Constructing Cospectral Non-isomorphic Signed Bipartite Graphs

Shupeng Li, Juan Liu, Hong Yang and Hong-Jian Lai

Abstract—Let \( S = (G, \sigma) \) be a signed graph, where \( \sigma \) is the sign function on the edges of the underlying graph \( G \). It is widely recognized that the adjacency spectrum alone cannot uniquely determine a signed graph. Therefore, it is of great interest to identify whether there exist any cospectral, non-isomorphic signed graphs within a specific class of signed graphs. In a significant contribution, Godsil et al. demonstrated that two components of \( G \) graphs. In a significant contribution, Godsil et al. demonstrated that two components of \( G \) cannot uniquely determine a signed graph. Therefore, it is of great interest to identify whether there exist any cospectral, non-isomorphic signed graphs within a specific class of signed graphs.

It is widely recognized that the adjacency spectrum alone can help us to generate multiple pairs of cospectral non-isomorphic signed bipartite graphs. In generalize Godsil’s result for two connected signed bipartite graphs, which can be either positive or negative. For simplicity, we assume that every edge of \( G \) is connected bipartite graphs, are cospectral if and only if at the minimum one of \( G_1 \) and \( G_2 \) is balanced. In this paper, we first generalize Godsil’s result for two connected signed bipartite graphs \( S_1 \) and \( S_2 \). Furthermore, we will define partitioned tensor product of two signed bipartite graphs, which will enable us to generate multiple pairs of cospectral non-isomorphic signed bipartite graphs.

Index Terms—Signed bipartite graph, adjacency matrix, GM-switching, partitioned tensor product.

I. INTRODUCTION

A signed graph is an ordered pair \( S = (G, \sigma) \) consisting of the underlying graph \( G \) with vertex set \( V(S) \), edge set \( E(S) \) and a mapping \( \sigma: E(S) \to \{1, -1\} \), called the signature. Each edge in the graph is associated with a value that can be either positive or negative. For simplicity, we assume that the graph \( G \) is simple, without multiple edges or self-loops. The adjacency matrix \( A(S) = (a_{ij}) \) for a signed graph \( S \) is a symmetric matrix with elements limited to \( 0, 1 \) and \( -1 \), where \( a_{ij} = \sigma(x_ix_j) \) when \( x_i \) and \( x_j \) are neighboring vertices, and \( a_{ij} = 0 \) otherwise. The adjacency spectrum of a signed graph \( S \) is the set of all eigenvalues of \( A(S) \) including multiplicities. Two signed graphs are cospectral for the adjacency matrices if they have the same adjacency spectrum.

Assume that \( S = (G, \sigma) \) is a signed graph with vertex set \( V(S) \) and edge set \( E(S) \). For a given vertex \( v \in V(S) \), we define \( d^+_S(v) \) as the count of positive edges that are incident to vertex \( v \) in \( S \), and \( d^-_S(v) \) as the count of negative edges that are incident to vertex \( v \) in \( S \). Additionally, we introduce \( d^+_S(v) = d^-_S(v) \) in this case.

If \( V(S) \) can be partitioned into two parts \( X \) and \( Y \) such that every edge of \( S \) has one end in \( X \) and the other end in \( Y \), then \( S \) is called the signed bipartite graph. We say that \( X \) and \( Y \) are the partite sets of \( S \). If \( |X| = |Y| \), then \( S \) is balanced. Furthermore, if \( S \) is a signed bipartite graph and its adjacency matrix can be expressed as \( \begin{bmatrix} 0 & B \ 
B^T & 0 \end{bmatrix} \), where \( 0 \) denotes the zero matrix, then the matrix \( B \) is the biadjacency matrix of \( S \).

Let \( A \) be a square symmetric matrix with elements limited to \( 0, 1 \) and \( -1 \) such that diagonal entries are zero. Suppose \( S_A \) denotes the signed graph whose adjacency matrix is given by \( A \). The Kronecker product of matrices \( A \) of size \( m \times n \) and \( B \) of size \( p \times q \), denoted by \( A \otimes B \), is defined as the block matrix of size \( mp \times nq \) constructed by replacing each entry \( a_{ij} \) of \( A \) with the matrix product \( a_{ij}B \). On the other hand, the partitioned tensor product of two partitioned matrices \( M = \begin{bmatrix} P & Q 
R & S \end{bmatrix} \) and \( H = \begin{bmatrix} A & B 
C & D \end{bmatrix} \), denoted as \( M \otimes H \), is defined as the block matrix \( \begin{bmatrix} P \otimes A & Q \otimes B 
R \otimes C & S \otimes D \end{bmatrix} \), where each block is obtained by taking the Kronecker product of the corresponding submatrices. These notions were introduced by Godsil et al. in [6].

Let \( S_1 = (G_1, \sigma_1) \) and \( S_2 = (G_2, \sigma_2) \) be two signed graphs. Their direct product is the signed graph \( S_1 \times S_2 \), whose vertex set is \( V(S_1) \times V(S_2) \), whose edges are all pairs \( (x_i, y_k)(x_j, y_l) \) with \( x_i, x_j \in E(S_1) \) and \( y_k, y_l \in E(S_2) \). The signature of the edge \( (x_i, y_k)(x_j, y_l) \) in \( S_1 \times S_2 \) is defined as \( \sigma((x_i, y_k)(x_j, y_l)) = \sigma_1(x_i, x_j)\sigma_2(y_k, y_l) \), where \( x_i, x_j \in E(S_1) \) and \( y_k, y_l \in E(S_2) \). The direct product construction can also be applied to signed bipartite graphs.

