CNF-Base Hypergraphs: (Dense) Maximal Non-Diagonality and Combinatorial Designs

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Abstract—**The class of maximal non-diagonal CNF-base hypergraphs resides below the hierarchy of diagonal base hypergraphs. It is extreme in the respect that its members are only one hyperedge away from diagonality. Here we prove a general criterion for maximal non-diagonality, provide connections to minimal diagonality, and elaborate the relationship to maximal satisfiable CNF formulas. Further, the stronger notion of dense maximal-non diagonality is studied and several concrete classes of BHGs of that property are constructed. We also provide non-diagonal base hypergraphs based on finite projective planes that become minimal diagonal and even non-simple via a specific retraction operation. Finally, a non-commutative joining operation for base hypergraphs is introduced and investigated. On that basis maximal non-diagonal base hypergraphs of arbitrary size can be constructed that especially are uniform.**

Index Terms—**hypergraph, CNF-satisfiability, orbit, transversal, finite-projective-plane**

I. Introduction

THE genuine and one of the most important NP-complete problems is the propositional satisfiability problem (SAT) for conjunctive normal form (CNF) formulas [6]. More precisely, SAT is the natural NP-complete problem and thus lies at the heart of computational complexity theory. Moreover, numerous computational problems can be encoded as equivalent instances of CNF-SAT via reduction [7]. From a theoretical point of view on the one hand subclasses are to be detected for which SAT can be decided efficiently. There are known several of them such as quadratic formulas, (extended and q-)Horn formulas, matching formulas, nested, co-nested formulas, and exact linear formulas etc. [2], [4], [5], [8], [10], [11], [12], [19], [22].

On the other hand it might be purposeful to reveal the structural aspects of CNF-SAT from diverse perspectives in order to attack the complexity issues among others. So here the focus lies on the concept of the (CNF-)base hypergraph which can be viewed as the projection of a collection of CNF formulas of a specific structure. A hierarchy of diagonal base hypergraphs has been proposed in [16], such that its ith level collects all instances with exactly i members in the orbit space of the diagonal, meaning unsatisfiable, fibre-transversals with respect to the action of the complementation group on clauses.

In the present paper the class of maximal non-diagonal base hypergraphs as introduced in [20] is studied further. Such instances are extreme among all members below the first level of the mentioned hierarchy: By definition they are only one hyperedge away from diagonality. Using variants of base hypergraphs defined via specific combinatorial designs it is shown in particular that there exist arbitrary large instances that are non-diagonal but not maximal non-diagonal, so the concept is far from trivial. The connection to minimal diagonal base hypergraphs is considered which are diagonal but none of their subhypergraphs have this property. Further, it is shown that not every maximal non-diagonal base hypergraph is derived from a minimal diagonal one. A general equivalent criterion is proven for maximal non-diagonality based on the concept of a minimal transversal meeting all minimal diagonal subhypergraphs of a given diagonal base hypergraph. Also the relationship to the concept of maximal satisfiable formulas as defined in [14] is exhibited.

Moreover a stronger notion, that of *dense* maximal non-diagonality is considered. Instances of that property become diagonal when adding *an arbitrary* further hyperedge of their smallest complete superhypergraph. Constructive existence results for dense maximal non-diagonal instances of arbitrary size are estabished, also showing that for every complete base hypergraph there are dense maximal non-diagonal subhypergraphs.

A retraction operation that was introduced for CNF formulas in [13] is transfered to base hypergraphs. Applying it to certain finite projective planes even yields minimal diagonal base hypergraphs that are non-simple, i.e., have more than one diagonal orbits. Those instances then are exploited for the construction of arbitrary large maximal non-diagonal base hypergraphs via iterating an enlargement step called *lifting*. This procedure also results in base hypergraphs of arbitrary size becoming members of the ith level of the diagonal hierarchy when adding a single edge, for certain integers $i > 1$.

Finally we define a non-commutative join-operation on base hypergraphs. Its structural properties are illuminated so that maximal non-diagonal base hypergraphs of arbitrary size can be constructed which are k-uniform, for every integer $k \geq 2$, so they are loopless and also Sperner. Thereby one even is lead to certain members of higher levels of the diagonal hierarchy.

Several concluding remarks reveal directions for future work on this topic.

II. Notation and Preliminaries

A Boolean or propositional variable, for short variable, x taking values from $\{0, 1\}$ can appear as a positive literal which is x or as a negative literal which is the negated variable $\overline{x}$. The negation of a literal is also termed as its *complement*, so complementation always means the negation of the underlying variable. Setting a literal to 1 means to set the corresponding variable accordingly. A clause c is a finite non-empty disjunction of literals over mutually distinct variables which is represented as a set $c = \{l_1, \ldots, l_k\}$, or for simplyfying the notation, as a no-order-imposing *sequence* of its literals: $c = l_1 \cdots l_k$. Occasionally $l(x) \in c$ is used to denote the literal over x in c.

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A conjunctive normal form formula, for short formula, C is a finite conjunction of different clauses and is considered as a set of these clauses $C = \{c_1, \ldots, c_m\}$. A formula has a *pure* literal if there is a variable occurring as the same literal in each clause. Let CNF be the collection of all formulas.

For a formula C (clause c), by $V(C)$ ($V(c)$) denote the set of variables occurring in C (c). Note that $|V(c)| = |c|$. As usual $|C|$ is the size, i.e., the cardinality and $\|C\| = \sum_{c \in C} |c|$ is the length of C.

Given $C \in \mathrm{CNF}$, SAT means to decide whether there is a (truth value) assignment $w : V(C) \to \{0,1\}$ such that (s.t.) there is no $c \in C$ all literals of which are set to 0. In that case w is a model of C, and $\mathcal{M}(C)$ is the space of all models of C. Let $\mathrm{SAT} \subseteq \mathrm{CNF}$ denote the collection of all formulas for which there is a model, and $\mathrm{UNSAT} := \mathrm{CNF} \setminus \mathrm{SAT}$.

Let V be a set of propositional variables, an assignment w can be regarded as the clause $\{x^{w(x)} : x \in V\}$ of length $|V|$, where $x^0 := \bar{x}$, $x^1 := x$. Similarly, for $b \subseteq V$, we identify the restriction $w|b =: w(b)$ with the clause $\{x^{w(x)} : x \in b\}$. The collection of all clauses over V of length $|V|$ is denoted as W_V which therefore also can be regarded as the set of all mappings $V \to \{0,1\}$. For a clause c we denote by c^γ the clause in which all its literals are complemented. In case of an assignment $w \in W_V$, we have the correspondence of w^γ to the assignment $1 - w : V \to \{0,1\}$ complementing all truth values. Similarly, let $C^\gamma = \{c^\gamma : c \in C\}$.

For $C \in \mathrm{CNF}$, and $\emptyset \neq U \subseteq V(C)$ the subformula $C(U) \subseteq C$ consists of all clauses possessing a literal over a variable in U. For short we identify $C(\{x\})$ with $C(x)$ whenever $x \in V(C)$, for which $o_C(x) := |C(x)|$ denotes its *occurrence number* in C. Restricting every $c \in C(U)$ to the literals over U, denoted as $c[U]$, yields the (U-)retraction $C[U]$ of C [13]. Observe that the satisfiability of $C[U]$ implies that of $C(U)$.

As introduced in [13] a formula C determines its base hypergraph (BHG) $\mathcal{H}(C) = (V(C), B(C))$ where $B(C) = \{V(c) : c \in C\}$. Let $C_b = \{c \in C : V(c) = b\} = C \cap W_b$ denote the fibre of C over b, thus $C = \bigcup_{b \in B(C)} C_b$. Also a given hypergraph $\mathcal{H} = (V, B)$ yields a BHG when regarding its vertices as Boolean variables s.t. for each $x \in V$ there is a (hyper)edge $b \in B$ containing x. Every $w \in W_V$ induces a clause set over B, namely $w(B) := \{w(b) : b \in B\}$. The size of $\mathcal{H}$ is $|\mathcal{H}| := |B|$ and $\|\mathcal{H}\| := \sum_{b \in B} |b|$ is its length. Occasionally the vertex set, edge set of an unspecified BHG $\mathcal{H}$ is refered to as $V(\mathcal{H}), B(\mathcal{H})$.

Given $\emptyset \neq U \subseteq V$ s.t. $b \cap U$, for $b \in B$ with $b \cap U \neq \emptyset$, all are mutually distinct, also the (U-)retraction $\mathcal{H}[U] := (U, B[U])$ of $\mathcal{H}$ can be defined, where as above $B[U] = \{b \cap U : b \in B, b \cap U \neq \emptyset\}$.

Let $\mathfrak{H}$ be the collection of all (CNF-)BHGs, and let $\mathfrak{H}^{\mathrm{con}}$ be the subclass of all connected instances. $\mathcal{H} = (V, B)$ is connected iff its bipartite incidence graph $I_\mathcal{H}$ is connected in the usual meaning. Here and henceforth 'iff' means if and only if. Recall that $I_\mathcal{H}$ has the vertex set decomposition $V \cup B$, and $v \in V$ is joined by an edge to $b \in B$ iff $v \in b$.

A BHG is linear if distinct hyperedges pairwise intersect in at most one vertex; if each of these intersections has size 1 the BHG even is exact linear. We use $\mathfrak{H}_{\mathrm{lin}}$, resp., $\mathfrak{H}_{\mathrm{xlin}}$ to denote the collections of all linear, resp., exact linear BHGs. A hypergraph is loopless if none of its hyperedges has size 1. A hypergraph is Sperner if no hyperedge is a proper subset of another one [3]. Adding a further edge $b \notin B(\mathcal{H})$ to the edge set of $\mathcal{H}$ sometimes is abbreviated by $\mathcal{H} \cup \{b\}$. In case $b = \{x\}$ is a loop, for simplicity we shall write $\mathcal{H} \cup \{x\}$ instead of $\mathcal{H} \cup \{\{x\}\}$.

A formula C s.t. $|C_b| = 1$, for all $b \in B(C)$, is (exact) linear if $\mathcal{H}(C)$ is (exact) linear [19]. Observe that a linear formula cannot contain a pair of complementary unit clauses.

As usual $K_\mathcal{H} := \bigcup_{b \in B(\mathcal{H})} W_b$ is the set of all clauses over $\mathcal{H}$. A $\mathcal{H}$-based formula is $C \subseteq K_\mathcal{H}$ s.t. $C_b \neq \emptyset$, for each $b \in B(\mathcal{H})$. Given a $\mathcal{H}$-based formula C with the additional property that $\bar{C}_b := W_b \setminus C_b \neq \emptyset$ holds, for each $b \in B(\mathcal{H})$, then its $\mathcal{H}$-based complement formula is $\bar{C} := K_\mathcal{H} \setminus C$.

A fibre-transversal, for short transversal, of $K_\mathcal{H}$ is a $\mathcal{H}$-based formula F with $|F_b| = 1$, for all $b \in B(\mathcal{H})$. Let its unique clause over b be refered to as F_b, so for simplicity the fibre may be identified with the clause it contains. Observe that, in general, $F(b) \neq F_b$. The set of all transversals of $K_\mathcal{H}$ is denoted as $\mathcal{F}(\mathcal{H})$.

