New Generalized Fuzzy Subalgebras of Hilbert Algebras

A. Iampan and N. Rajesh

***Abstract*—In this article, we present and investigate some of the features of novel generalized fuzzy subalgebras (FSAs) of Hilbert algebras called $(\in, \in \vee q_m)$-fuzzy subalgebras ($(\in, \in \vee q_m)$-FSAs) and also provide examples to support and oppose this idea. The level subsets of $(\in, \in \vee q_m)$-FSAs are used to describe them. There are also certain characterizations of $(\in, \in \vee q_m)$-FSAs developed. Moreover, we find that the Cartesian product of $(\in, \in \vee q_m)$-FSAs is still an $(\in, \in \vee q_m)$-FSA.**

***Index Terms*—Hilbert algebra, fuzzy subalgebra, $(\in, \in \vee q_m)$-fuzzy subalgebra, level subset, Cartesian product.**

I. Introduction and preliminaries

ZADEH [20] first suggested the notion of fuzzy sets (FSs). Fuzzy set theory has various applications in real-world settings, and many researchers have studied it. Several research works on the generalizations of FSs were done after the idea of FSs was introduced. [1], [3], [7] explore integrating FSs with some uncertainty techniques, such as soft and rough sets. The integration of FSs with other uncertainty strategies, such as soft sets and rough sets, has been researched in [6], [8], [9], [18], [22], [23], [24]. One of the expansions of FSs with more application is the notion of intuitionistic fuzzy sets (IFSs) proposed by Atanassov [2]. Applications of IFSs may be found in many different areas, such as multi-criteria decision-making, optimization problems, and medical diagnostics [10], [11], [12], [14], [17], [19].

Henkin [13] proposed the idea of Hilbert algebras in the early 1950s for various examinations of implication in intuitionistic and other non-classical logic. Diego [8] investigated these algebras specifically in the 1960s from an algebraic perspective. Busneag [5], [6] and Jun [14] both addressed Hilbert algebras, and it was realized that certain of their filters formed deductive systems. In Hilbert algebras, Dudek [9] studied the fuzzification of subalgebras, ideals, and deductive systems. Murali presented a definition of a fuzzy point that is a member of an FS under a natural equivalence on an FS in [15]. The concept of the quasi-coincidence of a fuzzy point with an FS, introduced in [16], was extremely important in creating many kinds of fuzzy subgroups.

Bhakat and Das [4] used the combined ideas of belonging and quasi-coincidence of a fuzzy point and an FS to establish a new type of fuzzy subgroups, $(\in, \in \vee q_m)$-fuzzy subgroups, in an earlier study. Rosenfeld's fuzzy subgroup's generalization, $(\in, \in \vee q_m)$-fuzzy subgroup, is significant and helpful.

The novel generalized FSAs of Hilbert algebras, which we refer to as $(\in, \in \vee q_m)$-FSAs, are introduced, and some of its key characteristics are studied in this research. We describe their level subsets $(\in, \in \vee q_m)$-FSAs. It is also demonstrated how certain $(\in, \in \vee q_m)$-FSAs may be characterized.

Let's study the idea of Hilbert algebras as it was initially introduced by Diego [8] in 1966 before we get started.

Definition I.1. [8] A *Hilbert algebra* is defined as a triplet denoted by $\mathcal{H} = (\mathcal{H}, *, 1)$, where $\mathcal{H} \neq \emptyset$, $*$ denotes a binary operation, and 1 is a fixed element of $\mathcal{H}$ that satisfies the axioms outlined below:

(i) $(\forall \varsigma, \dot{\varsigma} \in \mathcal{H})(\varsigma * (\dot{\varsigma} * \varsigma) = 1)$
(ii) $(\forall \varsigma, \dot{\varsigma}, \ddot{\varsigma} \in \mathcal{H})((\varsigma * (\dot{\varsigma} * \ddot{\varsigma})) * ((\varsigma * \dot{\varsigma}) * (\varsigma * \ddot{\varsigma})) = 1)$
(iii) $(\forall \varsigma, \dot{\varsigma} \in \mathcal{H})(\varsigma * \dot{\varsigma} = 1, \dot{\varsigma} * \varsigma = 1 \Rightarrow \varsigma = \dot{\varsigma})$.

We will replace a Hilbert algebra $\mathcal{H} = (\mathcal{H}, *, 1)$ with $\mathcal{H}$. It was established in [9] that the following was true.

(i) $(\forall \varsigma \in \mathcal{H})(\varsigma * \varsigma = 1)$
(ii) $(\forall \varsigma \in \mathcal{H})(1 * \varsigma = \varsigma)$
(iii) $(\forall \varsigma \in \mathcal{H})(\varsigma * 1 = 1)$
(iv) $(\forall \varsigma, \dot{\varsigma}, \ddot{\varsigma} \in \mathcal{H})(\varsigma * (\dot{\varsigma} * \ddot{\varsigma}) = \dot{\varsigma} * (\varsigma * \ddot{\varsigma}))$.

The binary relation $\leq$ in $\mathcal{H}$ is defined as

$$(\forall \varsigma, \dot{\varsigma} \in \mathcal{H})(\varsigma \leq y \Leftrightarrow \varsigma * \dot{\varsigma} = 1),$$

which is a partial order on $\mathcal{H}$ with 1 as the largest element.

Definition I.2. [21] Let $\emptyset \neq \Omega \subseteq \mathcal{H}$. Then Ω is called a *subalgebra* of $\mathcal{H}$, denoted with $\Omega \sqsubseteq \mathcal{H}$, if $\varsigma * \dot{\varsigma} \in \Omega, \forall \varsigma, \dot{\varsigma} \in \Omega$.

II. New fuzzy subalgebras of Hilbert algebras

Let $\mathcal{H} \neq \emptyset$. A *fuzzy set* (FS) [20] in $\mathcal{H}$ is defined to be a function $\hbar : \mathcal{H} \to [0, 1]$. An FS $\hbar$ in $\mathcal{H}$ of the form

$$\hbar(\dot{\varsigma}) = \begin{cases} t \in (0,1] & \text{if} \quad \dot{\varsigma} = \varsigma, \\ 0 & \text{otherwise}, \end{cases}$$

is said to be a *fuzzy point with support* ς *and value* t and is denoted by ς_t.

Let $\mathcal{H} \neq \emptyset$. For a fuzzy point ς_t and an FS $\hbar$ in $\mathcal{H}$, Pu and Liu [16] introduced the symbol $\varsigma_t \alpha \hbar$, where $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$. To say that $\varsigma_t \in \hbar$ (resp., $\varsigma_t q \hbar$), we mean $\hbar(\varsigma) \geq t$ (resp., $\hbar(\varsigma) + t > 1$), and in this case, ς_t is said to *belong to* (resp., *be quasi-coincident with*) an FS $\hbar$. To say that $\varsigma_t \in \vee q \hbar$ (resp., $\varsigma_t \in \wedge q \hbar$), we mean $\varsigma_t \in \hbar$ or $\varsigma_t q \hbar$ (resp., $\varsigma_t \in \hbar$ and $\varsigma_t q \hbar$). We assign the symbol $\varsigma_t \overline{\alpha} \hbar$ to the negation of $\varsigma_t \alpha \hbar$.