If every signed graph that has the same spectrum as \( S \) is also isomorphic to \( S \), then we say \( S \) is determined by its spectrum (DS). Otherwise, if there exist cospectral signed graphs that are not isomorphic to \( S \), then we say \( S \) has a cospectral mate or \( S \) is not determined by its spectrum (NDS). Godsil et al. [6] utilized the concept of partitioned tensor product to construct graphs that have same adjacency spectrum. Ji et al. [9] introduced a method of constructing cospectral bipartite graphs, which relies on adjacency and normalized Laplacian matrices and employs the unfolding technique. The notion of unfolding a bipartite graph initially introduced in [5] is further expanded upon by this construction method. This approach provided a more flexible and generalized framework for generating cospectral graphs. In recent research by Kannan et al. [10], bipartite graphs with the same eigenvalues for both adjacency and normalized Laplacian matrices were constructed using partitioned tensor product. For signed graphs, non-isomorphic Laplacian cospectral signed graphs were obtained by Ji et al. in [13] and used the operation of partial transpose. For more details, we refer to [2], [3], [4], [8], [9], [11], [12], [16].

The rest of the paper is organized as follows. In Section 2, we define the partitioned tensor product of two signed

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bipartite graphs and discuss the existence of cospectral non-isomorphic signed bipartite graphs. In Section 3, we use the GM-switching method to partitioned tensor product defined as Section 2, this enables us to construct additional examples of cospectral non-isomorphic signed bipartite graphs.

II. COSPECTRAL SIGNED BIPARTITE GRAPHS FOR PARTITIONED TENSOR PRODUCT

Next, we first discuss the presence of cospectral direct product for two signed graphs. Secondly, we define partitioned tensor product and give a sufficient and necessary condition of cospectral partitioned tensor product for two signed bipartite graphs. The following notations will be used in the rest of this paper.

Let \( S_1 = (G_1, \sigma_1) \) and \( S_2 = (G_2, \sigma_2) \) be two signed bipartite graphs with \( V(S_1) = X \cup Y \) and \( V(S_2) = U \cup W \), where \( X = \{x_1, x_2, \ldots, x_m\} \), \( Y = \{y_1, y_2, \ldots, y_n\} \), \( U = \{u_1, u_2, \ldots, u_p\} \) and \( W = \{w_1, w_2, \ldots, w_q\} \). Let \( B = (b_{ij})_{m \times n} \) and \( C = (c_{kl})_{p \times q} \) be the biadjacency matrices of \( S_1 \) and \( S_2 \), respectively, with
\[
b_{ij} = \begin{cases} \sigma_1(x_i y_j), & \text{if } x_i y_j \in E(S_1), \\ 0, \end{cases}
\]
and
\[
c_{kl} = \begin{cases} \sigma_2(u_k w_l), & \text{if } u_k w_l \in E(S_2), \\ 0, \end{cases}
\]
Then, \( S_1 \) and \( S_2 \) have adjacency matrices \( A(S_1) = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix} \) and \( A(S_2) = \begin{bmatrix} 0 & C \\ C^T & 0 \end{bmatrix} \), respectively, relative to vertex orderings \( x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n, u_1, u_2, \ldots, u_p, w_1, w_2, \ldots, w_q \), respectively. Define
\[
A(S_1)^\# = \begin{bmatrix} 0 & B^T \\ B & 0 \end{bmatrix}
\]
and
\[
A(S_2)^\# = \begin{bmatrix} 0 & C^T \\ C & 0 \end{bmatrix}.
\]

It is simple to verify that \( S_1 \times S_2 \) has adjacency matrix \( A(S_1) \otimes A(S_2) \) relative to the ordering \( (x_1, u_1), \ldots, (x_1, u_p), (x_1, y_1), \ldots, (x_1, y_n), (x_2, u_1), \ldots, (x_2, u_p), (x_2, y_1), \ldots, (x_2, y_n), \ldots, (x_m, u_1), \ldots, (x_m, u_p), (x_m, y_1), \ldots, (x_m, y_n) \in E(S_1) \times E(S_2) \) and a signature \( \sigma((x_i, u_k)(y_j, w_l)) = \sigma_1(x_i y_j) \sigma_2(u_k w_l) \), where \( x_i y_j \in E(S_1) \) and \( u_k w_l \in E(S_2) \);

(ii) the second signed bipartite graph of \( T_{S_1 \times S_2} \) has a vertex set \( (X \times U) \cup (Y \times W) \), an edge set \( \{(x_i, u_k)(y_j, w_l) | x_i y_j \in E(S_1), u_k w_l \in E(S_2) \} \) and a signature \( \sigma((x_i, u_k)(y_j, w_l)) = \sigma_1(x_i y_j) \sigma_2(u_k w_l) \), where \( x_i y_j \in E(S_1) \) and \( u_k w_l \in E(S_2) \).

By the definition of \( T_{S_1 \times S_2} \), the adjacency matrices of two signed bipartite graphs \( A(S_1) \otimes A(S_2) \) relative to the ordering \( (x_1, u_1), \ldots, (x_1, u_p), (x_2, u_1), \ldots, (x_2, u_p), \ldots, (x_m, u_1), \ldots, (x_m, u_p), (y_1, w_1), \ldots, (y_1, w_q), \ldots, (y_n, w_1), \ldots, (y_n, w_q) \) of its vertices and \( A(S_1) \otimes A(S_2)^\# \) relative to the ordering \( (x_1, w_1), \ldots, (x_1, w_q), (x_2, w_1), \ldots, (x_2, w_q), \ldots, (x_m, w_1), \ldots, (x_m, w_q), (y_1, u_1), \ldots, (y_1, u_p), \ldots, (y_n, u_1), \ldots, (y_n, u_p) \) of its vertices, respectively. Therefore
\[
T_{S_1 \times S_2} = \{S(A(S_1) \otimes A(S_2)), S(A(S_1) \otimes A(S_2)^\#)\}.
\]

Moreover, if \( S_1 \) and \( S_2 \) are connected, then \( S_1 \times S_2 \) will consist of exactly two components: \( S(A(S_1) \otimes A(S_2)) \) and \( S(A(S_1) \otimes A(S_2)^\#) \).

