Synchronization for Inertial Delayed Neural Networks Containing Discontinuous Activation Functions

Yanyan Wang and Zhilian Yan

Abstract—**The asymptotic and exponential synchronization problems are studied for inertial delayed neural networks containing discontinuous activation functions. By employing a variable replacement technique, the second-order differential models are transformed into first-order differential equations. A condition for the asymptotic synchronization of the drive-response models is derived using an appropriate Lyapunov functional and the LaSalle invariant principle. Then, through the use of another Lyapunov functional, a result concerning exponential synchronization is established. Finally, two examples are used to test the validity of the results achieved.**

Index Terms—**Activation function, asymptotic synchronization, exponential synchronization, neural network.**

I. INTRODUCTION

IN recent decades, delayed neural networks (DNNs) have garnered widespread attention due to their practical applications in various fields. Many scholars have devoted their efforts to studying the dynamics of DNNs, as phenomena like oscillation and bifurcation are heavily influenced by time delays [1, 2]. Numerous studies have been conducted on different kinds of DNNs, and the results of these studies can be found in the works of [3–9]. For instance, stability analysis for impulsive Cohen-Grossberg DNNs was discussed by Chaouki and Farah in [8], while novel asymptotic stability results for multi-delay neural networks were proposed by Faydasicok and Arik in [9].

The concept of drive-response synchronization, introduced by Pecora and Carroll [10] in 1990, has led to increased attention due to its practical applications. Different types of drive-response synchronization issues have been explored in Refs. [11–22]. For instance, Sambas et al. discussed adaptive synchronization for a novel chaotic system in [21], while Yan et al. investigated quasi-synchronization for memristive DNNs through hybrid event-triggered control in [22].

It is noteworthy that continuity of activation functions is a requirement in most studies. In [23], Forti and Nistri emphasized the importance and objectivity of discontinuous activations. Diode-like input-output activations were put to use by Kennedy and Chua in [24]. A few recent studies have started to focus on discontinuous activations. Building upon the concepts and results from [25], Liu, Zhang and Hu [26] investigated fixed-time synchronization, while Liu, Wang and Huang in [27] delved into multistability in activation functions with multiple discontinuous points.

Moreover, considering inertial terms is essential in DNNs, as the omission may hinder the understanding of chaotic behaviors and bifurcation [28, 29]. Consequently, scholars have turned their attention towards investigating the dynamic behavior of DNNs containing inertial terms. Synchronization and stability in such networks were studied in Refs. [30–32]. In [32], Huang, Cao and Liu investigated the adaptive synchronization problem for Cohen-Grossberg DNNs that included inertial terms, proposing an analytical scheme for the desired update law.

This paper focuses on studying the asymptotic and exponential synchronization problems of inertial DNNs (IDNNs) containing discontinuous activation functions. By employing a variable substitution method, the models are reformulated as first-order differential equations. We present a condition for the asymptotic synchronization of the drive-response IDNNs using an appropriate Lyapunov functional (LF) and the LaSalle invariant principle (LIP). Subsequently, we establish a result concerning exponential synchronization using another LF. The validity of our results is verified through two examples.

The remainder is structured as follows: In Section II, we describe the drive IDNN model, response IDNN model, assumptions, and necessary definitions and lemmas. Section III presents the two obtained results, and in Section IV, we provide two concrete examples to demonstrate the validity of our outcomes. Finally, Section V presents the concluding remarks.

II. PRELIMINARIES

Throughout, we denote by W^T and W^{-1} the transpose and inverse of any square matrix W, respectively, by diag$(\cdot)$ a diagonal matrix, and by $\mathbb{R}^m$ the m-dimensional Euclidean space.

Take account of the IDNN containing discontinuous activation functions as

$$\begin{aligned}\frac{d^2u(t)}{dt^2} =& - M\frac{du(t)}{dt} - L(u(t)) + Ph(u(t))\\ &+ Qh(u(t-\tau(t))) + J, \end{aligned}\tag{1}$$

where $u(t) = (u_1(t), \ldots, u_n(t))^T \in \mathbb{R}^n$ means the state vector, $\frac{d^2u(t)}{dt^2} = (\frac{d^2u_1(t)}{dt^2}, \ldots, \frac{d^2u_n(t)}{dt^2})^T$ denotes an inertial term; $M = \text{diag}(m_1, \ldots, m_n)$ with $m_i > 0, i = 1, \cdots, n$; $L = \text{diag}(l_1, \ldots, l_n)$ with $l_i > 0, i = 1, \cdots, n$; $P = (p_{ij})_{n\times n}$, $Q = (q_{ij})_{n\times n}$; $h(u(t)) = (h_1(u_1(t)), \ldots, h_n(u_n(t))^T$ stands for the activation function vector; $J = (J_1, \ldots, J_n)^T$

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denotes external input; $\tau(t)$ is the time delay with $0 \leq \tau(t) \leq \tau$ and $\dot{\tau}(t) \leq s < 1$, where τ and s are positive constants [33–36].