Definition II.1. An FS $\hbar$ in $\mathcal{H}$ is called an $(\in, \in \vee q)$-*fuzzy subalgebra* ($(\in, \in \vee q)$-FSA) of $\mathcal{H}$ if

$$(\forall \varsigma, \dot{\varsigma} \in \mathcal{H}, t_1, t_2 \in (0,1])(\varsigma_{t_1}, \dot{\varsigma}_{t_2} \in \hbar \Rightarrow (\varsigma * \dot{\varsigma})_{\min\{t_1, t_2\}} \in \vee q \hbar). \quad (1)$$

Manuscript received July 14, 2023; revised November 9, 2023.

This research project was supported by the Thailand Science Research and Innovation Fund and the University of Phayao (Grant No. FF67-UoE-Aiyared-Iampan).

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Remark II.2. Let $m \in [0,1)$ unless otherwise specified. By $\varsigma_t q_m \hbar$, we mean $\hbar(\varsigma)+t+m > 1$, $t \in (0, \frac{1-m}{2}]$. The notation $\varsigma_t \in \vee q_m \hbar$ means that $\varsigma_t \in \hbar$ or $\varsigma_t q_m \hbar$.

Definition II.3. An FS $\hbar$ in $\mathcal{H}$ is called an $(\in, \in \vee q_m)$-*fuzzy subalgebra* ($(\in, \in \vee q_m)$-FSA) of $\mathcal{H}$ if

$$(\forall \varsigma, \dot\varsigma \in \mathcal{H}, t_1, t_2 \in (0,1])(\varsigma_{t_1}, \dot\varsigma_{t_2} \in \hbar \Rightarrow (\varsigma * \dot\varsigma)_{\min\{t_1,t_2\}} \in \vee q_m \hbar). \qquad (2)$$

We note that different types of FSAs can be constructed for different values of $m \in [0,1)$. Hence, an $(\in, \in \vee q_m)$-FSA with $m = 0$ is called an $(\in, \in \vee q)$-FSA.

Example II.4. Let $\mathcal{H} = \{1, \varepsilon, \dot\varepsilon, \ddot\varepsilon, \grave\varepsilon\}$ with the following table:

$*$	ε	$\dot\varepsilon$	$\ddot\varepsilon$	$\grave\varepsilon$	1
ε	1	1	1	1	1
$\dot\varepsilon$	ε	1	$\ddot\varepsilon$	1	1
$\ddot\varepsilon$	ε	$\dot\varepsilon$	1	1	1
$\grave\varepsilon$	ε	$\dot\varepsilon$	$\ddot\varepsilon$	1	1
1	ε	$\dot\varepsilon$	$\ddot\varepsilon$	$\grave\varepsilon$	1

Then $\mathcal{H} = (\mathcal{H}, *, 1)$ is a Hilbert algebra. We define an FS $\hbar$ in $\mathcal{H}$ as follows:

$$\hbar(\varsigma) = \begin{cases} 0.7 & \text{if} \quad \varsigma = 1 \\ 0.8 & \text{if} \quad \varsigma = \varepsilon \\ 0.6 & \text{if} \quad \varsigma = \dot\varepsilon \\ 0.4 & \text{if} \quad \varsigma = \ddot\varepsilon \\ 0.4 & \text{if} \quad \varsigma = \grave\varepsilon \end{cases}$$

If $m = 0.2$, then $U(\hbar, t) = \{\varsigma \in \mathcal{H} \mid \hbar(\varsigma) \geq t\} = \mathcal{H}, \forall t \in (0, 0.4]$. Hence, $\hbar$ is an $(\in, \in \vee q_{0.2})$-FSA of $\mathcal{H}$.

Example II.5. Let $\mathcal{H} = \{1, \varepsilon, \dot\varepsilon, \ddot\varepsilon, \grave\varepsilon\}$ with the following table:

$*$	ε	$\dot\varepsilon$	$\ddot\varepsilon$	$\grave\varepsilon$	1
ε	1	$\dot\varepsilon$	$\dot\varepsilon$	$\grave\varepsilon$	1
$\dot\varepsilon$	ε	1	ε	$\grave\varepsilon$	1
$\ddot\varepsilon$	1	1	1	$\grave\varepsilon$	1
$\grave\varepsilon$	1	$\dot\varepsilon$	$\dot\varepsilon$	1	1
1	ε	$\dot\varepsilon$	$\ddot\varepsilon$	$\grave\varepsilon$	1

Then $\mathcal{H} = (\mathcal{H}, *, 1)$ is a Hilbert algebra. We define an FS $\hbar$ in $\mathcal{H}$ as follows:

$$\hbar(\varsigma) = \begin{cases} 0.45 & \text{if} \quad \varsigma = 1 \\ 0.41 & \text{if} \quad \varsigma = \varepsilon \\ 0.49 & \text{if} \quad \varsigma = \dot\varepsilon \\ 0.41 & \text{if} \quad \varsigma = \ddot\varepsilon \\ 0.41 & \text{if} \quad \varsigma = \grave\varepsilon. \end{cases}$$

If $m = 0.2$, we have

$$U(\hbar, t) = \begin{cases} \mathcal{H} & \text{if} \quad t \in (0, 0.4] \\ \{1, \varepsilon, \ddot\varepsilon, \grave\varepsilon\} & \text{if} \quad t \in (0.4, 0.45] \\ \{\dot\varepsilon\} & \text{if} \quad t \in (045, 0.49]. \end{cases}$$

Since $\{\dot\varepsilon\} \not\sqsubseteq \mathcal{H}$, so $U(\hbar, t) \not\sqsubseteq \mathcal{H}$ for $t \in (0.45, 0.49]$. Hence, $\hbar$ is not an $(\in, \in \vee q_{0.4})$-FSA of $\mathcal{H}$.

Proposition II.6. *Every* $(\in, \in)$*-FSA is an* $(\in, \in \vee q_m)$*-FSA.*

Proof: Straightforward. ∎

Remark II.7. The converse statement may not be true. Consider the $(\in, \in \vee q_{0.2})$-FSA of $\mathcal{H}$ defined in Example II.5. Then $\hbar$ is not an $(\in, \in)$-FSA of $\mathcal{H}$ since $\varepsilon_{0.71} \in \hbar$ and $\varepsilon_{0.75} \in \hbar$, but $(\varepsilon * \varepsilon)_{\min\{0.71, 0.75\}} = 0_{0.71} \overline{\in} \hbar$.

Theorem II.8. *An FS* $\hbar$ *in* $\mathcal{H}$ *is an* $(\in, \in \vee q_m)$*-FSA of* $\mathcal{H}$ *if and only if* $\hbar(\varsigma * \dot\varsigma) \geq \min\{\hbar(\varsigma), \hbar(\dot\varsigma), \frac{1-m}{2}\}$ *holds,* $\forall \varsigma, \dot\varsigma \in \mathcal{H}$.