An important type of transversals F over $\mathcal{H} = (V, B)$ are the compatible ones having the property $\bigcup_{b \in B} F_b \in W_V$, collected in $\mathcal{F}_{\mathrm{comp}}(\mathcal{H}) \subseteq \mathrm{SAT}$. Whereas a transversal F is diagonal if $F \cap F' \neq \emptyset$, for all $F' \in \mathcal{F}_{\mathrm{comp}}(\mathcal{H})$. Let $\mathcal{F}_{\mathrm{diag}}(\mathcal{H})$ be the subspace of all diagonal transversals of $K_\mathcal{H}$. Note that exactly the members of $\mathcal{F}_{\mathrm{diag}}(\mathcal{H})$ are unsatisfiable transversals. A BHG $\mathcal{H}$ is diagonal if $\mathcal{F}_{\mathrm{diag}}(\mathcal{H}) \neq \emptyset$, and it is minimal diagonal if no subhypergraph of $\mathcal{H}$ is diagonal. Let $\mathfrak{H}_{\mathrm{diag}}$ be the class of all diagonal BHGs, and $\mathfrak{H}_{\mathrm{mdiag}}$ denote the subcollection of all minimal diagonal instances.

The group G_V of variable complementation on clauses over V induces a corresponding action on the space of all CNF formulas over V [17]. The number of the orbits in the quotient space $\mathcal{F}_{\mathrm{diag}}(\mathcal{H})/G_V$ with respect to this action is denoted as $\delta(\mathcal{H})$ [15]; for short the term orbit is used in the sequel. It is $\delta = 0$ for all non-diagonal instances, collected in $\mathfrak{H}_0$. A BHG with $\delta = 1$ is called simple; all simple BHGs are collected in $\mathfrak{H}_{\mathrm{simp}}$. As defined in [16] let $\mathfrak{H}_i$ denote the set of all BHGs with $\delta \leq i$, and $\hat{\mathfrak{H}}_i \subsetneq \mathfrak{H}_{\mathrm{diag}}$ denote the set of those with $\delta = i$. So especially $\mathfrak{H}_{\mathrm{simp}} = \hat{\mathfrak{H}}_1$. We shall also make use of the further parameters of a BHG defined in [15]: $\beta(\mathcal{H}) = \|\mathcal{H}\| - |V|$, $\omega(\mathcal{H}) = 2^{\beta(\mathcal{H})}$ which is the number of all G_V-orbits of transversals over $\mathcal{H}$: $|\mathcal{F}(\mathcal{H})/G_V|$. As well as $\rho(\mathcal{H})$ [16] which is the fraction of all satisfiable but not compatible G_V-orbits in $\mathcal{F}(\mathcal{H})/G_V$, so $\omega(\mathcal{H}) = 1 + \delta(\mathcal{H}) + \rho(\mathcal{H})$.

We use $[n] = \{1, \ldots, n\}$, where n is a positive integer, and $\mathbb{N}$ for the set of all positive integers. For two finite sets A, B of equal cardinality let $\mathrm{Bij}(A, B)$ denote the collection of all bijections $A \to B$. As usual $S_n := \mathrm{Bij}([n], [n])$ denotes the symmetric group of all bijections on $[n]$, for $n \in \mathbb{N}$. If A, B are structured spaces, an isomorphism between both induces a bijection but not vice versa, in general. Especially the BHGs $\mathcal{H}_j = (V_j, B_j)$, $j \in [2]$, are isomorphic if there is $\sigma \in \mathrm{Bij}(V_1, V_2)$ s.t. $b \in B_1$ iff $\{\sigma(x) : x \in b\} \in B_2$.

Next we collect several useful properties of minimal unsatisfiable formulas. To that end the following result proven in [13] is needed, which characterizes the satisfiability of a formula C in terms of the compatible transversals in its based complement formula $\bar{C}$. Here a transversal of a $\mathcal{H}$-based formula $C \subset K_\mathcal{H}$ is a transversal of $K_\mathcal{H}$ that is contained in C.

Theorem 1: [13] *For $\mathcal{H} = (V, B)$, let $C \subset K_\mathcal{H}$ be a $\mathcal{H}$-*

based formula s.t. $\bar{C}$ is $\mathcal{H}$-based, too. Then C is satisfiable iff $\bar{C}$ admits a compatible transversal F. Moreover, the union of all clauses in F^γ is a model of C.

Recall that $C \in \mathrm{UNSAT}$ is minimal unsatisfiable if $C \setminus \{c\}$ is satisfiable, for every $c \in C$ [1]; then evidently $V(C) = V(C \setminus \{c\})$. We denote the class of exactly those instances by $\mathcal{I} \subset \mathrm{UNSAT}$.

Lemma 1: Let $C \in \mathcal{I}$ with $\mathcal{H}(C) =: \mathcal{H} = (V, B)$:

(i) *For every $w \in W_V$ one has $w(B) \cap C \neq \emptyset$.*
(ii) *There is $w \in W_V$, s.t. $|w(B) \cap C| = 1$.*
(iii) *For every $b \in B$ there is $w \in W_V$, s.t. $w(B) \cap C = \{w(b)\}$.*
(iv) *If $|B| > 1$ then, in general, there are $w \in W_V$ s.t. $|w(B) \cap C| > 1$, and $c \in C$ with $|\mathcal{M}(C \setminus \{c\})| > 1$.*
(v) *Let $w \in W_V$. Then $|w(B) \cap C| = 1$, and there is $b \in B$ s.t. $w(B) \cap C = \{w(b)\}$ iff $w^\gamma \in \mathcal{M}(C')$ where $C' := C \setminus \{w(b)\}$.*

PROOF. If $|B| = 1$ then $C = W_b = W_V$ is the only possible member of $\mathcal{I}$, for the unique $b = V \in B$. All assertions except for (iv) are evidently true. Now assume $|B| > 1$ then $\mathcal{H}(C) = \mathcal{H}(\bar{C})$ because $C \in \mathcal{I}$. As every $w \in W_V$ can be identified with a compatible transversal of $K_\mathcal{H}$, (i) is a direct consequence of Thm. 1, because $C \in \mathrm{UNSAT}$. Since (ii) is implied by (iii) let $b \in B$ and $c \in C_b$ be arbitrary. Then $C' := C \setminus \{c\} \in \mathrm{SAT}$ has a model $w' \in W_V$ s.t. $w'(b) = c^\gamma$, recall that $V = V(C) = V(C')$. Hence $w'^\gamma(b) = c \in C \cap \bar{C}'$ which clearly is the unique clause of C of that property. Moreover by Thm. 1 $w'^\gamma(B)$ can be identified with a compatible transversal of $\bar{C}'$ in case that $|C_b| > 1$ because then $\mathcal{H}(\bar{C}') = \mathcal{H}(C')$, so $w'^\gamma(B) \cap C = \{w'^\gamma(b)\}$. Finally, if $|C_b| = 1$ then $W_b \subset \bar{C}$ and $w'^\gamma(B \setminus \{b\})$ is a compatible transversal of $\bar{C}' \setminus W_b$ because $\mathcal{H}(\bar{C}' \setminus W_b) = \mathcal{H}(C')$, so (iii) and (ii) are verified.

Evidently $C = \{xy_1, xy_2, \bar{x}y_3, \bar{x}y_4, \bar{y}_1\bar{y}_2, \bar{y}_3\bar{y}_4\}$ belongs to $\mathcal{I}$. Set $w_0(B) := \{xy_1, xy_2, x\bar{y}_3, x\bar{y}_4, y_1y_2, \bar{y}_3\bar{y}_4\}$ then $|w_0(B) \cap C| = 3$, where $B := B(C)$. Let $c = \bar{y}_3\bar{y}_4$ and $w_1(B) := \{xy_1, x\bar{y}_2, xy_3, xy_4, y_1\bar{y}_2, y_3y_4\}$, $w_2(B) := \{x\bar{y}_1, xy_2, xy_3, xy_4, \bar{y}_1y_2, y_3y_4\}$ then $w_j \in \mathcal{M}(C \setminus \{c\})$, $j \in [2]$, implying (iv).

Let $w \in W_V$, $b \in B$ and set $C' := C \setminus \{w(b)\} \in \mathrm{SAT}$ because $C \in \mathcal{I}$. First assume $|C_b| > 1$ then $\mathcal{H}(C) = \mathcal{H}(C')$. Let $w(B) \cap C = \{w(b)\}$ then $w(B) \in \mathcal{F}_{\mathrm{comp}}(\bar{C}')$ because $\bar{C}' = \bar{C} \cup \{w(b)\}$ implying $w^\gamma \in \mathcal{M}(C')$ by Thm. 1. Conversely let $w^\gamma \in \mathcal{M}(C')$ then by the same theorem $w \in \mathcal{F}_{\mathrm{comp}}(\bar{C}')$. So $w(B) \cap C = \{w(b)\}$ because otherwise $w^\gamma \in \mathcal{M}(C)$ yielding a contradiction. Finally let $|C_b| = 1$ and $w(B) \cap C = \{w(b)\}$ then $w(B \setminus \{b\}) \in \mathcal{F}_{\mathrm{comp}}(\bar{C}' \setminus W_b)$ because $\mathcal{H}(\bar{C}' \setminus W_b) = \mathcal{H}(C')$, so $w^\gamma \in \mathcal{M}(C')$ by Thm. 1 recalling that $V(C') = V$. Conversely let $w^\gamma \in \mathcal{M}(C')$ then by the same theorem $w \in \mathcal{F}_{\mathrm{comp}}(\bar{C}' \setminus W_b)$. So $w(B) \cap C = \{w(b)\}$ because otherwise $w^\gamma \in \mathcal{M}(C)$ yielding a contradiction, finishing the argumentation. □

The next notion shall become purposeful below.

Definition 1: $\mathcal{H}$ is called a unique-vertex *BHG if every $b \in B(\mathcal{H})$ contains an* unique *vertex, hence not contained in another hyperedge. Moreover an* all-unique-vertex *BHG in addition has the property that all its edges are mutually vertex-disjoint.*

Proposition 1: For $\mathcal{H} = (V, B)$ denoting an appropriate unique-vertex *BHG one has:*

(1) *For each $m \in \mathbb{N}$ there is $\mathcal{H}$ s.t. $|B| = m$, and $\beta(\mathcal{H}) = (m-1)n$, for every $n \in \mathbb{N}$.*
(2) *An arbitrary $\mathcal{H}'$ can be modified to $\mathcal{H}$ s.t. $\beta(\mathcal{H}') = \beta(\mathcal{H})$.*
(3) *For $C \subset K_\mathcal{H}$ one has $C \in \mathrm{SAT}$ if every unique vertex of $\mathcal{H}$ becomes a pure literal in C. The latter especially is true if $C \in \mathcal{F}(\mathcal{H})$.*
(4) *One has $\mathcal{H} \in \mathfrak{H}_0$. If $\mathcal{H}$ in addition is an all-unique-vertex BHG then it is trivial $\beta(\mathcal{H}) = 0$, and vice versa.*

PROOF. For (1) let $U := \{u_j : j \in [m]\}$ and V_0 be an arbitrary vertex set of size n s.t. $U \cap V_0 = \emptyset$. Setting $b_j := u_j \cup V_0$, $j \in [m]$, $V = U \cup V_0$ and $B := \{b_j : j \in [m]\}$ provides $\mathcal{H} = (V, B)$ with the required properties. Assertions (2), (3), (4) are obvious. □

Let us mention a sometimes useful sufficient but, in general, not necessary criterion for the non-diagonality of a BHG $\mathcal{H}'$ relying on a BGH $\mathcal{H}$ that neither is a subhypergraph nor, in general, a retraction of $\mathcal{H}'$.

