Basins of Attraction of an Optimal Iterative Scheme for Solving Nonlinear Equations and Their Applications

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Abstract— **In this paper, we present an efficient iterative algorithm for finding the root of a nonlinear equation. We develop this approach by incorporating the weight function technique in the second step and applying Steffensen's method in the third step. The presented scheme has been shown to be optimal based on convergence analysis, with an order of convergence of eight. We evaluate the performance of our method across various application problems, including beam designing models, fractional conversion, ideal and non-ideal gas laws, Planck's constant, multifactor effects, blood rheology models, volume calculations in the van der Waals equation, and stirred tank reactors. It demonstrates exceptional performance when compared to other iterative approaches with similar convergence orders. Furthermore, we explore the dynamical and fractal behavior of the proposed method and existing methods by using several complex polynomials as test functions and analyzing the basins of attraction.**

Index Terms— **Nonlinear Equation, Optimal order, Iterative Method, Order of Convergence, Efficiency Index, Functional Evaluations, Dynamical Analysis.**

I. Introduction

FINDING the zeros of the nonlinear equation $h(x)=0$ is the most challenging problem in every science, engineering, and applied mathematics discipline. The simple roots of such equations cannot be found analytically since it is either computationally difficult or impossible. Then, we solve these equations using numerical techniques. So, to solve nonlinear equations, we require iterative techniques. The traditional Newton-Raphson's approach [NR] [6] is a well-known iterative approach for locating the root of such equations, denoted as

$$x_{n+1} = x_n - \frac{h(x_n)}{h'(x_n)}, \qquad n = 0,1,2,\ldots \tag{1}$$

This requires one step and involves two functional evaluations in each iteration. The convergence order of this approach is two, and the efficiency index is 1.414.

In this paper, we present an innovative eighth-order optimal iterative method. To show the usefulness of a recently developed system, we employed a variety of numerical examples and contrasted the results with those of other iterative processes of the same order. The recommended approach is efficient in terms of the number of iterations, function evaluations, and errors required to achieve a higher accuracy and efficiency index.

Using complex polynomials, we created a procedure in this study and looked at how it behaved dynamically. The attraction zones of the proposed method and other same-order current approaches are also illustrated and compared.

Some Basic Definitions

This section will discuss basic definitions of iterative methods for solving nonlinear equations.

Definition-1: Order of convergence [13]:

Consider x_0 be a root of the nonlinear function $h(x)$, $x \in R$. Let $\{x_{n+1}\}_{n=0}^{\infty}$ be a sequence that converges to x_0. Then, the order of convergence of the sequence p is defined as, if there exists a number $p \in R^{+}$ such that

$$\lim_{n\to\infty} \frac{x_{n+1} - x_0}{(x_n - x_0)^p} \approx K$$

For some $K \neq 0$, K is known as the asymptotic error constant.

If $p=1$, the sequence has linear convergence and

If $p=2,3$, the sequence has quadratic or cubic convergence.

Definition-2: Optimal order of convergence [16]:

The order for optimal iterative methods is defined as 2^{n-1}, where n is the function evaluations per iteration. It is known as Kung- Traub conjecture.

Definition-3: Computational order of convergence [15]:

Let x be a root of the nonlinear equation $h(x)=0$ and let x_{n-1}, x_n, x_{n+1} be convergence three successive iterations close to the root. Then it can be approximated as below

$$\rho = \frac{\log\left(\left|x_{n+1}-x\right| / \left|x_n - x\right|\right)}{\log\left(\left|x_n - x\right| / \left|x_{n-1} - x\right|\right)}$$

It is used to check the convergence order of a given iterative method.

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Definition-4: Weight function [9]:

The weight function is a fundamental variable that is sufficiently differentiable and defined on an interval. Iteration process behavior, order of convergence, and computational efficiency are the main goals of adding a weight function to a particular method.

Several popular optimal eighth-order iterative techniques for finding roots to nonlinear equations are already in use:

The following three-step optimal eighth-order iterative approach [PM] [11] was proposed by Prem Chand in 2019:

$$\begin{aligned} y_n &= x_n - \frac{h(x_n)}{h'(x_n)} \\ z_n &= x_n - \left(1 + 2\left(\frac{h(y_n)}{h(x_n)}\right)^2\right)\frac{h(x_n)+h(y_n)}{h'(x_n)} \\ x_{n+1} &= z_n - \frac{h(z_n)}{h'(x_n)}\left(\frac{1+2t_n+2u_n+3t_n^2}{1-s_n}\right) \end{aligned} \quad (2)$$

where

$$t_n = \frac{h(y_n)}{h(x_n)},\ u_n = \frac{h(z_n)}{h(x_n)},\ s_n = \frac{h(z_n)}{h(y_n)}$$

To resolve nonlinear equations in 2021, Sivakumar created the following optimal eighth-order scheme [PJ] [12]

$$\begin{aligned} y_n &= x_n - \frac{h(x_n)}{h'(x_n)} \\ z_n &= x_n - \frac{h(x_n)}{h'(x_n)}\left(\frac{h(x_n)-h(y_n)}{h(x_n)-2h(y_n)}\right) \\ w_n &= z_n - \frac{h(z_n)}{q'(z_n)} \end{aligned} \quad (3)$$

where

$$q'(z_n) = b_1 + 2b_2(z_n - x_n) + 3b_3(z_n - x_n)^2, b_1 = h'(x_n),$$

$$b_2 = \frac{h[y_n, x_n, x_n](z_n - x_n) - h[z_n, x_n, x_n](y_n - x_n)}{z_n - y_n},$$

$$b_3 = \frac{h[z_n, x_n, x_n] - h[y_n, x_n, x_n]}{z_n - y_n}.$$

Al-Subaihi proposed an ideal eighth-order iterative approach in 2019 [SB] [1], provided by

$$\begin{aligned} y_n &= x_n - \frac{h(x_n)}{h'(x_n)} \\ z_n &= y_n - \frac{h(x_n)+h(y_n)}{h(x_n)-h(y_n)}\frac{h(y_n)}{h'(x_n)} \\ x_{n+1} &= z_n - \frac{h(z_n)\,f[x_n, y_n]}{f[x_n, z_n]\,f[y_n, z_n]}\left(1 + \frac{h(z)}{h'(x)} - 2\left(\frac{h(y)}{h'(x)}\right)^3\right) \end{aligned} \quad (4)$$

Kung and Traub created the following optimal eighth-order approach [KT] [5] in 1974 for resolving nonlinear equations, is

$$\begin{aligned} y_n &= x_n - \frac{h(x_n)}{h'(x_n)} \\ z_n &= y_n - \frac{h(x_n)h(y_n)}{\left(h(x_n)-h(y_n)\right)^2}\frac{h(x_n)}{h'(x_n)} \\ x_{n+1} &= z_n - \frac{h(x_n)}{h'(x_n)}\frac{h(x_n)h(y_n)h(z_n)}{\left(h(x_n)-h(y_n)\right)^2}\frac{h(x_n)^2 + h(y_n)\left(h(y_n)-h(z_n)\right)}{\left(h(x_n)-h(z_n)\right)^2\left(h(y_n)-h(z_n)\right)} \end{aligned} \quad (5)$$

A derivative-free optimal iterative technique was proposed in 2020 by Ramandeep [RM] [3] and is as follows:

$$\begin{aligned} y_n &= x_n - \frac{h(x_n)}{h'(x_n)} \\ z_n &= y_n - \frac{h(y_n)}{h'(x_n)}\left(1 + 2u + 5u^2\right) \\ x_{n+1} &= z_n - \frac{h(z_n)}{h'(x_n)}\left(1 + 2u + t + 6u^2 + 4ut + t^2 + 6u^3 + 14u^2 t\right) \end{aligned} \quad (6)$$

where

$$u_n = \frac{h(y_n)}{h(x_n)},\ k_n = \frac{h(x_n)}{h'(x_n)},\ t_n = \frac{h(z_n)}{h(y_n)}$$