Proof: Let $\hbar$ be an $(\in, \in \vee q_m)$-FSA of $\mathcal{H}$. Assume $\hbar(\varsigma * \dot\varsigma) \geq \min\{\hbar(\varsigma), \hbar(\dot\varsigma), \frac{1-m}{2}\}$ is not true. Then $\exists \varsigma', \dot\varsigma' \in \mathcal{H}$ such that $\hbar(\varsigma' * \dot\varsigma') < \min\{\hbar(\varsigma'), \hbar(\dot\varsigma'), \frac{1-m}{2}\}$. If $\min\{\hbar(\varsigma'), \hbar(\dot\varsigma')\} < \frac{1-m}{2}$, then $\hbar(\varsigma' * \dot\varsigma') < \min\{\hbar(\varsigma'), \hbar(\dot\varsigma')\}$. Thus $\hbar(\varsigma' * \dot\varsigma') < t \leq \min\{\hbar(\varsigma'), \hbar(\dot\varsigma')\}$ for some $t \in (0,1]$. It follows that $\varsigma'_t \in \hbar$ and $\dot\varsigma'_t \in \hbar$, but $(\varsigma'_t * \dot\varsigma'_t) \overline{\in} \hbar$, which is contradictory. Moreover, $\hbar(\varsigma'_t * \dot\varsigma') + t < 2t < 1 - m$, and so $(\varsigma' * \dot\varsigma')_t \overline{q_m} \hbar$. Hence, $(\varsigma' * \dot\varsigma')_t \overline{\in \vee q_m} \hbar$, which is contradictory. On the other hand, if $\min\{\hbar(\varsigma'), \hbar(\dot\varsigma')\} \geq \frac{1-m}{2}$, then $\hbar(\varsigma') \geq \frac{1-m}{2}$, $\hbar(\dot\varsigma') \geq \frac{1-m}{2}$ and $\hbar(\varsigma' * \dot\varsigma') < \frac{1-m}{2}$. Thus $\varsigma'_{\frac{1-m}{2}} \in \hbar$ and $\dot\varsigma'_{\frac{1-m}{2}} \in \hbar$, but $(\varsigma' * \dot\varsigma')_{\frac{1-m}{2}} \overline{\in} \hbar$. Also $\hbar(\varsigma' * \dot\varsigma') + \frac{1-m}{2} < \frac{1-m}{2} + \frac{1-m}{2} = 1 - m$, that is, $(\varsigma' * \dot\varsigma')_{\frac{1-m}{2}} \overline{q_m} \hbar$. Hence, $(\varsigma' * \dot\varsigma')_{\frac{1-m}{2}} \overline{\in \vee q_m \hbar}$, which is contradictory. Hence, $\hbar(\varsigma * \dot\varsigma) \geq \min\{\hbar(\varsigma), \hbar(\dot\varsigma), \frac{1-m}{2}\}$ holds, $\forall \varsigma, \dot\varsigma \in \mathcal{H}$.

Conversely, assume $\hbar(\varsigma * \dot\varsigma) \geq \min\{\hbar(\varsigma), \hbar(\dot\varsigma), \frac{1-m}{2}\}$ holds, $\forall \varsigma, \dot\varsigma \in \mathcal{H}$. Let $\varsigma, \dot\varsigma \in \mathcal{H}$ and $t_1, t_2 \in (0,1]$ be such that $\varsigma_{t_1} \in \hbar$ and $\dot\varsigma_{t_2} \in \hbar$. Then $\hbar(\varsigma * \dot\varsigma) \geq \min\{\hbar(\varsigma), \hbar(\dot\varsigma), \frac{1-m}{2}\} \geq \min\{t_1, t_2, \frac{1-m}{2}\}$. Assume $t_1 \leq \frac{1-m}{2}$ or $t_2 \leq \frac{1-m}{2}$. Then $\hbar(\varsigma * \dot\varsigma) \geq \min\{t_1, t_2\}$, which implies that $(\varsigma * \dot\varsigma)_{\min\{t_1,t_2\}} \in \hbar$. Now suppose $t_1 > \frac{1-m}{2}$ and $t_2 > \frac{1-m}{2}$. Then $\hbar(\varsigma * \dot\varsigma) \geq \frac{1-m}{2}$, and thus $\hbar(\varsigma * \dot\varsigma) + \min\{t_1, t_2\} > \frac{1-m}{2} + \frac{1-m}{2} = 1 - m$, that is, $(\varsigma * \dot\varsigma)_{\min\{t_1,t_2\}} q_m \hbar$. Hence, $(\varsigma * \dot\varsigma)_{\min\{t_1,t_2\}} \in \vee q_m \hbar$, and consequently, $\hbar$ is an $(\in, \in \vee q_m)$-FSA of $\mathcal{H}$. ∎

Theorem II.9. *An FS* $\hbar$ *in* $\mathcal{H}$ *is an* $(\in, \in \vee q_m)$*-FSA of* $\mathcal{H}$ *if and only if each level set* $\emptyset \neq U(\hbar, t) \sqsubseteq \mathcal{H}, \forall t \in (0, \frac{1-m}{2}]$.

Proof: Assume an FS $\hbar$ is an $(\in, \in \vee q)$-FSA of $\mathcal{H}$. Let $t \in (0, \frac{1-m}{2}]$ and $\varsigma, \dot\varsigma \in U(\hbar, t)$. Then $\hbar(\varsigma) \geq t$ and $\hbar(\dot\varsigma) \geq t$. It follows from $\hbar(\varsigma * \dot\varsigma) \geq \min\{\hbar(\varsigma), \hbar(\dot\varsigma), \frac{1-m}{2}\}$ holds, $\forall \varsigma, \dot\varsigma \in \mathcal{H}$ that $\hbar(\varsigma * \dot\varsigma) \geq \min\{\hbar(\varsigma), \hbar(\dot\varsigma), \frac{1-m}{2}\} \geq \min\{t, \frac{1-m}{2}\} = t$, so that $\varsigma * \dot\varsigma \in U(\hbar, t)$. Hence, $U(\hbar, t) \sqsubseteq \mathcal{H}$.

Conversely, suppose $\emptyset \neq U(\hbar, t) \sqsubseteq \mathcal{H}, \forall t \in (0, \frac{1-m}{2}]$. If the condition $\hbar(\varsigma * \dot\varsigma) \geq \min\{\hbar(\varsigma), \hbar(\dot\varsigma), \frac{1-m}{2}\}$ holds, $\forall \varsigma, \dot\varsigma \in \mathcal{H}$ is not true, then $\exists \varepsilon, \dot\varepsilon \in \mathcal{H}$ such that $\hbar(\varepsilon * \dot\varepsilon) < \min\{\hbar(\varepsilon), \hbar(\dot\varepsilon), \frac{1-m}{2}\}$. Hence, we can take $t \in (0,1]$ such that $\hbar(\varepsilon * \dot\varepsilon) < t < \min\{\hbar(\varepsilon), \hbar(\dot\varepsilon), \frac{1-m}{2}\}$. Then $t \in (0, \frac{1-m}{2}]$ and $\varepsilon, \dot\varepsilon \in U(\hbar, t)$. Since $U(\hbar, t) \sqsubseteq \mathcal{H}$, $\varepsilon * \dot\varepsilon \in U(\hbar, t)$, so $\hbar(\varepsilon * \dot\varepsilon) \geq t$. This is contradictory. Therefore, $\hbar(\varsigma * \dot\varsigma) \geq \min\{\hbar(\varsigma), \hbar(\dot\varsigma), \frac{1-m}{2}\}$ holds, $\forall \varsigma, \dot\varsigma \in \mathcal{H}$ is valid, and so $\hbar$ is an $(\in, \in \vee q_m)$-FSA of $\mathcal{H}$. ∎

Theorem II.10. *Let* $\hbar$ *be an FS in* $\mathcal{H}$*. Then* $\emptyset \neq U(\hbar, t) \sqsubseteq \mathcal{H}, \forall t \in (\frac{1-m}{2}, 1]$ *if and only if* $\max\{\hbar(\varsigma * \dot\varsigma), \frac{1-m}{2}\} \geq \min\{\hbar(\varsigma), \hbar(\dot\varsigma)\}, \forall \varsigma, \dot\varsigma \in \mathcal{H}$.