We call \( T_{S_1 \times S_2} \) the partitioned tensor product of two signed bipartite graphs \( S_1 \) and \( S_2 \).

We start by recalling some properties of Kronecker product of matrices in the following.

**Proposition 2.1:** [15] Let \( M_1 = (m_{ij}^{(1)})_{p_1 \times q_1}, M_2 = (m_{ij}^{(2)})_{p_2 \times q_2}, H_1 = (h_{ij}^{(1)})_{s_1 \times t_1}, \) and \( H_2 = (h_{ij}^{(2)})_{s_2 \times t_2} \). Then each of the following holds:
(i) if both \( M_1 \) and \( H_1 \) are orthogonal matrices, then \( M_1 \otimes H_1 \) is also orthogonal matrix;
(ii) \( (M_1 \otimes H_1)^T = M_1^T \otimes H_1^T \);
(iii) if \( q_1 = p_2 \) and \( t_1 = s_2 \), then \( (M_1 \otimes H_1)(M_2 \otimes H_2) = (M_1 M_2 \otimes (H_1 H_2)) \);
(iv) if \( p_1 = p_2 \) and \( q_1 = q_2 \), then \( (M_1 + M_2) \otimes H = M_1 \otimes H + M_2 \otimes H \).

Next, we shall also develop a number of similar properties of partitioned tensor product for two partitioned matrices. All these will be applied in our arguments.

**Proposition 2.2:** Let \( M_1 = (m_{ij}^{(1)})_{p_1 \times q_1}, M_2 = (m_{ij}^{(2)})_{p_2 \times q_2}, H_1 = (h_{ij}^{(1)})_{s_1 \times t_1}, \) and \( H_2 = (h_{ij}^{(2)})_{s_2 \times t_2} \). Then each of the following holds:
(i) \( (M_1 \otimes H_1)^T = M_1^T \otimes H_1^T \);
(ii) let \( M_1 = \begin{bmatrix} 0 & S_1 \\ S_1 & 0 \end{bmatrix} \) and \( H_1 = \begin{bmatrix} A_1 & 0 \\ 0 & D_1 \end{bmatrix} \), or
\[
M_1 = \begin{bmatrix} 0 & Q_1 \\ Q_1 & 0 \end{bmatrix} \quad \text{and} \quad H_1 = \begin{bmatrix} A_1 & 0 \\ 0 & D_1 \end{bmatrix}.
\]

(iii-1) if \( q_1 = p_2 \) and \( t_1 = s_2 \), then \( (M_1 \otimes H_1)(M_2 \otimes H_2) = (M_1 M_2 \otimes (H_1 H_2)) \);
(iv-2) if both \( M_1 \) and \( H_1 \) are orthogonal matrices, then \( M_1 \otimes H_1 \) is also orthogonal matrix.

**Proof:** For (i), let \( M_1 = \begin{bmatrix} P_1 & Q_1 \\ R_1 & S_1 \end{bmatrix} \) and \( H_1 = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \). Then by Proposition 2.1 (ii),
\[
(M_1 \otimes H_1)^T = \begin{bmatrix} P_1^T \otimes A_1^T & R_1^T \otimes C_1^T \\ Q_1^T \otimes B_1^T & S_1^T \otimes D_1^T \end{bmatrix} = \begin{bmatrix} P_1^T & R_1^T \\ Q_1^T & S_1^T \end{bmatrix} \otimes \begin{bmatrix} A_1^T & C_1^T \\ B_1^T & D_1^T \end{bmatrix} = M_1^T \otimes H_1^T.
\]
For (ii), we assume that $M_1 = \begin{bmatrix} P_1 & 0 \\ 0 & S_1 \end{bmatrix}$ and $H_1 = \begin{bmatrix} A_1 & 0 \\ 0 & D_1 \end{bmatrix}$. If $q_1 = p_2$ and $t_1 = s_2$, we can define $M_2 = \begin{bmatrix} P_2 & Q_2 \\ R_2 & S_2 \end{bmatrix}$ and $H_2 = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}$ such that the number of columns of $P_1$ is equal to the number of rows of $P_2$, the number of columns of $S_1$ is equal to the number of rows of $S_2$, the number of columns of $A_1$ is equal to the number of rows of $A_2$ and the number of columns of $D_1$ is equal to the number of rows of $D_2$. Then, by Proposition 2.1 (iii),

\[
(M_1 \otimes H_1)(M_2 \otimes H_2) = \begin{bmatrix} P_1 \otimes A_1 & 0 \\ 0 & S_1 \otimes D_1 \end{bmatrix} \begin{bmatrix} P_2 \otimes A_2 & Q_2 \otimes B_2 \\ 0 & S_2 \otimes D_2 \end{bmatrix} = \begin{bmatrix} (P_1 \otimes A_1)(P_2 \otimes A_2) & (P_1 \otimes A_1)(Q_2 \otimes B_2) \\ (S_1 \otimes D_1)(R_2 \otimes C_2) & (S_1 \otimes D_1)(S_2 \otimes D_2) \end{bmatrix} \begin{bmatrix} (P_1, P_2) \otimes (A_1, A_2) & (P_1, Q_2) \otimes (A_1, B_2) \\ (S_1, R_2) \otimes (D_1, C_2) & (S_1, S_2) \otimes (D_1, D_2) \end{bmatrix} = \begin{bmatrix} (P_1, P_2) \otimes (A_1, A_2) & (P_1, Q_2) \otimes (A_1, B_2) \\ (S_1, R_2) \otimes (D_1, C_2) & (S_1, S_2) \otimes (D_1, D_2) \end{bmatrix} = (M_1M_2) \otimes (H_1H_2).
\]

Thus (ii-1) holds.

