $R = \begin{bmatrix} R_1' & \mathbf{0} \\ \mathbf{0} & R_2' \end{bmatrix}$ or R = $\begin{bmatrix} \mathbf{0} & R_1' \\ R_2' & \mathbf{0} \end{bmatrix}$, where R_i' is a permutation matrix for $i \in \{1, 2\}$, which can lead to

$$R^T \begin{bmatrix} \mathbf{0} & B \otimes C \\ B^T \otimes C^T & \mathbf{0} \end{bmatrix} R = \begin{bmatrix} \mathbf{0} & B \otimes C^T \\ B^T \otimes C & \mathbf{0} \end{bmatrix}.$$

If $R = \begin{bmatrix} R_1' & \mathbf{0} \\ \mathbf{0} & R_2' \end{bmatrix}$, then we have $R_1'^T(B \otimes C)R_2' = B \otimes C^T$, which implies that C is a square matrix. According to Lemma 2.2, it is possible to find two permutation matrices R_1 and R_2, which can lead to $R_1^T C R_2 = C^T$. Similarly, if $R = \begin{bmatrix} \mathbf{0} & R_1' \\ R_2' & \mathbf{0} \end{bmatrix}$, then we have $R_1'^T(B \otimes C)R_2' = B^T \otimes C$, which implies that B is a square matrix. By applying Lemma 2.2, we can find two permutation matrices R_1 and R_2, which can lead to $R_1^T B R_2 = B^T$.

Suppose that it is possible to find two permutation matrices R_1 and R_2, which can lead to $R_1^T B R_2 = B^T$. Let $R = \begin{bmatrix} \mathbf{0} & R_1 \\ R_2 & \mathbf{0} \end{bmatrix}$. Now $R^T A(S_1) R = A(S_1)$. Clearly, R is a permutation matrix. Set $F = \begin{bmatrix} \mathbf{0} & I_p \\ I_q & \mathbf{0} \end{bmatrix}$, then $F^T A(S_2) F = A(S_2)^{\#}$. Let $P = R \underline{\otimes} F$. Clearly, P is a permutation matrix. Now,

$$\begin{aligned} P^T(A(S_1)\underline{\otimes}A(S_2))P &= (R^T \underline{\otimes} F^T)(A(S_1)\underline{\otimes}A(S_2))(R\underline{\otimes}F) \\ &= (R^T A(S_1) R)\underline{\otimes}(F^T A(S_2) F) \\ &= A(S_1)\underline{\otimes}A(S_2)^{\#}. \end{aligned}$$

It should be noted that in the second step, Proposition 2.2 (i) and $(ii$-$1)$ are utilized. Therefore, the signed bipartite graphs $S_{A(S_1)\underline{\otimes}A(S_2)}$ and $S_{A(S_1)\underline{\otimes}A(S_2)^{\#}}$ are isomorphic.

Similarly, if it is possible to find two permutation matrices R_1 and R_2, which can lead to $R_1^T C R_2 = C^T$, then the signed bipartite graphs $S_{A(S_1)\underline{\otimes}A(S_2)}$ and $S_{A(S_1)\underline{\otimes}A(S_2)^{\#}}$ are isomorphic. This completes the proof of the theorem. ■

The following corollary follows from Theorem 2.2 and Theorem 2.3.

Corollary 2.1: Assume that S_1 and S_2 are two connected signed bipartite graphs, and assume that S_1 or S_2 are balanced. Then two signed bipartite graphs in $\mathcal{T}_{S_1 \underline{\times} S_2}$ are cospectral and non-isomorphic if and only if it is impossible to find permutation matrices R_1 and R_2, which can lead to $R_1^T B R_2 = B^T$ and $R_1^T C R_2 = C^T$.

III. Cospectral signed bipartite graphs with GM-switching

Next, we provide some constructions of cospectral signed bipartite graphs using the notion of partitioned tensor product of two signed bipartite graphs and GM-switching.

In 2019, Belardo, et al. [1] constructed signed graphs by GM-switching as follows.

