Asynchronous Energy-to-peak Control for 2D Roesser-type Markov Jump Systems

Qing Chen, Yu Zhang, Chengyi Han, Ling Chen, and Jianping Zhou

***Abstract*—This paper studies asynchronous energy-to-peak control for 2D Roesser-type Markov jump systems (RTMJSs). Given the practical challenge of obtaining the system state, output-feedback is utilized for closing the control loop instead of state feedback. The asynchronous behavior between the controlled plant and the controller is represented by a hidden Markov model. A sufficient condition for the stochastic stability and energy-to-peak disturbance attenuation (EPDA) of the closed-loop RTMJS is established using a Lyapunov function and Schur's complement. Under fixed conditional probability, a scheme for asynchronous output-feedback controller design is proposed, employing variable replacement and matrix transformation to cope with the nonlinear terms. The present work is then expanded to the uncertain conditional probability scenario and a related controller design method is developed. Finally, the effectiveness of the proposed design methods is demonstrated through a thermal process.**

***Index Terms*—2D Roesser-type system, Markov jump system, output-feedback, energy-to-peak control, hidden Markov model.**

I. Introduction

OVER the past few decades, two-dimensional (2D) systems have been successfully applied in a wide range of fields, such as thermal processes, gas absorption, image data processing, metal pressing processes, and digital filtering; see, e.g., [1–5]. One of the most critical characteristics of 2D systems is that their state propagates along two independent directions, namely the horizontal direction and the vertical direction, which makes the structure of 2D systems more complex than that of one-dimensional systems. Currently, there are many models to describe 2D systems, among which the Fornasini-Marchesini (FM) second model [6] and the Roesser model [2] are the most commonly used models. The Roesser model was first used for image processing. Compared with the FM second model, the Roesser model has a simple structure and is easy to apply [7].

In certain 2D systems, frequent abrupt variations in parameters or structures occur, which can be aptly described by Markov jump processes. In such cases, these systems can be modeled as 2D Markov jump systems. Presently, 2D Roesser-type Markov jump systems (RTMJSs) have received strong attention, and many research results have been published. For instance, stability analysis for such systems was carried out in [8, 9], where some stochastic stability conditions are presented in terms of matrix inequalities. Control design for RTMJSs was investigated in [10–13], where different control laws were introduced based on the Lyapunov stability theory.

It is worth noting that the controllers designed in [10–13] operate synchronously with the controlled plant, whereby their mode changes align with those of the plant. This synchronous design approach is often impractical, especially in networked environments where the controller may struggle to accurately discern the mode of the plant, leading to the controller mode mismatches with the plant mode. In contrast, asynchronous controllers may offer a more practical solution for engineering applications [14–18]. Furthermore, the impact of external disturbances is not considered in these references. However, external disturbances often occur in actual systems and have a significant impact on them [19–21]. To maintain the stability of the controlled plant despite external disturbances and obtain a smaller energy peak at the same time, energy-to-peak control has proven to be an effective solution for various control systems [22–24]. Notably, to our knowledge, asynchronous energy-to-peak control for RTMJSs remains unexplored in the existing literature.

In practical applications, complete system state information may be unattainable, while system output is often readily measurable and usable [25–28]. Motivated by the above discussion, this paper considers the asynchronous energy-to-peak control for 2D RTMJSs via output-feedback. The problem to be solved is to design an asynchronous output-feedback controller so that in the case of disturbance attenuation, the resulting closed-loop system (CLS) is stochastically stable (SS) with energy-to-peak disturbance attenuation (EPDA) level. The asynchronous behavior between the controlled object and the controller is represented by a hidden Markov model (HMM) as in [29, 30]. A sufficient condition for the stochastic stability and EPDA of the closed-loop RTMJS is established using a Lyapunov function and Schur's complement (SC). Under completely known conditional probability, to address the impact of nonlinear terms, an asynchronous output-feedback controller is designed by introducing a relaxation matrix and employing variable replacement. The present work is then expanded to the uncertain conditional probability scenario and a related controller design method is developed. Finally, a thermal process is used as an example to verify the effectiveness of the proposed asynchronous design methods.

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II. Preliminaries

The notations utilized throughout in this paper, unless otherwise stated, adhere to established conventions as detailed in [31, 32].

In this paper, we examine the following discrete-time 2D RTMJS:

$$\begin{cases} x_{i,j}^1 = A_{\vartheta_{i,j}} x_{i,j} + B_{\vartheta_{i,j}} u_{i,j} + E_{\vartheta_{i,j}} w_{i,j}, \\ y_{i,j} = C_{\vartheta_{i,j}} x_{i,j}, \end{cases} \tag{1}$$

where

$$x_{i,j}^1 = \begin{bmatrix} x_{i+1,j}^h \\ x_{i,j+1}^v \end{bmatrix}, \, x_{i,j} = \begin{bmatrix} x_{i,j}^h \\ x_{i,j}^v \end{bmatrix}.$$

Here, $x_{i,j}^h \in \mathbb{R}^{n_h}$ and $x_{i,j}^v \in \mathbb{R}^{n_v}$ represent the horizontal state and the vertical state, respectively; $u_{i,j} \in \mathbb{R}^{n_u}$, $w_{i,j} \in \mathbb{R}^{n_w}$, and $y_{i,j} \in \mathbb{R}^{n_y}$, respectively, are control input, external disturbance, and controlled output. $A_{\vartheta_{i,j}}$, $B_{\vartheta_{i,j}}$, $C_{\vartheta_{i,j}}$, and $E_{\vartheta_{i,j}}$, are pre-known system matrices of appropriate dimensions. The Markov parameter $\vartheta_{i,j}$ takes value in the finite set $\mathcal{V}_1 = \{1, 2, \ldots, v_1\}$ with transition probability matrix $\Lambda = \{\lambda_{pq}\}$, and satisfies the following transition probabilities:

