The Local Resolving Dominating Set of Comb Product Graphs

Reni Umilasari, Liliek Susilowati*, Slamin, Savari Prabhu and Osaye J. Fadekemi

***Abstract*—The local resolving dominating set studied in this paper is a notion that combines two concepts in graph theory, the local metric dimension and dominations in graphs. Let G and H be connected graphs of orders n and m, respectively; and x a vertex in H hereafter referred to as a linkage vertex. The comb product of G and H denoted by $G \triangleright H$, is a graph obtained by taking one copy of G and n copies of H and attaching the i-th copy of H at the vertex x to the i-th vertex of G. In this paper, we determine the local resolving dominating set of the comb products $G \triangleright S_n$ with two different linkage vertices, $G \triangleright K_n$, $G \triangleright K_{m,n}$ and $G \triangleright C_m$ graphs.**

***Index Terms*—local resolving set, domination, comb product, metric dimension**

I. Introduction

SEVERAL authors have worked on combining the notion of metric dimension with other graph theoretical concepts to generate a new concept, amongst others are local metric dimension [1], [2], adjacency metric dimension [3], and strong metric dimension [4]. For an ordered set $W = \{w_1, w_2, \ldots, w_k\}$ of k distinct vertices in a nontrivial connected graph G, the representation of a vertex v of G with respect to W is the k-vector $r(v|W) = \{d(v, w_1), d(v, w_2), \ldots, d(v, w_k)\}$ where $d(v, w_i)$ is the distance between v and w_i for $1 \leq i \leq k$. The minimum number of positive integer k is called the metric dimension of G, denoted by $dim(G)$. For every pair of adjacent vertices $uv \in E(G)$, W is said to be a local resolving set of G if $r(v|W) = r(u|W)$. The minimum k for which G has a local resolving set is the local dimension of G, denoted by $dim_l(G)$.

Many applications of the metric dimension can be found in the literature. For example, Khuller et al. in [5] used the concept of metric dimension to study the concept of robot navigation in Euclidean space, Wahyudi [6] made use of metric dimension to obtain the minimum placement of fire sensor installation, and Saifudin et. al. [7] applied it to the study of automatic aircraft navigation in fire protected forest areas.

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The notion of domination is one that is very relevant to date, and it is still being studied in various forms. A dominating set S is defined as a subset of $V(G)$ in which every vertex of G not in S is connected and has a distance of one to S. The lowest cardinality among all dominating sets in G is called the domination number of graph G, denoted by $\gamma(G)$.

Some of the varieties of domination that have been studied include total dominator edge chromatic number [9], power domination [10], independent domination [11], k-distance domination [17], and paired domination [12]. The conceptualization of domination in graphs has been widely applied to the study of many real-life situations. Most recently, Saifudin and Umilasari studied the application of connected dominations in graphs in determining the ideal placement for an ATM machine in a community [13], the best position for a security post of a zoo [14], and in the establishment of bulog regional sub-division in east java [15]. The concept of a resolving dominating set introduced by Brigham [16], is one that combines the notion of metric dimension and dominating set. This terminology was also called the metric locating dominating set by Henning and Oellermann [18].

The resolving dominating set is a set of vertices that satisfies the definition of both dominating set and resolving set. A lowest cardinality of the resolving dominating set is referred to as the resolving dominating number. The combination of metric dimension and the dominating set was done by L. Susilowati et al. in [19] and [20], and it is referred to as the dominant metric dimension, while R. Umilasari et al. in [21] combined the local metric dimension and dominating set. In other words, resolving dominating set and dominant metric dimension is the same concept that can be used interchangeably. Informally, let G be a connected graph. An ordered set $W \subseteq V(G)$ is said to be a dominant resolving set of G if W is a resolving set and a dominating set of G. The dominant resolving set with lowest cardinality is named a dominant basis of G, while the cardinality of a dominant basis is called the dominant metric dimension of G, denoted by $Ddim(G)$ [19].

The dominant local metric dimension which will be discussed in this paper is a combination of the two concepts in graph theory: the local metric dimension and dominating set. Formally, the definition of the dominant local metric dimension is presented in Definition 1. In addition, some existing results related to the dominant local metric dimension are also presented below.

Definition 1.1: [22] Given a connected graph G. An ordered set $W_l = \{w_1, w_2, \ldots, w_n\} \subseteq V(G)$ is called a dominant local resolving set if W_l is a local resolving set and a dominating set of G. The dominant local resolving set with lowest cardinality is called a dominant local basis.