Definition 3.1: [1] Suppose that $S = (G, \sigma)$ is a signed graph and π is a partition of its vertex set $V(S)$ into sets $\{U_1, U_2, \ldots, U_t, W\}$, where $|U_i| = n_i$ with $i = 1, 2, \ldots, t$ and $|W| = d$. The i-th *net-degree* of a vertex v_i is defined as the difference between the number of positive edges and the number of negative edges that connect vertex v_i to the vertices in U_i. Suppose that each of the following holds for any integers i and j with $1 \le i, j \le t$:

(i) any two vertices in U_i have the same j-th net-degree;

(ii) for any vertex $v \in W$:

$(ii$-$1)$ either vertex v has an equal number of positive and negative edges connecting it to U_i;

$(ii$-$2)$ or vertex v is connected by positive edges to half of the vertices in U_i and no edges are connected to other vertices;

$(ii$-$3)$ or vertex v is connected by negative edges to half of the vertices in U_i and and no edges are connected to other vertices;

$(ii$-$4)$ or vertex v is connected to all vertices in U_i by positive edges;

$(ii$-$5)$ or vertex v is connected to all vertices in U_i by negative edges.

Next, the signed graph S^{GM} obtained from S utilizing local switching with respect to the partition π can be described as follows. For each vertex $v \in W$ and $1 \le i \le t$, the following operations are performed:

(i) if the i-th net-degree of a vertex v_i is equal to 0, then the sign of any edge between v and a vertex in U_i is reversed;

(ii) if vertex v is connected by positive edges to half of the vertices in U_i and no edges are connected to other vertices, then existing positive edges connecting v to U_i are deleted. Instead, v is connected to the other $\frac{n_i}{2}$ vertices of U_i using new positive edges;

(iii) if vertex v is connected by negative edges to half of the vertices in U_i and no edges are connected to other vertices, then existing negative edges connecting v to U_i are deleted. Instead, v is connected to the other $\frac{n_i}{2}$ vertices of U_i using new negative edges.

We say that S^{GM} is constructed utilizing *GM-switching* of signed graph S.

Next, we introduce a special family of signed bipartite graph as follows. Let $\mathcal{S}$ be a signed bipartite graph family such that each signed bipartite graph $S \in \mathcal{S}$ with partite sets X and Y if and only if $V(S)$ can be partitioned into $t+1$ vertex subsets $X_1, X_2, \ldots, X_t$ and Y with $X = X_1 \cup X_2 \cup \cdots \cup X_t$ and $|X_i| = n_i$ satisfying Definition 3.1 (ii). Since S is the signed bipartite graph with partite subsets X and Y, and $X = X_1 \cup X_2 \cup \cdots \cup X_t$, we have that for each $1 \le i$, $j \le t$, j-th net-degree of any vertex in X_i is 0. Hence S satisfies Definition 3.1 (i). Therefore, S satisfies Definition 3.1.

Let the signed bipartite graph S^{GM} be constructed from S utilizing GM-switching with respect to the partition $\{X_1, X_2, \ldots, X_t, Y\}$ is obtained as above. Belardo, et al. [1] proved that S and S^{GM} are cospectral.

Suppose S_1^{GM} and S_2^{GM} are constructed by GM-switching for signed bipartite graphs S_1 and S_2, respectively. Let $\widetilde{B}$ and $\widetilde{C}$ be the biadjacency matrices of S_1^{GM} and S_2^{GM}, respectively. Then $A(S_1^{GM}) = \begin{bmatrix} \mathbf{0} & \widetilde{B} \\ \widetilde{B}^T & \mathbf{0} \end{bmatrix}$ and $A(S_2^{GM}) = \begin{bmatrix} \mathbf{0} & \widetilde{C} \\ \widetilde{C}^T & \mathbf{0} \end{bmatrix}$, hence S_1^{GM} and S_2^{GM} have adjacency matrices $A(S_1^{GM})$ and $A(S_2^{GM})$, respectively, relative to vertex orderings $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n$ and $u_1, u_2, \ldots, u_p, w_1, w_2, \ldots, w_q$, respectively. Define $A(S_1)^* = A(S_1^{GM})$ and $A(S_2)^* = A(S_2^{GM})$.

For any positive integer ℓ, let

$$Q_\ell = \frac{2}{\ell} J_\ell - I_\ell,$$

where J_ℓ represents the $\ell \times \ell$ matrix whose entries are all equal to 1 and I_ℓ represents the identity matrix of order ℓ.

Recently, Belardo et al. presented several properties of the matrix Q_ℓ in [1].