$$\begin{aligned} \lambda_{pq} &= \Pr\{\vartheta_{i+1,j} = q \,|\, \vartheta_{i,j} = p\} \\ &= \Pr\{\vartheta_{i,j+1} = q \,|\, \vartheta_{i,j} = p\}, \end{aligned} \tag{2}$$

where $p, q \in \mathcal{V}_1$, which satisfies $0 \leq \lambda_{pq} \leq 1$ and $\sum_{q=1}^{v_1} \lambda_{pq} = 1$ [33–35]. The boundary condition of 2D RTMJSs (1) is defined as follows:

$$\begin{aligned} X_0 &= \left\{x_{0,j}^h, \, x_{i,0}^v \,|\, i, j = 0, 1, \ldots \right\}, \\ \Gamma_0 &= \{\vartheta_{0,j}, \, \vartheta_{i,0} \,|\, i, j = 0, 1, \ldots \}, \end{aligned}$$

and define the zero boundary condition as $x_{0,j}^h = 0, x_{i,0}^v = 0 \; (i, j = 0, 1, \ldots)$. An assumption about the boundary condition is introduced:

Assumption 1. [7] The boundary condition X_0 of 2D RTMJS (1) satisfies that

$$\mathbb{E}\left\{\sum_{i=0}^{\infty}\left(\|x_{0,i}^h\|^2 + \|x_{i,0}^v\|^2\right)\right\} < \infty.$$

The asynchronous output-feedback controller to be designed takes the structure as

$$u_{i,j} = K_{\varrho_{i,j}} y_{i,j}, \tag{3}$$

where $K_{\varrho_{i,j}}$ represent the controller gains, whice depend on the parameter $\varrho_{i,j} \in \mathcal{V}_2$ and $\mathcal{V}_2 = \{1, 2, \ldots, v_2\}$. The controller mode is changed by the variable $\varrho_{i,j}$, which satisfies

$$\Pr\{\varrho_{i,j} = s \,|\, \vartheta_{i,j} = p\} = \pi_{ps},$$

where $0 \leq \pi_{ps} \leq 1$, $\sum_{s=1}^{v_2} \pi_{ps} = 1$, for $p \in \mathcal{V}_1, s \in \mathcal{V}_2$, and the conditional probability matrix $\Pi = \{\pi_{ps}\}$.

In what follows, we will simplify some symbols: these parameters $\vartheta_{i,j}, \vartheta_{i+1,j}$(or $\vartheta_{i,j+1}$), and $\varrho_{i,j}$ will be replaced by subscripts p, q, and s, respectively. For instance, A_p is short for $A_{\vartheta_{i,j}}$.

By applying controller (3) to 2D RTMJS (1), the resulting CLS is as

$$\begin{cases} x_{i,j}^1 = \bar{A}_{ps} x_{i,j} + E_p w_{i,j}, \\ y_{i,j} = C_p x_{i,j}, \end{cases} \tag{4}$$

where $\bar{A}_{ps} = A_p + B_p K_s C_p$. Below, two necessary definitions and two lemmas are introduced:

Definition 1. [36] CLS (4) with $w_{i,j} \equiv 0$ is said to be SS if the following condition holds:

$$\mathbb{E}\left\{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left(\|x_{i,j}\|^2\right)\right\} < \infty.$$

Definition 2. [37] Given a positive scalar μ, CLS (4) is said to have an EPDA level, for any $w_{i,j} \in l_2\{[0,\infty), [0,\infty)\}$ under the zero boundary condition,

$$\sup_{i\geq 0}\sup_{j\geq 0} \mathbb{E}\left\{\|y_{i,j}\|^2\right\} \leq \mu^2 \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \mathbb{E}\left\{\|w_{i,j}\|^2\right\}.$$

Lemma 1. [38] For any two matrices M_1 and $M_2 > 0$ with appropriate dimensions, the following inequality holds:

$$-M_1^T M_2^{-1} M_1 \leq M_2 - M_1^T - M_1.$$

Lemma 2. [39] For any scalar $\epsilon > 0$ and real matrices M_1, M_2 of suitable size,

$$2 M_1 M_2 \leq \epsilon^{-1} M_1^T M_1 + \epsilon M_2^T M_2.$$

III. Main results

In this section, we will investigate SS with EPDA level for CLS (4), and then, based on this, propose an asynchronous output-feedback controller design approach. First, we provide the following result.

Theorem 1. Given a positive scalar μ, if there are positive definite matrices $P_p = \text{diag}\left\{P_p^h, P_p^v\right\}$ and Q_{ps}, matrices K_s, for all $p \in \mathcal{V}_1$, $s \in \mathcal{V}_2$ such that

$$\sum_{s=1}^{v_2} \pi_{ps} Q_{ps} - P_p < 0, \tag{5}$$

$$\begin{bmatrix} -\bar{P}_p^{-1} & \bar{A}_{ps} \\ * & -Q_{ps} \end{bmatrix} < 0 \tag{6}$$

hold, where $\bar{P}_p = \sum_{q=1}^{v_1} \lambda_{pq} P_q$, then CLS (4) is SS.