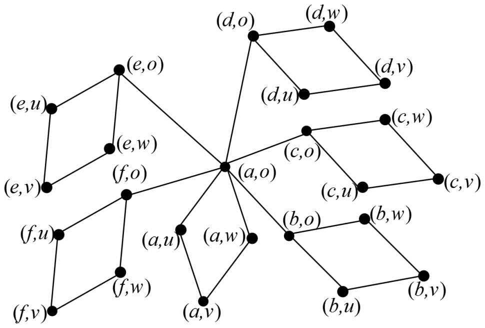

Fig. 1. Comb product graph of $S_6 \triangleright C_4$.

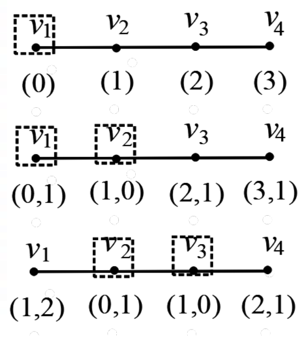

Fig. 2. Illustration to select the dominant local metric dimension of graph

The number of vertices in a dominant local basis of G is named the dominant local metric dimension and is denoted by $Ddim_l(G)$.

Lemma 1.2: [22] Let G be a connected graph and $W_l \subseteq V(G)$ be an ordered set. For every $v_i, v_j \in W_l$, $r(v_i|W_l) \neq r(v_j|W_l)$, for $i \neq j$.

Lemma 1.3: [23] Let G be a connected graph. If there is no local dominant resolving set with cardinality p, then every $S \subseteq V(G)$ with $|S| < p$ is not a local dominant resolving set.

Theorem 1.4: [22]

1) If $n \geq 4$, then $Ddim_l(C_n) = \gamma(C_n)$.
2) $Ddim_l(G) = 1$ if and only if $G \equiv S_n$, $n \geq 3$
3) Let G be a connected graph of order $n \geq 2$, $Ddim_l(G) = n-1$ if and only $if G \equiv K_n$, $n \geq 2$.
4) Let $K_{m,n}$ be a complete bipartite graph of order $m, n \geq 2$. The dominant local metric dimension of $K_{m,n}$ is $Ddim_l(K_{m,n}) = \gamma(K_{m,n})$.

Graphs studied in this research are $G \triangleright S_n$ with two linkage vertex differences, $G \triangleright K_n, G \triangleright K_{m,n}$ and $G \triangleright C_m$ graphs. Formally, the definition of comb product of two graphs is defined as follows.

Definition 1.5: [24] Let G and H be connected graphs and x is a vertex of $V(H)$ hereafter referred as linkage vertex. The comb product of G and H denoted by $G \triangleright H$ is a graph obtained by taking one copy of G and n copies of H and attaching vertex x from the i-th copy of graph H at the i-th vertex from the graph G, n be the order of G. This definition can be denoted as follows:

$$V(G \triangleright H) = \{(a,v)|a \in V(G); v \in V(H)\} \text{ and }$$
$$(a,v)(b,w) \in E(G \triangleright H) \text{ if } a = b \text{ and } v,w \in E(H) \text{ or }$$
$$ab \in E(G) \text{ and } v = w = x.$$

An example of a comb product graph between the Star graph S_6 with cycle graph C_4 Figure is shown in 1.

II. SOME ILLUSTRATIONS

This section shows the steps to determine the local resolving dominating set or dominant local metric dimension of graphs.

1) Select the comb product of any two graphs (say, $G \triangleright H$).
2) Observe the local metric dimension (W_l) and dominating number $[\gamma(G \triangleright H)]$ of the graph. Either of the two can be obtained first. That is, we obtain either (W_l) or $(\gamma(G \triangleright H))$ first. However, in this paper, we decided to determine the local metric dimension first, and then proceeded to the third step.
3) Choose the elements of the local resolving set of the graphs with lowest cardinality (that is the local basis of the graphs).
4) If every two adjacent vertices of the graph have different representations to the local resolving set, it is easy to see that the local resolving set can be selected as the minimum dominating set of the graphs. Then, there are two possibilities as follows:
   a. If yes, it means W_l is a dominant local resolving set. Then, $Ddim_l(G \triangleright H) = |W_l|$.
   b. If not, we revert back to the second step by changing or adding the element of W_l.

Figure 2 gives an illustration of how to select the dominant local metric dimension of a graph. The figure shows an example of a path graph (P_4) which has the cardinality of dominant local basis of two by choosing v_2 and v_3. This is because selecting $W_l = \{v_1\}$ will fail, since v_1 cannot dominate v_3 and v_4. This is same if we select the $W_l = \{v_1, v_2\}$ because v_4 cannot be dominated by v_1 or v_2. Therefore, $W_l = \{v_2, v_3\}$ is the dominant local basis of P_4 or $Ddim_l(P_4) = 2$.

III. THE LOCAL RESOLVING DOMINATING SET

For this part, we show the dominant local metric dimension of comb product graphs, the studied graphs are $G \triangleright S_n$ with two linkage vertex differences, $G \triangleright K_n$, $G \triangleright K_{m,n}$ and $G \triangleright C_m$.