Proposition 3.1: [1] Let $Q_\ell = \frac{2}{\ell} J_\ell - I_\ell$, and $\mathbf{x} = (x_i)_{i=1,2,\ldots,\ell}$ be a vector with entries in $\{0, 1, -1\}$. Then

each of the following holds:
(i) Q_ℓ is orthogonal and symmetric;
(ii) if a vector $\mathbf{x}$ with a sum of entries equal to 0, then we have $Q_\ell \mathbf{x} = -\mathbf{x}$;
(iii) if ℓ is an even integer and $\mathbf{x}$ is a vector with half of its elements being 0 and the other half being 1, then we have $Q_\ell \mathbf{x} = \mathbf{1}_\ell - \mathbf{x}$;
(iv) if ℓ is an even integer and $\mathbf{x}$ is a vector with half of its elements being 0 and the other half being -1, then we have $Q_\ell \mathbf{x} = -\mathbf{1}_\ell - \mathbf{x}$;
(v) if $\mathbf{x}$ is the vector $\mathbf{1}_\ell$, then we have $Q_\ell \mathbf{x} = \mathbf{x}$;
(vi) if $\mathbf{x}$ is the vector $-\mathbf{1}_\ell$, then we have $Q_\ell \mathbf{x} = \mathbf{x}$.

Let $S_1, S_2 \in \mathcal{S}$. Then $V(S_1)$ can be partitioned into t_1+1 vertex subsets $X_1, X_2, \ldots, X_{t_1}$ and Y with $X = X_1 \cup X_2 \cup \cdots \cup X_{t_1}$, $|Y| = n$, $|X_j| = m_j$ for $j \in \{1, 2, \ldots, t_1\}$ and $\sum_{j=1}^{t_1} m_j = m$. And $V(S_2)$ can be partitioned into t_2+1 vertex subsets $U_1, U_2, \ldots, U_{t_2}$ and W with $U = U_1 \cup U_2 \cup \cdots \cup U_{t_2}$, $|W| = q$, $|U_k| = p_k$ for $k \in \{1, 2, \ldots, t_2\}$ and $\sum_{k=1}^{t_2} p_k = p$. Let

$$B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_{t_1} \end{bmatrix} \quad and \quad C = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_{t_2} \end{bmatrix},$$

where B_j are $m_j \times n$ for $j \in \{1, 2, \ldots, t_1\}$ and C_k are $p_k \times q$ for $k \in \{1, 2, \ldots, t_2\}$. We get that the adjacency matrices of S_1 and S_2 are

$$A(S_1) = \begin{bmatrix} \mathbf{0} & B \\ B^T & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & B_1 \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & B_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & B_{t_1} \\ B_1^T & B_2^T & \cdots & B_{t_1}^T & \mathbf{0} \end{bmatrix}$$

and

$$A(S_2) = \begin{bmatrix} \mathbf{0} & C \\ C^T & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & C_1 \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & C_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & C_{t_2} \\ C_1^T & C_2^T & \cdots & C_{t_2}^T & \mathbf{0} \end{bmatrix},$$

respectively.

Let $Q_{A(S_1)} = diag(Q_{m_1}, Q_{m_2}, \ldots, Q_{m_{t_1}})$ and $Q_{A(S_2)} = diag(Q_{p_1}, Q_{p_2}, \ldots, Q_{p_{t_2}})$. According to Proposition 3.1, we can obtain

$$\begin{aligned} & Q_{A(S_1)} B \\ = & \begin{bmatrix} Q_{m_1} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & Q_{m_2} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & Q_{m_{t_1-1}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & Q_{m_{t_1}} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_{t_1-1} \\ B_{t_1} \end{bmatrix} \\ = & \begin{bmatrix} Q_{m_1} B_1 \\ Q_{m_2} B_2 \\ \vdots \\ Q_{m_{t_1-1}} B_{t_1-1} \\ Q_{m_{t_1}} B_{t_1} \end{bmatrix} \\ = & \widetilde{B} \end{aligned}$$

and

$$\begin{aligned} & Q_{A(S_2)} C \\ = & \begin{bmatrix} Q_{p_1} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & Q_{p_2} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & Q_{p_{t_2-1}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & Q_{p_{t_2}} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_{t_2-1} \\ C_{t_2} \end{bmatrix} \\ = & \begin{bmatrix} Q_{p_1} C_1 \\ Q_{p_2} C_2 \\ \vdots \\ Q_{p_{t_2-1}} C_{t_2-1} \\ Q_{p_{t_2}} C_{t_2} \end{bmatrix} \\ = & \widetilde{C}. \end{aligned}$$

In the remainder of this section, we proviede some constructions of cospectral non-isomorphic signed bipartite graphs using the notion of partitioned tensor product for signed bipartite graphs and GM-switching.