Proof: Consider a Lyapunov function as follows:

$$V_{i,j} = x_{i,j}^T P_p x_{i,j} \tag{7}$$

and define

$$\Delta V_{i,j} = x_{i,j}^{1T} P_q x_{i,j}^1 - x_{i,j}^T P_p x_{i,j}. \tag{8}$$

Then, according to CLS (4) with $w_{i,j} \equiv 0$, it has

$$\Delta V_{i,j} = x_{i,j}^T (\bar{A}_{ps}^T P_q \bar{A}_{ps} - P_p) x_{i,j}. \tag{9}$$

Taking the expectation operation of (9), one has

$$\mathbb{E}\{\Delta V_{i,j}\} = \mathbb{E}\left\{x_{i,j}^T\left(\sum_{s=1}^{v_2} \pi_{ps} \bar{A}_{ps}^T \bar{P}_p \bar{A}_{ps} - P_p\right) x_{i,j}\right\}. \tag{10}$$

By means of the SC, (6) can be rewritten as

$$\bar{A}_{ps}^T \bar{P}_p \bar{A}_{ps} < Q_{ps}. \tag{11}$$

It is deduced from (10) and (11) that

$$\begin{aligned} \mathbb{E}\{\Delta V_{i,j}\} &< \mathbb{E}\left\{x_{i,j}^T\left(\sum_{s=1}^{v_2} \pi_{ps} Q_{ps} - P_p\right) x_{i,j}\right\} \\ &\leq -\alpha \mathbb{E}\left\{\|x_{i,j}\|^2\right\}, \end{aligned} \tag{12}$$

where α is the minimum eigenvalue of $P_p - \sum_{s=1}^{v_2} \pi_{ps} Q_{ps}$. It follows from (5) that $\alpha > 0$. According to (12), one gets

$$\mathbb{E}\left\{\sum_{i=0}^{d_1}\sum_{j=0}^{d_2}\|x_{i,j}\|^2\right\} \leq -\frac{1}{\alpha}\mathbb{E}\left\{\sum_{i=0}^{d_1}\sum_{j=0}^{d_2}\Delta V_{i,j}\right\}, \tag{13}$$

where d_1 and d_2 are arbitrary positive integers. By substituting $x_{i,j}^1 = \begin{bmatrix} x_{i+1,j}^{hT} & x_{i,j+1}^{vT} \end{bmatrix}^T$ and $P_p = \operatorname{diag}\{P_p^h, P_p^v\}$ into $\Delta V_{i,j}$, it can be established that

$$\begin{aligned}
&\sum_{i=0}^{d_1}\sum_{j=0}^{d_2}\Delta V_{i,j} \\
&= \sum_{j=0}^{d_2}\left[x_{d_1+1,j}^{hT} P_{\vartheta_{d_1+1,j}}^h x_{d_1+1,j}^h - x_{0,j}^{hT} P_{\vartheta_{0,j}}^h x_{0,j}^h\right] \\
&+ \sum_{i=0}^{d_1}\left[x_{i,d_2+1}^{vT} P_{\vartheta_{i,d_2+1}}^v x_{i,d_2+1}^v - x_{i,0}^{vT} P_{\vartheta_{i,0}}^v x_{i,0}^v\right].
\end{aligned} \tag{14}$$

Combining (13) and (14), it yields

$$\begin{aligned}
&\mathbb{E}\left\{\sum_{i=0}^{d_1}\sum_{j=0}^{d_2}\|x_{i,j}\|^2\right\} \\
&\leq \frac{1}{\alpha}\mathbb{E}\Big\{\sum_{j=0}^{d_2} x_{0,j}^{hT} P_{\vartheta_{0,j}}^h x_{0,j}^h + \sum_{i=0}^{d_1} x_{i,0}^{vT} P_{\vartheta_{i,0}}^v x_{i,0}^v\Big\}.
\end{aligned} \tag{15}$$

Denote β as the maximum eigenvalue between P_p^h and P_p^v, and let d_1 and d_2 tend to infinity in (15). Then

$$\mathbb{E}\left\{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\|x_{i,j}\|^2\right\} \leq \frac{\beta}{\alpha}\mathbb{E}\left\{\sum_{i=0}^{\infty}\left(\|x_{i,0}^h\|^2 + \|x_{0,i}^v\|^2\right)\right\}$$

can be obtained. Recalling Assumption 1, it is evident that

$$\mathbb{E}\left\{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\|x_{i,j}\|^2\right\} < \infty.$$

Consequently, HMM-based CLS (4) with $w_{i,j} \equiv 0$ is SS. ■

Theorem 2. Given a positive scalar μ, if there are positive definite matrices $P_p = \operatorname{diag}\{P_p^h, P_p^v\}$ and Q_{ps}, matrices K_s for all $p \in \mathcal{V}_1$ and $s \in \mathcal{V}_2$ such that (5) and

$$\begin{bmatrix} -\bar{P}_p^{-1} & \bar{A}_{ps} & E_p \\ * & -Q_{ps} & 0 \\ * & * & -I \end{bmatrix} < 0, \tag{16}$$

$$\begin{bmatrix} -P_p & C_p^T \\ * & -\mu^2 I \end{bmatrix} < 0 \tag{17}$$

hold, then CLS (4) is SS with EPDA level.

Proof: By using SC, it can be easily obtained (6) from (16). Therefore, according to Theorem 1, CLS (4) with $w_{i,j} = 0$ is SS. Next, we just need to prove that CLS (4) has EPDA level μ. For this purpose, we introduce the following performance index:

$$\begin{aligned}
\mathbb{J} &= \mathbb{E}\left\{V_{i,j} - \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} w_{i,j}^T w_{i,j}\right\} \\
&\leq \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\mathbb{E}\left\{\Delta V_{i,j} - w_{i,j}^T w_{i,j}\right\}.
\end{aligned} \tag{18}$$

Based on (8), it can be obtained that

$$\Delta V_{i,j} = \hat{x}_{i,j}^T(\hat{A}_{ps}^T P_q \hat{A}_{ps} - P_p)\hat{x}_{i,j}, \tag{19}$$

where $\hat{A}_{ps} = \begin{bmatrix} \bar{A}_{ps} & E_p \end{bmatrix}$ and $\hat{x}_{i,j} = \begin{bmatrix} x_{i,j}^T & w_{i,j}^T \end{bmatrix}^T$.