Theorem 3.1: Let G be a non-trivial connected graph. If S_n is a star graph with $n \geq 2$, then

$$Ddim_l(G \triangleright S_n) = |V(G)| \times Ddim_l(S_n).$$

Proof. Let G be a non-trivial connected graph with $V(G) = \{u_i|i = 1,2,3,\ldots,m\}$. Let S_n be a star graph with $V(S_n) = \{a\} \cup \{b_j|j = 1,2,3,\ldots,n-1\}$ and $E(S_n) = \{ab_j|j = 1,2,3,\ldots,n-1\}$ for $n \geq 2$. Let $(S_n)_i$ be the i-th copy of S_n for $i = 1,2,3,\ldots,m$. The dominant local metric dimension of $G \triangleright S_n$ is divided into the following two ways of proof.

a. **Case 1:** a is a linkage vertex
   The vertex set of $G \triangleright S_n$ is $V(G \triangleright S_n) = \{a_{0i}|i = 1,2,3,\ldots,m\} \cup \{b_{ij}|i = 1,2,3,\ldots,m, j = 1,2,3,\ldots,n-1\}$. Since a is a linkage vertex, by Theorem (1.4), $B = \{a\}$ is a dominant local basis of S_n and $B_i = \{a_{0i}\}$ is a dominant local basis of

$(S_n)_i$, for every $i = 1, 2, 3, \ldots, m$ with $|B_i| = |B|$. Choose $W_l = \bigcup_{i=1}^{m} B_i = \{a_{0i} | i = 1, 2, 3, \ldots, m\}$ so that $|W_l| = m$. We show below that the representation of every adjacent vertex of $G \triangleright S_n$ is different.

i Based on Lemma (1.2), $r(a_{0i}|W_l) \neq r(a_{0k}|W_l)$ for every $a_{0i}, a_{0k} \in W_l$ with $i \neq k$.

ii Since $a_{0i} \in B_i$, an element 0 is present in the $r(a_{0i}|B_i)$, whereas $d(b_{ij}, a_{0i}) = 1$ and $b_{ij} \notin B_i$ so an element 0 is absent in the $r(b_{ij}|B_i)$. Therefore, $r(a_{0i}|B_i) \neq r(b_{ij}|B_i)$. Furthermore, since $B_i \subseteq W_l$ we have $r(b_{ij}|W_l) \neq r(a_{0i}|W_l)$.

b. **Case 2:** a is not a linkage vertex
The vertex set of $G \triangleright S_n$ is $V(G \triangleright S_n) = \{u_{0i} | i = 1, 2, 3, \ldots, m, u_i \in V(G)\} \cup \{a_{1i} | i = 1, 2, 3, \ldots, m, a \in V(S_n)\} \cup \{b_{ij} | i = 1, 2, 3, \ldots, m, j = 1, 2, 3, \ldots, n-2\}$. Let $b_{n-1} \in V(S_n)$ be a linkage vertex or a vertex that is attached to every vertex in $V(G)$ and denoted by u_{0i} for every $i = 1, 2, 3, \ldots, m$ in the $V(G)$. By Theorem (1.4), $B = \{a\}$ is a dominant local basis of S_n and $B_i = \{a_{0i}\}$ is a dominant local basis of $(S_n)_i$, for every $i = 1, 2, 3, \ldots, m$, implying $|B_i| = |B|$. Choose $W_l = \bigcup_{i=1}^{m} B_i = \{a_{1i} | i = 1, 2, 3, \ldots, m\}$ so that$|W_l| = m$. We show below that representation of every adjacent vertex of $G \triangleright S_n$ is different.

i for $u_{0i}, u_{0k} \in V(G \triangleright S_n) \backslash W_l$ with $i \neq k$.
Since G is a connected graph satisfying $d(u_{0k}, a) = d(u_{0k}, u_{0i}) + d(u_{0i}, a)$ for every $a \in B_i$, then $d(a, u_{0i}) \neq d(a, u_{0k})$ implies $r(u_{0i}|B_i) \neq r(u_{0k}|B_i)$. Thus since $B_i \subseteq W_l$, then $r(u_{0i}|W_l) \neq r(u_{0k}|W_l)$.

ii u_{0i} with $a_{1i} \in B_i$.
Since $a_{1i} \in B_i$, an element 0 exists in the $r(a_{1i}|B_i)$ whereas $d(u_{0i}, a_{1i}) = 1$ and $u_{0i} \notin B_i$ then the element 0 is also absent in the $r(u_{0i}|B_i)$. Therefore, $r(a_{1i}|B_i) \notin r(u_{0i}|B_i)$.

iii b_{ij} with $a_{1i} \in B_i$.
Since $a_{1i} \in B_i$, the element 0 exists in $r(a_{1i}|B_i)$ whereas $d(b_{ij}, a_{1i}) = 1$ and $b_{ij} \notin B_i$. Thus the element 0 is not in the $r(b_{ij}|B_i)$. Therefore $r(a_{1i}|B_i) \neq r(b_{ij}|B_i)$.