Using the weight function technique, Sharifi suggested a three-point iterative method [MS] [14] of an optimal eighth-order convergence in 2016, is provided by

$$\begin{aligned} y_n &= x_n - \frac{h(x_n)}{h'(x_n)} \\ z_n &= x_n + \frac{1}{h'(x_n)}\left(\frac{h^2(x_n)}{h(y_n) - h(x_n)} - \frac{h^2(y_n)}{h(x_n)}\right) \\ x_{n+1} &= z_n - \frac{h(z_n)}{h'(x_n)}F(x_n, y_n z_n)\left(1 + 2\frac{h(z_n)}{h(x_n)}\right) \end{aligned} \quad (7)$$

where

$$F(x_n, y_n z_n) = \frac{\left\{\begin{aligned} &h^3(y_n)\left(h(x_n) - 10h(y_n)\right) + 4h^2(x_n)\left(h^2(y_n)\right) \\ &+ h(x_n)h(y_n) \end{aligned}\right\}}{h(x_n)\left(2h(x_n) - h(y_n)\right)^2\left(h(y_n) - h(z_n)\right)}$$

The format of this research paper is as follows: Section I discusses the fundamental definitions that served as the foundation for this work and numerous existing methodologies are examined based on a review of the literature survey. Section II develops an effective three-step iterative process using the weight function technique and Steffensen's approximation. Section III investigates the order of convergence of the proposed approach. Real-world applications in science and engineering are included in Section IV of the developed scheme, along with numerical comparisons of the recommended technique with several existing, optimal approaches of the same order. The polynomiographs of various test functions using basins of attraction are described in Section V. The conclusions are provided in Section VI.

II. Eighth-Order Convergent Method

Consider x^* is an exact root of the nonlinear equation $h(x) = 0$ where $h(x)$ is continuous and has well-defined first-order derivatives. Let x_n be the root of the n[th] approximation and is

$$x^* = x_n + \varepsilon_n \quad (8)$$

where ε_n is the error. Thus, we have

$$h\left(x^*\right) = 0 \quad (9)$$

writing $h(x^*)$ by Taylor's series about x_n, we get

$$h(x^*) = h(x_n) + (x^* - x_n)h'(x_n) + \frac{(x^* - x_n)^2}{2!}h''(x_n) + \ldots$$

$$h(x^*) = h(x_n) + \varepsilon_n h'(x_n) + \frac{\varepsilon_n^2}{2!}h''(x_n) + \ldots \quad (10)$$

here higher powers of ε_n are neglected from ε_n^3 onwards. Using (9) and (10), we have

$$\varepsilon_n^2 h''(x_n) + 2\varepsilon_n h'(x_n) + 2h(x_n) = 0$$

$$\varepsilon_n = \left[-2h'(x_n) \pm \sqrt{4h'(x_n) - 8h(x_n)h''(x_n)}\right] \div 2h''(x_n)$$

$$\varepsilon_n = \frac{-2h(x_n)}{h'(x_n)} \cdot \frac{1}{1 + \sqrt{1 - 2\rho_n}} \quad (11)$$

On substituting x^* by x_{n+1} in (9) and from (11), we get

$$x_{n+1} = y_n - H(\tau)\left[\frac{2h(y_n)}{h'(y_n)} \cdot \frac{1}{1 + \sqrt{1 - 2\rho_n}}\right] \quad (12)$$

where, $y_n = x_n - \frac{h(x_n)}{h'(x_n)}$,

$$h'(y_n) = 2h[y_n, x_n] - h'(x_n), \qquad \rho_n = \frac{h'(x_n) - h'(y_n)}{h'(x_n)},$$

and $H(\tau) = 1 - \tau$, and $\tau = \frac{h(y_n)}{h(x_n)}$ is the weight function. Equation (12) has optimal fourth- order convergence, with three functional evaluations per iteration.

Now, we wish to enhance the order of convergence of the technique by taking equation (1) as the first step, equation (12) as the second step, and a modified version of Newton's method as the third step. To make the technique optimal, we further reduce the functional evaluations to four by taking the following Steffensen's approximation:

$$h'(z_n) = \frac{\left(h(z_n + h(z_n))\right) - \left(h(z_n - h(z_n))\right)}{2h(z_n)}. \quad (13)$$

Algorithm: The iterative scheme is computed as x_{n+1}

1. $y_n = x_n - \frac{h(x_n)}{h'(x_n)}$

2. $z_n = y_n - H(\tau)\left[\frac{2h(y_n)}{h'(y_n)} \cdot \frac{1}{1 + \sqrt{1 - 2\rho_n}}\right]$

   where, $\rho_n = \frac{h'(x_n) - h'(y_n)}{h'(x_n)}$,

   $h'(y_n) = 2h[y_n, x_n] - h'(x_n), H(\tau) = 1 - \tau$ and $\tau = \frac{h(y_n)}{h(x_n)}$

3. $x_{n+1} = z_n - \frac{h(z_n)}{h'(z_n)}$

   where, $h'(z_n) = \frac{\left(h(z_n + h(z_n))\right) - \left(h(z_n - h(z_n))\right)}{2h(z_n)}$ (14)

III. Convergence Criteria

Theorem [4, 8]: Suppose $x_0 \in D$ be a single zero of a sufficiently differentiable function h for an open interval D. If x_0 is the neighborhood of x^*. Then the algorithm (14) has an optimal eighth-order convergence with error equation,

$$\varepsilon_{n+1} = (-c_2^2 c_3^2)\varepsilon_n^8 + o\left(\varepsilon_n^9\right).$$

Proof: Let the single root of $h(x) = 0$ be x^* and $x^* = x_n + \varepsilon_n$. Thus,

$$h(x^*) = 0$$

Through Taylor's series, writing $h(x_n)$ about x^*, we get

$$h(x_n) = h'(x^*)\left(\varepsilon_n + c_2\varepsilon_n^2 + c_3\varepsilon_n^3 + c_4\varepsilon_n^4 + \ldots\right) \quad (15)$$

$$h'(x_n) = h'(x^*)\left(1 + 2c_2\varepsilon_n + 3c_3\varepsilon_n^2 + 4c_4\varepsilon_n^3 + \ldots\right) \quad (16)$$

Dividing (15) by (16), we get

$$\frac{h(x_n)}{h'(x_n)} = \varepsilon_n - c_2\varepsilon_n^2 - \left(2c_3 - 2c_2^2\right)\varepsilon_n^3 - \left(3c_4 - 7c_2c_3 + 4c_2^3\right)\varepsilon_n^4 + \ldots \quad (17)$$

Substituting (17) in the first step of (14), we obtain

$$y_n = x^* + c_2\varepsilon_n^2 + \left(2c_3 - 2c_2^2\right)\varepsilon_n^3 + \left(3c_4 - 7c_2c_3 + 4c_2^3\right)\varepsilon_n^4 + \ldots \quad (18)$$

Expanding $h(y_n)$ about x^* by using Taylor series, we get

$$h(y_n) = h'(x^*)\left(c_2\varepsilon_n^2 + \left(2c_3 - 2c_2^2\right)\varepsilon_n^3 + \left(3c_4 - 7c_2c_3 + 5c_2^3\right)\varepsilon_n^4 + \ldots\right) \quad (19)$$

$$h'(y_n) = h'(x^*)\left(1 + \left(2c_2^2 - c_3\right)\varepsilon_n^2 + \left(6c_2c_3 - 4c_2^3 - 2c_4\right)\varepsilon_n^3 + \ldots\right) \quad (20)$$

Dividing (19) by (20), we get

$$\frac{h(y_n)}{h'(y_n)} = c_2\varepsilon_n^2 + \left(2c_3 - 2c_2^2\right)\varepsilon_n^3 + \left(3c_2^3 - 6c_2c_3 + 3c_4\right)\varepsilon_n^4 + \ldots \quad (21)$$

From (16) and (20), we obtain

$$\rho_n = 2c_2\varepsilon_n + \left(4c_3 - 6c_2^2\right)\varepsilon_n^2 + \left(6c_4 + 16c_2^3 - 20c_2c_3\right)\varepsilon_n^3 + \ldots \quad (22)$$