Applying (18) and (19), it can be established that

$$\begin{aligned}
\mathbb{J} &\leq \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\mathbb{E}\left\{\Delta V_{i,j} - w_{i,j}^T w_{i,j}\right\} \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\mathbb{E}\Big\{\hat{x}_{i,j}^T\Big(\sum_{s=1}^{v_2}\pi_{ps}\hat{A}_{ps}^T \bar{P}_p \hat{A}_{ps} \\
&\quad + \operatorname{diag}\{-P_p, -I\}\Big)\hat{x}_{i,j}\Big\}.
\end{aligned} \tag{20}$$

According to (5), the following inequality can be further deduced from (20):

$$\begin{aligned}
\mathbb{J} &\leq \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\mathbb{E}\Big\{\hat{x}_{i,j}^T\sum_{s=1}^{v_2}\pi_{ps}\Big(\hat{A}_{ps}^T \bar{P}_p \hat{A}_{ps} \\
&\quad + \operatorname{diag}\{-Q_{ps}, -I\}\Big)\hat{x}_{i,j}\Big\}.
\end{aligned} \tag{21}$$

According to SC, we can get from (16) and (21) that $\mathbb{J} \leq 0$. It implies that

$$V_{i,j} \leq \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} w_{i,j}^T w_{i,j}. \tag{22}$$

On the other hand, by applying the SC to (17), one gets

$$C_p^T C_p - \mu^2 P_p < 0. \tag{23}$$

By the means of (4), (22), and (23) that for any $i, j > 0$, one has

$$y_{i,j}^T y_{i,j} \leq \mu^2 V_{i,j} \leq \mu^2 \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} w_{i,j}^T w_{i,j},$$

namely

$$\sup_{i\geq 0}\sup_{j\geq 0}\mathbb{E}\left\{\|y_{i,j}\|^2\right\} \leq \mu^2 \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\mathbb{E}\left\{\|w_{i,j}\|^2\right\},$$

which implies the desired EPDA level. The proof is completed. ■

Though Theorem 2 gives a sufficient condition, due to the existence of nonlinear terms, it is difficult to directly design a controller. Next, we will further study the design methods of the asynchronous output-feedback controller.

First, we consider the controller design under completely known conditional probability.

Theorem 3. Given a positive scalar $\hat{\mu}$, if there are positive definite matrices $\hat{P}_p = \operatorname{diag}\left\{\hat{P}_p^h, \hat{P}_p^v\right\}$ and $\hat{Q}_{ps}$, matrices H_s, Y_s, G_s for all $p \in \mathcal{V}_1$, $s \in \mathcal{V}_2$ such that

$$\begin{bmatrix} -\hat{P}_p & \hat{\mathcal{P}}_{p\pi} \\ * & -\hat{\mathcal{Q}}_p \end{bmatrix} < 0, \tag{24}$$

$$\begin{bmatrix} -\hat{\mathcal{P}}_p & \hat{\Upsilon}_{12} \\ * & \hat{\Upsilon}_{22} \end{bmatrix} < 0, \tag{25}$$

$$\begin{bmatrix} -\hat{P}_p & \hat{P}_p^T C_p^T \\ * & -\hat{\mu} I \end{bmatrix} < 0, \tag{26}$$

$$G_s^T + G_s > 0, \tag{27}$$

$$C_p H_s = G_s C_p \tag{28}$$

hold, where

$$\hat{\mathcal{P}}_{p\pi} = \left[\sqrt{\pi_{p1}}\hat{P}_p \quad \sqrt{\pi_{p2}}\hat{P}_p \quad \ldots \quad \sqrt{\pi_{pv_2}}\hat{P}_p\right],$$
$$\hat{\mathcal{Q}}_p = \operatorname{diag}\left\{\hat{Q}_{p1}, \hat{Q}_{p2}, \ldots, \hat{Q}_{pv_2}\right\},$$
$$\hat{\mathcal{P}}_p = \operatorname{diag}\left\{\hat{P}_1, \hat{P}_2, \ldots, \hat{P}_{v_1}\right\},$$
$$\hat{\Upsilon}_{12} = \begin{bmatrix} \sqrt{\lambda_{p1}}(A_pH_s + B_pY_sC_p) & \sqrt{\lambda_{p1}}E_p \\ \sqrt{\lambda_{p2}}(A_pH_s + B_pY_sC_p) & \sqrt{\lambda_{p2}}E_p \\ \vdots & \vdots \\ \sqrt{\lambda_{pv_1}}(A_pH_s + B_pY_sC_p) & \sqrt{\lambda_{pv_1}}E_p \end{bmatrix},$$
$$\hat{\Upsilon}_{22} = \operatorname{diag}\left\{\hat{Q}_{ps} - H_s^T - H_s, -I\right\},$$

then CLS (4) is SS with EPDA level $\mu = \sqrt{\hat{\mu}}$. The control gains are given by

$$K_s = Y_sG_s^{-1}, s \in \{1, 2, \ldots, v_2\}. \tag{29}$$

Proof: Condition (27) guarantees that the matrix G_s is invertible. We introduce the following symbols for all $p \in \mathcal{V}_1$, $s \in \mathcal{V}_2$:

$$\hat{P}_p = P_p^{-1}, \ \hat{Q}_{ps} = Q_{ps}^{-1}, \ \hat{\mu} = \mu^2.$$

By pre-multiplying and post-multiplying (24) with

$$\operatorname{diag}\{P_p, \underbrace{I, \ldots, I}_{v_2}\},$$

it can be rewritten as

$$\begin{bmatrix} -P_p & \hat{\varPi}_p \\ * & -\hat{\mathcal{Q}}_p \end{bmatrix} < 0, \tag{30}$$

where $\hat{\varPi}_p = \left[\sqrt{\pi_{p1}}I \quad \sqrt{\pi_{p2}}I \quad \ldots \quad \sqrt{\pi_{pv_2}}I\right]$. Utilizing the SC, it is obvious that (30) is equivalent to (5).