Lemma 2: Let $\mathcal{H} = (V, B), \mathcal{H}' = (V', B')$ with $V \subseteq V'$, $|B| = |B'|$ and $\varphi \in \mathrm{Bij}(B, B')$ s.t. $b \subseteq \varphi(b)$, for all $b \in B$. Then $\mathcal{H} \in \mathfrak{H}_0$ implies $\mathcal{H}' \in \mathfrak{H}_0$.

PROOF. Let $\mathcal{H} \in \mathfrak{H}_0$ and suppose there is $F' \in \mathcal{F}_{\mathrm{diag}}(\mathcal{H}')$. For every $c' \in F'$ let $c := c'[\varphi^{-1}(V(c'))] \in K_\mathcal{H}$ then the collection F of the resulting clauses is a member of $\mathcal{F}(\mathcal{H})$ because $\varphi^{-1}(V(c')) \in B$. By assumption $F \in \mathrm{SAT}$ has a model $w \in W_V$. Evidently with $c \subseteq c'$ also c' is solved by w, for all $c' \in F'$ yielding a contradiction. □

III. Uniform-, Regular-, or Finite-Projective-Plane-BHGs and Non-Diagonality by Retraction

As indicated in the introduction, BHGs parameterized analogous to combinatorial designs might be helpful to answer existence questions in our context. In this section we provide several classes of minimal diagonal BHGs on that basis. To that end, $\mathcal{H}$ is called *regular* or more precisely r-regular, if there is $r \in \mathbb{N}$ s.t. every vertex is contained in exactly r edges. Then $\mathcal{H}$ has the degree $\deg(\mathcal{H}) = r$. Similarly, if every edge has a fixed size k, then $\mathcal{H}$ is k-*uniform*, $k \in \mathbb{N}$. In particular, for $k = 2$ a BHG becomes a simple, undirected graph: each edge has size 2. We shall call a 2-uniform BHG that is isomorphic to a cycle a *cycle*-BHG, which therefore also is 2-regular.

Lemma 3: Let $C \in \mathrm{CNF}$ with regular BHG $\mathcal{H}(C)$. Then $o_C(x) = \deg(\mathcal{H}(C))$, for all $x \in V(C)$ iff C is a transversal over $\mathcal{H}(C)$.

PROOF. Let $\mathcal{H} := \mathcal{H}(C) = (V, B)$ then, for $x \in V$, $C(x) = \{c \in C : x \in V(c)\} = \bigcup_{b \in B: x \in b} C_b$ as disjoint union. Thus $o_C(x) = |C(x)| \geq |\{b \in B : x \in b\}| = \deg(\mathcal{H}(C))$, and equality holds true iff there is no $b \in B$ s.t. $|C_b| > 1$ hence iff C is a transversal. □

A special case occurs if $\mathcal{H}$ is r-regular and k-uniform, and even more specific, if both parameters are equal: $r = k$. In the last case and if in addition $\mathcal{H} = (V, B) \in \mathfrak{H}_{\mathrm{xlin}} \subset \mathfrak{H}_0$, then one has $|V| = m(k) = |B|$, where $m(k) := 1 + k(k-1)$ [19]. Moreover $\mathcal{H}$ then is isomorphic to a finite projective plane of order $o = k - 1$, denoted as $\mathrm{FPP}(k-1)$, for $k \in \mathbb{N}$, $k \geq 2$, [19]. Thereby the lines of the plane are identified with the hyperedges and the points of the plane with the vertices of $\mathcal{H}$ which in turn are viewed as Boolean variables. The base case here is given by $\mathrm{FPP}(1)$ which is isomorphic to a cycle-BHG of 3 vertices, and 3 edges. It is well known (cf. e.g. [21], [23]) that a finite projective plane of order o at least exists if o is the cardinality of a finite field, hence is a prime power. Recall that the case $o = 2$ is named as the Fano plane. In terms of combinatorial designs, $\mathrm{FPP}(o)$ is a Steiner system, because every pair of points, i.e., vertices occurs in exactly one block, resp. line, resp. hyperedge. That crucial property throughout refered to as the *unique-pair-rule* turns out to be useful.

In the sequel we shall call $\mathcal{H}$ a $\mathrm{FPP}(o)$*-BHG* if each of its connected components can be identified with $\mathrm{FPP}(o)$, where those are defined over mutually disjoint point sets. Then $\mathcal{H}$ is disconnected if it has more than one component. Similarly $\mathcal{H}(x)$ is called a x*-connected* $\mathrm{FPP}(o)$*-BHG* if it is derived from a $\mathrm{FPP}(o)$-BHG by replacing exactly one (arbitrary) vertex of each component by the new vertex x, refered to as the *connecting vertex*, yielding a connected BHG. Observe that also every x-connected $\mathrm{FPP}(k-1)$-BHG obeys the unique-pair-rule, and that every vertex has the degree k, except for x. The degree of x is $k \cdot s$, if s is the number of the $\mathrm{FPP}(k-1)$ instances involved.

Example 1: Below left: hyperedges of a FPP(2)-BHG $\mathcal{H}$ of two (disjoint) components. Below right: x-connected FPP(2)-BHG $\mathcal{H}(x)$ derived form $\mathcal{H}$ by replacing x_1 resp. x_2 by the connecting vertex x of degree $3 \cdot 2$.

$$\mathcal{H}: \begin{array}{l} \{x_1 y_1 z_1, \\ x_1 u_1 v_1, \\ x_1 p_1 q_1, \\ y_1 u_1 p_1, \\ y_1 v_1 q_1, \\ z_1 u_1 q_1, \\ z_1 v_1 p_1\} \end{array} \cup \begin{array}{l} \{x_2 y_2 z_2 \\ x_2 u_2 v_2, \\ x_2 p_2 q_2, \\ y_2 u_2 p_2, \\ y_2 v_2 q_2, \\ z_2 u_2 q_2, \\ z_2 v_2 p_2\} \end{array} \qquad \mathcal{H}(x): \begin{array}{l} \{x y_1 z_1, \\ x u_1 v_1, \\ x p_1 q_1, \\ y_1 u_1 p_1, \\ y_1 v_1 q_1, \\ z_1 u_1 q_1, \\ z_1 v_1 p_1\} \end{array} \cup \begin{array}{l} \{x y_2 z_2, \\ x u_2 v_2, \\ x p_2 q_2, \\ y_2 u_2 p_2, \\ y_2 v_2 q_2, \\ z_2 u_2 q_2, \\ z_2 v_2 p_2\} \end{array}$$

Next 2-uniform and small minimal diagonal BHGs are defined by members of the classes above that shall become useful. Result (ii) below is remarkable in the sense that it provides an even non-simple BHG. Recall that a minimal diagonal BHG does not contain any proper sub-BHG which is diagonal. Thm. 6 in [16] states that $\mathcal{H}$ is minimal diagonal iff $\mathcal{F}_{\mathrm{diag}}(\mathcal{H}) \subseteq \mathcal{I}$.

Lemma 4: A BHG isomorphic to one of the following structures is loopless, uniform, Sperner and minimal diagonal:

(i) *two x-connected* $\mathrm{FPP}(1)$*-components being simple,*
(ii) *the retraction $\mathcal{H}[V \setminus b]$, where $\mathcal{H} = (V, B)$ is a one-component* $\mathrm{FPP}(2)$*-BHG, and $b \in B$ is arbitrary; moreover then $\delta(\mathcal{H}[V \setminus b]) = 3$.*

PROOF. The looplessness and Spernerian property in both assertions directly are implied by the k-uniformity, for $k \in \{2, 3\}$. Let $\mathcal{H} = (V, B)$ be the x-connected union of two $\mathrm{FPP}(1)$, namely $\mathcal{H}_j = (V_j, B_j)$, $j \in [2]$, with $V_1 = \{x\} \cup \{y_1, y_2\}$, $V_2 = \{x\} \cup \{y_3, y_4\}$, and $B_1 = \{xy_1, xy_2, y_1y_2\}$, $B_2 = \{xy_3, xy_4, y_3y_4\}$. Hence $V = V_1 \cup V_2$, $B = B_1 \cup B_2$. The transversal $\{xy_1, xy_2, \bar{x}y_3, \bar{x}y_4, \bar{y}_1\bar{y}_2, \bar{y}_3\bar{y}_4\} \in \mathrm{UNSAT}$ already used in the proof of La. 1 shows that $\mathcal{H} \in \mathfrak{H}_{\mathrm{diag}}$. The assertion then follows from the proof of Thm. 8 in [16].

Regarding (ii), consider the first component of the set of hyperedges on the left hand side in Example 1. Let the corresponding BHG be $\mathcal{H} = (V, B)$ which is non-diagonal and isomorphic to $\mathrm{FPP}(2)$. For simplicity dropping the index 1 from the vertex-symbols yields $V = \{x, y, z, u, v, p, q\}$. Evidently the retractions $\mathcal{H}[V \setminus b]$, for all $b \in B$, are isomorphic. Thus it suffices to establish the assertion for $\mathcal{H}' = (V', B') := \mathcal{H}[V \setminus \{x, y, z\}]$ with $B' := \{uv, up, vp, vq, pq, uq\}$. The transversal $F_0' := \{pu, \bar{u}v, \bar{v}p, \bar{v}q, \bar{p}\bar{q}, uq\} \in \mathrm{UNSAT}$ shows that $\mathcal{H}' \in \mathfrak{H}_{\mathrm{diag}}$.

Let $\mathcal{H}' =: \mathcal{H}_1' \cup \mathcal{H}_2'$ with $B_1' := \{uv, up, vp\}$ and $B_2' := \{vq, pq, uq\}$. So $\mathcal{H}_1'$ is isomorphic to $\mathrm{FPP}(1)$ hence is a cycle-BHG. A member of $\mathcal{F}(\mathcal{H}_1')$ can fix at most one variable among $\{u, v, p\}$ to have the same assignment in each of its models. Such a variable is called a *backbone (BB)*: The *cycle-pattern* $pu, \bar{u}v, \bar{v}p$ contained in F_0' e.g. fixes only p to 1. Also for u, v there are analogous cyle patterns fixing exactly one of them.