The activation functions are required to meet the following hypotheses:

$(H1)$: For any $i \in \{1, \ldots, n\}$, h_i is bounded and non-decreasing on $\mathbb{R}$;

$(H2)$: For any $i \in \{1, \ldots, n\}$, h_i is continuous on $\mathbb{R}$ except a countable set of points of discontinuity σ_{ki}, where there exist right and left limits $h_i(\sigma_{ki}^{+})$ and $h_i(\sigma_{ki}^{-})$;

$(H3)$: Denote $K[g(u)] = \left[K[g_1(u_1)], \ldots, K[g_n(u_n)]\right]^T$, where $K[g_i(u_i)] = [g_i(u_i^{-}), g_i(u_i^{+})]$. For any $i \in \{1, \ldots, n\}$, there are constants $\sigma_i \geq 0$, $\omega_i \geq 0$ such that

$$sup \mid \chi_i - \lambda_i \mid \leq \sigma_i \mid s - \tilde{s} \mid + \omega_i,$$

where $\chi_i \in K[g_i(s)]$, $\lambda_i \in K[g_i(\tilde{s})]$.

According to the definition of a solution in the sense of Filippov, the following differential inclusion system is given:

$$\begin{aligned}\frac{du(t)}{dt} \in M^{-1}[-\frac{d^2u(t)}{dt^2} - L(u(t)) + PK[g(u(t))] \\ + QK[g(u(t-\tau(t)))] + J].\end{aligned}$$

Based on measurable selection lemma given in [23], we have

$$\begin{aligned}\frac{du(t)}{dt} = M^{-1}[-\frac{d^2u(t)}{dt^2} - L(u(t)) + P\gamma(t) \\ + Q\gamma(t-\tau(t)) + J]\end{aligned} \tag{2}$$

for almost all t, where

$$\begin{aligned}\gamma(t) &= (\gamma_1(t), \ldots, \gamma_n(t))^T \in K[g(u(t))], \\ \gamma(t-\tau(t)) &= (\gamma_1(t-\tau(t)), \ldots, \gamma_n(t-\tau(t)))^T \\ &\in K[g(u(t-\tau(t)))].\end{aligned}$$

The corresponding response IDNN is offered below:

$$\begin{aligned}\frac{d^2\tilde{u}(t)}{dt^2} = -M\frac{d\tilde{u}(t)}{dt} - L(\tilde{u}(t)) + Ph(\tilde{u}(t)) \\ + Qh(\tilde{u}(t-\tau(t))) + J] + C(t),\end{aligned} \tag{3}$$

where $C(t) = (c_1(t), \ldots, c_n(t))^T$ denotes an appropriate control input vector.

Denote the following variable substitution:

$$y(t) = \frac{du(t)}{dt} + Au(t),\ \tilde{y}(t) = \frac{d\tilde{u}(t)}{dt} + A\tilde{u}(t), \tag{4}$$

where $A = \mathrm{diag}(a_1, \ldots, a_n)$ with $a_i > 0$, $i = 1, \cdots, n$, and it will be designed in Section III.

Employing (2) and (4), we have

$$\begin{cases}\frac{du(t)}{dt} = y(t) - Au(t), \\ \frac{dy(t)}{dt} = -A(A-M)u(t) + (A-M)y(t) - Lu(t) \\ \qquad + P\gamma(t) + Q\gamma(t-\tau(t)) + J.\end{cases} \tag{5}$$