If both $M_1$ and $H_1$ are orthogonal matrices, then by Proposition 2.2 (i) and (ii-1),

\[
(M_1 \otimes H_1)(M_2 \otimes H_2)^T = (M_1 \otimes H_1)(M_1^T \otimes H_1^T) = (M_1M_1^T) \otimes (H_1H_1^T) = I_{P_1} \otimes I_{S_1} = I_{M_1}.
\]

So $M_1 \otimes H_1$ is also an orthogonal matrix.

The same holds true when $M_1 = \begin{bmatrix} 0 & Q_1 \\ R_1 & 0 \end{bmatrix}$ and $H_1 = \begin{bmatrix} 0 & B_1 \\ C_1 & 0 \end{bmatrix}$, respectively.

Next, we will discuss the existence of cospectral non-isomorphic signed bipartite graphs in $T_{S_1 \times S_2}$. It is evident that $S_1$ and $S_2$ are cospectral if and only if the corresponding adjacency matrices exhibit orthogonal similarity.

Theorem 2.2: Assume that $S_1$ and $S_2$ are two signed bipartite graphs. Then two signed bipartite graphs of $T_{S_1 \times S_2}$ are cospectral if and only if $S_1$ or $S_2$ is balanced.

Proof: Assume first that $S_1$ or $S_2$ is balanced. We just need to show that the matrices $A(S_1) \otimes A(S_2)$ and $A(S_1) \otimes A(S_2)^\#$ are orthogonally similar, which implies that $S_{A(S_1) \otimes A(S_2)}$ and $S_{A(S_1) \otimes A(S_2)^\#}$ are cospectral. Therefore, two signed bipartite graphs in $T_{S_1 \times S_2}$ are cospectral.

If $m = n$, then it is possible to find two orthogonal matrices $R_1$ and $R_2$, which can lead to $R_1^T B R_2 = B^T$. We can define $R = \begin{bmatrix} 0 & R_1 \\ R_2 & 0 \end{bmatrix}$. Now, we have $R^T A(S_1) R = A(S_1)$. Let $P = \begin{bmatrix} 0 & R_1 \odot I_p \\ R_2 \odot I_q & 0 \end{bmatrix}$. It follows that $F^T A(S_2) F = A(S_2)$. Clearly, both $R$ and $F$ are orthogonal matrices.

Let $P \otimes F = \begin{bmatrix} 0 & R_1 \odot I_p \\ R_2 \odot I_q & 0 \end{bmatrix}$. By Proposition 2.2 (ii-1), $P$ is an orthogonal matrix. Now,

\[
P^T (A(S_1) \otimes A(S_2)) P = (R \otimes F)^T (A(S_1) \otimes A(S_2)) (R \otimes F) = (R^T A(S_1)) R \otimes (F^T A(S_2) F) = A(S_1) \otimes A(S_2)^\#.
\]

Be aware that the second step uses Proposition 2.2 (ii) and (ii-1). Therefore, we can conclude that $S_{A(S_1) \otimes A(S_2)}$ and $S_{A(S_1) \otimes A(S_2)^\#}$ are cospectral.

If $p = q$, then it is possible to find two orthogonal matrices $F_1$ and $F_2$, which can lead to $F_1^T C F_2 = C^T$. Clearly, it is also possible to find a permutation matrix $R = \begin{bmatrix} I_m & 0 \\ 0 & I_n \end{bmatrix}$, which can lead to $R^T A(S_1) R = A(S_1)$.

Let $F = \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix}$. We have, $F^T A(S_2) F = A(S_2)^\#$. It is evident that both $R$ and $F$ are orthogonal matrices.

Let $P = R \otimes F = \begin{bmatrix} I_m \otimes F_1 & 0 \\ 0 & I_n \otimes F_2 \end{bmatrix}$. By Proposition 2.2 (ii-1), $P$ is an orthogonal matrix. Now,

\[
P^T (A(S_1) \otimes A(S_2)) P = (R \otimes F)^T (A(S_1) \otimes A(S_2)) (R \otimes F) = (R^T A(S_1)) R \otimes (F^T A(S_2) F) = A(S_1) \otimes A(S_2)^\#.
\]

It should be noted that in the second step, Proposition 2.2 (i) and (ii-1) are utilized. As a result, we can conclude that $S_{A(S_1) \otimes A(S_2)}$ and $S_{A(S_1) \otimes A(S_2)^\#}$ are cospectral.

In contrast, suppose $S_{A(S_1) \otimes A(S_2)}$ and $S_{A(S_1) \otimes A(S_2)^\#}$ are cospectral. This implies that $m p + n q = m q + n p$, which further simplifies to $(m - n) (p - q) = 0$. From this, we can conclude that $m = n$ or $p = q$. Therefore, we have proved the theorem.

Hammack et al. [7] established a cancellation law for $(0, 1)$-matrices. This result can be further specialized to $(0, 1, -1)$-matrices and the following lemma may be proven using a similar approach as Lemma 3 in [7].

Lemma 2.2: Assume that $A_1, A_2$ and $C$ are $(0, 1, -1)$-matrices for which $C \neq 0$, and $A_1$ is square and has at least one nonzero entry in each row. Suppose it is possible to find two permutation matrices $Q_1$ and $R_1$, which can lead to $Q_1 (C \otimes A_1) R_1 = C \otimes A_2$. Then it is possible to find two permutation matrices $Q_2$ and $R_2$, which can lead to $Q_2 A_1 R_2 = A_2$. Also, if $Q_1 (A_1 \otimes C) R_1 = A_2 \otimes C$, then it is also possible to find two permutation matrices $Q_2$ and $R_2$, which can lead to $Q_2 A_1 R_2 = A_2$.