Proof: Suppose $\emptyset \neq U(\hbar, t) \sqsubseteq \mathcal{H}$. Assume $\max\{\hbar(\varsigma * \dot\varsigma), \frac{1-m}{2}\} < \min\{\hbar(\varsigma), \hbar(\dot\varsigma)\} = t$ for some $\varsigma, \dot\varsigma \in \mathcal{H}$, then $t \in (\frac{1-m}{2}, 1], \hbar(\varsigma * \dot\varsigma) < t, \varsigma \in U(\hbar, t)$ and $\dot\varsigma \in U(\hbar, t)$. Since $\varsigma, \dot\varsigma \in U(\hbar, t)$ and $U(\hbar, t) \sqsubseteq \mathcal{H}$, $\varsigma * \dot\varsigma \in U(\hbar, t)$, which is contradictory.

The converse is straightforward. ∎

Theorem II.11. *Let* $\hbar$ *be an* $(\in, \in \vee q_m)$*-FSA of* $\mathcal{H}$*. If it satisfies* $\hbar(\varsigma) < \frac{1-m}{2}, \forall \varsigma \in \mathcal{H}$*, then it is an FSA of* $\mathcal{H}$.

Proof: Let $\varsigma, \dot\varsigma \in \mathcal{H}$ and $t_1, t_2 \in (0,1]$ be such that $\varsigma_{t_1} \in \hbar$ and $\dot\varsigma_{t_2} \in \hbar$. Then $\hbar(\varsigma) \geq t_1$ and $\hbar(\dot\varsigma) \geq t_2$. It follows from Theorem II.8 that $\hbar(\varsigma * \dot{\varsigma}) > \min\{\hbar(\varsigma), \hbar(\dot{\varsigma}), \frac{1-m}{2}\} = \min\{\hbar(\varsigma), \hbar(\dot{\varsigma})\} = \min\{t_1, t_2\}$, so $(\varsigma * \dot{\varsigma})_{\min\{t_1,t_2\}} \in \hbar$. Hence, $\hbar$ is an FSA of $\mathcal{H}$. ■

Theorem II.12. *If $0 \leq m < n < 1$, then each $(\in, \in \vee q_m)$-FSA of $\mathcal{H}$ is an $(\in, \in \vee q_n)$-FSA of $\mathcal{H}$.*

Proof: Let $\hbar$ be an $(\in, \in \vee q_m)$-FSA of $\mathcal{H}$ and let $\varsigma, \dot{\varsigma} \in \mathcal{H}$. Then $\hbar(\varsigma * \dot{\varsigma}) > \min\{\hbar(\varsigma), \hbar(\dot{\varsigma}), \frac{1-m}{2}\} \geq \min\{\hbar(\varsigma), \hbar(\dot{\varsigma}), \frac{1-n}{2}\}$. Thus from Theorem II.8, $\hbar$ is an $(\in, \in \vee q_n)$-FSA of $\mathcal{H}$. ■

Note that an $(\in, \in \vee q_n)$-FSA may not be an $(\in, \in \vee q_m)$-FSA for $0 \leq m < n < 1$.

Theorem II.13. *$\emptyset \neq \dot{\Omega} \sqsubseteq \mathcal{H}$ if and only if $\hbar_{\dot{\Omega}}$ is an $(\in, \in \vee q_m)$-FSA of $\mathcal{H}$.*

Proof: Let $\dot{\Omega} \sqsubseteq \mathcal{H}$. Then $\chi_{\dot{\Omega}}(\varsigma) = 1$ for $\varsigma \in \dot{\Omega}$ and $\chi_{\dot{\Omega}}(\varsigma) = 0$ for $\varsigma \notin \dot{\Omega}$. Thus $U(\hbar_{\dot{\Omega}}, t) = \dot{\Omega}, \forall t \in (0, \frac{1-m}{2}]$. Hence, by Theorem II.9, $\chi_{\dot{\Omega}}$ is an $(\in, \in \vee q_m)$-FSA of $\mathcal{H}$.

Conversely, suppose $\hbar_{\dot{\Omega}}$ is an $(\in, \in \vee q_m)$-FSA of $\mathcal{H}$. Then $\hbar(\varsigma * \dot{\varsigma}) > \min\{\chi_{\dot{\Omega}}(\varsigma), \chi_{\dot{\Omega}}(\dot{\varsigma}), \frac{1-m}{2}\} = \min\{1, \frac{1-m}{2}\} = \frac{1-m}{2}, \forall \varsigma, \dot{\varsigma} \in \mathcal{H}$. Since $m \in [0, 1)$, it follows that $\chi_{\dot{\Omega}}(\varsigma * \dot{\varsigma}) = 1$, so $\varsigma * \dot{\varsigma} \in \dot{\Omega}$. Hence, $\dot{\Omega} \sqsubseteq \mathcal{H}$. ■

Theorem II.14. *For every $\dot{\Omega} \sqsubseteq \mathcal{H}$ and every $t \in (0, \frac{1-m}{2}]$, $\exists (\in, \in \vee q_m)$-FSA $\hbar$ of $\mathcal{H}$ such that $U(\hbar, t) = \dot{\Omega}$.*

Proof: Let $\hbar$ be an FS in $\mathcal{H}$ defined by

$$\hbar(\varsigma) = \begin{cases} t & \text{if} \quad \varsigma \in \dot{\Omega} \\ 0 & \text{otherwise,} \end{cases}$$

where $t \in (0, \frac{1-m}{2}]$. Obviously, $U(\hbar, t) = \dot{\Omega}$. Assume $\hbar(\varsigma * \dot{\varsigma}) < \min\{\hbar(\varsigma), \hbar(\dot{\varsigma}), \frac{1-m}{2}\}$ for some $\varsigma, \dot{\varsigma} \in \mathcal{H}$. Since $\hbar(\mathcal{H}) = \{0, t\}$, it follows that $\hbar(\varsigma * \dot{\varsigma}) = 0$ and $\min\{\hbar(\varsigma), \hbar(\dot{\varsigma}), \frac{1-m}{2}\} = t$. Hence, $\hbar(\varsigma) = \hbar(\dot{\varsigma}) = t$, and so $\varsigma, \dot{\varsigma} \in \dot{\Omega}$. Since $\dot{\Omega} \sqsubseteq \mathcal{H}$, $\varsigma * \dot{\varsigma} \in \dot{\Omega}$. Thus $\hbar(\varsigma * \dot{\varsigma}) = t$, which is contradictory. Therefore, $\hbar(\varsigma * \dot{\varsigma}) \geq \min\{\hbar(\varsigma), \hbar(\dot{\varsigma}), \frac{1-m}{2}\}, \forall \varsigma, \dot{\varsigma} \in \mathcal{H}$. By Theorem II.8, $\hbar$ is an $(\in, \in \vee q_m)$-FSA of $\mathcal{H}$. ■