Both cases 1 and 2, it can be concluded that W_l is a local resolving set of $G \triangleright S_n$. Further, because every vertex $a_{0i} \in W_l$ is adjacent to every vertex u_{0i} and $b_{ij} \in V(G \triangleright S_n)$ for $i = 1, 2, 3, \ldots, m$, so W_l is a dominating set. Therefore, $W_l = \{a_{0i} | i = 1, 2, 3, \ldots, m\}$ is a dominant local resolving set of $G \triangleright S_n$. Next, take any $S \subseteq V(G \triangleright S_n)$ with $|S| < |W_l|$. Let $|S| = |W_l| - 1$, so i is present such that $|B_i| = 0$ for a $(S_n)_i$, so there is a vertex in the $(S_n)_i$ which is not adjacent to S. Therefore, S is not a dominant local resolving set of $G \triangleright S_n$. Based on Lemma 1.3, any set T with $|T| < |S|$ is not a dominant local resolving set of $G \triangleright S_n$. Therefore, $W_l = \bigcup_{i=1}^{m} B_i$ is a dominant local basis of $G \triangleright S_n$. Further, since B_i is a dominant local basis of $(S_n)_i$ with $|B_i| = Ddim_l(S_n)$, then $Ddim_l(G \triangleright S_n) = |W_l| = |V(G)| \times Ddim_l(S_n)$ for $n \geq 2$ if the linkage vertex is dominant vertex in the S_n or not.

Figure III (a) showed a graph $C_6 \triangleright S_4$ where the linkage vertex is a dominant vertex of S_3 whereas Figure III (b) showed an example of $C_6 \triangleright S_5$ with a linkage vertex that is not a dominant vertex of S_4. In both figures, the elements of the dominant local resolving set are represented by some vertices with a square box.

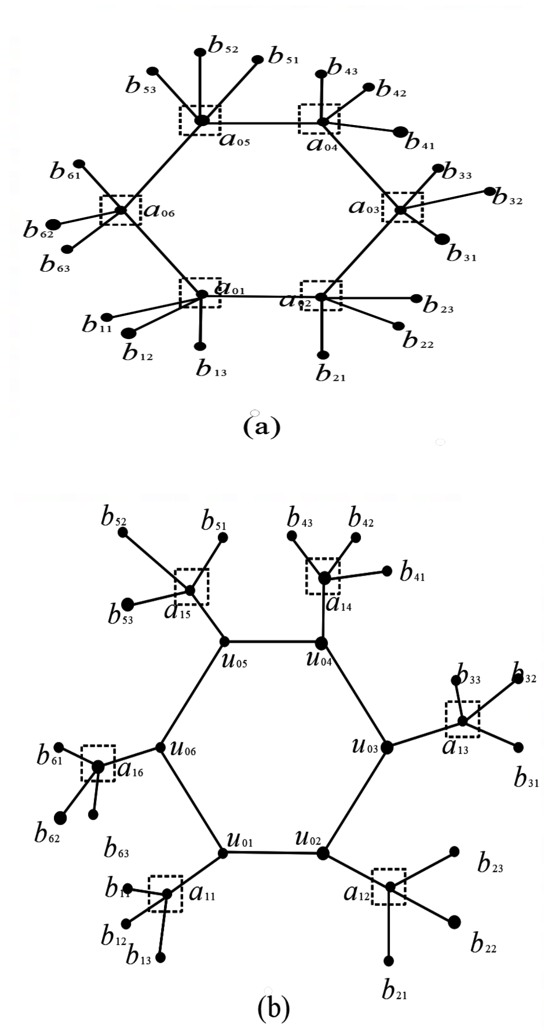

Fig. 3. (a.) Graph $C_6 \triangleright S_4$, (b.) Graph $C_6 \triangleright S_5$

We present the proof for the dominant local metric dimension of the graph $G \triangleright K_n$. It is important to note that since the complete graph K_n is a graph that is $(n-1)$-regular, the selection of the linkage vertex can be chosen from any of its vertices.