Next, we verify that (16) can be guaranteed by (25). Based on Lemma 1, it has

$$-H_s^T\hat{Q}_{ps}^{-1}H_s \leq \hat{Q}_{ps} - H_s^T - H_s,$$

the following inequality can be obtained from (25):

$$\begin{bmatrix} -\hat{\mathcal{P}}_p & \hat{\Upsilon}_{12} \\ * & \tilde{\Upsilon}_{22} \end{bmatrix} < 0, \tag{31}$$

where $\tilde{\Upsilon}_{22} = \operatorname{diag}\left\{-H_s^TQ_{ps}H_s, -I\right\}$.

Combining (28) and (29), pre-multiplying and post-multiplying (31) by

$$\operatorname{diag}\{\underbrace{I, \ldots, I}_{v_1}, H_s^{-1}, I\},$$

we can obtain (16). Pre-multiplying and post-multiplying (26) by $\operatorname{diag}\{P_p, I\}$, we can conclude that (26) and (17) are equivalent. Therefore, the proof is completed. ∎

Next, the controller design method is expanded to the uncertain conditional probability scenario. In this case, π_{ps} satisfies $\sum_{s=1}^{v_2}\pi_{ps} = 1$ and $0 \leq \underline{\pi}_{ps} \leq \pi_{ps} \leq \overline{\pi}_{ps} \leq 1$ for all $p \in \mathcal{V}_1$ and $s \in \mathcal{V}_2$, where $\underline{\pi}_{ps}$ and $\overline{\pi}_{ps}$ are known parameters, respectively, standing for the lower and upper bounds of π_{ps}.

Denoting

$$\rho_{ps} = \frac{\overline{\pi}_{ps} + \underline{\pi}_{ps}}{2}, \sigma_{ps} = \frac{\overline{\pi}_{ps} - \underline{\pi}_{ps}}{2},$$

we can obtain

$$\pi_{ps} = \rho_{ps} + \eta_{ps}\sigma_{ps}, \tag{32}$$

where η_{ps} is unknown and belongs to [-1, 1].

Based on Theorem 2, we can get the following Theorem:

Theorem 4. Given a positive scalar $\hat{\mu}$, if there are positive definite matrices $\hat{P}_p = \operatorname{diag}\left\{\hat{P}_p^h, \hat{P}_p^v\right\}$ and Q_{ps}, matrices K_s, $\epsilon_s > 0$, for all $p \in \mathcal{V}_1$, $s \in \mathcal{V}_2$ such that

$$\begin{bmatrix} \Phi_1 & \Phi_2 & \Phi_3 \\ * & \Phi_4 & 0 \\ * & * & \Phi_5 \end{bmatrix} < 0, \tag{33}$$
$$\begin{bmatrix} \Theta_1 & \Theta_2 \\ * & \Theta_3 \end{bmatrix} < 0, \tag{34}$$
$$\begin{bmatrix} -\hat{P}_p & \hat{P}_p^TC_p^T \\ * & -\hat{\mu}I \end{bmatrix} < 0, \tag{35}$$

hold, where

$$\Phi_1 = \hat{P}_p - 2I + \sum_{s=1}^{v_2}\frac{1}{2}\epsilon_s\sigma_{ps}^2I,$$
$$\Phi_2 = \left[\sqrt{\rho_{p1}}Q_{p1} \quad \sqrt{\rho_{p2}}Q_{p2} \quad \cdots \quad \sqrt{\rho_{pv_2}}Q_{pv_2}\right],$$
$$\Phi_3 = \left[Q_{p1}^T \quad Q_{p2}^T \quad \cdots \quad Q_{pv_2}^T\right],$$
$$\Phi_4 = \operatorname{diag}\left\{-Q_{p1}, -Q_{p2}, \ldots, -Q_{pv_2}\right\},$$
$$\Phi_5 = \operatorname{diag}\left\{-2\epsilon_1I, -2\epsilon_2I, \ldots, -2\epsilon_{v_2}I\right\},$$
$$\Theta_1 = \operatorname{diag}\left\{-\hat{P}_1, -\hat{P}_2, \ldots, -\hat{P}_{v_1}\right\},$$
$$\Theta_2 = \begin{bmatrix} \sqrt{\lambda_{p1}}(A_p + B_pK_sC_p) & \sqrt{\lambda_{p1}}E_p \\ \sqrt{\lambda_{p2}}(A_p + B_pK_sC_p) & \sqrt{\lambda_{p2}}E_p \\ \vdots & \vdots \\ \sqrt{\lambda_{pv_1}}(A_p + B_pK_sC_p) & \sqrt{\lambda_{pv_1}}E_p \end{bmatrix},$$
$$\Theta_3 = \operatorname{diag}\left\{-Q_{ps}, -I\right\},$$

then CLS (4) is SS with EPDA level $\mu = \sqrt{\hat{\mu}}$ and controller (7) with gains K_s, $s \in \{1, 2, \ldots, v_2\}$.

Proof: According to Lemma 1, we have the following inequality:

$$-P_p \leq \hat{P}_p - 2I.$$

By SC, inequality (33) is equivalent to

$$\sum_{s=1}^{v_2}\rho_{ps}Q_{ps} + \sum_{s=1}^{v_2}\frac{1}{2}\epsilon_s\sigma_{ps}^2I + \sum_{s=1}^{v_2}\frac{1}{2}\epsilon_s^{-1}Q_{ps}^TQ_{ps} - P_p < 0.$$

Furthermore, we can get from Lemma 2 that

$$\eta_{ps}\sigma_{ps}Q_{ps} \leq \frac{1}{2}\epsilon_s\sigma_{ps}^2I + \frac{1}{2}\epsilon_s^{-1}Q_{ps}^TQ_{ps}.$$

Thus, we have

$$\sum_{s=1}^{v_2}(\rho_{ps} + \eta_{ps}\sigma_{ps})Q_{ps} - P_p < 0. \tag{36}$$

From (32) and (36), we can obtain (5).