It is worth pointing out that

$$\mathcal{T}_{S_1 \underline{\times} S_2} = \{S_{A(S_1) \underline{\otimes} A(S_2)}, S_{A(S_1) \underline{\otimes} A(S_2)^{\#}}\},$$

$$\mathcal{T}_{S_1^{GM} \underline{\times} S_2} = \{S_{A(S_1)^* \underline{\otimes} A(S_2)}, S_{A(S_1)^* \underline{\otimes} A(S_2)^{\#}}\},$$

$$\mathcal{T}_{S_1 \underline{\times} S_2^{GM}} = \{S_{A(S_1) \underline{\otimes} A(S_2)^*}, S_{A(S_1) \underline{\otimes} (A(S_2)^*)^{\#}}\}$$

and

$$\mathcal{T}_{S_1^{GM} \underline{\times} S_2^{GM}} = \{S_{A(S_1)^* \underline{\otimes} A(S_2)^*}, S_{A(S_1)^* \underline{\otimes} (A(S_2)^*)^{\#}}\}.$$

Next, we give the result in the following.

Theorem 3.1: Assume that S_1 and S_2 are two signed bipartite graphs. Then each of the following holds:
(i) if $S_1 \in \mathcal{S}$, then two signed bipartite graphs $S_1 \times S_2$ and $S_1^{GM} \times S_2$ are cospectral;
(ii) if $S_2 \in \mathcal{S}$, then two signed bipartite graphs $S_1 \times S_2$ and $S_1 \times S_2^{GM}$ are cospectral;
(iii) if $S_1, S_2 \in \mathcal{S}$, then four signed bipartite graphs $S_1 \times S_2$, $S_1^{GM} \times S_2$, $S_1 \times S_2^{GM}$ and $S_1^{GM} \times S_2^{GM}$ are mutually cospectral.

Theorem 2.2 implies the following theorem.

Theorem 3.2: Assume that S_1 and S_2 are two signed bipartite graphs. Then each of the following holds:
(i) if $S_1 \in \mathcal{S}$, then two signed bipartite graphs of $\mathcal{T}_{S_1^{GM} \underline{\times} S_2}$ are cospectral if and only if S_1 or S_2 is balanced;
(ii) if $S_2 \in \mathcal{S}$, then two signed bipartite graphs of $\mathcal{T}_{S_1 \underline{\times} S_2^{GM}}$ are cospectral if and only if S_1 or S_2 is balanced;
(iii) if $S_1, S_2 \in \mathcal{S}$, then two signed bipartite graphs of $\mathcal{T}_{S_1^{GM} \underline{\times} S_2^{GM}}$ are cospectral if and only if S_1 or S_2 is balanced.

The following two lemmas are essential for our further discussion.

Lemma 3.1: Assume that S_1 and S_2 are two signed bipartite graphs, where $S_1 \in \mathcal{S}$. Then the signed bipartite graphs $S_{A(S_1) \underline{\otimes} A(S_2)}$ and $S_{A(S_1)^* \underline{\otimes} A(S_2)}$ are cospectral.

Proof: We only need to demonstrate that the matrices $A(S_1) \underline{\otimes} A(S_2)$ and $A(S_1)^* \underline{\otimes} A(S_2)$ are orthogonally similar, and hence $S_{A(S_1) \underline{\otimes} A(S_2)}$ and $S_{A(S_1)^* \underline{\otimes} A(S_2)}$ are cospectral. Assume that S_1^{GM} is constructed by GM-switching for signed bipartite graph S_1.