Theorem II.15. *The intersection of any family of $(\in, \in \vee q_m)$-FSAs of $\mathcal{H}$ is an $(\in, \in \vee q_m)$-FSA of $\mathcal{H}$.*

Proof: Let $\{\hbar_i \mid i \in \mathcal{I}\}$ be a family of $(\in, \in \vee q_m)$-FSAs of $\mathcal{H}$ and let $\hbar = \bigcap_{i \in \mathcal{I}} \hbar_i$. Then

$$\begin{aligned} \hbar(\varsigma * \dot{\varsigma}) &= \sup_{i \in \mathcal{I}} \hbar_i(\varsigma * \dot{\varsigma}) \\ &\geq \sup_{i \in \mathcal{I}} \min\{\hbar_i(\varsigma), \hbar_i(\dot{\varsigma}), \frac{1-m}{2}\} \\ &= \min\{\sup_{i \in \mathcal{I}} \hbar_i(\varsigma), \sup_{i \in \mathcal{I}} \hbar_i(\dot{\varsigma}), \frac{1-m}{2}\} \\ &= \min\{\bigcap_{i \in \mathcal{I}} \hbar_i(\varsigma), \bigcap_{i \in \mathcal{I}} \hbar_i(\dot{\varsigma}), \frac{1-m}{2}\} \\ &= \min\{\hbar(\varsigma), \hbar(\dot{\varsigma}), \frac{1-m}{2}\}. \end{aligned}$$

Hence, by Theorem II.8, $\hbar$ is an $(\in, \in \vee q_m)$-FSA of $\mathcal{H}$. ■

The union of two $(\in, \in \vee q_m)$-FSAs of $\mathcal{H}$ is not an $(\in, \in \vee q_m)$-FSA, in general.

Theorem II.16. *The union of ordered family of $(\in, \in \vee q_m)$-FSAs of $\mathcal{H}$ is an $(\in, \in \vee q_m)$-FSA of $\mathcal{H}$.*

Proof: Let $\{\hbar_i \mid i \in \mathcal{I}\}$ be an ordered family of $(\in, \in \vee q_m)$-FSAs of $\mathcal{H}$, that is, $\hbar_i \subseteq \hbar_j$ or $\hbar_j \subseteq \hbar_i, \forall i, j \in \mathcal{I}$. Then for $\hbar = \bigcup_{i \in \mathcal{I}} \hbar_i$, we have

$$\begin{aligned} \hbar(\varsigma * \dot{\varsigma}) &= \inf_{i \in \mathcal{I}} \hbar_i(\varsigma * \dot{\varsigma}) \\ &\geq \inf_{i \in \mathcal{I}} \min\{\hbar_i(\varsigma), \hbar_i(\dot{\varsigma}), \frac{1-m}{2}\} \\ &= \min\{\inf_{i \in \mathcal{I}} \hbar_i(\varsigma), \inf_{i \in \mathcal{I}} \hbar_i(\dot{\varsigma}), \frac{1-m}{2}\} \\ &= \min\{\bigcup_{i \in \mathcal{I}} \hbar_i(\varsigma), \bigcup_{i \in \mathcal{I}} \hbar_i(\dot{\varsigma}), \frac{1-m}{2}\} \\ &= \min\{\hbar(\varsigma), \hbar(\dot{\varsigma}), \frac{1-m}{2}\}. \end{aligned}$$

Thus $\inf_{i \in \mathcal{I}} \min\{\hbar_i(\varsigma), \hbar_i(\dot{\varsigma}), \frac{1-m}{2}\} \leq \sup_{i \in \mathcal{I}} \min\{\hbar_i(\varsigma), \hbar_i(\dot{\varsigma}), \frac{1-m}{2}\}$. Suppose $\inf_{i \in \mathcal{I}} \min\{\hbar_i(\varsigma), \hbar_i(\dot{\varsigma}), \frac{1-m}{2}\} \neq \bigcup_{i \in \mathcal{I}} \min\{\hbar_i(\varsigma), \hbar_i(\dot{\varsigma}), \frac{1-m}{2}\}$. Then $\exists s \in [0, 1]$ such that

$$\begin{aligned} &\inf_{i \in \mathcal{I}} \min\{\hbar_i(\varsigma), \hbar_i(\dot{\varsigma}), \frac{1-m}{2}\} < s \\ &< \sup_{i \in \mathcal{I}} \min\{\hbar_i(\varsigma), \hbar_i(\dot{\varsigma}), \frac{1-m}{2}\}. \end{aligned}$$

Since $\hbar_i \subseteq \hbar_j$ or $\hbar_j \subseteq \hbar_i, \forall i, j \in \mathcal{I}$, $\exists k \in \mathcal{I}$ such that $s < \min\{\hbar_i(\varsigma), \hbar_i(\dot{\varsigma}), \frac{1-m}{2}\}$. On the other hand, $\min\{\hbar_i(\varsigma), \hbar_i(\dot{\varsigma}), \frac{1-m}{2}\} > s, \forall i \in \mathcal{I}$, which is contradictory. Hence,

$$\begin{aligned} &\inf_{i \in \mathcal{I}} \min\{\hbar_i(\varsigma), \hbar_i(\dot{\varsigma}), \frac{1-m}{2}\} \\ &= \min\{\bigcup_{i \in \mathcal{I}} \hbar_i(\varsigma), \bigcup_{i \in \mathcal{I}} \hbar_i(\dot{\varsigma}), \frac{1-m}{2}\} \\ &= \min\{\hbar(\varsigma), \hbar(\dot{\varsigma}), \frac{1-m}{2}\}. \end{aligned}$$

Therefore, $\hbar$ is an $(\in, \in \vee q_m)$-FSA of $\mathcal{H}$. ■

Theorem II.17. *For any finite strictly increasing chain of a subalgebra of $\mathcal{H}$, $\exists (\in, \in \vee q_m)$-FSA $\hbar$ of $\mathcal{H}$ whose level subalgebras are precisely the members of the chain with $\hbar_{\frac{1-m}{2}} = \mathcal{H}_0 \subset \mathcal{H}_1 \subset \ldots \subset \mathcal{H}_n = \mathcal{H}$.*

Proof: Let $\{t_i \in (0, \frac{1-m}{2}] \mid i = 1, 2, \ldots, n\}$ be such that $\frac{1-m}{2} > t_1 > t_2 > t_3 > \ldots > t_n$. Consider the FS $\hbar$ defined by

$$\hbar(\varsigma) = \begin{cases} \frac{1-m}{2} & \text{if} \quad \varsigma \in \mathcal{H}_0 \\ t_k & \text{if} \quad \varsigma \in \mathcal{H}_k \backslash \mathcal{H}_{k-1}, k = 1, 2, \ldots, n. \end{cases}$$