Then we get the following theorem.

Theorem 2.3: Assume that $S_1$ and $S_2$ are connected signed bipartite graphs whose biadjacency matrices are $B$ and $C$, respectively. Then two signed bipartite graphs of $T_{S_1 \times S_2}$ are isomorphic if and only if it is possible to find two permutation matrices $R_1$ and $R_2$, which can lead to $R_1^T B R_2 = B^T$ or $R_1^T C R_2 C = C^T$.

Proof: Assuming the given condition, it is possible to find a permutation matrix $R$ that can be expressed in one of $R = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}$ or $R = \begin{bmatrix} 0 & R'_i \\ R'_i & 0 \end{bmatrix}$, where $R'_i$ is a permutation matrix for $i \in \{1, 2\}$, which can lead to

\[
R^T \begin{bmatrix} 0 & B \otimes C \\ B^T \otimes C & 0 \end{bmatrix} R = \begin{bmatrix} 0 & B \otimes C \\ B^T \otimes C & 0 \end{bmatrix}.
\]
If \( R = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} \), then we have \( R_1^T (B \otimes C) R_2 = B \otimes C^T \), which implies that \( C \) is a square matrix. According to Lemma 2.2, it is possible to find two permutation matrices \( R_1 \) and \( R_2 \), which can lead to \( R_1^T C R_2 = C^T \). Similarly, if \( R = \begin{bmatrix} 0 & R_1 \\ R_2 & 0 \end{bmatrix} \), then we have \( R_1^T (B \otimes C) R_2 = B^T \otimes C \), which implies that \( B \) is a square matrix. By applying Lemma 2.2, we can find two permutation matrices \( R_1 \) and \( R_2 \), which can lead to \( R_1^T B R_2 = B^T \).

Suppose that it is possible to find two permutation matrices \( R_1 \) and \( R_2 \), which can lead to \( R_1^T B R_2 = B^T \). Let

\[
R = \begin{bmatrix} 0 & R_1 \\ R_2 & 0 \end{bmatrix}.
\]

Now \( R^T A(S_1) R = A(S_1) \). Clearly, \( R \) is a permutation matrix. Set \( F = \begin{bmatrix} I_p & 0 \\ I_q & 0 \end{bmatrix} \), then

\[
F^T A(S_2) F = A(S_2)^\#.
\]

Let \( P = R \otimes F \). Clearly, \( P \) is a permutation matrix now.

\[
P^T (A(S_1) \otimes A(S_2)) P = (R^T \otimes F^T) (A(S_1) \otimes A(S_2)) (R \otimes F) = \left( R^T A(S_1) R \right) \otimes \left( F^T A(S_2) F \right) = \left( A(S_1) \otimes A(S_2) \right)^\#.
\]

It should be noted that in the second step, Proposition 2.2 (i) and (ii-1) are utilized. Therefore, the signed bipartite graphs \( S_{A(S_1) \otimes A(S_2)} \) and \( S_{A(S_1) \otimes A(S_2)^\#} \) are isomorphic.

Similarly, if it is possible to find two permutation matrices \( R_1 \) and \( R_2 \), which can lead to \( R_1^T C R_2 = C^T \), then the signed bipartite graphs \( S_{A(S_1) \otimes A(S_2)} \) and \( S_{A(S_1) \otimes A(S_2)^\#} \) are isomorphic. This completes the proof of the theorem.

The following corollary follows from Theorem 2.2 and Theorem 2.3.

Corollary 2.1: Assume that \( S_1 \) and \( S_2 \) are two connected signed bipartite graphs, and assume that \( S_1 \) or \( S_2 \) are balanced. Then two signed bipartite graphs in \( T_{S_1 \otimes S_2} \) are cospectral and non-isomorphic if and only if it is impossible to find permutation matrices \( R_1 \) and \( R_2 \), which can lead to \( R_1^T B R_2 = B^T \) and \( R_1^T C R_2 = C^T \).

III. COSPECTRAL SIGNED BIPARTITE GRAPHS WITH GM-SWITCHING

Next, we provide some constructions of cospectral signed bipartite graphs using the notion of partitioned tensor product of two signed bipartite graphs and GM-switching.

In 2019, Belardo, et al. [1] constructed signed graphs by GM-switching as follows.

Definition 3.1: [1] Suppose that \( S = (G, \pi) \) is a signed graph and \( \pi \) is a partition of its vertex set \( V(S) \) into sets \( \{U_1, U_2, \ldots, U_t, W\} \), where \( |U_i| = n_i \) with \( i = 1, 2, \ldots, t \) and \( |W| = d \). The \( i \)-th net-degree of a vertex \( v_i \) is defined as the difference between the number of positive edges and the number of negative edges that connect vertex \( v_i \) to the vertices in \( U_i \). Suppose that each of the following holds for any integers \( i \) and \( j \) with \( 1 \leq i, j \leq t \):

(i) any two vertices in \( U_i \) have the same \( j \)-th net-degree;
(ii) for any vertex \( v \in W \):

(ii-1) either vertex \( v \) has an equal number of positive and negative edges connecting it to \( U_i \);
(ii-2) or vertex \( v \) is connected by positive edges to half of the vertices in \( U_i \) and no edges are connected to other vertices;

(iii) or vertex \( v \) is connected by negative edges to half of the vertices in \( U_i \) and no edges are connected to other vertices;

(iv) or vertex \( v \) is connected by positive edges to half of the vertices in \( U_i \) and no edges are connected to other vertices.