Theorem 3.2: Let G be a non-trivial connected graph. If $n \geq 3$, then

$$Ddim_l(G \triangleright K_n) = |V(G)| \times (Ddim_l(K_n) - 1).$$

Proof. Let G be a non-trivial connected graph with $V(G) = \{u_i | i = 1, 2, 3, dots, m\}$ and K_n a complete graph with $V(K_n) = \{v_j | j = 1, 2, 3, \ldots, n\}$ and $E(K_n) = \{v_i v_j | i, j = 1, 2, 3, \ldots, n, i \neq j\}$ for $n \geq 3$. Let $(K_n)_i$ be the i-th copies of K_n for $i = 1, 2, 3, \ldots, m$. The vertex set of $G \triangleright K_n$ is $V(G \triangleright K_n) = \{v_{0i} | i = 1, 2, 3, \ldots, m, u_i \in V(G)\}$ $\cup \{v_{ij} | i = 1, 2, 3, \ldots, m, j = 2, 3, \ldots, n\}$. Without loss of generality, let v_1 be a linkage vertex of K_n. Then, $B = \{v_j | j = 2, 3, 4, \ldots, n\}$ is a dominant local basis of K_n as described in Theorem (1.4), and B_i is a dominant basis of $(K_n)_i$ so that for every $i = 1, 2, 3, \ldots, m$,

$|B_i| = |B|$. Choose $W_l = \bigcup_{i=1}^{m}\{B_i - \{v_i n\}\}$ so that $|W_l| = m((n-1)-1)$. By Lemma (1.2), $r(v_{ij}|W_l) \neq r(v_{lk}|W_l)$ for every $v_{ij}, v_{lk} \in W_l$ with $ij \neq lk$ and so $V(G \triangleright K_n)W_l = \{v_{0i}, v_{in}|i = 1,2,3,\dots,m\}$. Take any 2 adjacent vertices in $V(G\triangleright K_n)W_l$. Based on each possibility, we showed that the representation of every 2 adjacent vertices is different.

i For $v_{0i}, v_{0j} \in V(G \triangleright K_n)\backslash W_l$ with $i \neq j$.
Since $d(v_{0j}, v) = d(v_{0j}, v_{0i}) + d(v_{0i}, v)$ for every $v \in B_i$, $d(v, v_{0i}) \neq d(v, u_{0j})$ and $r(v_{0i}|B_i) \neq r(v_{0j}|B_i)$. Thus, $B_i \subseteq W_l$ implies $r(v_{0i}|W_l) \neq r(v_{0j}|W_l)$.

ii For $v_{0i}, v_{0j} \in V(G \triangleright K_n)\backslash W_l$ with $v_{ij} \in B_i$.
Since $v_{ij} \in B_i$, there exists element 0 in $r(v_{ij}|B_i)$, whereas $d(v_{0i}, v_{ij}) = 1$ and $v_{0i} \notin B_i$. Then the element 0 is not in $r(v_{0i}|B_i)$. Therefore, $r(v_{ij}|B_i) \notin r(v_{0i}|B_i)$ and since $B_i \subseteq W_l$, it follows that $r(v_{0i}|W_l) \neq r(v_{ij}|W_l)$.

iii For $v_{0i}, v_{in} \in V(G \triangleright K_n)\backslash W_l$.
Since for $i \neq j$, $d(v_{0i}, v_{0j}) \neq d(v_{in}, v_{0j})$, then every $v \in B_j - \{v_{jn}\}$ leads to $d(v, v_{0i}) \neq d(v, v_{in})$ so that $r(v_{0i}B_j - \{v_{jn}\}) \neq r(v_{in}|B_j - \{v_{jn}\})$. Thus $B_j - \{v_{jn}\} \subseteq W_l$ yields that $r(v_{0i}|W_l) \neq r(v_{in}|W_l)$.

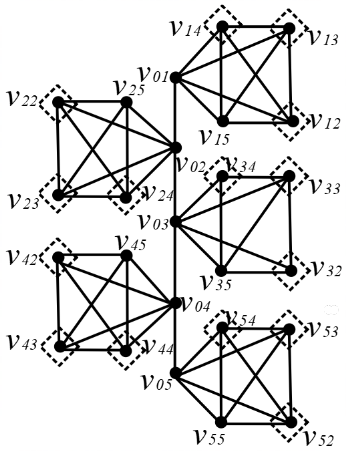

Fig. 4. $Ddim_l(P_5 \triangleright K_5) = 15$

Therefore W_l is a local resolving set of $G \triangleright K_n$. Now every vertex $v_{ij} \in W_l$ is adjacent to every vertex v_{0i} and $v_{in} \in V(G \triangleright K_n)$ for $i = 1,2,3,\dots,m$. Then W_l is a dominating set implying $W_l = \{v_{ij}|i = 1,2,3,\dots,m\}$ is a dominant local resolving set of $G \triangleright K_n$. Next, take any $S \subseteq V(G \triangleright K_n)$ with $|S| < |W_l|$. Let $|S| = |W_l| - 1$, so i is such that S contains $|B_i| - 2$ elements of $(K_n)_i$. As a result, there are 2 vertices in the $(K_n)_i$ having the same representation as S, and so S is not a local resolving set of $G \triangleright K_n$. By Lemma (1.3), any set T with $|T| < |S|$ is not a dominant local resolving set of $G \triangleright K_n$. Therefore, $W = \bigcup_{i=1}^{m}\{B_i - \{v_{in}\}\}$ is a dominant local basis of $G \triangleright K_n$. Furthermore, since B_i is a dominant local basis of $(K_n)_i$ with $|B_i| = Ddim_l(K_n)$, $Ddim_l(G \triangleright K_n) = |W_l| = |V(G)| \times (Ddim_l(K_n) - 1)$ for $n \geq 3$.