Let $R = \begin{bmatrix} Q_{A(S_1)} & \mathbf{0} \\ \mathbf{0} & I_n \end{bmatrix}$, where $Q_{A(S_1)} = diag(Q_{m_1}, Q_{m_2}, \ldots, Q_{m_{t_1}})$, by Proposition 3.1 ($i$), R is an orthogonal matrix and symmetric matrix. Then

$$\begin{aligned} & R^T A(S_1) R \\ = & \begin{bmatrix} Q_{A(S_1)} & \mathbf{0} \\ \mathbf{0} & I_n \end{bmatrix} \begin{bmatrix} \mathbf{0} & B \\ B^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} Q_{A(S_1)} & \mathbf{0} \\ \mathbf{0} & I_n \end{bmatrix} \\ = & \begin{bmatrix} \mathbf{0} & Q_{A(S_1)} B \\ B^T Q_{A(S_1)} & \mathbf{0} \end{bmatrix} \\ = & A(S_1)^*. \end{aligned}$$

Obviously, it is possible to find two identity matrices I_p and I_q, which can lead to $I_p C I_q = C$. Define $F = \begin{bmatrix} I_p & \mathbf{0} \\ \mathbf{0} & I_q \end{bmatrix}$. Clearly, F is an orthogonal matrix. Now, we have $F^T A(S_2) F = A(S_2)$.

Let $P = R \underline{\otimes} F = \begin{bmatrix} Q_{A(S_1)} \otimes I_p & \mathbf{0} \\ \mathbf{0} & I_n \otimes I_q \end{bmatrix}$. By Proposition 2.2 ($ii$-2), P is an orthogonal matrix. Now,

$$\begin{aligned} P^T (A(S_1) \underline{\otimes} A(S_2)) P &= (R \underline{\otimes} F)^T (A(S_1) \underline{\otimes} A(S_2)) (R \underline{\otimes} F) \\ &= (R^T A(S_1) R) \underline{\otimes} (F^T A(S_2) F) \\ &= A(S_1)^* \underline{\otimes} A(S_2). \end{aligned}$$

Note that the second step utilizes Proposition 2.2 (i) and (ii-1). Thus $S_{A(S_1) \underline{\otimes} A(S_2)}$ and $S_{A(S_1)^* \underline{\otimes} A(S_2)}$ are cospectral. ■

Lemma 3.2: Assume that S_1 and S_2 are two signed bipartite graphs, where S_1 and $S_2 \in \mathcal{S}$. Then the signed bipartite graphs $S_{A(S_1) \underline{\otimes} A(S_2)}$, $S_{A(S_1) \underline{\otimes} A(S_2)^*}$, $S_{A(S_1)^* \underline{\otimes} A(S_2)}$ and $S_{A(S_1)^* \underline{\otimes} A(S_2)^*}$ are mutually cospectral.

Proof: Let

$$R = \begin{bmatrix} I_m & \mathbf{0} \\ \mathbf{0} & I_n \end{bmatrix}$$

and

$$F = \begin{bmatrix} Q_{A(S_2)} & \mathbf{0} \\ \mathbf{0} & I_q \end{bmatrix},$$

where $Q_{A(S_2)} = diag(Q_{p_1}, Q_{p_2}, \ldots, Q_{p_{t_2}})$ and $P = R \underline{\otimes} F$. By a similar proof to that of Lemma 3.1, we can obtain $P^T (A(S_1) \underline{\otimes} A(S_2)) P = A(S_1) \underline{\otimes} A(S_2)^*$, which implies that $S_{A(S_1) \underline{\otimes} A(S_2)}$ and $S_{A(S_1) \underline{\otimes} A(S_2)^*}$ are cospectral. It follows by Lemma 3.1 that the signed bipartite graphs $S_{A(S_1) \underline{\otimes} A(S_2)}$, $S_{A(S_1)^* \underline{\otimes} A(S_2)}$ and $S_{A(S_1) \underline{\otimes} A(S_2)^*}$ are mutually cospectral. $S_{A(S_1) \underline{\otimes} A(S_2)^*}$ and $S_{A(S_1)^* \underline{\otimes} A(S_2)^*}$ are cospectral by Lemma 3.1. Hence, the signed bipartite graphs $S_{A(S_1) \underline{\otimes} A(S_2)}$, $S_{A(S_1) \underline{\otimes} A(S_2)^*}$, $S_{A(S_1)^* \underline{\otimes} A(S_2)}$ and $S_{A(S_1)^* \underline{\otimes} A(S_2)^*}$ are mutually cospectral. ■

We are now able to formulate our main results.

Theorem 3.3: Assume that S_1 and S_2 are two signed bipartite graphs, where S_1 and $S_2 \in \mathcal{S}$. Then any two signed bipartite graphs in $\mathcal{T}_{S_1 \underline{\times} S_2} \cup \mathcal{T}_{S_1^{GM} \underline{\times} S_2} \cup \mathcal{T}_{S_1 \underline{\times} S_2^{GM}} \cup \mathcal{T}_{S_1^{GM} \underline{\times} S_2^{GM}}$ are cospectral if and only if S_1 or S_2 is balanced.