Next, the signed graph \( S^{GM} \) obtained from \( S \) utilizing local switching with respect to the partition \( \pi \) can be described as follows. For each vertex \( v \in W \) and \( 1 \leq \ell \leq t \), the following operations are performed:

(i) if the \( \ell \)-th net-degree of a vertex \( v_i \), \( 1 \leq i \leq t \), is reversed.

Assume that \( \pi \) is a permutation matrix. Set \( R = R \otimes F \).

Let \( P = R \otimes F \). Clearly, \( P \) is a permutation matrix now.

\[
F^T A(S_2) F = A(S_2)^\#
\]

Since \( F^T A(S_1) F = A(S_1) \), then we have \( F^T A(S_2) F = A(S_2)^\# \).

Therefore, the signed graph \( S^{GM} \) constructed utilizing GM-switching of signed graph \( S \).

Next, we introduce a special family of signed bipartite graph as follows. Let \( S \) be a signed bipartite graph family such that each signed bipartite graph \( S \in S \) with partite sets \( X \) and \( Y \) if and only if \( V(S) \) can be partitioned into \( t + 1 \) vertex subsets \( X_1, X_2, \ldots, X_t, Y \) and \( X = X_1 \cup X_2 \cup \cdots \cup X_t, Y \) and \( |X_i| = n_i \) satisfying Definition 3.1 (ii). Since \( S \) is the signed bipartite graph with partite subsets \( X \) and \( Y \), we have that for each \( 1 \leq i \), \( 1 \leq j \leq t \), \( j \)-th net-degree of any vertex in \( X_i \) is 0. Hence \( S \) satisfies Definition 3.1 (i). Therefore, \( S \) satisfies Definition 3.1.

Let the signed bipartite graph \( S^{GM} \) be constructed from \( S \) utilizing GM-switching with respect to the partition \( \{X_1, X_2, \ldots, X_t, Y\} \) obtained as above. Belardo, et al. [1] proved that \( S \) and \( S^{GM} \) are cospectral.

Suppose \( S_1^{GM} \) and \( S_2^{GM} \) are constructed by GM-switching for signed bipartite graphs \( S_1 \) and \( S_2 \), respectively. Let \( \tilde{B} \) and \( \tilde{C} \) be the biadjacency matrices of \( S_1^{GM} \) and \( S_2^{GM} \), respectively. Then \( A(S_1^{GM}) = \begin{bmatrix} 0 & \tilde{B} \\ \tilde{B}^T & 0 \end{bmatrix} \) and \( A(S_2^{GM}) = \begin{bmatrix} 0 & \tilde{C} \\ \tilde{C}^T & 0 \end{bmatrix} \), hence \( S_1^{GM} \) and \( S_2^{GM} \) have adjacency matrices \( A(S_1^{GM}) \) and \( A(S_2^{GM}) \), respectively, relative to vertex orderings \( x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n \) and \( u_1, u_2, \ldots, u_p, w_1, w_2, \ldots, w_q \), respectively. Define \( A(S_1) = A(S_1^{GM}) \) and \( A(S_2) = A(S_2^{GM}) \).

For any positive integer \( \ell \), let \( Q_{\ell} = \frac{2}{\ell} J_{\ell} - I_{\ell} \), where \( J_{\ell} \) represents the \( \ell \times \ell \) matrix whose entries are all equal to 1 and \( I_{\ell} \) represents the identity matrix of order \( \ell \).

Recently, Belardo et al. presented several properties of the matrix \( Q_{\ell} \) in [1].

Proposition 3.1: [1] Let \( Q_{\ell} = \frac{2}{\ell} J_{\ell} - I_{\ell} \), and \( x = (x_i)_{i=1,2,\ldots,\ell} \) be a vector with entries in \( \{0, 1, -1\} \). Then
each of the following holds:

(i) $Q_x$ is orthogonal and symmetric;

(ii) if a vector $x$ with a sum of entries equal to 0, then we have $Q_t x = -x$;

(iii) if $t$ is an even integer and $x$ is a vector with half of its elements being 0 and the other half being 1, then we have $Q_t x = 1_t - x$;

(iv) if $t$ is an even integer and $x$ is a vector with half of its elements being 0 and the other half being -1, then we have $Q_t x = -1_t - x$;

(v) if $x$ is the vector $1_t$, then we have $Q_t x = x$;

(vi) if $x$ is the vector $-1_t$, then we have $Q_t x = x$.

Let $S_1, S_2 \in S$. Then $V(S_1)$ can be partitioned into $t_1 + 1$ vertex subsets $X_1, X_2, \ldots, X_t$, and $Y$ with $X = X_1 \cup X_2 \cup \cdots \cup X_{t_1}$, $|Y| = n_1$, $|X_j| = m_j$ for $j \in \{1, 2, \ldots, t_1\}$ and $\sum_{j=1}^{t_1} m_j = m$. And $V(S_2)$ can be partitioned into $t_2 + 1$ vertex subsets $U_1, U_2, \ldots, U_{t_2}$ and $W$ with $U = U_1 \cup U_2 \cup \cdots \cup U_{t_2}$, $|W| = q$, $|U_k| = p_k$ for $k \in \{1, 2, \ldots, t_2\}$ and $\sum_{k=1}^{t_2} p_k = p$. Let

$$B = \begin{bmatrix} B_1 & & \\ & \vdots & \\ B_{t_1} & & \\ & & B_2 & \\ & & & B_t \\ & & & & \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_{t_2} \end{bmatrix},$$

where $B_j$ are $m_j \times n$ for $j \in \{1, 2, \ldots, t_1\}$ and $C_k$ are $p_k \times q$ for $k \in \{1, 2, \ldots, t_2\}$. We get the adjacency matrices of $S_1$ and $S_2$ are

$$A(S_1) = \begin{bmatrix} 0 & \cdots & 0 & B_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & B_{t_1} \\ B_1^T & B_2^T & \cdots & B_{t_1}^T \end{bmatrix}$$

and

$$A(S_2) = \begin{bmatrix} 0 & \cdots & 0 & C_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & C_{t_2} \\ C_1^T & C_2^T & \cdots & C_{t_2}^T \end{bmatrix}$$

respectively.