To clarify the result in Theorem (3.2), Figure 4 is an example of $P_5 \triangleright K_5$ where $Ddim_l(P_5 \triangleright K_5) = 15$. The elements of the dominant local basis are indicated by some vertices with a square box. Next, we observe the dominant local metric dimension of $G \triangleright K_{m,n}$.

Theorem 3.3: Let G be a non-trivial connected graph. If $m, n \geq 2$, then

$$Ddim_l(G \triangleright K_{m,n}) = |V(G)| \times Ddim_l(K_{m,n}).$$

Proof. Let G be a non-trivial connected graph, the vertex set is $V(G) = \{u_k|k = 1,2,3,\dots,p\}$ for $p \geq 2$ and $K_{m,n}$ is a complete bipartite graph with the vertex set is $V(K_{m,n}) = \{a_i|i = 1,2,3,\dots,m\} \cup \{b_j|j = 1,2,3,\dots,n\}$ and the edge set is $E(K_{m,n}) = \{a_ib_j|i = 1,2,3,\dots,m; j = 1,2,3,\dots,n\}$ for $m \geq 2$ and $n \geq 2$. Without reducing the generality of the proof, let a_1 is a linkage vertex graph $K_{m,n}$. The k-th copy of $K_{m,n}$ with $k = 1,2,3,\dots,p$ is referred as $(K_{m,n})_k$ with $V((K_{m,n})_k) = \{a_{ki}|i = 1,2,3,\dots,m\} \cup \{b_{kj}|j = 1,2,\dots,n\}$, for every $k = 1,2,3,\dots,p$. The vertex set of graph $G\triangleright K_{m,n}$ is $V(G\triangleright K_{m,n}) = \{a_{ki}, b_{kj}|k = 1,2,3,\dots,p,\ i = 1,2,3,\dots,m, j = 1,2,3,\dots,n\}$, and edge set there of $E(G \triangleright K_{m,n}) = \{a_{k1}a_{l1}|u_ku_l \in E(G)\} \cup \{a_{ki}b_{kj}|a_ib_j \in E(K_{m,n}), k = 1,2,3,\dots,p, i = 1,2,3,\dots,m, j = 1,2,3,\dots,n\}$. Let $B = \{a_1, b_1\}$ is a dominant local basis of $K_{m,n}$ as described in the Theorem 1.4 [22], and B_k is a dominant local basis of $(K_{m,n})_k$ so that for every $k = 1,2,3,\dots,p$ occur $|B_k| = |B|$. Choose $W_l = \bigcup_{i=1}^{m} B_i$ with $B_k = \{a_{k1}, b_{k1}\}$ for every $k = 1,2,3,\dots,p$, so $|W_l| = p\times 2$. By Lemma (1.2), obtained that $r(a_{k1}|W_l) \neq r(b_{k1}|W_l)$ for every $a_{k1}, b_{k1} \in W_l, k = 1,2,3,\dots,p$ with $a_{k1} \neq b_{k1}$. Further, take any 2 adjacent vertices of $V(G\triangleright K_n)W_l$. on each possibility, it is shown that the representation of every 2 adjacent vertices are different.

i For $a_{k1}a_{l1} \in W_l$ with $k \neq l$. Since G was a connected graph then $d(a_{l1}, v) = d(a_{l1}, a_{k1}) + d(a_{k1}, v)$ for every $v \in B_k$ so $d(v, a_{k1}) \neq d(v, a_{l1})$ caused $r(a_{k1}|B_{0k}) \neq r(a_{l1}|B_k)$. Since $B_k \subseteq W_l$ then $r(a_{k1}|W_l) \neq r(a_{l1}|W_l)$.

ii For $a_{ki}, b_{kj} \in V(G \triangleright K_{m,n})\backslash W_l$, $a_{ki}b_{kj} \in E(K_{m,n})_k$ for $k = 1,2,\dots,p$. Since $B_k = \{a_{k1}, b_{k1}\}$ is a local basis of the $(K_{m,n})_k$ then $(a_{ki}|B_k) = (2,1)$ and $r(b_{kj}|B_k) = (1,2)$ for every $i = 2,3,\dots,m, j = 2,3,\dots,n$, so $r(a_{ki}|B_i) \neq r(b_{kj}|B_k)$. Further, since $B_k \subseteq W_l$ then $r(a_{ki}|W_l) \neq r(b_{kj}|W_l)$.