Proof: Note that

$$\begin{aligned} & (\mathcal{T}_{S_1 \underline{\times} S_2}) \cup (\mathcal{T}_{S_1^{GM} \underline{\times} S_2}) \cup (\mathcal{T}_{S_1 \underline{\times} S_2^{GM}}) \cup (\mathcal{T}_{S_1^{GM} \underline{\times} S_2^{GM}}) \\ = & \{S_{A(S_1) \underline{\otimes} A(S_2)}, S_{A(S_1) \underline{\otimes} A(S_2)^\#}, S_{A(S_1)^* \underline{\otimes} A(S_2)}, \\ & S_{A(S_1)^* \underline{\otimes} A(S_2)^\#}, S_{A(S_1) \underline{\otimes} A(S_2)^*}, S_{A(S_1) \underline{\otimes} (A(S_2)^*)^\#}, \\ & S_{A(S_1)^* \underline{\otimes} A(S_2)^*}, S_{A(S_1)^* \underline{\otimes} (A(S_2)^*)^\#}\}. \end{aligned}$$

First, suppose that S_1 or S_2 is balanced. $S_{A(S_1) \underline{\otimes} A(S_2)}$, $S_{A(S_1) \underline{\otimes} A(S_2)^*}$, $S_{A(S_1)^* \underline{\otimes} A(S_2)}$ and $S_{A(S_1)^* \underline{\otimes} A(S_2)^*}$ are mutually cospectral by Lemma 3.2. It follows by Theorem 2.2 and Theorem 3.2 that any two signed bipartite graphs in $(\mathcal{T}_{S_1 \underline{\times} S_2}) \cup (\mathcal{T}_{S_1^{GM} \underline{\times} S_2}) \cup (\mathcal{T}_{S_1 \underline{\times} S_2^{GM}}) \cup (\mathcal{T}_{S_1^{GM} \underline{\times} S_2^{GM}})$ are cospectral.

Conversely, assume that any two signed bipartite graphs in $\mathcal{T}_{S_1 \underline{\times} S_2} \cup \mathcal{T}_{S_1^{GM} \underline{\times} S_2} \cup \mathcal{T}_{S_1 \underline{\times} S_2^{GM}} \cup \mathcal{T}_{S_1^{GM} \underline{\times} S_2^{GM}}$ are cospectral, we conclude that $S_{A(S_1) \underline{\otimes} A(S_2)}$ and $S_{A(S_1) \underline{\otimes} A(S_2)^\#}$ are cospectral. Hence S_1 or S_2 is balanced by Theorem 2.2. ■

Theorem 3.4: Assume that S_1 and S_2 are two connected signed bipartite graphs, where $S_1 \in \mathcal{S}$. Then the signed bipartite graphs $S_{A(S_1) \underline{\otimes} A(S_2)}$ and $S_{A(S_1)^* \underline{\otimes} A(S_2)}$ are isomorphic if and only if it is possible to find two permutation matrices R_1 and R_2, which can lead to $R_1^T (B \otimes C) R_2 = (Q_{A(S_1)} B) \otimes C$ or $R_1^T (B \otimes C) R_2 = (Q_{A(S_1)} B)^T \otimes C^T$.

Proof: First, assume that it is possible to find a permutation matrix R, which can lead to either $R = \begin{bmatrix} R_1 & \mathbf{0} \\ \mathbf{0} & R_2 \end{bmatrix}$ or $R = \begin{bmatrix} \mathbf{0} & R_1 \\ R_2 & \mathbf{0} \end{bmatrix}$, where R_i is the permutation matrix with $i \in \{1, 2\}$, such that

$$\begin{aligned} & R^T \begin{bmatrix} \mathbf{0} & B \otimes C \\ B^T \otimes C^T & \mathbf{0} \end{bmatrix} R \\ = & \begin{bmatrix} \mathbf{0} & (Q_{A(S_1)} B) \otimes C \\ (Q_{A(S_1)} B)^T \otimes C^T & \mathbf{0} \end{bmatrix}. \end{aligned}$$

If $R = \begin{bmatrix} R_1 & \mathbf{0} \\ \mathbf{0} & R_2 \end{bmatrix}$, then $R_1^T (B \otimes C) R_2 = (Q_{A(S_1)} B) \otimes C$.