Similarly, by employing (4), the equivalent model to IDNN (3) is offered below:

$$\begin{cases}\frac{d\tilde{u}(t)}{dt} = \tilde{y}(t) - A\tilde{u}(t), \\ \frac{d\tilde{y}(t)}{dt} = -A(A-M)\tilde{u}(t) + (A-M)\tilde{y}(t) - L\tilde{u}(t) \\ \qquad + P\tilde{\gamma}(t) + Q\tilde{\gamma}(t-\tau(t)) + J + C(t).\end{cases} \tag{6}$$

where

$$\begin{aligned}\tilde{\gamma}(t) &= (\tilde{\gamma}_1(t), \ldots, \tilde{\gamma}_n(t))^T \in K[g(\tilde{u}(t))], \\ \tilde{\gamma}(t-\tau(t)) &= (\tilde{\gamma}_1(t-\tau(t)), \ldots, \tilde{\gamma}_n(t-\tau(t)))^T \\ &\in K[g(\tilde{u}(t-\tau(t)))].\end{aligned}$$

For the drive-response IDNNs (5) and (6), taking account of error signals $e(t) = \tilde{u}(t) - u(t)$ and $\tilde{e}(t) = \tilde{y}(t) - y(t)$, and setting $\beta(t) = \tilde{\gamma}(t) - \gamma(t)$, $\beta(t-\tau(t)) = \tilde{\gamma}(t-\tau(t)) - \gamma(t-\tau(t))$, we have

$$\begin{cases}\frac{de(t)}{dt} = \tilde{e}(t) - Ae(t), \\ \frac{d\tilde{e}(t)}{dt} = -A(A-M)e(t) + (A-M)\tilde{e}(t) - Le(t) \\ \qquad + P\beta(t) + Q\beta(t-\tau(t)) + C(t).\end{cases}$$

Finally, we recall the relevant definitions and lemma.

Definition 1. *Drive IDNN (1) and response IDNN (3) are asymptotically synchronized if*

$$\lim_{t\to+\infty}(\tilde{u}_i(t)) - u_i(t)) = 0,\ i = 1, \ldots, n.$$

Definition 2. *Drive IDNN (1) and response IDNN (3) are exponentially synchronized if there are constants $\theta > 0$ and $\omega > 0$ such that*

$$\begin{aligned}\sum_{i=1}^{n} |\tilde{u}_i(t)) - u_i(t)| \leq \omega e^{-\theta t}(\mid \tilde{u}_i(0)) - u_i(0) \mid \\ + \mid \tilde{y}_i(0)) - y_i(0) \mid),\ i = 1, \ldots, n.\end{aligned}$$

Lemma 1. *[37] (Young's Inequality) For any constants a_1, a_2, w, and r satisfying $a_1 > 0$, $a_2 > 0$, $w > 1$, and $\frac{1}{w} + \frac{1}{r} = 1$,*

$$a_1 a_2 \leq \frac{1}{w}a_1^w + \frac{1}{r}a_2^r$$

holds.

III. Main Results

First, the problem about the asymptotic synchronization is discussed in this section. The condition guaranteeing the asymptotic synchronization is given below:

Theorem 1. *Under Hypotheses (H_1)–(H_3), drive-response IDNNs (1) and (3) with discontinuous activation functions are asymptotically synchronized if parameters a_i, $i = 1, 2, \ldots, n$, are offered below:*

$$a_i = \frac{1 + \sigma_i^2 \sum_{j=1}^{n} \mid p_{ji} \mid + \frac{\sigma_i^2}{1-s} \sum_{j=1}^{n} \mid q_{ji} \mid}{2}, \tag{7}$$

and the controller is designed as

$$\begin{aligned}c_i(t) = &-(\mid 1 - a_i^2 + a_i m_i \mid + l_i) \mid e_i(t) \mid sign(\tilde{e}_i(t)) \\ &-\Big((a_i - m_i) + \frac{\sum_{j=1}^{n} \mid p_{ij} \mid + \sum_{j=1}^{n} \mid q_{ij} \mid}{2}\Big)\tilde{e}_i(t) \\ &-\Big(\sum_{j=1}^{n} \mid p_{ji} \mid + \frac{1}{1-s}\sum_{j=1}^{n} \mid q_{ji} \mid\Big) \\ &(\frac{1}{2\tilde{e}_i(t)}\omega_i^2 + \sigma_i\omega_i\frac{\mid e_i(t) \mid}{\tilde{e}_i(t)}),\ i = 1, 2, \ldots, n.\end{aligned} \tag{8}$$