From (22), on simplification

$$\left(1 + \sqrt{1 - 2\rho_n}\right)^{-1} = \frac{1}{2}\left(\begin{array}{l}1 + c_2\varepsilon_n + \left(2c_3 - c_2^2\right)\varepsilon_n^2 + \left(3c_4 - 2c_2c_3\right)\varepsilon_n^3 \\ + \left(37c_2^4 + 8c_3^2 + 12c_2c_4 - 52c_2^2c_3\right)\varepsilon_n^4\end{array}\right) \quad (23)$$

Using (21) and (23), we get

$$\frac{2h(y_n)}{h'(y_n)}\left(\frac{1}{1 + \sqrt{1 - 2\rho_n}}\right) = c_2\varepsilon_n^2 + \left(2c_3 - c_2^2\right)\varepsilon_n^3 + \left(3c_4 - 2c_2c_3\right)\varepsilon_n^4 \ldots \quad (24)$$

and $H(\tau) = 1 - \frac{h(y_n)}{h(x_n)} = \left(\begin{array}{l}1 - c_2\varepsilon_n - \left(2c_3 - 3c_2^2\right)\varepsilon_n^2 - \\ \left(3c_4 - 10c_2c_3 + 8c_2^3\right)\varepsilon_2^3 + \ldots\end{array}\right)$ (25)

From the second step of (14), we get

$$z_n = x^* + K_1\varepsilon_n^4 + K_2\varepsilon_n^5 + K_3\varepsilon_n^6 + \ldots \quad (26)$$

where $K_1 = -c_2c_3$, $K_2 = c_2c_4 - c_3^2 + c_2^4$,

$$K_3 = c_2c_5 + 6c_2^2c_4 + 4c_2c_3^2 + 5c_2^3c_3 - c_2^5 - c_3c_4 - 13c_2c_3c_4$$

Expanding $h(z_n)$ about x^* by using Taylor expansion as follows:

$$h(z_n) = h'(x^*)\left(K_1\varepsilon_n^4 + K_2\varepsilon_n^5 + K_3\varepsilon_n^6 + K_4\varepsilon_n^7 + K_5\varepsilon_n^8 ...\right) \quad (27)$$

$$h'(z_n) = h'(x*)\left(1 + 2K_1\varepsilon_n^4 + ...\right) \quad (28)$$

Substituting (18), (24), and (25) are in the third step of (14), we get

$$\varepsilon_{n+1} = (-c_2^2c_3^2)\varepsilon_n^8 + o\left(\varepsilon_n^9\right)$$

Hence, the order of convergence of the proposed algorithm is eight with four functional evaluations and the efficiency index is $E.I = 8^{1/4} = 1.6817$. It is denoted with (MR).

IV. Numerical Applications Of The Proposed Method

We performed a few numerical simulations to demonstrate how the proposed approach (MR) performed with eighth-order convergence employing problems from real-world applications. All numerical calculations are carried out by the mp-math library in PYTHON using an Intel(R) Core (TM) i5-10210U processor running at 2.11 GHz and a 64-bit operating system. The halting criteria is $\left|f(x_n)\right| < \varepsilon$, where the tolerance is set to $\varepsilon = 10^{-199}$, and the required accuracy is set to 690 decimal places. The analogy of the efficiency index is shown in Table I. We estimated the zero for two alternative initial assumptions (x_0) for each application problem. Tables II show the numerical results obtained and contrast them with those of other approaches. Each table includes starting estimates x_0, the number of iterations (n), error values for the first five iterations (E1, E2, E3, E4, and E5), and the number of function evaluations ($\left|h(x_{n+1})\right|$).

Some real-life applications:

To show the efficiency of the new optimal eighth-order method MR, this section presents a few real-world application problems from various fields. It compares the results to some well-known existing methods in Table II. The efficiency index is shown in Table I.

TABLE I
COMPARISION OF EIFFICIENCY INTEX

Methods	P	N	E. I
PM	8	4	1.6817
PJ	8	4	1.6817
SB	8	4	1.6817
KT	8	4	1.6817
RM	8	4	1.6817
MS	8	4	1.6817
MR	8	4	1.6817

Where P is the order of convergence, N is the number of functional evaluations per iteration, and E.I is the efficiency- index.

Application 1. (Beam Designing Model, [11])

The following nonlinear equation represents the depth of embedment in a sheet-pile wall:

$$h_1(x) = \frac{x^3 + 2.87x^2 - 10.28}{4.62} - x$$

The approximated root is 2.0021187789538272.

Application 2. (Fractional Conversion, [11])

The equation has the following form when the nitrogen is transformed to ammonia fractionally when fed hydrogen under certain conditions:

$$h_2(x) = \frac{8(4-x)^2 x^2}{(6-3x)^2(2-x)} - 0.186$$

It can be reduced in polynomial form as

$$h_2(x) = x^4 - 7.79075x^3 + 14.7445x^2 + 2.511x - 1.674$$

The real root is 0.2777595428417206.

Application 3. (Ideal and Non-Ideal Gas Laws, [11])

The ideal gas equation is given by

$$\left(p + {an^2}/{x^2}\right)(x - nb) = nRT$$

where a, b, are constants characterizing the strength of the attractions, the size of the gas particle, and p, x, n, T, R, are constants describing pressure, volume, number of moles, temperature, and universal constant of the gas. The molal volume is calculated by solving the equation.

$$h_3(x) = \left(p + \frac{c_1}{x^2}\right)(x - c_2) - RT$$

We take parameter values when the root of the nonlinear equation is 24.5125881284415006.

Application 4. (Plank's Constant, [12])

Plank's radiation law problem determines the energy density within an isothermal blackbody and is represented by

$$\varphi(\lambda) = \frac{8\pi ch\lambda^{-5}}{e^{ch/\lambda kT} - 1}$$

Here, taking $x = ch/\lambda kT$, the above equation becomes

$$1 - \frac{x}{5} = e^{-x}$$

Let us define

$$h_4(x) = e^{-x} - 1 + \frac{x}{5}$$

The root of the equation is given by 4.96511423174427630.

Application 5. (Study of Multifactor effect, [17])

When two parallel plates are placed in an air gap, an electron's moment is given by

$$x(t) = x_0 + \left(v_0 + eE_0(\mathrm{m}\,w)^{-1}\sin\left(wt_0 + \eta\right)\right)\left(t - t_0\right) + eE_0\left(\mathrm{m}\,w^2\right)^{-1}\left(\cos\left(wt_0 + \eta\right) + \sin\left(wt_0 + \eta\right)\right)$$

It can be reduced in nonlinear form as

$$h_5(x) = x - 0.5\cos x + \frac{\pi}{4}$$

The above equation has a simple root at $x^* \approx -0.309466139208214$.

Application 6. (Blood rheology model, [2])

Blood rheology is a branch of science that studies the physical properties and flow characteristics of blood. Since blood is a non-Newtonian fluid, it is referred to as a Caisson fluid. We employ the following function in the form of a nonlinear equation to study the plug flow of Caisson fluids:

$$H = 1 - \frac{16}{7}\sqrt{u} + \frac{4}{3}u - \frac{1}{21}u^4$$

Where flow rate reduction is computed by H. Take $H = 0.40$, we have

$$h_6(x) = \frac{1}{441}x^8 - \frac{8}{63}x^5 - 0.0571428571x^4 + \frac{16}{9}x^2 - 3.624489796x + 0.3$$

The root of the nonlinear equation $h_6(x) = 0$ is 0.0864335580522467.

Application 7. (Volume of van der Waals equation, [2])

Van der Waals' equation of a non-ideal gas is represented by

$$h(V) = pV^3 - n(RT + bp)V^2 + n^2aV - n^3ab$$

Put $V = x$ and by giving particular values to the parameters, the above equation is of the form

$$h_7(x) = 40x^3 - 95.26535116x^2 + 35.28x - 5.6998368$$

The above equation has three roots in which one is real, that is $x^* \approx 1.9707842194070294$.