Each edge of B_2' contains q thus both $\mathcal{H}_1'$, $\mathcal{H}_2'$ are exact linear. If a transversal $F' \in \mathcal{F}(\mathcal{H}')$ has a pure literal over q hereby all its clauses over B_2 can be satisfied. All clauses over B_1 can then be satisfied independently by the exact linearity. Thus it suffices to consider transversals having property $(*)$, namely exactly two equal literals over q are set to solve the corresponding clauses over B_2. In the remaining clause over B_2 thereby the assignment of exactly one in $\{u, v, p\}$ is fixed.

Now first let any $b \in B_1'$ be removed from B' and let $F \in \mathcal{F}(\mathcal{H}' \setminus \{b\})$ be arbitrary of property $(*)$. Since $\mathcal{H}_1' \setminus \{b\}$ contains each of $\{u, v, p\}$, there remains a distinct variable for each of the remaining F-clauses over B_1', hence $F \in \mathrm{SAT}$.

Next remove any $b \in B_2'$ from B', and let $F \in \mathcal{F}(\mathcal{H}' \setminus \{b\})$ be arbitrary of property $(*)$. W.l.o.g. assume that F contains a BB in $\{u, v, p\}$. Case (1): b contains the BB, then assign one of $\{u, v, p\}$ except for the BB s.t. its clause over B_2' is solved. The remaining clause can be satisfied by q, and the last variable in $\{u, v, p\}$ can be assigned accordingly to solve all B_1'-clauses.

Case (2) b does not contain the BB, so it occurs in a clause of F over $B_2' \setminus \{b\}$, which must be solved by q. The remaining clause can then be satisfied by one of $\{u, v, p\}$ which is no BB. Since the BB solves two clauses over B_1' there is left one in $\{u, v, p\}$ for solving the last clause over B_1'. Hence in either case $F \in \mathrm{SAT}$ establishing the minimal diagonality of $\mathcal{H}'$.

Observe that $F_0' \neq \hat{F}_0 := \{uv, u\bar{p}, \bar{v}p, vq, \bar{p}q, \bar{u}\bar{q}\} \in \mathcal{F}_{\mathrm{diag}}(\mathcal{H}')$ contains a cycle-pattern over B_1' that fixes u to 1. Moreover it provides a bifurcation (cf. [15]) for p at the clauses $up, \bar{v}p \in F_0'$ respectively $u\bar{p}, \bar{v}p \in \hat{F}_0$. Thus $\delta(\mathcal{H}') \geq 2$ implying $\mathcal{H}' \in \mathfrak{H}_{\mathrm{mdiag}} \setminus \mathfrak{H}_{\mathrm{simp}}^{\mathrm{con}}$.

The last cycle-pattern over B_1' that fixes v to 1 provides a last inequivalent diagonal transversal. Indeed it is easily verified that it admits a distinct bifurcation for either of $\{u, v\}$ regarding $F_0', \hat{F}_0$. So $\delta(\mathcal{H}') = 3$. □

Notice that the previous result shows that a non-diagonal BHG might be modifiable yielding a diagonal one by a retraction involving exactly one edge. So the following concept is motivated:

Definition 2: $\mathcal{H} = (V, B) \in \mathfrak{H}_0$ is called retraction non-diagonal *if there exists $b \in B$, $b \neq V$, s.t. either $\mathcal{H}[V \setminus b] \in \mathfrak{H}_{\mathrm{diag}}$, or there is $F \in \mathcal{F}(\mathcal{H})$ s.t. $F[V \setminus b] \in \mathrm{UNSAT}$. Further, for integer $i > 1$, $\mathcal{H}$ is* strict retraction i-non-diagonal *if there exists $b \in B$, $b \neq V$, s.t. $\mathcal{H}[V \setminus b] \in \hat{\mathfrak{H}}_i$. Let $\mathfrak{H}_{\mathrm{retnd}}$ ($\mathfrak{H}_{\mathrm{retnd}}^{(i)} \subsetneq \mathfrak{H}_{\mathrm{retnd}}$) denote the class of all (strict) retraction (i)-non-diagonal BHGs.*

Observe that the first alternative in the previous definition is well-defined as long as $b' \setminus b$ are mutually distinct, for all

$b' \in B$ with $b' \setminus b \neq \emptyset$.

If e.g. $\mathcal{H} = (V, B)$ is isomorphic to FPP(1) then there is no $b \in B$ s.t. $\mathcal{H}[V \setminus b]$ is defined. But obviously there is $F \in \mathcal{F}(\mathcal{H})$ s.t. $F[V \setminus b] \equiv \{x, \bar{x}\} \in$ UNSAT, where $\{x\} = V \setminus b$. In that perspective any FPP(1)-BHG is retraction non-diagonal. On behalf of the unique-pair-rule it is easy to verify that $\mathcal{H}[V \setminus b]$ is well-defined, for every $b \in B$, if $\mathcal{H}$ is any FPP(o)-BHG, for $o \geq 2$ appropriate. In particular one has:

Proposition 2: Let $\mathcal{H} = (V, B)$ be

(1) *an arbitrary* FPP(2)*-BHG then $\mathcal{H} \in \mathfrak{H}_{\mathrm{retnd}}$, even $\mathcal{H}[V \setminus b] \in \mathfrak{H}_{\mathrm{diag}}$, for each $b \in B$, and $\mathcal{H}[V \setminus b] \in \mathfrak{H}_{\mathrm{mdiag}}$, for each $b \in B$ if $\mathcal{H}$ is connected,*

(2) *an arbitrary x-connected* FPP(2)*-BHG then $\mathcal{H} \in \mathfrak{H}_{\mathrm{retnd}}$, even $\mathcal{H}[V \setminus b] \in \mathfrak{H}_{\mathrm{diag}}$, for each $b \in B$, and even $\mathcal{H}[V \setminus b] \in \mathfrak{H}^{\mathrm{con}}_{\mathrm{diag}}$, for each $b \in B$ with $x \notin b$.*

PROOF. Let $s \in \mathbb{N}$, $\mathcal{H}_j = (V_j, B_j)$ be isomorphic to FPP(2), $j \in [s]$, $\mathcal{H} = \bigcup_{j \in [s]} \mathcal{H}_j$ and $b \in B$ be arbitrary. For statement (1), let $\mathcal{H}$ be a disjoint union. Then there is $j' \in [s]$ unique with $b \in B_{j'}$ hence $\mathcal{H}[V \setminus b] = \mathcal{H}_{j'}[V_{j'} \setminus b] \cup \bigcup_{j \in [s] \setminus \{j'\}} \mathcal{H}_j$. Since the first component here is diagonal by La. 4 (ii), the assertion follows by symmetry. Also the connected case is clear by the same lemma because then $s = 1$.

For statement (2) let the components of $\mathcal{H}$ mutually have in common only the connecting vertex x. As above there is $j' \in [s]$ unique with $b \in B_{j'}$. If $x \notin b$ then $x \in V \setminus b$ and so $\mathcal{H}[V \setminus b] = \mathcal{H}_{j'}[V_{j'} \setminus b] \cup \bigcup_{j \in [s] \setminus \{j'\}} \mathcal{H}_j \in \mathfrak{H}^{\mathrm{con}}_{\mathrm{diag}}$. Since $b \cap V_j = \{x\}$, for all $j \in [s] \setminus \{j'\}$, if $x \in b$ one has $\mathcal{H}[V \setminus b] = \mathcal{H}_{j'}[V_{j'} \setminus b] \cup \bigcup_{j \in [s] \setminus \{j'\}} \mathcal{H}_j[V_j \setminus \{x\}] \in \mathfrak{H}_{\mathrm{diag}}$ which is disconnected as a disjoint union. For both alternatives again La. 4 (ii) was used ensuring that the first components are diagonal. □

Remark 1: Not every FPP(o)*-BHG is retraction non-diagonal: Let e.g. $\mathcal{H} = (V, B)$ be a one-component* FPP(3)*-BHG hence non-diagonal, and $b \in B$ be arbitrary. The edge set of its retraction $\mathcal{H}' = (V', B') := \mathcal{H}[V \setminus b]$ is (isomorphic to) the one shown below left:*

$$B' = \begin{array}{l} \{a_{11}\,a_{12}\,a_{13}, \\ a_{21}\,a_{22}\,a_{23}, \\ a_{31}\,a_{32}\,a_{33}, \\ a_{11}\,a_{21}\,a_{31}, \\ a_{12}\,a_{22}\,a_{32}, \\ a_{13}\,a_{23}\,a_{33}, \\ a_{11}\,a_{23}\,a_{32}, \\ a_{12}\,a_{21}\,a_{33}, \\ a_{13}\,a_{22}\,a_{31}, \\ a_{11}\,a_{22}\,a_{33}, \\ a_{12}\,a_{23}\,a_{31}, \\ a_{13}\,a_{21}\,a_{32}\} \end{array} \qquad \hat{B} = \begin{array}{l} \{a_{11}\,a_{12}\,a_{13}, \\ a_{21}\,a_{22}, \\ a_{31}\,a_{32}, \\ a_{11}\,a_{21}\,a_{31}, \\ a_{22}\,a_{32}, \\ a_{23}\,a_{33}, \\ a_{11}\,a_{23}\,a_{32}, \\ a_{12}\,a_{33}, \\ a_{13}\,a_{31}, \\ a_{11}\,a_{22}\,a_{33}, \\ a_{12}\,a_{23}, \\ a_{13}\,a_{21}\} \end{array}$$

Hence $\mathcal{H}'$ containing 12 edges and $9 = 13 - 4$ variables is 3-uniform and 4-regular. Let $\hat{\mathcal{H}} = (V', \hat{B})$ be obtained from $\mathcal{H}'$ by reducing 8 of its edges about exactly one variable, s.t. V' and the remaining edges remain unaltered; $\hat{B}$ is shown above right. Assume that $\hat{\mathcal{H}} \in \mathfrak{H}_0$ then La. 2 implies that also $\mathcal{H}'$ remains in $\mathfrak{H}_0$. So $\mathcal{H}$ cannot be retraction non-diagonal because all these retractions are isomorphic. To verify this assumption consider the two variable-disjoint cycle-sub-BHGs of $\hat{\mathcal{H}}$ with the edge sets:

$$\begin{aligned} \hat{B}_1 &:= \{a_{32}a_{22},\, a_{22}a_{21},\, a_{21}a_{13},\, a_{13}a_{31},\, a_{31}a_{32}\} \\ \hat{B}_2 &:= \{a_{23}a_{12},\, a_{12}a_{33},\, a_{33}a_{23}\} \end{aligned}$$

By cycle-patterns at most one variable in either of them can be fixed as a BB in a transversal F, so there are two cases. (i): Both fixed variables occur together with a_{11} in a common remaining clause. This situation is given e.g. via setting

$$\begin{aligned} F_1 &:= \{a_{32}\bar{a}_{22},\, a_{22}\bar{a}_{21},\, a_{21}\bar{a}_{13},\, a_{13}\bar{a}_{31},\, a_{31}a_{32}\} \\ F_2 &:= \{a_{23}\bar{a}_{12},\, a_{12}\bar{a}_{33},\, a_{33}a_{23}\} \end{aligned}$$

exactly fixing a_{32} in F_1, and a_{23} in F_2 to 1 thereby solving 2 clauses of either of F_1, F_2. Setting $a_{11}\bar{a}_{23}\bar{a}_{32}$ fixes a_{11} to 1. Negating a_{11} in the 3 remaining clauses containing it, there remain seven 2-uniform unsolved clauses. There always is a variable, say a_{21} having 3 occurrences, of which exactly 2 belong to the same literal: one remaining in F_1, and the second where a_{21} occurs together with a_{31} which are solved via setting a_{21} accordingly. It is easy to verify that each of the remaining clauses can be satisfied by a seperate variable. The other variants of case (i) behave analogously.