Proof: To prove the conclusion that we want to obtain, the following LF is given:

$$V_1(t) = e^T(t)e(t) + \tilde{e}^T(t)\tilde{e}(t)$$

$$+\frac{1}{(1-s)}\sum_{i=1}^{n}\sum_{j=1}^{n}\mid q_{ij}\mid\int_{t-\tau(t)}^{t}\beta_j^2(z)dz.$$

Computing the derivative of $V_1(t)$, the authors have

$$\begin{aligned}\frac{dV_1(t)}{dt} =& \sum_{i=1}^{n}\Big(2e_i(t)\dot{e}_i(t)+2\tilde{e}_i(t)\dot{\tilde{e}}_i(t)\Big)\\ &+\frac{1}{1-s}\sum_{i=1}^{n}\sum_{j=1}^{n}\mid q_{ij}\mid(\beta_j(t))^2\\ &-\frac{1(1-\dot{\tau}(t))}{1-s}\sum_{i=1}^{n}\sum_{j=1}^{n}\mid q_{ij}\mid(\beta_j(t-\tau(t)))^2\\ =& \sum_{i=1}^{n}\Big[2e_i(t)(\tilde{e}_i(t)-a_ie_i(t))\\ &+2\tilde{e}_i(t)(-a_i(a_i-m_i)e_i(t)\\ &+(a_i-m_i)\tilde{e}_i(t)-l_ie_i(t)+\sum_{j=1}^{n}p_{ij}\beta_j(t)\\ &+\sum_{j=1}^{n}q_{ij}\beta_j(t-\tau(t))+c_i(t))\Big]\\ &+\frac{1}{1-s}\sum_{i=1}^{n}\sum_{j=1}^{n}\mid q_{ij}\mid(\beta_j(t))^2\\ &-\frac{(1-\dot{\tau}(t))}{1-s}\sum_{i=1}^{n}\sum_{j=1}^{n}\mid q_{ij}\mid(\beta_j(t-\tau(t)))^2.\end{aligned}$$

Applying Hypotheses (H_1)–(H_3), (8), and Lemma 1, we get

$$\begin{aligned}\frac{dV_1(t)}{dt} \le& \sum_{i=1}^{n}\Big[-2a_ie_i^2(t)+2e_i(t)\tilde{e}_i(t)\\ &-2a_i(a_i-m_i)e_i(t)\tilde{e}_i(t)\\ &+2(a_i-m_i)\tilde{e}_i^2(t)-2l_ie_i(t)\tilde{e}_i(t)\\ &+2\sum_{j=1}^{n}p_{ij}\beta_j(t)\tilde{e}_i(t)+2\sum_{j=1}^{n}q_{ij}\beta_j(t-\tau(t))\tilde{e}_i(t))\\ &+2c_i(t)\tilde{e}_i(t)\Big]+\frac{1}{1-s}\sum_{i=1}^{n}\sum_{j=1}^{n}\mid q_{ij}\mid(\beta_j(t))^2\\ &-\sum_{i=1}^{n}\sum_{j=1}^{n}\mid q_{ij}\mid(\beta_j(t-\tau(t)))^2\\ \le& \sum_{i=1}^{n}\Big[-2a_ie_i^2(t)+2(a_i-m_i)\tilde{e}_i^2(t)\\ &+2\mid 1-a_i^2+a_im_i+l_i\mid\mid e_i(t)\mid\mid\tilde{e}_i(t)\mid\\ &+\sum_{j=1}^{n}\mid p_{ij}\mid(\beta_j^2(t)+\tilde{e}_i^2(t))\\ &+\sum_{j=1}^{n}\mid q_{ij}\mid(\beta_j^2(t-\tau(t))+\tilde{e}_i^2(t))\\ &+2c_i(t)\tilde{e}_i(t)\Big]+\frac{1}{1-s}\sum_{i=1}^{n}\sum_{j=1}^{n}\mid q_{ij}\mid(\beta_j(t))^2\\ &-\sum_{i=1}^{n}\sum_{j=1}^{n}\mid q_{ij}\mid(\beta_j(t-\tau(t)))^2\\ \le& \sum_{i=1}^{n}\Big[(\sigma_i^2\sum_{j=1}^{n}\mid p_{ji}\mid+\frac{\sigma_i^2}{1-s}\sum_{j=1}^{n}\mid q_{ji}\mid-2a_i)e_i^2(t)\\ &+2(a_i-m_i+\sum_{j=1}^{n}\mid p_{ij}\mid+\sum_{j=1}^{n}\mid q_{ij}\mid)\tilde{e}_i^2(t)\\ &+2\mid 1-a_i^2+a_im_i+l_i\mid\mid e_i(t)\mid\mid\tilde{e}_i(t)\mid\\ &+\Big(\sum_{j=1}^{n}\mid p_{ji}\mid+\frac{1}{1-s}\sum_{j=1}^{n}\mid q_{ji}\mid\Big)(\omega_i^2\\ &+2\sigma_i\omega_i\mid e_i(t)\mid)+2c_i(t)\tilde{e}_i(t)\Big].\end{aligned}$$