Application 8. (Stirred Tank Reactor, [10])

At rates of β and $q - \beta$, respectively, the reactor gets materials from the stirred tank. The following are the equipment improvements for mixed reactions:

$$H_1 + H_2 \rightarrow H_3;\ H_3 + H_2 \rightarrow H_4;\ H_4 + H_2 \rightarrow H_5;\ H_4 + H_2 \rightarrow H_6$$

The nonlinear polynomial equation shown below was found by Douglas et al. [7] during their initial examination of this sophisticated control system:

$$\frac{2.98 * (x + 2.25)}{(x + 1.45) * (x + 2.85)^2 * (x + 4.35)} = \frac{1}{G_c}$$

where G_c is the gain of the proportional controller. By taking $G_c = 0$, we have

$$h_8(x) = x^4 + 11.50x^3 + 47.49x^2 + 83.06325x - 51.23266875 = 0$$

The root of $h_8(x)$ is -1.45.

TABLE-II
COMPARISON OF SUCCESSIVE ERRORS

Method	n	$\lvert x_1 - x_0 \rvert$	$\lvert x_2 - x_1 \rvert$	$\lvert x_3 - x_2 \rvert$	$\lvert x_4 - x_3 \rvert$	$\lvert h(x_{n+1}) \rvert$	C. P. U
h1(x)	x_0	1.6					
PM	4	0.401753	0.000366	3.15E-29	9.49E-230	3.88E-229	0.00602
PJ	4	0.402125	6.35E-06	7.69E-45	3.58E-356	1.46E-355	0.00522
SB	4	0.401917	0.000202	7.85E-32	4.08E-251	1.67E-250	0.00717
KT	4	0.402236	0.000117	1.20E-33	1.41E-265	5.78E-265	0.00611
RM	5	0.384636	0.017482	6.70E-18	6.70E-157	1.59E-689	0.00527
MS	5	0.400477	0.001642	4.92E-25	3.00E-197	1.59E-689	0.00731
MR	4	0.402112	6.32E-06	2.41E-45	3.98E-395	1.63E-394	0.00513
h1(x)	x_0	3.8					
PM	5	1.766047	0.031834	9.25E-14	5.30E-106	1.58E-689	0.00754
PJ	5	1.7917	0.006181	6.11E-21	5.70E-165	1.59E-689	0.00672
SB		Divergent					
KT		Divergent					
RM	5	1.774966	0.022915	1.40E-17	5.10E-154	2.68E-689	0.00672
MS	5	1.770605	0.027276	1.71E-16	6.50E-129	1.59E-689	0.00805
MR	5	1.801014	0.003132	8.91E-24	3.76E-188	4.40E-690	0.00745
h2(x)	x_0	0.5					
PM	4	0.222232	8.53E-06	1.33E-39	4.52E-310	4.06E-309	0.00652
PJ	4	0.222239	1.38E-06	3.74E-47	5.89E-372	9.53E-371	0.00538
SB	4	0.222235	5.40E-06	1.38E-41	2.55E-326	3.29E-325	0.00701
KT	4	0.222235	5.19E-06	1.08E-41	3.67E-327	3.30E-326	0.00688
RM	4	0.222237	3.48E-06	1.66E-45	4.52E-360	4.06E-359	0.00527
MS	4	0.222234	6.10E-06	2.82E-41	5.85E-324	5.26E-323	0.00698
MR	4	0.222240	6.41E-07	1.44E-50	9.44E-400	8.48E-399	0.00532
h2(x)	x_0	0.2					
PM	4	Divergent					
PJ	4	0.07776	1.04E-08	3.92E-64	8.60E-508	1.39E-506	0.00672
SB		Divergent					
KT		Divergent					
RM	4	0.077759	8.98E-08	3.29E-58	1.07E-461	9.61E-461	0.00698
MS	4	0.077759	8.81E-08	5.32E-56	9.45E-442	8.49E-441	0.00823
MR	4	0.07776	8.42E-10	1.27E-73	3.42E-584	3.07E-583	0.00651

h3(x)	x_0	-5					
PM	4	29.59136	0.078771	3.69E-28	8.84E-239	8.79E-239	0.00635
PJ	4	29.54655	0.033965	4.25E-31	2.57E-262	2.55E-262	0.00577
SB	4	29.55738	0.044796	3.99E-30	1.61E-254	1.59E-254	0.00697
KT	4	29.56122	0.048628	7.69E-30	3.04E-252	3.02E-252	0.00619
RM	4	29.51775	0.005163	1.21E-37	1.08E-314	1.07E-314	0.00597
MS	4	29.56122	0.048632	7.74E-30	3.22E-252	3.20E-252	0.00631
MR	4	29.51276	0.000171	1.07E-51	2.53E-429	2.51E-429	0.00558
h3(x)	x_0	10					
PM	4	14.51259	8.32E-07	5.90E-68	3.73E-557	3.71E-557	0.00745
PJ	4	14.51259	6.34E-07	6.35E-69	6.42E-565	6.39E-565	0.00672
SB		Divergent					
KT	4	14.51259	7.26E-07	1.94E-68	4.93E-561	4.90E-561	0.00714
RM	4	14.51259	5.02E-07	9.69E-70	1.86E-571	1.85E-571	0.00672
MS	4	14.51259	7.48E-07	2.45E-68	3.29E-560	3.27E-560	0.00805
MR	4	14.51259	3.64E-09	4.44E-89	1.31E-689	6.13E-689	0.00702
h4(x)	x_0	7					
PM		Divergent					
PJ	4	2.034886	1.39E-08	4.62E-72	7.00E-580	6.76E-580	0.00685
SB	4	2.034886	1.53E-08	1.34E-71	4.73E-576	4.57E-576	0.00654
KT		Divergent					
RM	4	2.034886	1.51E-08	7.84E-72	4.23E-578	4.08E-578	0.00628
MS	4	2.034886	1.55E-08	1.51E-71	1.28E-575	1.24E-575	0.00731
MR	4	2.034886	7.80E-10	2.93E-83	1.17E-670	1.13E-670	0.00614
h4(x)	x_0	14					
PM	4	9.034885	3.84E-07	2.63E-60	1.26E-485	1.22E-485	0.00754
PJ	4	9.034885	3.76E-07	1.34E-60	1.65E-488	3.30E-488	0.00672
SB	4	9.034885	3.83E-07	2.06E-60	1.43E-486	1.38E-486	0.00786
KT		Divergent					
RM	4	9.034885	4.09E-07	2.33E-60	2.57E-486	2.48E-486	0.00672
MS		Divergent					
MR	4	9.034886	1.60E-08	9.07E-73	9.82E-587	9.48E-587	0.00745
h5(x)	x_0	0.5					
PM	4	0.809400	6.59E-05	6.30E-37	4.41E-293	3.74E-293	0.00482
PJ	4	0.809469	2.41E-06	4.18E-50	3.49E-400	2.96E-400	0.00467
SB	4	0.809438	2.81E-05	6.06E-41	2.81E-326	2.38E-326	0.00514
KT	4	0.809439	2.70E-05	1.29E-40	3.46E-323	2.94E-323	0.00551
RM	4	0.809456	9.67E-06	4.21E-45	5.49E-360	4.66E-360	0.00496
MS	4	0.809432	3.39E-05	1.82E-39	1.23E-313	1.04E-313	0.00478
MR	4	0.809465	7.53E-07	2.06E-54	4.47E-435	5.48E-435	0.00454
h5(x)	x_0	-2					
PM	4	Divergent					
PJ	4	1.691365	0.000831	8.46E-30	9.76E-238	8.28E-238	0.00579
SB	4	1.692022	0.001489	3.76E-27	6.17E-216	6.17E-216	0.00704
KT	4	1.692443	0.001909	8.06E-26	8.17E-205	8.17E-205	0.00712
RM	5	1.694103	0.003570	1.46E-24	1.10E-195	5.47E-691	0.00618
MS	4	1.691815	0.001281	7.43E-27	9.61E-213	8.15E-213	0.00675
MR	4	1.691115	0.000581	2.59E-31	3.99E-250	3.39E-250	0.00564
h6(x)	x_0	0.1					
PM	4	0.186433	1.67E-07	1.92E-55	5.86E-439	1.94E-438	0.00389
PJ	4	0.186434	7.33E-09	7.61E-68	4.07E-540	3.41E-539	0.00372
SB	4	0.186433	6.97E-08	5.48E-59	8.00E-468	2.66E-467	0.00369
KT		Divergent					
RM	4	0.186434	1.06E-09	1.35E-76	9.51E-612	3.15E-611	0.00404
MS		Divergent					
MR	4	0.186434	6.01E-11	2.03E-87	1.71E-692	2.05E-691	0.00364
h6(x)	x_0	-0.6					
PM	4	0.685932	0.000501	1.26E-27	2.02E-216	6.71E-216	0.00564
PJ	4	0.686373	6.05E-05	1.63E-36	4.59E-289	1.52E-288	0.00556
SB	4	0.686133	0.000301	6.63E-30	3.67E-235	1.22E-234	0.00603
KT	4	0.686176	0.000258	2.23E-30	6.91E-239	2.29E-238	0.00591
RM	4	0.686252	0.000181	9.57E-35	5.99E-277	1.99E-276	0.00582
MS	4	0.686063	0.000370	2.87E-30	3.61E-239	1.19E-238	0.00625
MR	4	0.686433	4.18E-07	1.11E-56	2.71E-453	8.97E-453	0.00509
h7(x)	x_0	-1.2					
PM	5	0.249423	0.000577	1.16E-23	3.20E-181	2.62E-689	0.00798
PJ	4	0.249945	5.5E-05	2.24E-33	1.72E-262	9.76E-260	0.00765
SB	4	0.249671	0.000329	3.53E-26	6.22E-202	3.54E-201	0.00786
KT	4	0.249724	0.000276	1.08E-26	5.79E-206	3.29E-205	0.00759
RM	4	0.249766	0.000234	1.96E-30	1.22E-238	6.93E-238	0.00762
MS		Divergent					
MR	4	0.249991	8.65E-06	1.76E-40	5.11E-318	2.90E-317	0.00754