Therefore W_l is a local resolving set of $G \triangleright K_{m,n}$. Further, since for every $k = 1,2,3,\dots,p$ the vertex $a_{k1} \in W_l$ is adjacent to every vertex b_{kj} and $b_{k1} \in W_l$ is adjacent to every vertex $a_{ki} \in V(G \triangleright K_{m,n})$ for every $i = 2,3,\dots,m, j = 2,3,\dots,n$, so W_l is a dominating set. Therefore, $W_l = \{a_{k1}, b_{k1}\}$ for every $k = 1,2,3,\dots,p$ is a dominant local resolving set of $G \triangleright K_{m,n}$. Next, take any $S \subseteq V(G \triangleright K_{m,n})$ with $|S| < |W_l|$. Let $|S| = |W_l| - 1$, so k is present such that S contained $|B_k| - 1$ element of $(K_n)_k$. As a result, there is a vertex in $(K_{m,n})_k$ which is not connected to S, and so S is not a dominating set of $G\triangleright K_{m,n}$. By Lemma (1.3), any set T with $|T| < |S|$ is not a dominant local resolving set of $G\triangleright K_{m,n}$. Therefore, $W_l = \{a_{k1}, b_{k1}\}$ is a dominant local basis of $G\triangleright K_n$. Moreover, since B_i is a dominant local basis of $(K_{m,n})_i$ with $|B_i| = Ddim_l(K_{m,n})$, $Ddim_l(G \triangleright K_{m,n}) = |W_l| = |V(G)| \times (Ddim_l(K_{m,n}))$ for $m, n \geq 2$.

Figure 5 is an example of the graph $P_3 \triangleright K_{3,3}$ with $Ddim_l(K_{m,n})$ which has dominant local metric dimensions

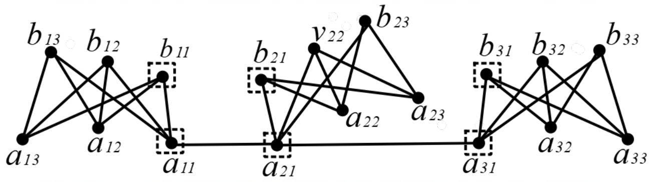

Fig. 5. $Ddim_l(P_3 \triangleright K_{3,3}) = 6$

equal six.

In the next theorem for $Ddim_l(G \triangleright C_m)$, we use a similar proof to that of the dominant local metric dimension of $G \triangleright K_n$. This is because the cycle graph (C_m) is also a regular graph and the selection of the linkage vertex can be chosen from any vertex in the cycle graph.

Theorem 3.4: Let G be a non-trivial connected graph. If $n \geq 3$, then

$$Ddim_l(G \triangleright C_m) = |V(G)| \times Ddim_l(C_m).$$

Proof. Let G be a non-trivial connected graph with $V(G) = \{v_i | i = 1, 2, 3, \ldots, n\}$. The vertex set of the cycle graph is given as $V(C_m) = \{u_j | j = 1, 2, 3, \ldots, m\}$ and the edge set is $E(C_m) = \{u_j u_{j+1} | j = 1, 2, 3, \ldots, m - 1\} \cup \{u_1 u_m\}$. We represent the i-th copy of C_m with $i = 1, 2, 3, \ldots, n$ as $(C_m)_i$ where $V((C_m)_i) = \{v_{ij} | j = 2, 3, \ldots, m\}$ for every $i = 1, 2, 3, \ldots, n$. The vertex set of $G \triangleright C_n$ is denoted as $V(G \triangleright C_m) = \{v_{ij} | i = 1, 2, 3, \ldots n, j = 2, 3, \ldots, m\}$, and the edge set as $E(G \triangleright C_m) = \{v_{i1} v_{j1} | u_i u_j \in E(G), i = 1, 2, 3, \ldots n\ , j = 2, 3, \ldots, m\} \cup \{v_{ij} v_{ij+1} | v_i v_j \in E(G)\} \cup \{v_{1j} v_{1m}\}$. Without loss of generality, let v_1 be a linkage vertex of C_m, B be a dominant basis of C_m and B_i be a dominant basis of $(C_m)_i$. Accordingly, for every $i = 1, 2, 3, \ldots, n$, occur $|B_i| = |B|$. Choose $W_l = \bigcup_{i=1}^{n} \{B_i\}$ by considering the following cases of m.

a. For $m \neq 0(\mathrm{mod}3)$,
Choose $W_l = \{v_{i1}, v_{i4}, v_{i7}, v_{i10}, \ldots, v_{i(3j-2)}, v_{im}\}$, so that $|W_l| = n \times \lceil \frac{m}{3} \rceil$.
b. For $m \equiv 0(\mathrm{mod}3)$,
Choose $W_l = \{v_{i1}, v_{i4}, v_{i7}, v_{i10}, \ldots, v_{i(3j-2)}\}$, so that $|W_l| = n \times \lceil \frac{m}{3} \rceil$.