From (7), the following inequality is given:

$$\frac{dV_1(t)}{dt}\le-\sum_{i=1}^{n}e_i^2(t).$$

Hence, $\frac{dV_1}{dt}=0$ is equivalent to $e_i(t)=0$ for $i=1,2,\dots,n$. Utilizing the LIP, the drive and response IDNNs (1) and (3) are asymptotically synchronized. ∎

Remark 1. Compared to the continuous activation functions considered in Ref. [32], the discontinuous activation functions under the present consideration are more realistic.

Sometimes, the convergence speed of the asymptotic synchronization does not satisfy the practical needs. In order to improve the convergence speed, the exponential synchronization of IDNN models (1) and (3) is studied. The condition ensuring such a type of synchronization with exponential convergence speed is offered below:

Theorem 2. *Under Hypotheses (H_1)–(H_3), drive-response IDNNs (1) and (3) with discontinuous activation functions are exponentially synchronized if parameters $a_i, i=1,2,\dots,n$, are offered below:*

$$a_i=\sigma_i\sum_{j=1}^{n}\mid p_{ji}\mid+1, \tag{9}$$

and the controller is designed as

$$\begin{aligned}c_i(t)=&[a_i(a_i-m_i)+l_i]sign(\tilde{e}_i(t))e_i(t)\\ &-[2+(a_i-m_i)]\mid\tilde{e}_i(t)\mid\\ &-\sigma_i\sum_{j=1}^{n}\mid q_{ji}\mid\mid e_i(t-\tau(t))\mid\\ &-\omega_i\Big(\sum_{j=1}^{n}\mid p_{ji}\mid+\sum_{j=1}^{n}\mid q_{ji}\mid\Big),\\ &i=1,2,\dots,n.\end{aligned}\tag{10}$$

Proof: Define a LF as follow:

$$V_2(t)=\sum_{i=1}^{n}e^{\theta t}\Big[\mid\int_{u_i(t)}^{\tilde{u}_i(t)}dt\mid+\mid\int_{y_i(t)}^{\tilde{y}_i(t)}dt\mid\Big],$$

where $0<\theta<1$. Computing the derivative of $V_2(t)$, we have

$$\begin{aligned}&\frac{dV_2(t)}{dt}\\ =&\sum_{i=1}^{n}\Big[e^{\theta t}(sign(e_i(t))\frac{de_i(t)}{dt}+sign(\tilde{e}_i(t))\frac{d\tilde{e}_i(t)}{dt})\\ &+\theta(\mid e_i(t)\mid+\mid\tilde{e}_i(t)\mid)\Big]\\ =&e^{\theta t}\sum_{i=1}^{n}\Big[sign(e_i(t))(\tilde{e}_i(t))-a_ie_i(t))\\ &+sign(\tilde{e}_i(t))(-a_i(a_i-m_i)e_i(t)\end{aligned}$$

$$+(a_i-m_i)\tilde{e}_i(t)-l_ie_i(t)+\sum_{j=1}^{n}p_{ij}\beta_j(t)$$
$$+\sum_{j=1}^{n}q_{ij}\beta_j(t-\tau(t))+c_i(t))$$
$$+\theta(|\ e_i(t)\ |+|\ \tilde{e}_i(t)\ |)\Big].$$