It is easy to conclude that (16) is equivalent to (34).

Pre-multiplying and post-multiplying (35) by $\operatorname{diag}\{P_p, I\}$, we can derive (17). Therefore, the proof is completed. ∎

Remark 1. We use different methods to deal with the nonlinear terms in Theorems 3 and 4. In Theorem 3, where the conditional probability is completely known, the nonlinear terms are dealt with by congruence transformation and the introduction of a relaxation matrix. In Theorem 4, to deal with the scenario of uncertain conditional probability, new decision variables ϵ_s $(s \in \{1, 2, \ldots, v_2\})$ need to be introduced. At this point, we use the inequality scaling method to deal with the nonlinear terms.

IV. Application Examples

In this section, we give a practical example to verify the effectiveness of the designed asynchronous output-feedback controller.

Consider a thermal process described by the following partial differential equation [8]:

$$\frac{\partial T_{x,t}}{\partial x} = -\frac{\partial T_{x,t}}{\partial t} - a_{\vartheta_{x,t}} T_{x,t} + b_{\vartheta_{x,t}} u_{x,t}, \tag{37}$$

where $T_{x,t}$ represents the temperature at space $x \in [0, x_f]$ and time $t \in [0, \infty)$, $u_{x,t}$ is the control input, $a_{\vartheta_{x,t}}$ and $b_{\vartheta_{x,t}}$ are the heat transfer coefficients and $\vartheta_{x,t}$ is the switching parameter obeying the Markov chain. For given Δx and Δt, denote

$$\begin{aligned} T_{i,j} &= T_{i\Delta x, j\Delta t}, \\ \frac{\partial T_{x,t}}{\partial x} &\simeq \frac{T_{i,j} - T_{i-1,j}}{\Delta x}, \\ \frac{\partial T_{x,t}}{\partial t} &\simeq \frac{T_{i,j+1} - T_{i,j}}{\Delta t}. \end{aligned}$$

Then, (37) can be discretized as

$$\begin{aligned} T_{i,j+1} = &\left(1 - \frac{\Delta t}{\Delta x} - a_{\vartheta_{i,j}} \Delta t\right) T_{i,j} \\ &+ \frac{\Delta t}{\Delta x} T_{i-1,j} + b_{\vartheta_{i,j}} \Delta t u_{i,j}. \end{aligned} \tag{38}$$

Define $x^h_{i,j} = T_{i-1,j}$ and $x^v_{i,j} = T_{i,j}$. Then (38) can be transformed into the form of (1), as follows:

$$\begin{bmatrix} x^h_{i+1,j} \\ x^v_{i,j+1} \end{bmatrix} = A_{\vartheta_{i,j}} \begin{bmatrix} x^h_{i,j} \\ x^v_{i,j} \end{bmatrix} + B_{\vartheta_{i,j}} u_{i,j},$$

where

$$A_{\vartheta_{i,j}} = \begin{bmatrix} 0 & 1 \\ \frac{\Delta t}{\Delta x} & 1 - \frac{\Delta t}{\Delta x} - a_{\vartheta_{i,j}} \Delta t \end{bmatrix}, \; B_{\vartheta_{i,j}} = \begin{bmatrix} 0 \\ b_{\vartheta_{i,j}} \Delta t \end{bmatrix}.$$

In addition, we consider 2D RTMJS (1) with the following parameters:

$$\Delta x = 0.12, \Delta t = 0.1, a_1 = -2.5, a_2 = 0.5, b_1 = 1, b_2 = 0.5.$$

Then, the relevant system matrices of 2D RTMJS (1) are listed as follows:

Mode 1:

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0.8333 & 0.4167 \end{bmatrix}, \; B_1 = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix},$$
$$C_1 = \begin{bmatrix} 1 & 0.5 \end{bmatrix}, \; E_1 = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}.$$

Mode 2:

$$A_2 = \begin{bmatrix} 0 & 1 \\ 0.8333 & 0.1167 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0 \\ 0.05 \end{bmatrix},$$
$$C_2 = C_1, \; E_2 = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}.$$

The transition probability matrix Λ is given as follows:

$$\Lambda = \begin{bmatrix} 0.3 & 0.7 \\ 0.6 & 0.4 \end{bmatrix}.$$

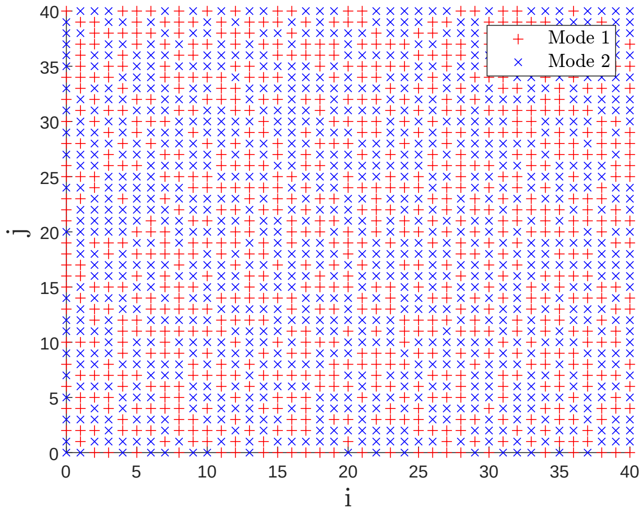

Fig. 1. Switching signal of the 2D RTMJS.