(ii): Both fixed variables occur together with a_{11} in distinct clauses. This situation is given e.g. via permuting and setting

$$\begin{aligned} F'_1 &:= \{a_{22}\bar{a}_{32},\, a_{32}\bar{a}_{31},\, a_{31}\bar{a}_{13},\, a_{13}\bar{a}_{21},\, a_{21}a_{22}\} \\ F'_2 &:= \{a_{12}\bar{a}_{23},\, a_{23}\bar{a}_{33},\, a_{33}a_{12}\} \end{aligned}$$

fixing a_{22}, a_{12} to 1 solving 2 clauses of either of F'_1, F'_2. In this case a_{11} cannot be fixed in either of its 4 occurrences; of which at least 2 must be of the same literal. Thus fixing a_{11} accordingly solves two of these clauses. In the remaining, a_{11} is removed yielding a current formula of 6 clauses and 6 variables. It is easy to check that for every clause there remains a variable for solving it. The further variants of case (ii) behave analogously. In summary one obtains $\hat{\mathcal{H}} \in \mathfrak{H}_0$. □

Regarding the general case one has:

Proposition 3: For every $i \in \mathbb{N}$ s.t. $\hat{\mathfrak{H}}_i \neq \emptyset$ there is a retraction non-diagonal BHG, which even is strict retraction i-non-diagonal BHG if $i > 1$.

PROOF. Let $\mathcal{H}' = (V', B') \in \hat{\mathfrak{H}}_i \neq \emptyset$ be of size m s.t. its edges have a fixed labeling. Let B be obtained from B' by adding to $b'_j \in B'$ the new vertex u_j yielding $b_j \in B$, for all $j \in [m]$, finally add the edge $b_0 := \{u_1, \ldots, u_j\}$. Moreover set $V := V' \cup b_0$ and $\mathcal{H} := (V, B)$, then evidently $\mathcal{H}[V \setminus b_0] = \mathcal{H}[V'] = \mathcal{H}'$ is well-defined, for all $i \in \mathbb{N}$. Let $F \in \mathcal{F}(\mathcal{H})$ be arbitrary. Solve its clause c_m over b_m by any of its literals except for $l(u_m) \in c_m$, hence u_m is free for solving $c_0 \in F$ over b_0. Then u_j independently solves $c_j \in F$ over b_j, for all $j \in [m-1]$. So $\mathcal{H}' \in \mathfrak{H}_0$ implying its (strict) retraction (i-)non-diagonality. □

IV. MAXIMAL NON-DIAGONALITY VERSUS MINIMAL DIAGONALITY

As a proposal for clarifying the structure of all non-diagonal BHGs that hence reside below the hierarchy of the diagonal instances, here we focus on the notion of maximal non-diagonality as defined below. BHGs of that property are extreme in the non-diagonal range as they are exactly one hyperedge away from diagonality. Especially it shall be shown also that there are arbitrary large non-diagonal BHGs which are not maximal non-diagonal, so the concept is far from trivial. Moreover Prop. 4 in [18] states a criterion for a non-diagonal BHG becoming a diagonal instance of i diagonal orbits, for $i > 0$ arbitrary, by adding exactly one further edge.

Definition 3: A non-diagonal $\mathcal{H} = (V, B)$ is called maximal non-diagonal *if there is a diagonal BHG $\mathcal{H}' = (V, B')$, with $B \subseteq B'$, s.t. for every $b \in B' \setminus B$ one has $\delta(\mathcal{H} \cup \{b\}) >$*

0. *Then $\mathcal{H}$ also is called maximal non-diagonal with respect to (wrt.) $\mathcal{H}'$. Let $\mathfrak{H}_{\rm maxnd}$ denote the class of all maximal non-diagonal BHGs.*

Equivalently, a non-diagonal $\mathcal{H} = (V, B)$ is maximal non-diagonal iff there is $b \subseteq V$, $b \notin B$ s.t. $\mathcal{H} \cup \{b\} \in \mathfrak{H}_{\rm diag}$. The non-trivial existence of a maximal non-diagonal BHG can be established on the basis of the members of $\mathfrak{H}_{\rm diag}$, among which also are loopless instances according to La. 4 (i), (ii).

Lemma 5: Let $\mathcal{H} \in \mathfrak{H}_{\rm mdiag}$ then $\mathcal{H} \setminus \{b\} \in \mathfrak{H}_{\rm maxnd}$, for every $b \in B(\mathcal{H})$. □

According to Cor. 3 in [16] stating $\mathfrak{H}^{\rm con}_{\rm simp} \subseteq \mathfrak{H}_{\rm mdiag}$ one obtains:

Corollary 1: Let $\mathcal{H} \in \mathfrak{H}^{\rm con}_{\rm simp}$ be loopless then $\mathcal{H} \setminus \{b\} \in \mathfrak{H}_{\rm maxnd}$ is loopless, for every $b \in B(\mathcal{H})$. □

It is not difficult to verify that every non-diagonal BHG containing at least two loops is maximal non-diagonal. So the loopless case is of specific interest. The next lemma provides a general necessary and sufficient criterion for a loopless, maximal non-diagonal BHG.

Lemma 6: Let $\mathcal{H} \in \mathfrak{H}_0$ be loopless. Then $\mathcal{H} \in \mathfrak{H}_{\rm maxnd}$ iff there is $x \in V(\mathcal{H})$ s.t. $\mathcal{H} \cup \{x\} \in \mathfrak{H}_{\rm diag}$.

PROOF. The sufficiency directly follows from the definition because looplessness is assumed. Next let $\mathcal{H} = (V, B) \in \mathfrak{H}_{\rm maxnd}$ and $b \subseteq V$, $b \notin B$ s.t. $\mathcal{H} \cup \{b\} \in \mathfrak{H}_{\rm diag}$. So there is $F \in \mathcal{F}(\mathcal{H})$ and $c_F \in W_b$ s.t. for all $w \in \mathcal{M}(F)$, $w(b) = c^\gamma_F \in W_b$. Since $|b| \geq 1$ there is $x \in V \cap b$ with $l(x) \in c_F$ and $w(x) = \overline{l(x)}$, for all $w \in \mathcal{M}(F)$, hence $\mathcal{H} \cup \{x\} \in \mathfrak{H}_{\rm diag}$. □

Observe that FPP(1)-BHGs are maximal non-diagonal. But the classes $\mathrm{FPP}(o)$, $o \geq 2$ appropriate, and their x-connected enlargements in fact provide concrete arbitrary large, connected and non-diagonal BHGs which fail to be maximal non-diagonal:

Theorem 2: Let $\mathcal{H}$ be any x-connected $\mathrm{FPP}(o)$-BHG of $s \in \mathbb{N}$ components, where o is an arbitrary prime power. Then $\mathcal{H} \in \mathfrak{H}_0 \setminus \mathfrak{H}_{\rm maxnd}$.

PROOF. Let o be a fixed prime power, hence $k := o + 1 \geq 3$ and first assume that $s = 1$. So $\mathcal{H} = (V, B)$ is loopless and isomorphic to $\mathrm{FPP}(k-1)$ which then exists. Recall that $\mathcal{H}$ obeys the unique-pair-rule. As stated above $\mathcal{H} \in \mathfrak{H}_{\rm xlin}$ implying its non-diagonality. Evidently $I_\mathcal{H}$ is k-regular and so it admits a partition of its edge set into k edge-disjoint perfect matchings [9]. Thus for arbitrary $x \in V$, $b \in B$ with $x \in b$, there is a perfect matching $M(x, b)$ of $I_\mathcal{H}$ that contains $\{x, b\}$.

Relying on La. 6 let $x \in V$ and set $\mathcal{H}' := \mathcal{H} \cup \{x\}$. For arbitrary $F' \in \mathcal{F}(\mathcal{H}')$ its clause over $\{x\}$ must be assigned as forced. First assume that a further clause c' of F' containing x is solved hereby. Any fixed perfect matching $M := M(x, V(c'))$ of $I_\mathcal{H}$ ensures that for every remaining clause c, hence $V(c) \in B$, there is a unique variable $x \neq v \in V$ with $\{v, V(c)\} \in M$ for solving c independently. So $F' \in \mathrm{SAT}$.

If no further clause containing x is solved by the initial assignment, every occurrence of x is removed from those clauses as all these literals are assigned 0. Let $\hat{F}$ be the resulting formula, where $\hat{F}_1$ collects all reduced clauses having size $k - 1 \geq 2$, and in which each variable of $\hat{F}$ occurs exactly once. Further, let $\hat{F}_2$ collect all those clauses of F' which remained unaltered so far. Thus $\mathcal{H}(\hat{F}_2) \in \mathfrak{H}_{\rm xlin}$.

Case (1): There is $y \in V(\hat{F})$ occurring as the same literal in exactly one clause, say $c_1 \in \hat{F}_1$, and in at least one, say $c_2 \in \hat{F}_2$. Assigning y accordingly solves both of them. One has $\mathcal{H}_2 := \mathcal{H}(\hat{F}_2 \setminus \{c_2\}) \in \mathfrak{H}_{\rm xlin}$ and according to Thm. 13 in [19], $I_{\mathcal{H}_2}$ admits a perfect matching.

Assume that there even is a perfect matching M_2 of $I_{\mathcal{H}_2}$ not using the variables in $V(c_2)$. Since $x, y \in V(c_1)$ no further variable of c_2 occurs in c_1, hence we have the admissible enlargement $M := M_2 \cup \{\{x, V(c_1)\}, \{y, V(c_2)\}\}$ as a partial perfect matching of $I_\mathcal{H}$. Because of the unique-pair-rule, for every $c_1 \neq c \in \hat{F}_1$, there is $y \neq v_c \in V(c_2) \cap V(c)$ unique providing the corresponding matching edge $\{v_c, V(c)\}$ for accordingly enlarging the current M yielding a perfect matching of $I_\mathcal{H}$. So $\hat{F} \in \mathrm{SAT}$ where y is the only variable solving two clauses, so also $F' \in \mathrm{SAT}$.

The existence of M_2 as required above remains to be verified. Let $x \neq y_j, j \in [k-1]$, be the remaining variables in $V(c_1)$ and w.l.o.g. we may assume that $y_{k-1} := y$, further for convenience set $c(k-1) := c_2$. In $\mathcal{H} \setminus \{V(c_1)\}$ thus in $\mathcal{H}(\hat{F}_2)$ every y_j occurs in exactly $k-1$ further positions, let the corresponding set of hyperedges be refered to as $B(y_j)$, $j \in [k-1]$. Then $V(c_2) = V(c(k-1)) \in B(y_{k-1})$. Fix an arbitrary $V(c(1)) \in B(y_1)$, the *leading edge*, and match it to y_1. Let $u \neq y_1$ be the unique variable in $V(c_2) \cap V(c(1))$.