By Lemma 1.2, $r(u|W_l) \neq r(v|W_l)$ for every $u, v \in W_l$ with $u \neq v$. Next, we take any 2 adjacent vertices of $V(G \triangleright C_m) W_l$ and show that the representation of every 2 adjacent vertices is different from W_l.

i. For $v_{i1}, v_{j1} \in W_l$ with $i \neq j$.
Since $d(v_{j1}, v) = d(v_{j1}, v_{i1}) + d(v_{i1}, v)$ for every $v \in B_i$, $d(v, v_{i1}) \neq d(v, v_{j1})$ and $r(v_{i1}|B_i) \neq r(v_{j1}|B_i)$. Since $B_i \subseteq W_l$, $r(v_{i1}|W_l) \neq r(v_{j1}|W_l)$.
ii. For $v_{ij}, v_{ik} \in V(G \triangleright C_m)$ with $j \neq k$.
Since B_i is a dominant local basis of $(C_m)_i$, it is clear that $r(v_{ij}|B_i) \neq r(v_{ik}|B_i)$. Thus $B_i \subseteq W_l$, and so $r(v_{ij}|W_l) \neq r(v_{ik}|W_l)$.

Based on the above descriptions, W_l is a local resolving set of $G \triangleright C_m$. Moreover, since $E(G \triangleright C_m) = \{v_{i1} v_{j1} | u_i u_j \in E(G), i = 1, 2, 3, \ldots n, j = 2, 3, \ldots, m\}$ $\{v_{ij} v_{ij+1} | v_i v_j \in E(G)\} \cup \{v_{1j} v_{1m}\}$, $v_{i(3j-1)}$ is adjacent to $v_{i(3j-2)}$ and $v_{i(3j)}$ is adjacent to $v_{i(3(j+1)-2)}$. Then W_l is a dominating set of G_m. Therefore, W_l is a dominant local resolving set of $G \triangleright C_m$. By taking any $S \subseteq V(G \triangleright C_m)$ with $|S| < |W_l|$, let $|S| = |W_l| - 1$, so i is present such that S contains $|B_i| - 1$ elements of $(C_m)_i$. Since B_i is a dominant basis of $(C_m)_i$, S is not a local resolving set or S is not a dominating set. Thus S is not a dominant local resolving set of $G \triangleright C_m$. By Lemma (1.3), any set T with $|T| < |S|$ is not a dominant local resolving set of $G \triangleright C_m$. Therefore, $W_l = \bigcup_{i=1}^{m} \{B_i\}$ is a dominant local basis of $G \triangleright C_m$. In addition, B_i is a dominant local basis of $(C_m)_i$ with $|B_i| = Ddim_l(C_m)$ and so $Ddim_l(G \triangleright C_m) = |W_l| = |V(G)| \times Ddim_l(C_m)$.

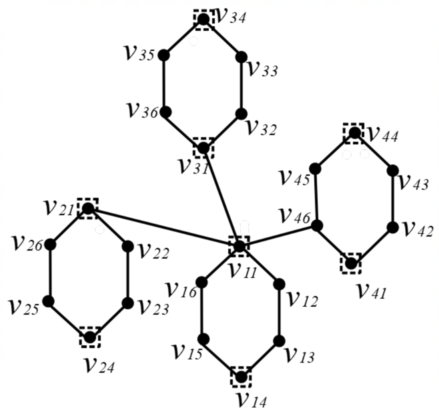

Fig. 6. $Ddim_l(S_4 \triangleright C_6) = 8$

Figure 6 shows an example of the graph $S_4 \triangleright C_6$. By Theorem 3.4, the graph has $Ddim_l(S_4 \triangleright C_6) = 8$, and the elements of the dominant local basis are indicated by some vertices with a square box.

IV. Conclusion And Prospects

After observe and analyze some graphs, we have obtained results on the dominant local metric dimension involving the comb product of 2 connected graphs, say G and H. For H, we have considered the star graph, complete graph, complete bipartite graph, and cycle graph. For these graphs, we have shown that the local resolving dominating set of $G \triangleright H$ depends on the selection of the linkage vertices, which in turn determines the value of $Ddim_l(H)$. For the next research, we can find the computer algorithm to determine the local resolving set of any graph. Besides that, we can also apply this theory into some real life problems.

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