According to Hypotheses (H_1)–(H_3), (9), and (10), the authors get

$$\begin{aligned}\frac{dV_2(t)}{dt} \le\ & e^{\theta t}\Big[(-a_i+\sigma_i\sum_{j=1}^{n}|\ p_{ji}\ |+\theta)\ |\ e_i(t)\ | \\ & -[a_i(a_i-m_i)+l_i]sign(\tilde{e}_i(t))e_i(t) \\ & +[1+\theta+(a_i-m_i)]\ |\ \tilde{e}_i(t)\ | \\ & +\sigma_i\sum_{j=1}^{n}|\ q_{ji}\ ||\ e_i(t-\tau(t))\ | \\ & +\omega_i(\sum_{j=1}^{n}|\ p_{ji}\ |+\sum_{j=1}^{n}|\ q_{ji}\ |)+c_i\Big] \\ \le\ & e^{\theta t}\Big[(-a_i+\sigma_i\sum_{j=1}^{n}|\ p_{ji}\ |+\theta)\ |\ e_i(t)\ | \\ & -[a_i(a_i-m_i)+l_i]sign(\tilde{e}_i(t))e_i(t) \\ & +[2+(a_i-m_i)]\ |\ \tilde{e}_i(t)\ | \\ & +\sigma_i\sum_{j=1}^{n}|\ q_{ji}\ ||\ e_i(t-\tau(t))\ | \\ & +\omega_i(\sum_{j=1}^{n}|\ p_{ji}\ |+\sum_{j=1}^{n}|\ q_{ji}\ |)+c_i\Big] \\ =\ & -(1-\theta)e^{\theta t}\ |\ e_i(t)\ |\le 0.\end{aligned}$$

Hence,

$$\begin{aligned}V_2(t)\le V_2(0)=\ & \sum_{i=1}^{n}(|\ \tilde{u}_i(0))-u_i(0)\ | \\ & +|\ \tilde{y}_i(0)-y_i(0)\ |).\end{aligned} \tag{11}$$

Meanwhile,

$$\begin{aligned}V_2(t)\ =\ & e^{\theta t}\sum_{i=1}^{n}\Big(\ |\ \tilde{u}_i(t))-u_i(t)\ |+|\ \tilde{y}_i(t)-y_i(t)\ |\ \Big) \\ \ge\ & e^{\theta t}\sum_{i=1}^{n}|\ \tilde{u}_i(t))-u_i(t)\ |\ .\end{aligned} \tag{12}$$

From (11) and (12), we can conclude:

$$\begin{aligned}& \sum_{i=1}^{n}|\ \tilde{u}_i(t))-u_i(t)\ | \\ \le\ & e^{-\theta t}(|\ \tilde{u}_i(0))-u_i(0)\ |+|\ \tilde{y}_i(0))-y_i(0)\ |).\end{aligned}$$

Thus, the drive IDNN (1) and response IDNN (3) are exponentially synchronized. ■

Remark 2. In contrast to the condition outlined in Theorem 1, the criterion presented in Theorem 2 guarantees the exponential synchronization of IDNNs (1) and (3), potentially leading to a faster convergence.

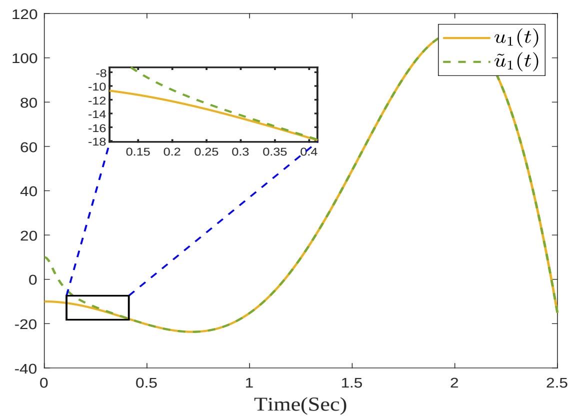

Fig. 1. Transient behavior of $u_1(t)$ and $\tilde{u}_1(t)$.

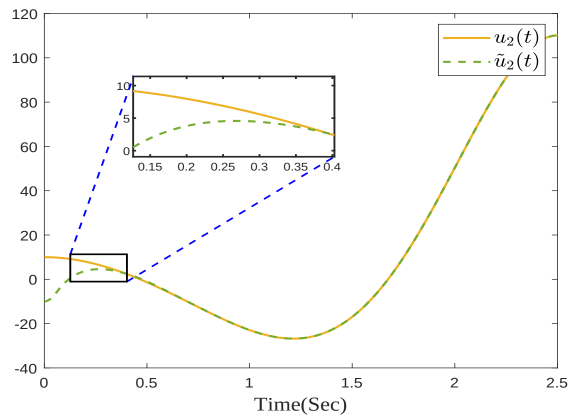

Fig. 2. Transient behavior of $u_2(t)$ and $\tilde{u}_2(t)$.