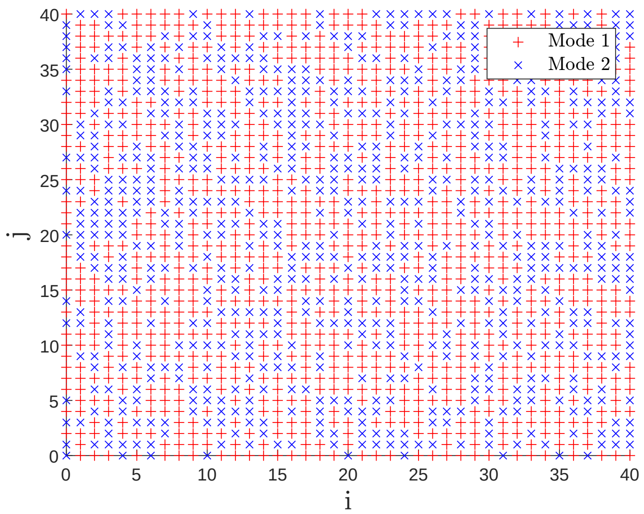

Fig. 2. Switching signal of the asynchronous controller.

Next, we consider two cases concerning completely known conditional probability and uncertain conditional probability, respectively:

Case 1: Completely Known Conditional Probability

The conditional probability matrix Π is given by

$$\Pi = \begin{bmatrix} 0.4 & 0.6 \\ 0.8 & 0.2 \end{bmatrix}.$$

By utilizing the method given in Theorem 3 and obtaining the following asynchronous output-feedback controller gains:

$$K_1 = -10.6748, K_2 = -9.2180,$$

and the optimized EPDA level $\mu^* = 0.3298$.

Case 2: Uncertain Conditional Probability

Suppose the conditional probability matrix Π are given as

$$\Pi = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix},$$

where $\pi_{11} \in [0.4 + \beta, 1], \pi_{12} \in [0, 0.6], \pi_{21} \in [0, 0.8]$, and $\pi_{22} \in [0.2 + \beta, 1]$, with β being an adjustable parameter. Here, we set $\beta = 0.05$. By solving Theorem 4 we can obtain the following control gains:

$$K_1 = -10.1416, K_2 = -11.4966,$$

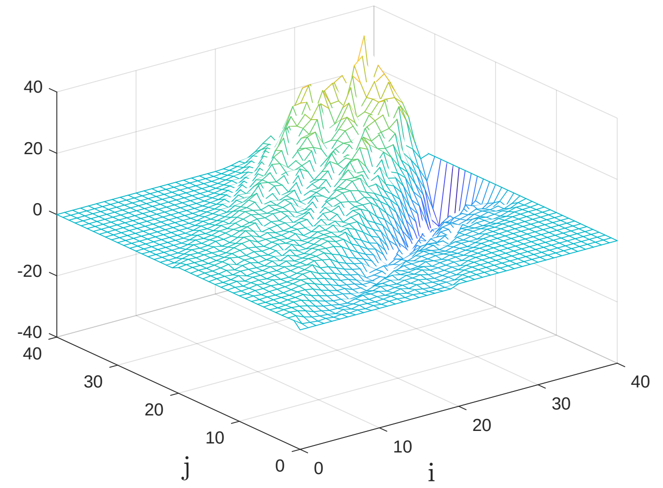

Fig. 3. Horizontal state $x^h_{i,j}$ without control input.

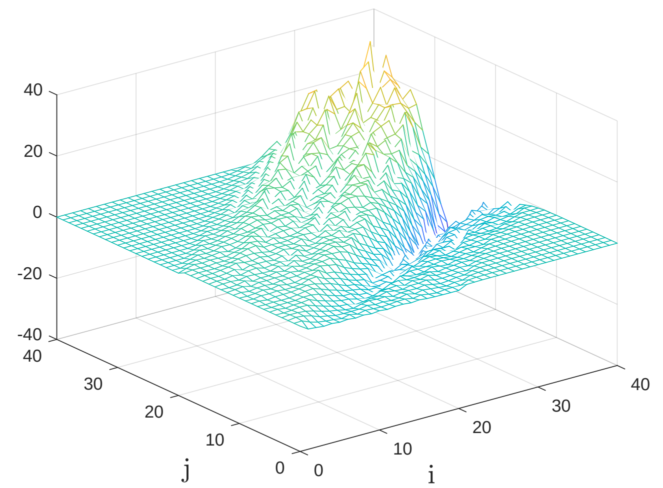

Fig. 4. Vertical state $x^v_{i,j}$ without control input.

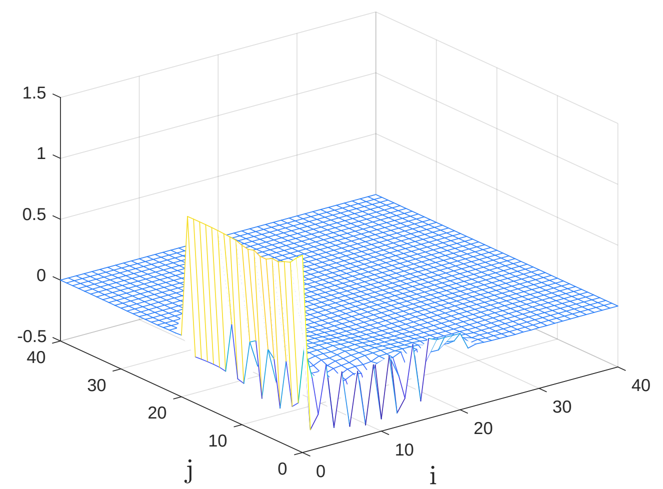

Fig. 5. Horizontal state $x^h_{i,j}$ with control input.

and the optimized EPDA level $\mu^* = 1.1718$. Since there is a feasible solution, we can conclude according to Theorem 4 that 2D RTMJS (1) is SS with EPDA level.