For every $j \in [k-2]$, there is exactly one leading edge $V(c(j)) \in B(y_j)$ uniquely determined via $u \in V(c(j))$ that is matched to y_j. The edge $c_2 = c(k-1)$ matched to $y_{k-1} := y$ clearly is special because its variables are not used. Evidently $V(c(j)) \cap V(c(j')) = \{u\}$, for all distinct $j, j' \in [k-1]$. Therefore either of the $k - 2 \geq 1$ variables in $V(c(j)) \setminus \{u, y_j\}$ is not used so far and occurs in exactly one of $B(y_{j+1}) \setminus V(c(j+1))$ to which it is matched, for every $j \in [k-2]$. Thus an appropriate perfect matching M_2 of $I_{\mathcal{H}_2}$ as required is provided.

Case (2): Every variable of $\hat{F}$ occurs as the same literal in exactly $k-1$ positions of $\hat{F}_2$, and in exactly one position of $\hat{F}_1$ as the complemented literal, say in $\tilde{c}_y$. Assign y s.t. all $k-1$ clauses in $\hat{F}_2$ containing y are solved. Fix a single one of them, say $c'_y \in \hat{F}_2$. The literal over y in $\tilde{c}_y$ is assigned 0, hence $\tilde{c}_y \in \hat{F}_1$ currently fails to have an occurrence of y. One has $\mathcal{H}_y := \mathcal{H}(\hat{F}_2 \setminus \{c'_y\}) \in \mathfrak{H}_{\rm xlin}$ and as shown above $I_{\mathcal{H}_y}$ admits a perfect matching M_y not using the variables in $V(c'_y)$.

Further, every $c \in \hat{F}_1 \setminus \{\tilde{c}_y\}$ can be matched to the unique variable determined via $V(c'_y) \cap V(c)$ enlarging M_y accordingly. Since $k - 2 \geq 1$ (recall that only x, y are removed) at least there is $u \in V(\tilde{c}_y)$, so $\{u, V(\tilde{c}_y)\}$ can be set as the final matching edge providing a perfect matching of $I_\mathcal{H}$. So $\hat{F} \in \mathrm{SAT}$, where y is the only variable solving $k-1$ clauses simultaneouly, implying $F' \in \mathrm{SAT}$.

Now let $s \geq 2$, $\mathcal{H}_j = (V_j, B_j)$, $j \in [s]$, be the components of the x-connected $\mathcal{H} = (V, B)$ which is loopless and so La. 6 can be used.

(i): Let $\mathcal{H}' := \mathcal{H} \cup \{x\}$, $F' \in \mathcal{F}(\mathcal{H}')$ be arbitrary and $F'_j := F'|B_j \in \mathcal{F}(\mathcal{H}_j)$, $j \in [s]$. Assign x as forced by the unit clause, and w.l.o.g. let $\mathcal{H}_j$, $j \in [p']$ with $s' \leq s$ appropriate, be those components of $\mathcal{H}$ s.t. F'_j has a clause c_j with $x \in V(c_j)$ that is solved by the assignment of x. For every $j \in [s']$, one therefore has a perfect matching $M_j(x, V(c_j))$ of $I_{\mathcal{H}_j}$ solving F'_j. So, $F' \in \mathrm{SAT}$ if $s' = s$.

Otherwise over the components $\mathcal{H}_j$, all literals over x in the clauses of F'_j are assigned 0; thus $F'_j \in \mathrm{SAT}$ according to the proof of the corresponding case for $r = 1$, for all

$j \in [s] \setminus [s']$. Since $V(F'_j) \setminus \{x\}$, $j \in [s] \setminus [s']$, are mutually disjoint one obtains $F' \in \mathrm{SAT}$.

(ii): Let $\mathcal{H}' := \mathcal{H} \cup \{u\}$, for $x \neq u \in V$, and let $F' \in \mathcal{F}(\mathcal{H}')$ be arbitrary. Then there is $j_u \in [s]$ unique with $u \in V_{j_u}$, let $s' := s-1$. As in the proof for $s = 1$ there is a partial assignment w_{j_u} solving $F'_{j_u} := F'|B_{j_u} \in \mathcal{F}(\mathcal{H}_{j_u})$. Here x is the unique variable fixed by w_{j_u} also occurring in $F' \setminus F'_{j_u}$ which therefore is satisfiable either according to the case (i) if $s' \geq 2$, or to the proof for $s' = 1$. □

Formally $\mathfrak{H}_{\mathrm{maxnd}}$ decomposes into the following subclasses.

Definition 4: Let $i \in \mathbb{N}$, $\mathcal{H} = (V, B)$ with $\delta(\mathcal{H}) = 0$, and $\mathcal{H}' = (V, B')$ with $B' \supset B$, $\delta(\mathcal{H}') \geq i$.

1. *$\mathcal{H}$ is called* maximal i-non-diagonal *(wrt. $\mathcal{H}'$) if for every $b \in B' \setminus B$ one has $\delta(\mathcal{H} \cup \{b\}) \in [i]$. Let $\mathfrak{H}^{(i)}_{\mathrm{maxnd}}$ denote the class of all* maximal i-non-diagonal *BHGs.*
2. *$\mathcal{H}$ is called* strict *maximal i-non-diagonal (wrt. $\mathcal{H}'$) if for all $b \in B' \setminus B$ one has $\delta(\mathcal{H} \cup \{b\}) = i$. Set $\hat{\mathfrak{H}}^{(i)}_{\mathrm{maxnd}} \subseteq \mathfrak{H}^{(i)}_{\mathrm{maxnd}}$ for all such instances.*

Remark 2: So far it is unknown whether the classes $\hat{\mathfrak{H}}_i$ are non-trivial, for every $i \in \mathbb{N}$. As shown in [18] there are infinitely many i s.t. $\hat{\mathfrak{H}}_i \neq \emptyset$. Similarly, the non-trivial existence of the classes $\hat{\mathfrak{H}}^{(i)}_{\mathrm{maxnd}}$ as defined above needs to be established. First results in this direction can be found in Thm. 6, resp. Cor. 4, below. According to Prop. 4 in [18] a strict maximal i-non-diagonal BHG $\mathcal{H} = (V, B)$ wrt. $\mathcal{H}' = (V, B') \in \hat{\mathfrak{H}}_{i'}$, for $i' \geq i$ appropriately, must have the property that for each $b \in B' \setminus B$ there are exactly i distinct orbits in $\mathcal{F}(\mathcal{H})/G_V$ s.t. for each orbit there is a member F and a clause $c_F \in W_b$ with $w(b) = c_F$, for all $w \in \mathcal{M}(F)$. Then $\delta(\mathcal{H} \cup \{b\}) = i$ is ensured, hence $\mathcal{H} \in \hat{\mathfrak{H}}^{(i)}_{\mathrm{maxnd}}$.

Let $\mathcal{H}_j \in \mathfrak{H}_{\mathrm{mdiag}}$, $j \in [2]$, with $V(\mathcal{H}_1) \cap V(\mathcal{H}_2) = \emptyset$ then obviously $\mathcal{H}_1 \cup \mathcal{H}_2 \notin \mathfrak{H}_{\mathrm{mdiag}}$. On that basis using e.g. La. 4 one obtains large, loopless maximal non-diagonal BHGs failing to be subhypergraphs of minimal diagonal instances.

Proposition 4: Let $s \in \mathbb{N}$, $\mathcal{H}_j = (V_j, B_j) \in \mathfrak{H}_{\mathrm{mdiag}}$, $j \in [s]$, be (loopless) mutually vertex-disjoint, and $\mathcal{H}' = (V, B') := \bigcup_{j \in [s]} \mathcal{H}_j$. Set $B := B' \setminus \{b_j : j \in [s]\}$, for a fixed selection $b_j \in B_j$, $j \in [s]$, and $\mathcal{H} := (V, B)$ then:

1. *$\mathcal{H} \in \mathfrak{H}_{\mathrm{maxnd}}$ (is loopless).*
2. *Let $i \in \mathbb{N}$ and $\mathcal{H}_j \in \hat{\mathfrak{H}}_i$, s.t. $\omega(\mathcal{H}_j) = \omega \in \mathbb{N}$, for all $j \in [s]$, then $\delta(\mathcal{H}') \geq \ell$ and $\mathcal{H} \in \hat{\mathfrak{H}}^{(\ell)}_{\mathrm{maxnd}}$ (is loopless) where $\ell = i\omega^{s-1}$.*

PROOF. $\mathcal{H}_j \in \mathfrak{H}_{\mathrm{mdiag}}$ implies $\delta(\mathcal{H}_j \setminus \{b_j\}) = 0$, $j \in [s]$, so $\delta(\mathcal{H}) = 0$. (1) is obvious. Regarding (2) by assumption $\delta(\mathcal{H}_j) = i, \omega(\mathcal{H}_j) = \omega$, for all $j \in [s]$. Since the $\mathcal{H}_j$ are mutually vertex-disjoint, we can apply La. 1 (ii) in [16] hence $\delta(\mathcal{H}') = \prod_{j \in [s]} \omega(\mathcal{H}_j) - \prod_{j \in [s]} (\omega(\mathcal{H}_j) - \delta(\mathcal{H}_j)) = \omega^s - (\omega - i)^s$. Obviously $i < \omega$, hence $\delta(\mathcal{H}') = \omega^s[1 - (1 - i/\omega)^s]$ $\geq \omega^s[1 - (1 - i/\omega)] = \ell$. Adding to B exactly one arbitrary of the edges $b_j \in B_j$, $j \in [s]$, say $b_{j'}$, yields $\mathcal{H} \cup \{b_{j'}\} = \mathcal{H}_{j'} \cup \bigcup_{j \in [s] \setminus \{j'\}} (\mathcal{H}_j \setminus \{b_j\})$. Again using the cited lemma, one obtains $\delta(\mathcal{H} \cup \{b_{j'}\}) = \omega^s - \omega^{s-1}(\omega - i) = \ell$. □

Recall that due to La. 4 (ii) it is ensured that there are $i > 1$ s.t. $\hat{\mathfrak{H}}_i \cap \mathfrak{H}_{\mathrm{mdiag}} \neq \emptyset$ even containing loopless instances. The next result also provides connected members:

Theorem 3: For fixed $s \in \mathbb{N}$, and mutually vertex-disjoint $\mathcal{H}_j = (V_j, B_j) \in \mathfrak{H}_{\mathrm{mdiag}}$, $j \in [s]$, let $\mathcal{H}_0 = (V_0, B_0) := \bigcup_{j \in [s]} \mathcal{H}_j$, choose $b_0 \subset V_0$ s.t. $|b_0 \cap V_j| = 1$, for all $j \in [s]$, and let $y \notin V_0$. Then $\mathcal{H}' = (V, B') := \mathcal{H}_0 \cup \{b\} \in \mathfrak{H}^{\mathrm{con}}$ is diagonal, where $b := b_0 \cup \{y\}$. Further, for a fixed selection $b_j \in B_j$, $j \in [s]$, and $B := \{b\} \cup \bigcup_{j \in [s]} (B_j \setminus \{b_j\})$ one has $\mathcal{H} := (V, B) \in \mathfrak{H}_{\mathrm{maxnd}}$. Moreover if $b_j \cap b_0 = \emptyset$, for all $j \in [s]$, then $\mathcal{H} \in \mathfrak{H}^{\mathrm{con}}$.