Let $\varsigma, \dot{\varsigma} \in \mathcal{H}$ be such that $\varsigma \in \mathcal{H}_i \backslash \mathcal{H}_{i-1}$ and $\dot{\varsigma} \in \mathcal{H}_j \backslash \mathcal{H}_{j-1}$, where $1 \leq i, j \leq n$. If $i \geq j$, then $\varsigma \in \mathcal{H}_i$ and $\dot{\varsigma} \in \mathcal{H}_i$, so $\varsigma * \dot{\varsigma} \in \mathcal{H}_i$. Thus $\hbar(\varsigma * \dot{\varsigma}) \geq t_i = \min\{t_i, t_j\} = \min\{\hbar(\varsigma), \hbar(\dot{\varsigma}), \frac{1-m}{2}\}$. If $i < j$, then $\varsigma \in \mathcal{H}_j$ and $\dot{\varsigma} \in \mathcal{H}_j$, so $\varsigma * \dot{\varsigma} \in \mathcal{H}_j$. Thus $\hbar(\varsigma * \dot{\varsigma}) \leq t_j = \min\{t_i, t_j\} = \min\{\hbar(\varsigma), \hbar(\dot{\varsigma}), \frac{1-m}{2}\}$. Hence, $\hbar$ is an $(\in, \in \vee q_m)$-FSA of $\mathcal{H}$. ■

Definition II.18. Let $\mathcal{H} \neq \emptyset$. For any FS $\hbar$ in $\mathcal{H}$ and $t \in (0, 1]$, we define two sets $[\hbar]_t = \{\varsigma \in \mathcal{H} \mid \varsigma_t \in \vee q_m \hbar\}$ and $Q(\hbar, t) = \{\varsigma \in \mathcal{H} \mid \varsigma_t q_m \hbar\}$. Then $[\hbar]_t = U(\hbar, t) \cup Q(\hbar, t)$.

Theorem II.19. *An FS $\hbar$ in $\mathcal{H}$ is an $(\in, \in \vee q_m)$-FSA of $\mathcal{H}$ if and only if $[\hbar]_t \sqsubseteq \mathcal{H}, \forall t \in (0, 1]$.*

Proof: Assume $\hbar$ is an $(\in,\in\vee q_m)$-FSA of $\mathcal{H}$ and let $\varsigma,\dot{\varsigma}\in[\hbar]_t$ for $t\in(0,1]$. Then $(\varsigma,t)\in\vee q_m\hbar$ and $(\dot{\varsigma},t)\in\vee q_m\hbar$, that is, $\hbar(\varsigma)>1$ or $\hbar(\varsigma)+t>1-m$, and $\hbar(\dot{\varsigma})>1$ or $\hbar(\dot{\varsigma})+t>1-m$. By Theorem II.8, we have $\hbar(\varsigma*\dot{\varsigma})\geq\min\{\hbar(\varsigma),\hbar(\dot{\varsigma}),\frac{1-m}{2}\}$.

Case 1: If $\hbar(\varsigma)\geq t$ and $\hbar(\dot{\varsigma})\geq t$, then $\hbar(\varsigma*\dot{\varsigma})\geq\min\{\hbar(\varsigma),\hbar(\dot{\varsigma}),\frac{1-m}{2}\}=\frac{1-m}{2}$. Hence, $\hbar(\varsigma*\dot{\varsigma})+t>\frac{1-m}{2}+\frac{1-m}{2}=1-m$, and so $(\varsigma*\dot{\varsigma},t)q_m\hbar$. If $t\leq\frac{1-m}{2}$, then $\hbar(\varsigma*\dot{\varsigma})\geq\min\{\hbar(\varsigma),\hbar(\dot{\varsigma}),\frac{1-m}{2}\}=\frac{1-m}{2}\geq t$, and thus $(\varsigma*\dot{\varsigma},t)\in\hbar$. Hence, $(\varsigma*\dot{\varsigma},t)\in\vee q_m\hbar$. Therefore, $\varsigma*\dot{\varsigma}\in[\hbar]_t$.

Case 2: If $\hbar(\varsigma)\geq t$ and $\hbar(\dot{\varsigma})+t>1-m$. If $t>\frac{1-m}{2}$, then $\hbar(\varsigma*\dot{\varsigma})\geq\min\{\hbar(\varsigma),\hbar(\dot{\varsigma}),\frac{1-m}{2}\}=\hbar(\dot{\varsigma})\wedge\frac{1-m}{2}>(1-m-t)\wedge\frac{1-m}{2}=1-m-t$, and so $(\varsigma*\dot{\varsigma},t)q_m\hbar$. If $t\leq\frac{1-m}{2}$, then $\hbar(\varsigma*\dot{\varsigma})\geq\min\{\hbar(\varsigma),\hbar(\dot{\varsigma}),\frac{1-m}{2}\}\geq\min\{t,1-m-t,\frac{1-m}{2}\}=t$. Hence, $(\varsigma*\dot{\varsigma},t)\in\hbar$, and hence $(\varsigma*\dot{\varsigma},t)\in\vee q_m\hbar$. Therefore, $\varsigma*\dot{\varsigma}\in[\hbar]_t$.

Case 3: If $\hbar(\varsigma)+t>1-m$ and $\hbar(\dot{\varsigma})\geq t$. If $t>\frac{1-m}{2}$, then $\hbar(\varsigma*\dot{\varsigma})\geq\min\{\hbar(\varsigma),\hbar(\dot{\varsigma}),\frac{1-m}{2}\}=\hbar(\varsigma)\wedge\frac{1-m}{2}>(1-m-t)\wedge\frac{1-m}{2}=1-m-t$, and so $(\varsigma*\dot{\varsigma},t)q_m\hbar$. If $t\leq\frac{1-m}{2}$, then $\hbar(\varsigma*\dot{\varsigma})\geq\min\{\hbar(\varsigma),\hbar(\dot{\varsigma}),\frac{1-m}{2}\}\geq\min\{1-m-t,t,\frac{1-m}{2}\}=t$. Hence, $(\varsigma*\dot{\varsigma},t)\in\hbar$, and hence $(\varsigma*\dot{\varsigma},t)\in\vee q_m\hbar$. Therefore, $\varsigma*\dot{\varsigma}\in[\hbar]_t$.

Case 4: If $\hbar(\varsigma)+t>1-m$ and $\hbar(\dot{\varsigma})+t>1-m$. If $t>\frac{1-m}{2}$, then $\hbar(\varsigma*\dot{\varsigma})\geq\min\{\hbar(\varsigma),\hbar(\dot{\varsigma}),\frac{1-m}{2}\}>(1-m-t)\wedge\frac{1-m}{2}=1-m-t$, and so $(\varsigma*\dot{\varsigma},t)q_m\hbar$. If $t\leq\frac{1-m}{2}$, then $\hbar(\varsigma*\dot{\varsigma})\geq\min\{\hbar(\varsigma),\hbar(\dot{\varsigma}),\frac{1-m}{2}\}\geq\min\{1-m-t,t,\frac{1-m}{2}\}\geq(1-m-t)\wedge\frac{1-m}{2}=\frac{1-m}{2}\geq t$. Hence, $(\varsigma*\dot{\varsigma},t)\in\hbar$, and hence $(\varsigma*\dot{\varsigma},t)\in\vee q_m\hbar$. Therefore, $\varsigma*\dot{\varsigma}\in[\hbar]_t$.