Let $Q_A(S_1) = \text{diag}(Q_{m_1}, Q_{m_2}, \ldots, Q_{m_{t_1}})$ and $Q_A(S_2) = \text{diag}(Q_{p_1}, Q_{p_2}, \ldots, Q_{p_{t_2}})$. According to Proposition 3.1, we can obtain

$$Q_A(S_1) B = \begin{bmatrix} Q_{m_1} & 0 & \cdots & 0 & 0 \\ 0 & Q_{m_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & Q_{m_{t_1}} & 0 \\ 0 & 0 & \cdots & 0 & Q_{m_{t_1}} \\ Q_{m_1} B_1 & Q_{m_2} B_2 \\ \vdots \\ Q_{m_{t_1}} B_{t_1} \\ Q_{m_{t_1}} B_{t_1} \\ B_1 \\ B_2 \\ \vdots \\ B_{t_1} \\ B_{t_1} \end{bmatrix}$$

and

$$Q_A(S_2) C = \begin{bmatrix} Q_{p_1} & 0 & \cdots & 0 & 0 \\ 0 & Q_{p_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & Q_{p_{t_2}} & 0 \\ 0 & 0 & \cdots & 0 & Q_{p_{t_2}} \\ Q_{p_1} C_1 & Q_{p_2} C_2 \\ \vdots \\ Q_{p_{t_2}} C_{t_2} \\ Q_{p_{t_2}} C_{t_2} \end{bmatrix}$$

In the remainder of this section, we provide some constructions of cospectral non-isomorphic signed bipartite graphs using the notion of partitioned tensor product for signed bipartite graphs and GM-switching.

It is worth pointing out that

$$T_{S_1 \otimes S_2} = \{ S_{A(S_1) \otimes A(S_2)}, S_{A(S_1) \otimes A(S_2)^\star} \},$$

$$T_{S_1 \otimes S_2}^{\star \star} = \{ S_{A(S_1) \otimes A(S_2)^\star}, S_{A(S_1) \otimes A(S_2)^\star} \},$$

$$T_{S_1 \otimes S_2}^{\star \star \star} = \{ S_{A(S_1) \otimes A(S_2)^\star^\star}, S_{A(S_1) \otimes A(S_2)^\star^\star} \}$$

and

$$T_{S_1 \otimes S_2}^{\star \star \star \star} = \{ S_{A(S_1) \otimes A(S_2)^\star^\star^\star}, S_{A(S_1) \otimes A(S_2)^\star^\star^\star} \}.$$
Let $R = \begin{bmatrix} Q_{A(S_1)} & 0 \\ 0 & I_n \end{bmatrix}$, where $Q_{A(S_1)} = \text{diag}(Q_{m_1}, Q_{m_2}, \ldots, Q_{m_k})$, by Proposition 3.1 (i), $R$ is an orthogonal matrix and symmetric matrix. Then
\[
R^T A(S_1) R = \begin{bmatrix} Q_{A(S_1)} & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} B \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = A(S_1)^*.
\]

Obviously, it is possible to find two identity matrices $I_p$ and $I_q$, which lead to $I_p C I_q = C$. Define $F = \begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix}$. Clearly, $F$ is an orthogonal matrix. Now, we have $F^T A(S_1) F = A(S_2)$.

Let $P = R \otimes F = \begin{bmatrix} Q_{A(S_1)} & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} 0 & I_q \end{bmatrix}$. By Proposition 2.2 (ii-1), $P$ is an orthogonal matrix. Now,
\[
P^T (A(S_1) \otimes A(S_2)) P = (R \otimes F)^T (A(S_1) \otimes A(S_2)) (R \otimes F) = (R^T A(S_1) R) \otimes (F^T A(S_2) F) = A(S_1)^* \otimes A(S_2)^*.
\]

Note that the second step utilizes Proposition 2.2 (i) and (ii-1). Thus $S_{A(S_1) \otimes A(S_2)}$ and $S_{A(S_1)^* \otimes A(S_2)}$ are cospectral.

**Lemma 3.2:** Assume that $S_1$ and $S_2$ are two signed bipartite graphs, where $S_1$ and $S_2 \in S$. Then the signed bipartite graphs $S_{A(S_1) \otimes A(S_2)}$, $S_{A(S_1)^* \otimes A(S_2)}$, and $S_{A(S_1)^* \otimes A(S_2)}$ are mutually cospectral.

**Proof:** Let
\[
R = \begin{bmatrix} I_p & 0 \\ 0 & I_n \end{bmatrix}
\]
and
\[
F = \begin{bmatrix} Q_{A(S_2)} & 0 \\ 0 & I_q \end{bmatrix},
\]
where $Q_{A(S_2)} = \text{diag}(Q_{p_1}, Q_{p_2}, \ldots, Q_{p_k})$ and $P = R \otimes F$. By a similar proof to that of Proposition 3.1, we can obtain $P^T (A(S_1) \otimes A(S_2)) P = A(S_1) \otimes A(S_2)^*$, which implies that $S_{A(S_1) \otimes A(S_2)}$ and $S_{A(S_1)^* \otimes A(S_2)}$ are cospectral. It follows by Lemma 3.1 that the signed bipartite graphs $S_{A(S_1) \otimes A(S_2)}$, $S_{A(S_1)^* \otimes A(S_2)}$, and $S_{A(S_1)^* \otimes A(S_2)}$ are mutually cospectral. Hence, the signed bipartite graphs $S_{A(S_1)^* \otimes A(S_2)}$, $S_{A(S_1)^* \otimes A(S_2)}$, and $S_{A(S_1)^* \otimes A(S_2)}$ are mutually cospectral by Lemma 3.1.