PROOF. Since $\delta(\mathcal{H}_j) > 0$, $j \in [s]$, also $\delta(\mathcal{H}_0) > 0$, and $b_0 \notin B_j, j \in [s]$. Moreover each $\mathcal{H}_j$ is minimal diagonal, thus also connected. Adding the new edge $b = b_0 \cup \{y\}$ to $\mathcal{H}_0$ provides the connected BHG $\mathcal{H}' = \mathcal{H}_0 \cup \{b_0 \cup \{y\}\}$. Evidently $\mathcal{H}_0 \subsetneq \mathcal{H}'$ and because of the monotony of the mapping δ, as stated in Prop. 6 (1) in [18], one has $\delta(\mathcal{H}') > \delta(\mathcal{H}_0) > 0$. Since $|B_j| > 1$, b_j can be selected s.t. $b_j \cap b_0 = \emptyset$, for all $j \in [s]$, maintaining the connectedness of $\mathcal{H}$. Since the new variable y ensures that a clause over b can be satisfied independently and because of the fact that the instances $\mathcal{H}_j$, $j \in [s]$, are minimal diagonal, the rest of the theorem follows directly from Prop. 4 (1). □

Remark 3: Note that the new variable y added to b_0 above, in general, cannot be omitted. Consider e.g. $\mathcal{H}_j := (V_j, B_j)$ with $V_j := \{u_j, v_j\}$, $B_j := \{u_j, v_j, u_j v_j\}$, then $\mathcal{H}_j$ is a minimal diagonal BHG, $j \in [2]$. Set $b_0 := v_1 u_2$, and choose the selection $b_1 := u_1 \in B_1$, $b_2 := v_2 \in B_2$. Then $\mathcal{H} := (V, \hat{B})$ even is diagonal for $\hat{B} := \{b_0\} \cup \bigcup_{j \in [2]} B_j \setminus \{b_j\}$ instead of $B := \{b\} \cup \bigcup_{j \in [2]} B_j \setminus \{b_j\}$ where $b := b_0 \cup \{y\}$ as required in Thm. 3. Indeed, $\hat{B}$ contains the edges $v_1 u_2$, u_2, and v_1 yielding a simple, hence diagonal subhypergraph.

Theorem 4: There are loopless, Sperner, and even linear, maximal non-diagonal BHGs. There also exist exact linear and maximal non-diagonal BHGs.

PROOF. The first claim directly is implied by La. 4 (i) and La. 5. Regarding the second statement, let $V = \{u, x, y\}$ and $B = \{x, xy, yu, ux\}$. We claim that $\mathcal{H} = (V, B) \in \mathfrak{H}^{\mathrm{con}}_{\mathrm{simp}}$ where the connectedness is obvious. Observe that $\mathcal{H} \setminus \{x\}$ is a FPP(1)-BHG, so it is exact linear and non-diagonal [19]. Moreover it is easy to verify that all its transversals, for which x is a BB, belong to the same G_V-orbit. A diagonal transversal of $\mathcal{H}$ can occur only if x already is a BB of its restriction to $\mathcal{H} \setminus \{x\}$, e.g., $\{x, \bar{x}y, \bar{y}u, \bar{u}\bar{x}\} \in \mathcal{F}_{\mathrm{diag}}(\mathcal{H})$ which easily can be verified to belong to $\mathcal{I}$. An unsatisfiable formula in a distinct orbit can occur only if x is maintained as a BB, so there can be no bifurcation implying $\mathcal{H} \in \mathfrak{H}^{\mathrm{con}}_{\mathrm{simp}}$. Therefore by Lemma 5 $\mathcal{H} \setminus \{x\} \in \mathfrak{H}_{\mathrm{maxnd}}$. □

V. MAXIMAL NON-DIAGONALITY: THE GENERAL CASE

In this section further maximal non-diagonal BHGs are constructed which do not necessarily rely on minimal diagonal ones. It will become clear also that maximal non-diagonality and retraction non-diagonality are distinct structures. First a stronger version of maximal non-diagonality is considered, namely the *dense* maximal non-diagonality. To that end we begin with collecting several facts regarding the complete BHG over a fixed vertex set V, which is $\mathcal{K}_V := (V, 2^V \setminus \{\emptyset\})$. Observe that $\mathcal{K}_V \in \mathfrak{H}_{\mathrm{diag}} \setminus \mathfrak{H}_{\mathrm{mdiag}}$, if $|V| > 2$. We write $\mathcal{K}_n$ instead of $\mathcal{K}_V$ in case $|V| = n$.

Proposition 5: For $n \in \mathbb{N}$ one has

$$\omega(\mathcal{K}_n) = 2^{n(2^{n-1}-1)}, \quad \delta(\mathcal{K}_n) = \omega(\mathcal{K}_n) - \prod_{k \in [n]} (2^k - 1)^{\binom{n}{k}}$$

PROOF. Let $\alpha_n := \alpha(\mathcal{K}_n), \alpha \in \{\beta, \omega, \delta\}$ Recalling that for $\mathcal{H} = (V, B)$ it is $\beta(\mathcal{H}) = \sum_{b \in B} |b| - |V|$ and $\omega(\mathcal{H}) = 2^{\beta(\mathcal{H})}$

the first assertion follows from

$$\beta_n = -n + \sum_{k\in[n]} \binom{n}{k} k = -n + n \sum_{k\in[n]} \binom{n-1}{k-1} = n(2^{n-1} - 1)$$

The last assertion is true in case $n = 1$ directly yielding $\omega(\mathcal{K}_1) = 1$, $\delta(\mathcal{K}_1) = 0$. Next fix a vertex set V of $n \geq 2$ vertices and let $\mathcal{H}_1$ consist of the corresponding n loops only, then $\omega(\mathcal{H}_1) = 1, \delta(\mathcal{H}_1) = 0$. Let $1 < j \leq n$ be fixed, and let $\mathcal{H}_k$ be obtained from $\mathcal{H}_{k-1}$ by adding all edges of size k over V to it. Claim:

$$\delta(\mathcal{H}_k) = \omega(\mathcal{H}_k) - (2^k - 1)^{\binom{n}{k}}(\omega(\mathcal{H}_{k-1}) - \delta(\mathcal{H}_{k-1}))$$

meaning $\omega(\mathcal{H}_k) - \delta(\mathcal{H}_k) = (2^k - 1)^{\binom{n}{k}}(\omega(\mathcal{H}_{k-1}) - \delta(\mathcal{H}_{k-1}))$. Iterative insertion for $k = n$ to $k = 2$ yields $\delta_k = \delta(\mathcal{H}_k)$ with $\omega_n = \omega(\mathcal{H}_n)$ because $\mathcal{H}_n = \mathcal{K}_n$ and $\omega(\mathcal{H}_1) - \delta(\mathcal{H}_1) = 1$. For proving the claim it suffices to set $\omega_k(j)$, resp. $\delta_k(j)$, for the number of all orbits, resp. all diagonal orbits, after exactly j edges of size k have been added to $\mathcal{H}_{k-1}$, and to verify

$$\delta_k(j) = \omega_k(j) - (2^k - 1)^j(\omega(\mathcal{H}_{k-1}) - \delta(\mathcal{H}_{k-1}))$$

by induction on $j \in \mathbb{N}$. Finally insert $j(k) := \binom{n}{k}$, for which one has $\delta_k(j(k)) = \delta(\mathcal{H}_k)$. First recall Thm. 4 in [18] stating that adding a new edge b to a given $\mathcal{H} = (V, B)$ that already contains a loop for every $x \in b$ yields $\delta(\mathcal{H} \cup \{b\}) = 2^{|b|}\delta(\mathcal{H}) + \rho(\mathcal{H}) + 1$. Since $\rho(\mathcal{H}) + 1 = \omega(\mathcal{H}) - \delta(\mathcal{H})$ one has $\delta(\mathcal{H} \cup \{b\}) = (2^{|b|} - 1)\delta(\mathcal{H}) + \omega(\mathcal{H})$. Thus adding a zero in form of $2^{|b|}\omega(\mathcal{H}) - 2^{|b|}\omega(\mathcal{H})$ yields the induction base for $j = 1$ with the identifications: $\mathcal{H} = \mathcal{H}_{k-1}$, $|b| = k$, and $\omega(\mathcal{H} \cup \{b\}) = \omega_k(1) = 2^k\omega(\mathcal{H}_{k-1}) = 2^{|b|}\omega(\mathcal{H})$. By the same theorem we obtain via the induction hypothesis for fixed j

$$\delta_k(j + 1) = (2^k - 1)\delta_k(j) + \omega_k(j)$$

$$= (2^k - 1)\omega_k(j) - (2^k - 1)^{j+1}(\omega(\mathcal{H}_{k-1}) - \delta(\mathcal{H}_{k-1})) + \omega_k(j)$$

yielding the claim for $j + 1$, as $\omega_k(j + 1) = 2^k\omega_k(j)$. □

The stronger notion of dense maximal non-diagonality is formulated in precise terms as follows.

Definition 5: $\mathcal{H} = (V, B) \in \mathfrak{H}_{\mathrm{maxnd}}$ *is called* dense maximal non-diagonal (wrt. $\mathcal{K}_V$) *if* $\mathcal{H} \cup \{b\} \in \mathfrak{H}_{\mathrm{diag}}$, *for every* $b \in B(\mathcal{K}_V) \setminus B$. *If even for every such* b *one has* $\delta(\mathcal{H} \cup \{b\}) = i$, *for fixed* $i \in \mathbb{N}$, *we call* $\mathcal{H}$ dense maximal i-non-diagonal (wrt. $\mathcal{K}_V$).

Obviously every *dense* maximal (i-)non-diagonal BHG also is (strict) maximal (i-)non-diagonal. The converse, in general, is false, cf. e.g. Rem. 4, below. More precisely:

Proposition 6: Let $\mathcal{H} = (V, B)$ *be dense maximal non-diagonal wrt.* $\mathcal{K}_V$ *where* $|V| \geq 2$. *Then* $\mathcal{H}$ *is maximal non-diagonal wrt. every* $\mathcal{H}' \in \mathfrak{H}_{\mathrm{diag}}$, *with* $\mathcal{H} \subsetneq \mathcal{H}' \subseteq \mathcal{K}_V$ *and* $V(\mathcal{H}') = V$. □

The next result provides criteria for dense maximal non-diagonality.

Theorem 5: Let $\mathcal{H} = (V, B) \in \mathfrak{H}_0$ *with* $|V| \geq 2$.