Consequently, $[\hbar]_t\sqsubseteq\mathcal{H}$.

Conversely, let $\hbar$ be an FS in $\mathcal{H}$ and $t\in(0,1]$ be such that $[\hbar]_t\sqsubseteq\mathcal{H}$. If it is possible, let $\hbar(\varsigma*\dot{\varsigma})<t\leq\min\{\hbar(\varsigma),\hbar(\dot{\varsigma}),\frac{1-m}{2}\}$ for some $t\in(0,1)$. Then $\varsigma,\dot{\varsigma}\in U(\hbar,t)\subseteq[\hbar]_t$, which implies that $\varsigma*\dot{\varsigma}\in[\hbar]_t$. Hence, $\hbar(\varsigma*\dot{\varsigma})\in[\hbar]_t$ or $\hbar(\varsigma*\dot{\varsigma})+t+m>1$, a contradiction. Therefore, $\hbar(\varsigma*\dot{\varsigma})\geq\min\{\hbar(\varsigma),\hbar(\dot{\varsigma}),\frac{1-m}{2}\},\forall\varsigma,\dot{\varsigma}\in\mathcal{H}$. By Theorem II.8, $\hbar$ is an $(\in,\in\vee q_m)$-FSA of $\mathcal{H}$. ■

Theorem II.20. *Let $\hbar$ be a proper $(\in,\in\vee q_m)$-FSA of $\mathcal{H}$ having at least two values $t_1,t_2<\frac{1-m}{2}$. If all $[\hbar]_t,t\in(0,\frac{1-m}{2}]$, are subalgebras, then $\hbar$ can be decomposed into the union of two proper non-equivalent $(\in,\in\vee q_m)$-FSAs of $\mathcal{H}$.*

Proof: Let $\hbar$ be a proper $(\in,\in\vee q_m)$-FSA of $\mathcal{H}$ with values $\frac{1-m}{2}>t_1>t_2>\ldots>t_n$, where $n>2$. Let $\mathcal{H}_0=[\hbar]_{\frac{1-m}{2}}$ and $\mathcal{H}_k=[\hbar]_{t_k}$ for $k=1,2,\ldots,n$. Then $[\hbar]_{\frac{1-m}{2}}=\mathcal{H}_0\subset\mathcal{H}_1\subset\ldots\subset\mathcal{H}_n=\mathcal{H}$ is the chain of $(\in,\in\vee q_m)$-subalgebras. Consider two FSs $\lambda_1,\lambda_2\leq\hbar$ defined by

$$\lambda_1(\varsigma)=\begin{cases}t_1 & \text{if} \quad \varsigma\in\mathcal{H}_1\\ t_k & \text{if} \quad \varsigma\in\mathcal{H}_k\backslash\mathcal{H}_{k-1},k=2,\ldots,n\end{cases}$$

$$\lambda_2(\varsigma)=\begin{cases}\hbar(\varsigma) & \text{if} \quad \varsigma\in\mathcal{H}_0\\ t_2 & \text{if} \quad \varsigma\in\mathcal{H}_2\backslash\mathcal{H}_0\\ t_k & \text{if} \quad \varsigma\in\mathcal{H}_k\backslash\mathcal{H}_{k-1},k=3,\ldots,n.\end{cases}$$

Then λ_1 and λ_2 are $(\in,\in\vee q_m)$-FSAs of $\mathcal{H}$ with $\mathcal{H}_0\subset\mathcal{H}_1\subset\ldots\subset\mathcal{H}_n$ and $\mathcal{H}_0\subset\mathcal{H}_1\subset\ldots\subset\mathcal{H}_n$ being respectively chains of $(\in,\in\vee q_m)$-FSAs. Obviously, $\hbar=\lambda_1\cup\lambda_2$. Moreover, λ_1 and λ_2 are non-equivalent since $\mathcal{H}_0\neq\mathcal{H}_1$. ■

A mapping $\mathfrak{F}:(\mathcal{H},*,1_{\mathcal{H}})\to(\hat{\mathcal{H}},\star,1_{\hat{\mathcal{H}}})$ of Hilbert algebras is called a *homomorphism* if $\mathfrak{F}(\varsigma*\dot{\varsigma})=\mathfrak{F}(\varsigma)\star\mathfrak{F}(\dot{\varsigma}),\forall\varsigma,\dot{\varsigma}\in\mathcal{H}$. Note that if $\mathfrak{F}:\mathcal{H}\to\hat{\mathcal{H}}$ is a homomorphism of Hilbert algebras, then $\mathfrak{F}(1_{\mathcal{H}})=1_{\hat{\mathcal{H}}}$.

Theorem II.21. *Let $\mathfrak{F}:\mathcal{H}\to\hat{\mathcal{H}}$ be a homomorphism of Hilbert algebras. If $\hbar$ is an $(\in,\in\vee q_m)$-FSA of $\hat{\mathcal{H}}$, where $m\in(0,1)$, then $\mathfrak{F}^{-1}(\hbar)$ is an $(\in,\in\vee q_m)$-FSA of $\mathcal{H}$, where $\mathfrak{F}^{-1}(\hbar)=\hbar\circ f$.*

Proof: Let $\hbar$ be an $(\in,\in\vee q_m)$-FSA of $\hat{\mathcal{H}}$, where $m\in(0,1)$ and $\varsigma,\dot{\varsigma}\in\mathcal{H}$. Then

$$\begin{aligned}\mathfrak{F}^{-1}(\hbar)(\varsigma*\dot{\varsigma})&=\hbar(\mathfrak{F}(\varsigma*\dot{\varsigma}))\\&=\hbar(\mathfrak{F}(\varsigma)*\mathfrak{F}(\dot{\varsigma}))\\&\geq\min\{\hbar(\mathfrak{F}(\varsigma)),\hbar(\mathfrak{F}(\dot{\varsigma})),\frac{1-m}{2}\}\\&=\min\{\mathfrak{F}^{-1}(\hbar(\varsigma)),\mathfrak{F}^{-1}(\hbar(\dot{\varsigma})),\frac{1-m}{2}\}.\end{aligned}$$

Hence, $\mathfrak{F}^{-1}(\hbar)$ is an $(\in,\in\vee q_m)$-FSA of $\mathcal{H}$. ■

Definition II.22. Let $\mathfrak{F}:\mathcal{H}\to\hat{\mathcal{H}}$ be a function of Hilbert algebras. An $(\in,\in\vee q_m)$-FSA $\hbar$ of $\mathcal{H}$ is said to be $\mathfrak{F}$-*invariant* if $\mathfrak{F}(\varsigma)=\mathfrak{F}(\dot{\varsigma})$ implies that $\hbar(\varsigma)=\hbar(\dot{\varsigma}),\forall\varsigma,\dot{\varsigma}\in\mathcal{H}$.