IV. Numerical Example

The section gives concrete examples to validate the proposed results.

Example 1. *Take into account IDNN (1) with the following parameters:*

$$n=2,\ h_i(u)=2u+sign(u),\ i=1,\ 2,\ \tau(t)=0.2\text{sin}(t),$$
$$M=\begin{bmatrix}0.6 & 0\\ 0 & 1.7\end{bmatrix},\ L=\begin{bmatrix}0.6 & 0\\ 0 & 1.5\end{bmatrix},\ P=\begin{bmatrix}-0.4 & -7\\ 2 & -1.5\end{bmatrix},$$
$$Q=\begin{bmatrix}-0.3 & -0.7\\ -0.8 & -1.2\end{bmatrix},\ J=\begin{bmatrix}0.2\\ 0.1\end{bmatrix}.$$

Based on these parameters, it is determined that $s=0.2$, and Hypotheses $(H1)$–$(H3)$ hold true with $\sigma_i=2$ and $\omega_i=0$, $i=1,2$. Set $a_1=8.05$ and $a_2=9.65$. Then, it can be found that (7) is satisfied. According to Theorem 1, IDNNs (1) and (3) under controller (8) are asymptotically synchronized.

Example 2. *Take into account IDNN (1) with*

$$n=2,\ h_i(u)=4u+sign(u),\ i=1,\ 2,\ \tau(t)=0.6\text{sin}(t),$$
$$M=\begin{bmatrix}0.7 & 0\\ 0 & 1.5\end{bmatrix},\ L=\begin{bmatrix}0.8 & 0\\ 0 & 1.3\end{bmatrix},\ P=\begin{bmatrix}-0.2 & -6\\ 3 & -1.3\end{bmatrix},$$
$$Q=\begin{bmatrix}-0.1 & -0.5\\ -0.6 & -1\end{bmatrix},\ J=\begin{bmatrix}0.4\\ 0.5\end{bmatrix}.$$

Based on these parameters, it is determined that $s=0.6$, and Hypotheses $(H1)$–$(H3)$ hold true with $\sigma_i=4$ and $\omega_i=0$,

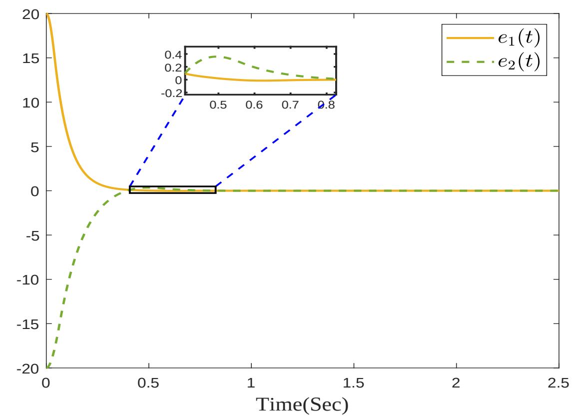

Fig. 3. Transient behavior of $e_1(t) = \tilde{u}_1(t) - u_1(t), e_2(t) = \tilde{u}_2(t) - u_2(t)$.

$i = 1, 2$. Set $a_1 = 13.8$ and $a_2 = 8.6$. Then, it can be found that (9) is satisfied. According to Theorem 1, drive and response IDNNs (1) and (3) under controller (10) are exponentially synchronized.

Fig. 1 and Fig. 2 describe the simulation results of the transient behavior for state variables u_1 and $\tilde{u}_1$, as well as u_2 and $\tilde{u}_2$. Fig. 3 further depicts the transient behavior of the synchronization error, offering validation to the theoretical analysis presented in Section III.

V. CONCLUSIONS

This study explores the asymptotic and exponential synchronization issues for IDNNs containing discontinuous activation functions. The IDNN models, represented as second-order equations, are converted into first-order differential equations by variable substitution. To derive an asymptotic synchronization condition for the drive-response IDNNs, an appropriate LF and the LIP from functional differential equations are employed. Subsequently, an exponential synchronization criterion for the drive-response IDNNs is established by applying another LF. In the end, two illustrative examples are offered to affirm the effectiveness and feasibility of the results achieved. Recently, networked control, recognized as an emerging research area, has gained growing attention within the automation community [38–42]. Our future research endeavors will focus on investigating the synchronization of IDNNs under networked control frameworks.

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