The switching signals of system mode and controller mode are depicted in Fig. 1 and Fig. 2. Next, we will further demonstrate the effectiveness of the design approach by comparing the evolutions of system states with and without

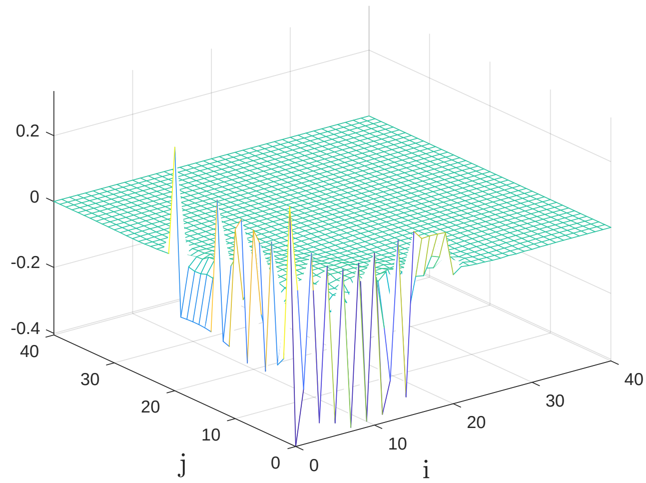

Fig. 6. Vertical state $x^v_{i,j}$ with control input.

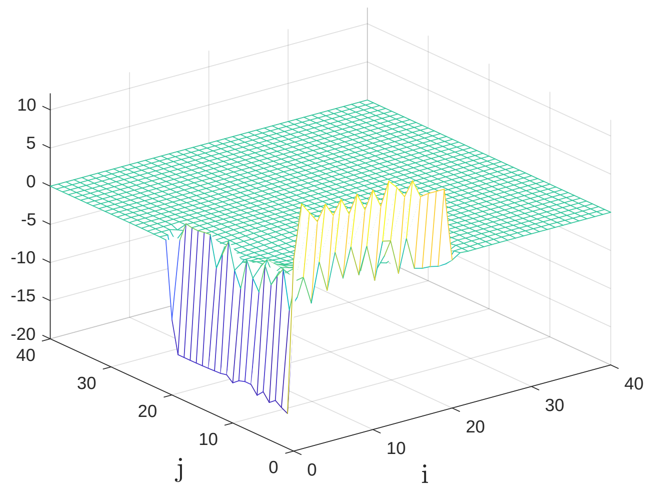

Fig. 7. Trajectory of the control input $u_{i,j}$.

control input. Therefore, we assume that the system has the following initial boundary condition:

$$x^h_{0,j} = \begin{cases} -1, & 0 \le j \le 19 \\ 0, & elsewhere \end{cases}$$
$$x^v_{i,0} = \begin{cases} 1, & 0 \le j \le 19 \\ 0, & elsewhere \end{cases}$$

and the external disturbance is given as

$$w_{i,j} = cos(0.1\pi(i+j))exp(-0.15(i+j)).$$

With these parameters, we obtain the state trajectories of the open-loop 2D RTMJS, which are displayed in Figs. 3 and 4. Clearly, the system under consideration is unstable without control input. The trajectories of the state of the HMM-based CLS are given in Figs. 5 and 6, and the trajectory of the control input is given in Fig. 7. Obviously, the trajectories of states $x^h_{i,j}$ and $x^v_{i,j}$ and control input $u_{i,j}$ of the CLS quickly converge to zero.

We introduce the following function to measure the EPDA level of HMM-based CLS:

$$\mu_{\hat{i},\hat{j}} = \sqrt{\frac{\sup\limits_{0\le i\le \hat{i}} \sup\limits_{0\le j\le \hat{j}} \mathbb{E}\{\|y_{i,j}\|^2\}}{\sum_{i=0}^{\hat{i}}\sum_{j=0}^{\hat{j}} \mathbb{E}\left\{\|w_{\hat{i},\hat{j}}\|^2\right\}}}.$$

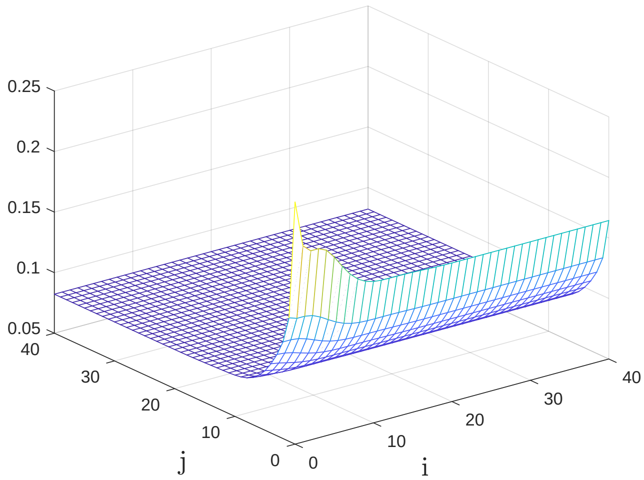

Fig. 8. Trajectory of $\mu_{\hat{i},\hat{j}}$.

Under the zero boundary condition, the state trajectory of μ is as shown in Fig. 8. It can be seen from the figure that the minimum upper bound of μ is 0.25, which is less than the optimal value $\mu^* = 1.1718$. The above results demonstrate that the designed asynchronous output-feedback controller can effectively stabilize the considered 2D RTMJS (1).

V. Conclusion

This paper studied the design issues of the asynchronous energy-to-peak controller of 2D RTMJS (1). Given the practical challenge of obtaining the system state, output-feedback was utilized for closing the control loop instead of state feedback. The asynchronous behavior between the 2D RTMJS (1) and the controller (3) was represented by HMM. A sufficient condition (see Theorem 2) for the stochastic stability and EPDA of HMM-based CLS (4) was established using a Lyapunov function and SC. Under the completely known conditional probability, an asynchronous output-feedback controller design method was proposed by introducing the relaxation matrix and using variable replacement (see Theorem 3). The present work was then expanded to the uncertain conditional probability scenario and a related controller design method was developed (see Theorem 4). Finally, a practical example of the thermal process was used to illustrate the effectiveness of the proposed design approaches.

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