h7(x)	x_0	-1.4					
PM	4	0.05	1.95E-08	1.99E-59	2.39E-467	1.36E-466	0.00459
PJ	4	0.05	6.79E-10	1.20E-72	1.17E-574	1.17E-574	0.00492
SB	4	0.05	6.90E-09	1.34E-63	2.67E-501	1.52E-500	0.00487
KT	4	0.05	6.59E-09	1.14E-63	8.85E-502	5.03E-501	0.00511
RM	4	0.05	5.11E-10	2.57E-75	1.07E-597	6.08E-597	0.00487
MS	4	0.05	5.26E-09	8.79E-65	5.40E-511	3.07E-510	0.00531
MR	4	0.05	1.08E-10	1.06E-79	8.61E-632	4.89E-631	0.00465
h8(x)	x_0	1.8					
PM	8	0.170607	0.000177	1.96E-28	1.89E-66	1.61E-218	0.00745
PJ	8	0.170787	2.84E-06	2.77E-36	2.66E-74	2.87E-224	0.00742
SB	8	0.170687	9.73E-05	1.17E-29	1.13E-67	1.21E-217	0.00766
KT	8	0.170837	5.32E-05	3.51E-31	3.37E-69	3.63E-219	0.00714
RM	8	0.162195	0.008589	1.35E-17	1.30E-55	1.39E-205	0.00694
MS	8	0.170011	0.000773	1.10E-24	1.06E-62	1.14E-212	0.00705
MR	5	0.170781	3.60E-06	2.38E-41	6.10E-120	5.09E-275	0.00654
h8(x)	x_0	2.2					
PM	8	0.229175	4.04E-05	1.39E-31	1.33E-69	1.44E-219	0.00602
PJ	8	0.229213	2.46E-06	1.81E-36	1.74E-74	1.49E-226	0.00595
SB	8	0.229196	1.97E-05	3.06E-33	2.94E-71	2.52E-223	0.00543
KT	8	0.229199	1.68E-05	5.79E-34	5.57E-72	5.99E-222	0.00578
RM	8	0.229209	6.36E-06	3.31E-35	3.19E-73	3.43E-223	0.00558
MS	8	0.229192	2.43E-05	1.39E-32	1.33E-70	1.43E-220	0.00548
MR	5	0.229216	4.34E-08	4.83E-49	2.50E-135	8.63E-306	0.00527

Where, *n* represents the number of iterations, $|x_1 - x_0|$ to $|x_4 - x_3|$ are error values of first four iterations, and $\left|h\left(x_{n+1}\right)\right|$ represents the functional evaluations.

The residual fall graph for nonlinear equations using simultaneous methods PM, PJ, SB, KT, RM, MS and MR in the above-mentioned real-world problems.