We are now able to formulate our main results.

**Theorem 3.3:** Assume that $S_1$ and $S_2$ are two signed bipartite graphs, where $S_1$ and $S_2 \in S$. Then any two signed bipartite graphs in $T_{S_1 \otimes S_2} \cup T_{S_{G^M} \otimes S_2} \cup T_{S_1 \otimes S_{G^M}}$ are cospectral if and only if $S_1$ or $S_2$ is balanced.

**Proof:** Note that
\[
(T_{S_1 \otimes S_2} \cup T_{S_{G^M} \otimes S_2} \cup T_{S_1 \otimes S_{G^M}} \cup T_{S_{G^M} \otimes S_{G^M}} = \{ S_{A(S_1) \otimes A(S_2)}, S_{A(S_1)^* \otimes A(S_2)} \}, S_{A(S_1)^* \otimes A(S_2)} \},
\]
$S_{A(S_1) \otimes A(S_2)}$, $S_{A(S_1)^* \otimes A(S_2)}$ and $S_{A(S_1)^* \otimes A(S_2)}$ are mutually cospectral by Theorem 2.2. It follows by Theorem 2.2 and Theorem 3.2 that any two signed bipartite graphs in $T_{S_1 \otimes S_2} \cup T_{S_{G^M} \otimes S_2} \cup T_{S_1 \otimes S_{G^M}}$ are cospectral.

Conversely, assume that any two signed bipartite graphs in $T_{S_1 \otimes S_2} \cup T_{S_{G^M} \otimes S_2} \cup T_{S_1 \otimes S_{G^M}}$ are cospectral, we conclude that $S_{A(S_1) \otimes A(S_2)}$ and $S_{A(S_1)^* \otimes A(S_2)}$ are cospectral. Hence $S_1$ or $S_2$ is balanced by Theorem 2.2.

**Theorem 3.4:** Assume that $S_1$ and $S_2$ are two connected signed bipartite graphs, where $S_1 \in S$. Then the signed bipartite graphs $S_{A(S_1) \otimes A(S_2)}$ and $S_{A(S_1)^* \otimes A(S_2)}$ are isomorphic if and only if is possible to find two permutation matrices $R_1$ and $R_2$, which can lead to $R_1^T (B \otimes C) R_2 = (Q_{A(S_1)}) B \otimes C$ or $R_2^T (B \otimes C) R_2 = (Q_{A(S_1)}) B \otimes C$.

**Proof:** First, assume that it is possible to find a permutation matrix $R$, which can lead to either $R = \begin{bmatrix} R_1 & 0 \\ R_2 & 0 \end{bmatrix}$, where $R_1$ is the permutation matrix with $i \in \{1, 2\}$, such that
\[
R^T \begin{bmatrix} 0 & B \otimes C \\ B^T \otimes C^T & 0 \end{bmatrix} R = \begin{bmatrix} (Q_{A(S_1)}) B \otimes C \\ 0 \end{bmatrix}.
\]

If $R_1 = \begin{bmatrix} R_1 & 0 \\ R_2 & 0 \end{bmatrix}$, then $R_1^T (B \otimes C) R_2 = (Q_{A(S_1)}) B \otimes C$. If $R_2 = \begin{bmatrix} R_1 & 0 \\ R_2 & 0 \end{bmatrix}$, then $R_2^T (B \otimes C) R_2 = (Q_{A(S_1)}) B \otimes C$.

Suppose it is possible to find two permutation matrices $R_1$ and $R_2$, which can lead to $R_1^T (B \otimes C) R_2 = (Q_{A(S_1)}) B \otimes C$.

Let $R = \begin{bmatrix} R_1 & 0 \\ R_2 & 0 \end{bmatrix}$. Clearly, $R$ is a permutation matrix. Now,
\[
R^T \begin{bmatrix} 0 & B \otimes C \\ B^T \otimes C^T & 0 \end{bmatrix} R = \begin{bmatrix} R_1 & 0 \\ R_2 & 0 \end{bmatrix}.
\]

Now we suppose that it is possible to find two permutation matrices $R_1$ and $R_2$, which can lead to $R_1^T (B \otimes C) R_2 = (Q_{A(S_1)}) B \otimes C$.

Thus the graphs $S_{A(S_1) \otimes A(S_2)}$ and $S_{A(S_1)^* \otimes A(S_2)}$ are isomorphic.
By utilizing Theorem 3.4, we can derive the following corollary.

**Corollary 3.1:** Assume that \( S_1 \) and \( S_2 \) are two connected signed bipartite graphs with \( S_1 \in \mathcal{S} \) and \( S_1^{\text{GM}} \) being also connected. Then the signed bipartite graphs \( S_{A(S_1) \otimes A(S_2)} \) and \( S_{A(S_1) \ast \otimes A(S_2)} \) are cospectral and non-isomorphic if and only if it is impossible to find permutation matrices \( R_1 \) and \( R_2 \), which can lead to
\[
R_1^T (B \otimes C) R_2 = (Q_{A(S_1)} B) \otimes C
\]
and
\[
R_1^T (B \otimes C) R_2 = (Q_{A(S_1)} B)^T \otimes C^T.
\]

By a similarly argument, any two signed bipartite graphs in \( T_{S_1 \times S_2} \cup T_{S_1 \times S_2}^{\text{GM}} \cup T_{S_1 \times S_2} \cup T_{S_1 \times S_2}^{\text{GM}} \) also have similar results, we omit these results and proofs.

**References**