Theorem II.23. *Let $\mathfrak{F}:\mathcal{H}\to\hat{\mathcal{H}}$ be a homomorphism of Hilbert algebras and $\hbar$ an $(\in,\in\vee q_m)$-FSA of $\mathcal{H}$. If $\hbar$ is $\mathfrak{F}$-invariant, then $\mathfrak{F}(\hbar)$ is an $(\in,\in\vee q_m)$-FSA of $\hat{\mathcal{H}}$, where*

$$\mathfrak{F}(\hbar)(\varsigma)=\begin{cases}\sup\limits_{\varsigma\in\mathfrak{F}^{-1}(\dot{\varsigma})} & \text{if} \quad \mathfrak{F}^{-1}(\dot{\varsigma})\neq\emptyset\\ 0 & \text{otherwise}.\end{cases}$$

Proof: Let $\dot{\varsigma}_1,\dot{\varsigma}_2\in\hat{\mathcal{H}}$. If $\mathfrak{F}^{-1}(\dot{\varsigma}_1)=\emptyset$ or $\mathfrak{F}^{-1}(\dot{\varsigma}_2)=\emptyset$, then the proof is obvious. Otherwise, let $\mathfrak{F}^{-1}(\dot{\varsigma}_1)\neq\emptyset$ and $\mathfrak{F}^{-1}(\dot{\varsigma}_2)\neq\emptyset$. Then $\exists\varsigma_1,\varsigma_2\in\mathcal{H}$ such that $\mathfrak{F}(\varsigma_1)=\dot{\varsigma}_1$ and $\mathfrak{F}(\varsigma_2)=\dot{\varsigma}_2$. Thus

$$\begin{aligned}&\mathfrak{F}(\hbar)(\dot{\varsigma}_1*\dot{\varsigma}_2)\\&=\sup_{\varsigma\in\mathfrak{F}^{-1}(\dot{\varsigma}_1*\dot{\varsigma}_2)}\hbar(\varsigma)\\&=\sup_{\varsigma\in\mathfrak{F}^{-1}(\mathfrak{F}(\varsigma_1)*\mathfrak{F}(\varsigma_2))}\hbar(\varsigma)\\&=\sup_{\varsigma\in\mathfrak{F}^{-1}(\mathfrak{F}(\varsigma_1*\varsigma_2))}\hbar(\varsigma)\\&=\hbar(\varsigma_1*\varsigma_2)\\&\geq\min\{\hbar(\varsigma_1),\hbar(\varsigma_2),\frac{1-m}{2}\}\\&=\min\{\sup_{\varsigma\in\mathfrak{F}^{-1}(\mathfrak{F}(\varsigma_1))}\hbar(\varsigma),\sup_{\varsigma\in\mathfrak{F}^{-1}(\mathfrak{F}(\varsigma_2))}\hbar(\varsigma),\frac{1-m}{2}\}\\&=\min\{\sup_{\varsigma\in\mathfrak{F}^{-1}(\dot{\varsigma}_1)}\hbar(\varsigma),\sup_{\varsigma\in\mathfrak{F}^{-1}(\dot{\varsigma}_2)}\hbar(\varsigma),\frac{1-m}{2}\}\\&=\min\{\mathfrak{F}(\hbar)(\dot{\varsigma}_1),\mathfrak{F}(\hbar)(\dot{\varsigma}_2),\frac{1-m}{2}\}.\end{aligned}$$

Hence, $\mathfrak{F}(\hbar)$ is an $(\in,\in\vee q_m)$-FSA of $\hat{\mathcal{H}}$. ■

Let $(\mathcal{H},*,1_{\mathcal{H}})$ and $(\hat{\mathcal{H}},\star,1_{\hat{\mathcal{H}}})$ be Hilbert algebras. Then $(\mathcal{H}\times\hat{\mathcal{H}},\diamond,(1_{\mathcal{H}},1_{\hat{\mathcal{H}}}))$ is defined by $(\varsigma,\dot{\varsigma})\diamond(\varepsilon,\dot{\varepsilon})=(\varsigma*\varepsilon,\dot{\varsigma}\star\dot{\varepsilon}),\forall\varsigma,\varepsilon\in\mathcal{H}$ and $\dot{\varsigma},\dot{\varepsilon}\in\hat{\mathcal{H}}$.

Let $\hbar_1$ and $\hbar_2$ be $(\in,\in\vee q_m)$-FSAs of $\mathcal{H}$, where $m\in(0,1)$. The Cartesian product of $\hbar_1$ and $\hbar_2$ is defined by $\hbar_1\times\hbar_2$, where $(\hbar_1\times\hbar_2)(\varsigma,\dot{\varsigma})=\min\{\hbar_1(\varsigma),\hbar_2(\dot{\varsigma})\},\forall\varsigma,\dot{\varsigma}\in\mathcal{H}$.

Theorem II.24. *Let $\hbar_1$ and $\hbar_2$ be $(\in,\in\vee q_m)$-FSAs of $\mathcal{H}$, where $m\in(0,1)$. Then $\hbar_1\times\hbar_2$ is an $(\in,\in\vee q_m)$-FSA of $\mathcal{H}\times\mathcal{H}$.*

Proof: Let $(\varsigma_1, \dot{\varsigma}_1), (\varsigma_2, \dot{\varsigma}_2) \in \mathcal{H} \times \mathcal{H}$. Then

$$\begin{aligned}
&(\hbar_1 \times \hbar_2)((\varsigma_1, \dot{\varsigma}_1) \diamond (\varsigma_2, \dot{\varsigma}_2)) \\
&= (\hbar_1 \times \hbar_2)(\varsigma_1 * \varsigma_2, \dot{\varsigma}_1 * \dot{\varsigma}_2) \\
&= \min\{\hbar_1(\varsigma_1 * \varsigma_2), \hbar_2(\dot{\varsigma}_1 * \dot{\varsigma}_2)\} \\
&\geq \min\{\min\{\hbar_1(\varsigma_1), \hbar_1(\varsigma_2), \frac{1-m}{2}\}, \\
&\quad \min\{\hbar_1(\dot{\varsigma}_1), \hbar_1(\dot{\varsigma}_2), \frac{1-m}{2}\}\} \\
&= \min\{\min\{\hbar_1(\varsigma_1), \hbar_2(\dot{\varsigma}_1), \frac{1-m}{2}\}, \\
&\quad \min\{\hbar_1(\varsigma_2), \hbar_1(\dot{\varsigma}_2), \frac{1-m}{2}\}\} \\
&= \min\{(\hbar_1 \times \hbar_2)(\varsigma_1, \dot{\varsigma}_1), (\hbar_1 \times \hbar_2)(\varsigma_2, \dot{\varsigma}_2), \frac{1-m}{2}\}.
\end{aligned}$$

Hence, $\hbar_1 \times \hbar_2$ is an $(\in, \in \vee q_m)$-FSA of $\mathcal{H} \times \mathcal{H}$. ∎

III. Conclusion

In this article, we introduced $(\in, \in \vee q_m)$-FSAs of Hilbert algebras and looked at some of their key characteristics. The level subsets of $(\in, \in \vee q_m)$-FSAs were used to describe them. Additionally, several descriptions of $(\in, \in \vee q_m)$-FSAs are developed. Moreover, we have found that the Cartesian product of $(\in, \in \vee q_m)$-FSAs is still an $(\in, \in \vee q_m)$-FSA.

In the near future, we will extend the study from this article to the ideal and compare the results with this article.

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