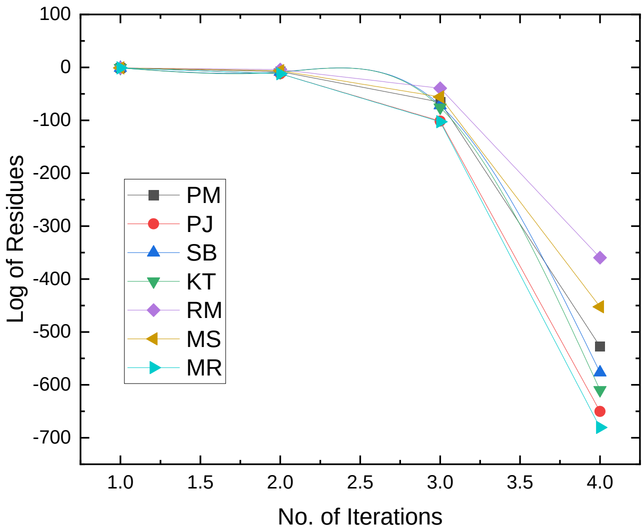


Fig. 1. $h_1(x)$ at x_0=1.6

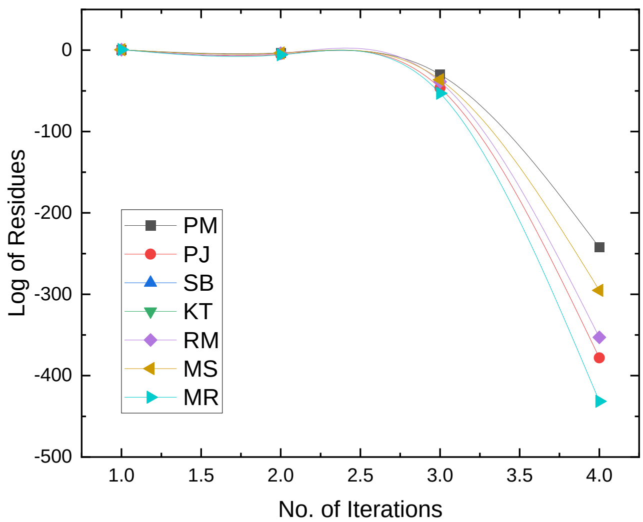


Fig. 2. $h_1(x)$ at x_0=3.8

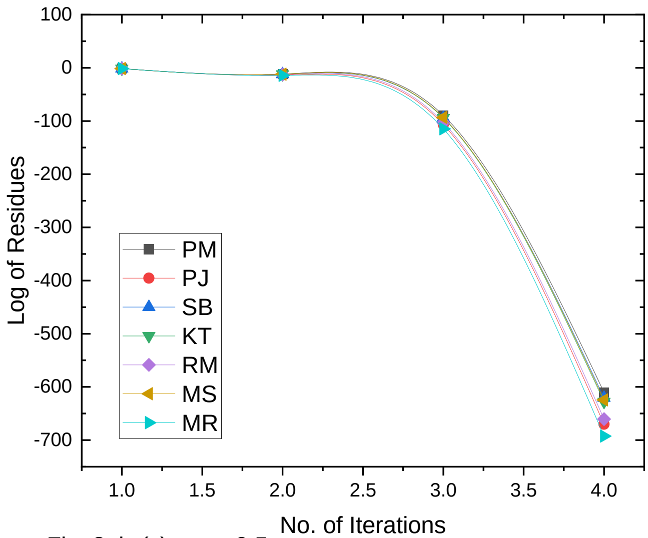


Fig. 3. $h_2(x)$ at x_0=0.5

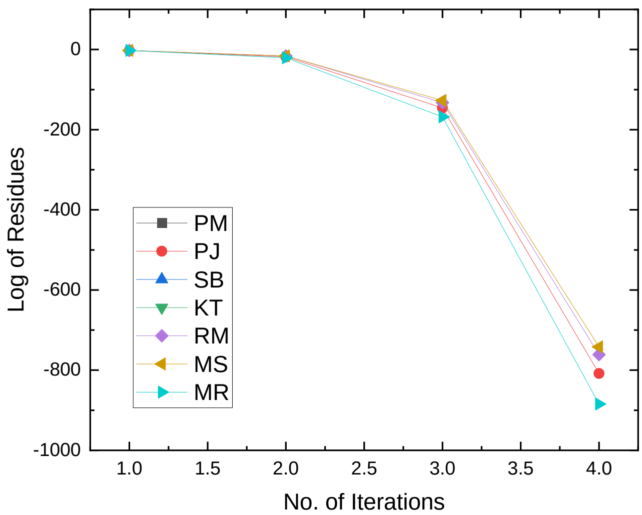


Fig. 4. $h_2(x)$ at x_0=0.2

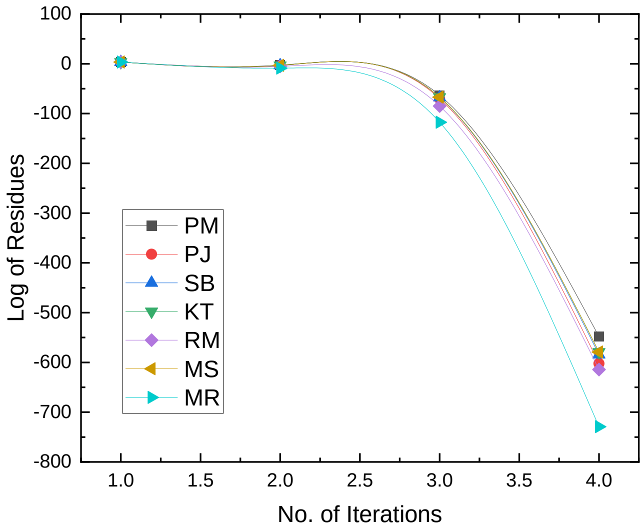


Fig. 5. $h_3(x)$ at x_0=-5

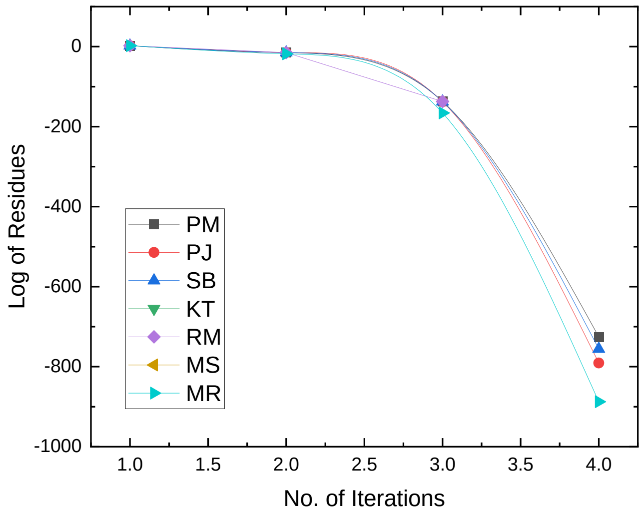


Fig. 8. $h_4(x)$ at x_0=14

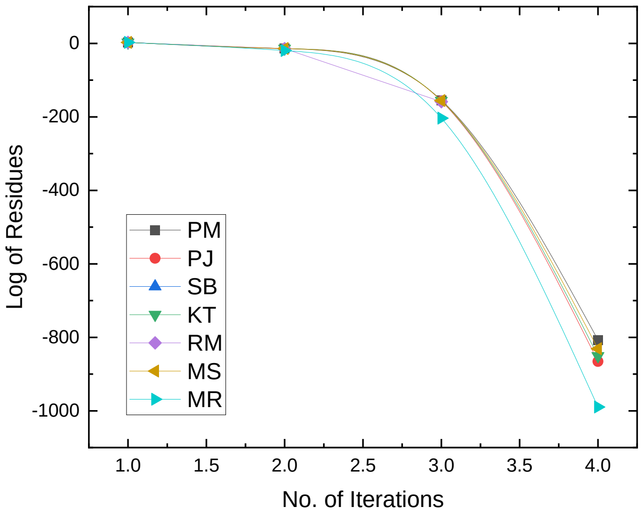


Fig. 6. $h_3(x)$ at x_0=10

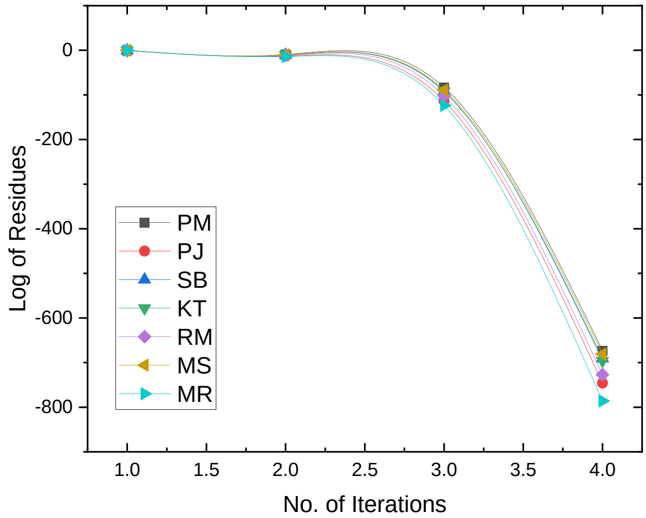


Fig. 9. $h_5(x)$ at x_0=0.5

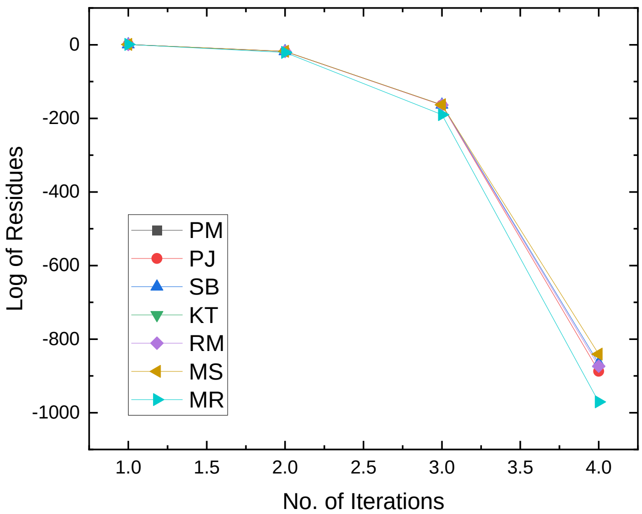


Fig. 7. $h_4(x)$ at x_0=7

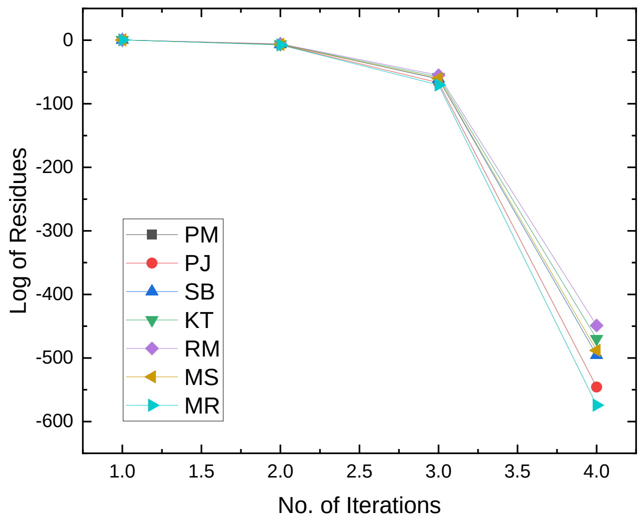


Fig. 10. $h_5(x)$ at x_0=2

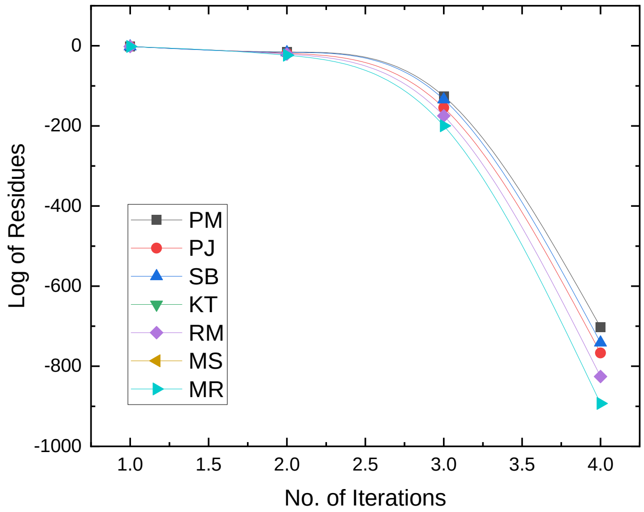


Fig. 11. $h_6(x)$ at x_0=0.1

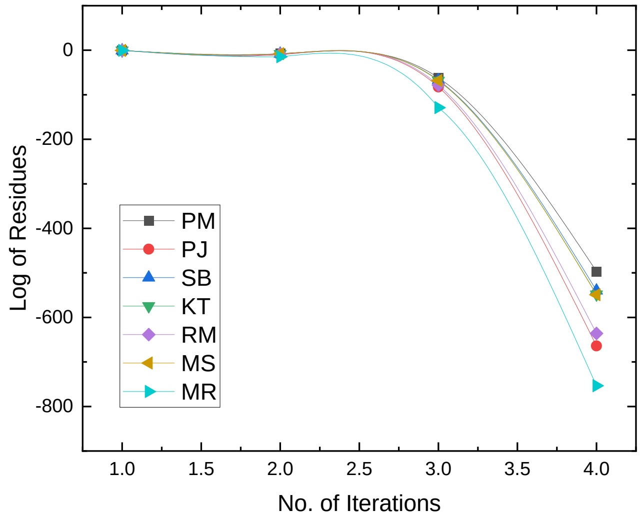


Fig. 12. $h_6(x)$ at x_0=-0.6

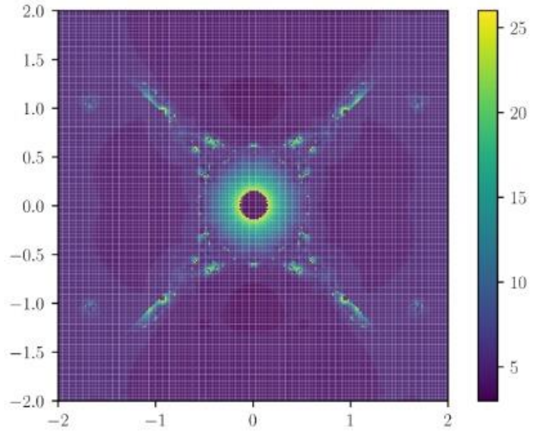
Using Origin Pro software for graphical comparisons, "Fig. 1" to "Fig. 12" show the graphical behaviour. The residual fall graph shows SS's suggested strategy requires fewer iterations to reach zero error than another current methods PM, PJ, SB, KT, RM, MS, and MR. The error graph demonstrates the effectiveness and speedy convergence of the proposed technique MR in this way.

V. Basins Of Attraction

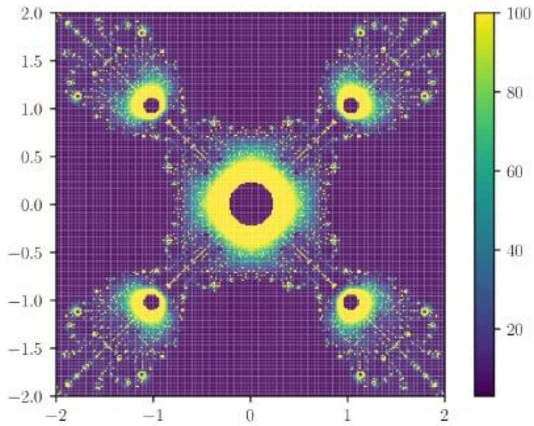