If $R = \begin{bmatrix} \mathbf{0} & R_1 \\ R_2 & \mathbf{0} \end{bmatrix}$, then $R_1^T (B \otimes C) R_2 = (Q_{A(S_1)} B)^T \otimes C^T$.

Suppose it is possible to find two permutation matrices R_1 and R_2, which can lead to $R_1^T (B \otimes C) R_2 = (Q_{A(S_1)} B) \otimes C$. Let $R = \begin{bmatrix} R_1 & \mathbf{0} \\ \mathbf{0} & R_2 \end{bmatrix}$. Clearly, R is a permutation matrix. Now,

$$\begin{aligned} & R^T (A(S_1) \underline{\otimes} A(S_2)) R \\ = & \begin{bmatrix} R_1^T & \mathbf{0} \\ \mathbf{0} & R_2^T \end{bmatrix} \begin{bmatrix} \mathbf{0} & B \otimes C \\ B^T \otimes C^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} R_1 & \mathbf{0} \\ \mathbf{0} & R_2 \end{bmatrix} \\ = & \begin{bmatrix} \mathbf{0} & R_1^T (B \otimes C) R_2 \\ R_2^T (B^T \otimes C^T) R_1 & \mathbf{0} \end{bmatrix} \\ = & \begin{bmatrix} \mathbf{0} & (Q_{A(S_1)} B) \otimes C \\ (Q_{A(S_1)} B)^T \otimes C^T & \mathbf{0} \end{bmatrix} \\ = & A(S_1)^* \underline{\otimes} A(S_2). \end{aligned}$$

Now we suppose that it is possible to find two permutation matrices R_1 and R_2, which can lead to $R_1^T (B \otimes C) R_2 = (Q_{A(S_1)} B)^T \otimes C^T$, let $R = \begin{bmatrix} \mathbf{0} & R_1 \\ R_2 & \mathbf{0} \end{bmatrix}$. Clearly, R is a permutation matrix. Now,

$$\begin{aligned} & R^T (A(S_1) \underline{\otimes} A(S_2)) R \\ = & \begin{bmatrix} \mathbf{0} & R_2^T \\ R_1^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0} & B \otimes C \\ B^T \otimes C^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0} & R_1 \\ R_2 & \mathbf{0} \end{bmatrix} \\ = & \begin{bmatrix} \mathbf{0} & R_2^T (B^T \otimes C^T) R_1 \\ R_1^T (B \otimes C) R_2 & \mathbf{0} \end{bmatrix} \\ = & \begin{bmatrix} \mathbf{0} & (Q_{A(S_1)} B) \otimes C \\ (Q_{A(S_1)} B)^T \otimes C^T & \mathbf{0} \end{bmatrix} \\ = & A(S_1)^* \underline{\otimes} A(S_2). \end{aligned}$$

Thus the graphs $S_{A(S_1) \underline{\otimes} A(S_2)}$ and $S_{A(S_1)^* \underline{\otimes} A(S_2)}$ are isomorphic. ■

By utilizing Theorem 3.4, we can derive the following corollary.

Corollary 3.1: Assume that S_1 and S_2 are two connected signed bipartite graphs with $S_1 \in \mathcal{S}$ and S_1^{GM} being also connected. Then the signed bipartite graphs $S_{A(S_1)\underline{\otimes}A(S_2)}$ and $S_{A(S_1)^*\underline{\otimes}A(S_2)}$ are cospectral and non-isomorphic if and only if it is impossible to find permutation matrices R_1 and R_2, which can lead to $R_1^T(B\otimes C)R_2 = (Q_{A(S_1)}B)\otimes C$ and $R_1^T(B\otimes C)R_2 = (Q_{A(S_1)}B)^T\otimes C^T$.

By a similarly argument, any two signed bipartite graphs in $\mathcal{T}_{S_1\underline{\times}S_2} \cup \mathcal{T}_{S_1^{GM}\underline{\times}S_2} \cup \mathcal{T}_{S_1\underline{\times}S_2^{GM}} \cup \mathcal{T}_{S_1^{GM}\underline{\times}S_2^{GM}}$ also have similar results, we omit these results and proofs.

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