One of the dynamical notions pertaining to the stability and convergence of roots for some iterative approaches is the idea of polynomiographs using a basin of attraction. A basin of attraction is a portion of $2D$ space where iterations will continue into the attractor no matter what initial prediction is made. We employ a square region over a mesh grid $[-2,2]\times[-2,2]\in C^2$ with tolerance $|x_{n+1}-x_n|<10^{-16}$ to create polynomiographs over the complex plane C. The assumption is that $N=100$ is the maximum number of iterations. It is possible to generate polynomiographs of different complex-valued polynomials over the complex plane C using the PYTHON programming. Roots can be given various colors, and convergence can be observed when the colors are altered. Fig.1 and Fig.2 shows the basins of attraction of the proposed method and comparison methods for the complex polynomials $f_1(z)=1-z^4$, $f_2(z)=1-z^{11}$.

The developed algorithm MR and other existing methods have the following basins:

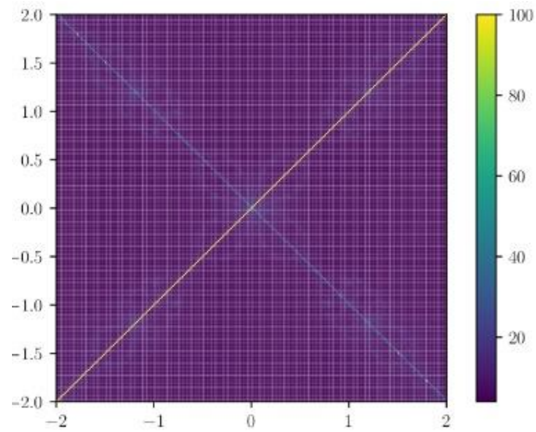
Example 1. $f_1(z)=1-z^4$



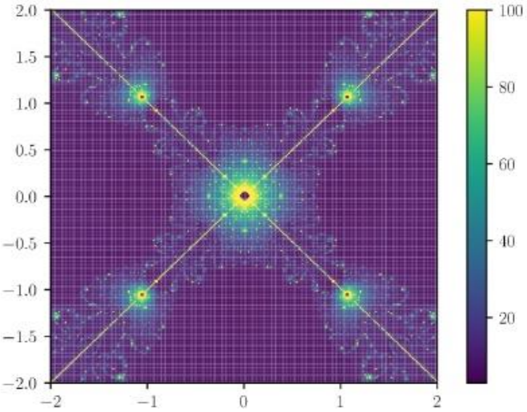
(a) MR



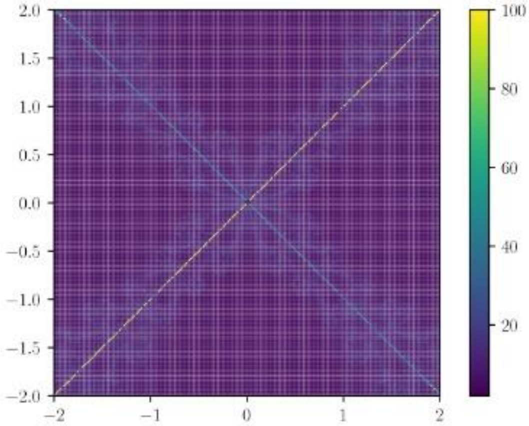
(b) PM



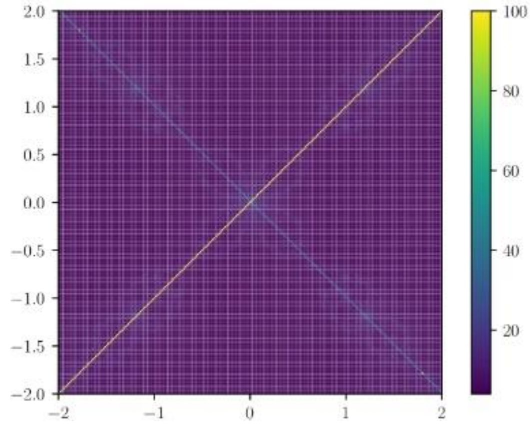
(c) PJ



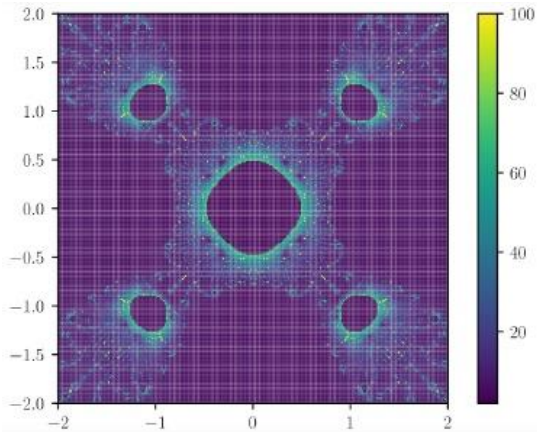
(d) SB



(e) KT



(f) RM



(g) MS

Fig. 1. The polynomiographs for the suggested methods (MR), PM, PJ, SB, KT, RM, MS for $f_1(z)$.

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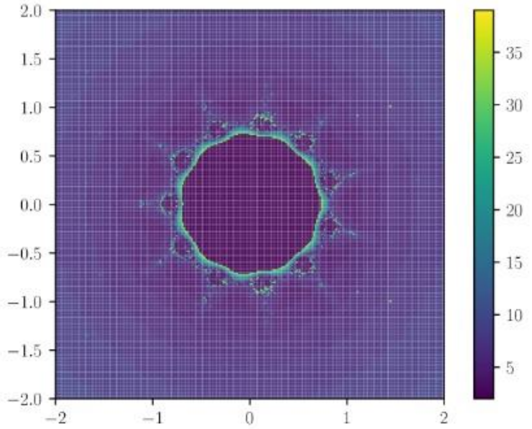
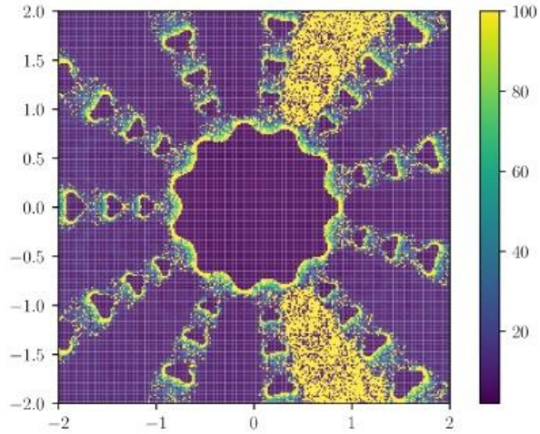
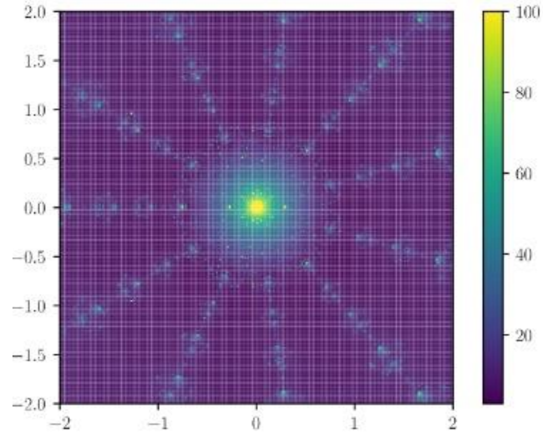
Example 2. $f_2(z) = 1 - z^{11}$



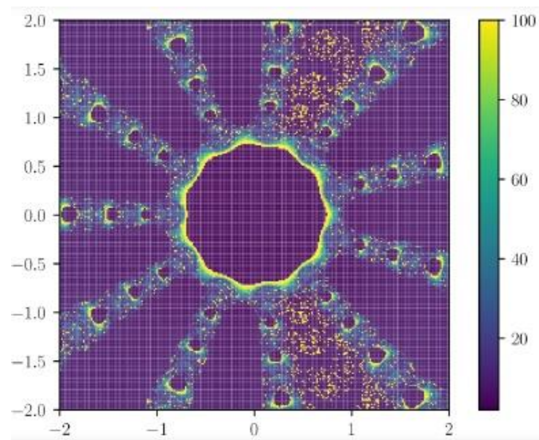
(a) MR



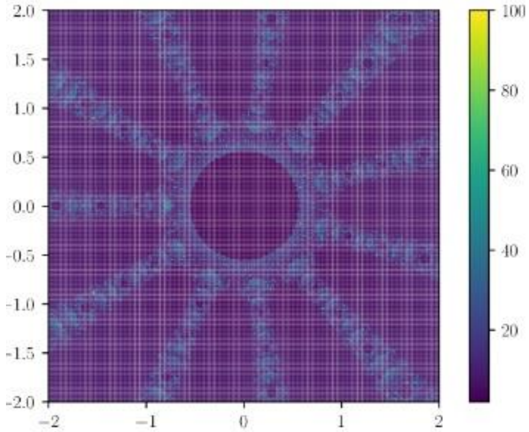
(b) PM



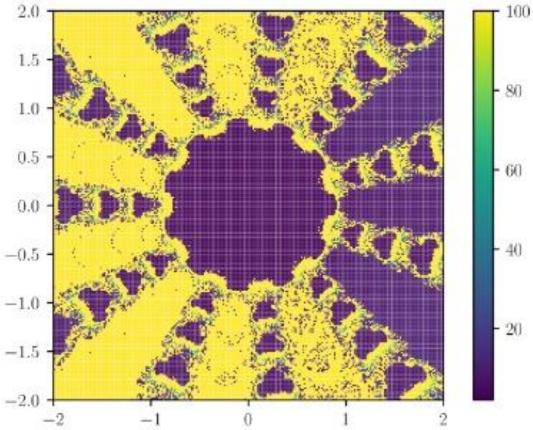
(c) PJ



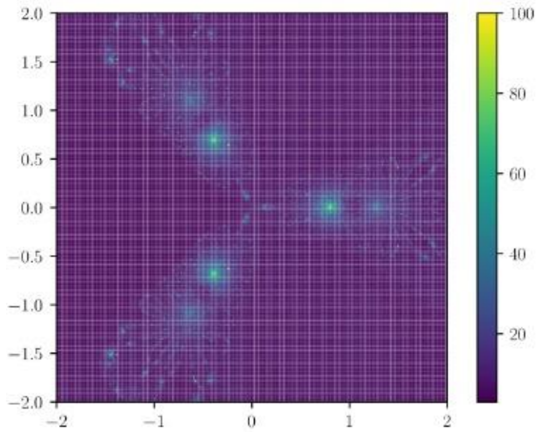
(d) SB



(e) KT



(f) RM



(g) MS

Fig. 2. The polynomiographs for the suggested methods (MR), PM, PJ, SB, KT, RM, MS for $f_2(z)$.

The fractal graphs illustrating the behavior of polynomials for the proposed MR method and other comparison methods are displayed in Fig. 1 and Fig. 2. The absence of chaotic behavior in the fractal graphs is indicative of the effective performance of the MR approach. On the other hand, methods PM, PJ, SB, KT, RM, and MS exhibit some unstable behavior near the boundary locations. This information underscores that, for all three polynomials considered, the proposed MR technique outperforms the others in terms of the number of iterations required to achieve convergence.

VI. Conclusions

In this paper, we have developed an eighth-order iterative method for efficiently locating the root of a nonlinear equation. This method is constructed by combining the composition technique, a modified generalized form of the Newton-Steffensen method, and the weight function approach. The analogy of these approaches is presented in Table I. In Table II, we provide a comprehensive comparison of the method's performance in terms of errors and iterations with respect to other existing methods. We have conducted a thorough analysis, contrasting our proposed methodology with several alternative eighth-order algorithms, and detailing the dynamic behavior of the method. The results depicted in Fig. 1 and Fig. 2 illustrate that the new algorithm's basins of attraction converge more rapidly and accurately in terms of iterations.

Acknowledgment

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References

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