(1) *$\mathcal{H}$ is dense maximal non-diagonal iff for all $b \subseteq V, b \notin B$ there is $F \in \mathcal{F}(\mathcal{H})$ and $c \in W_b$ s.t. $w(b) = c^\gamma \in W_b$ for all $w \in \mathcal{M}(F)$.*

(2) *$\mathcal{H}$ is dense maximal non-diagonal if there exists $F \in \mathcal{F}(\mathcal{H})$ with $|\mathcal{M}(F)| = 1$.*

PROOF. Evidently $\mathcal{K}_V \in \mathfrak{H}_{\mathrm{diag}}$ because it contains an instance of $\mathcal{K}_2 \in \mathfrak{H}^{\mathrm{con}}_{\mathrm{simp}}$. $\mathcal{H} \in \mathfrak{H}_0$ implies $B \subsetneq B(\mathcal{K}_V)$. Let $b \in B(\mathcal{K}_V) \setminus B$ be chosen arbitrarily and regarding (1) let $\mathcal{H}$ be dense maximal non-diagonal. Hence $\mathcal{H} \in \mathfrak{H}_{\mathrm{maxnd}}$ and $\mathcal{H} \cup \{b\} \in \mathfrak{H}_{\mathrm{diag}}$. So there is $F' \in \mathcal{F}_{\mathrm{diag}}(\mathcal{H} \cup \{b\})$ with $c := F'_b$ unique and $F := F' \setminus \{c\} \in \mathrm{SAT}$.

Suppose there is $w \in \mathcal{M}(F)$ with $w(b) \neq c^\gamma$, then $w(b) \cap c \neq \emptyset$ meaning that w also satisfies F' providing a contradiction. Reversely assume there is $F \in \mathcal{F}(\mathcal{H})$ and $c \in W_b$ s.t. $w(b) = c^\gamma \in W_b$, for all $w \in \mathcal{M}(F)$. Then $F \cup \{c\} \in \mathcal{F}_{\mathrm{diag}}(\mathcal{H} \cup \{b\})$ thus $\mathcal{H}$ is dense maximal non-diagonal providing (1).

If there is F having the single model w only then for a further b set $c := w(b)^\gamma$ so (2) is implied by (1). □

The following example is closely related to a minimal diagonal BHG based on the FPP(1); it also provides a loopless dense maximal non-diagonal instance.

Example 2: Due to La. 4 (i), the x-connected union $\mathcal{H}$ of two FPP(1)-components $\mathcal{H}_j = (V_j, B_j)$, $j \in [2]$, is minimal diagonal. For the notation refering to the proof of that lemma, recall $F = \{xy_1, xy_2, \bar{x}y_3, \bar{x}y_4, \bar{y}_1\bar{y}_2, \bar{y}_3\bar{y}_4\} \in \mathcal{I} \cap \mathcal{F}_{\mathrm{diag}}(\mathcal{H})$. Considering the decomposition $F = F_1 \cup F_2$ with $F_j \in \mathcal{F}(\mathcal{H}_j) \cap \mathrm{SAT}$, let $w_j \in \mathcal{M}(F_j)$ be arbitrary, $j \in [2]$, then $w_1(x) = 1$, and $w_2(x) = 0$. Setting $\mathcal{H}' = (V, B') = \mathcal{H} \setminus \{b' := xy_2\}$ one obtains $F' := F \setminus F_{b'} \in \mathcal{F}(\mathcal{H}') \cap \mathrm{SAT}$. Since F_2 is a subformula of F' we have $w'(x) = 0$ directly implying $w'(y_1) = 1 \Rightarrow w'(y_2) = 0$, for all $w' \in \mathcal{M}(F')$. Finally defining $\hat{\mathcal{H}} := \mathcal{H}' \cup \{y_2y_3\}$ admits the transversal $\hat{F} = F' \cup \{y_2y_3\} \in \mathcal{F}(\tilde{\mathcal{H}}) \cap \mathrm{SAT}$. Every model of $\hat{F}$ must be a model w' of F', thus one is forced to assign $w'(y_3) = 1$ implying $w'(y_4) = 0$. It follows that $|\mathcal{M}(\hat{F})| = 1$ thus $\hat{\mathcal{H}}$ is dense maximal non-diagonal according to Thm. 5 (2).

Remark 4: (1) *The previous example also proves the existence of a maximal non-diagonal BHG that is not dense: $\mathcal{H}' = (V, B') = \mathcal{H} \setminus \{xy_2\}$ which is in $\mathfrak{H}_{\mathrm{maxnd}}$ due to La. 5. But $b = y_2y_3 \subseteq V$, $b \notin B'$ yields $\mathcal{H}' \cup \{b\} \notin \mathfrak{H}_{\mathrm{diag}}$.*

(2) *The existence of dense maximal i-non-diagonality can be ensured at least for $i = 1$, and every $n \in \mathbb{N}$: Let $V = \{x_j : j \in [n]\}$, $B = \{\{x_j\} : j \in [n]\}$, so for $\mathcal{H} = (V, B)$ consisting of loops only one has $\mathcal{F}(\mathcal{H}) = \mathcal{F}_{\mathrm{comp}}(\mathcal{H})$ which therefore is non-diagonal. It even is dense maximal non-diagonal wrt. $\mathcal{K}_n$ relying on Thm. 5 (2). Adding an arbitrary edge $b \in B(\mathcal{K}_n) \setminus B$ yields $\mathcal{H}' := \mathcal{H} \cup \{b\}$. Since $\mathcal{H}$ contains all loops, again using Thm. 4 in [18] one has $\delta(\mathcal{H}') = 2^{|b|}\delta(\mathcal{H}) + \rho(\mathcal{H}) + 1 = 1$. So $\mathcal{H}$ is dense maximal 1-non-diagonal.*

(3) *Thus maximal non-diagonality can be derived from two extreme classes: Minimal diagonal instances on the one hand and complete instances on the other one. Moreover these classes overlap in the class of complete BHGs with exactly two vertices.*

Regarding the second remark above and refering to Def. 4 (2) one has:

Theorem 6: There are integers $i > 1$ s.t. the class of dense maximal i-non-diagonal BHGs is non-empty. Moreover for those i one has $\hat{\mathfrak{H}}^{(i)}_{\mathrm{maxnd}} \neq \emptyset$, and also $\mathfrak{H}^{(i)}_{\mathrm{retnd}} \neq \emptyset$.

PROOF. Consider $\mathcal{H} = (V, B) \in \mathfrak{H}^{\mathrm{con}}_{\mathrm{simp}} \subseteq \mathfrak{H}_{\mathrm{mdiag}}$ with $V = \{u, x, y\}$ as defined in the proof of Thm. 4. Therefore the connected $\mathcal{H}' := \mathcal{H} \cup \{uxy\} \in \mathfrak{H}_{\mathrm{diag}} \setminus \mathfrak{H}_{\mathrm{mdiag}}$ fulfills $\delta(\mathcal{H}') =: i > 1$. Moreover every transversal of $\mathcal{H}' \setminus \{x\}$ which is a FPP(1)-BHG has at most one BB. Thus $\hat{\mathcal{H}} := \mathcal{H}' \setminus \{x\} = (\mathcal{H} \setminus \{x\}) \cup \{uxy\} \in \mathfrak{H}_0$. Finally, adding either of the loops over u, x, y to $\hat{\mathcal{H}}$ yields a result isomorphic to $\mathcal{H}'$ implying that $\hat{\mathcal{H}}$ is dense maximal i-non-diagonal wrt. $\mathcal{K}_3$ (over V). The second assertion is clear by definition. Finally the last assertion immediately follows from Prop. 3. □

In fact $\delta(\mathcal{H}') = i = 8$ as considered in the previous proof: By the proof of Thm. 4 $\delta(\mathcal{H}) = 1$, further $\omega(\mathcal{H}) = 16$, hence $\rho(\mathcal{H}) = 14$. Enlarging $\mathcal{H}$ about $b := \{uxy\}$ to $\mathcal{H}'$ means 2^3 inequivalent extensions of a fixed representative of the diagonal orbit yielding at least 8 orbits in $\mathcal{F}_{\mathrm{diag}}(\mathcal{H}')$. Further, the three edges of size 2 in $\mathcal{H}$ can contribute exactly 1 BB by a cycle-pattern in a satisfiable transversal of $\mathcal{H}$. The loop also provides at most one further BB. Hence at most two of the three variables in b can be fixed in either such transversal. Thus it remains satisfiable by enlarging it over b, therefore $\delta(\mathcal{H}') = 8$.

The next result, similar to Prop. 4, can be verified analogously.

Proposition 7: Let $s \in \mathbb{N}$, $\mathcal{H}_j = (V_j, B_j) \in \mathfrak{H}_{\mathrm{maxnd}}$, $j \in [s]$, be mutually vertex-disjoint, $\mathcal{H} := \bigcup_{j\in[s]} \mathcal{H}_j$, and $\hat{\mathcal{H}} := \bigcup_{j\in[s]} \mathcal{K}_{V_j}$.

(1) *If $\mathcal{H}_j = (V_j, B_j)$ is dense maximal non-diagonal for all $j \in [s]$, then $\mathcal{H}$ is maximal non-diagonal wrt. $\hat{\mathcal{H}}$.*
(2) *If $\mathcal{H}_j = (V_j, B_j)$ is dense maximal i-non-diagonal, and $\omega(\mathcal{H}_j) = \omega \in \mathbb{N}$, for all $j \in [s]$, then $\mathcal{H}$ is strict maximal ℓ-non-diagonal wrt. $\hat{\mathcal{H}}$, with $\ell = i\omega^{s-1}$.* □

Towards generalizing the previous discussion the following concept is crucial.

Definition 6: Let $\mathcal{H} = (V, B) \in \mathfrak{H}_{\mathrm{diag}}$. A subhypergraph $\tilde{\mathcal{H}} \subseteq \mathcal{H}$ with $\tilde{\mathcal{H}} \in \mathfrak{H}_{\mathrm{mdiag}}$ is called a diagonal germ. *Let $\mathfrak{G}(\mathcal{H})$ be the collection of all diagonal germs of $\mathcal{H}$. Any $T \subset B$ is called a* transversal of diagonal germs (TDG) *of $\mathcal{H}$ if $T \cap B(\tilde{\mathcal{H}}) \neq \emptyset$ for every $\tilde{\mathcal{H}} \in \mathfrak{G}(\mathcal{H})$. A TDG T of $\mathcal{H}$ is* minimal *if it does not contain a proper TDG of $\mathcal{H}$.*

Example 3: Consider the BHG $\mathcal{H} = (V, B) \in \mathfrak{H}_{\mathrm{diag}}$ with $V = \{u, v, x, y\}$, $B = \{u, v, x, y, uv, ux, xy\}$ that is not minimal diagonal. Then one has $\mathfrak{G}(\mathcal{H}) = \{\mathcal{H}_l = (V_l, B_l) : l \in [6]\}$, where

$$\begin{aligned} &B_1 := \{x, y, xy\}, && B_2 := \{u, v, uv\}, \\ &B_3 := \{x, u, xu\}, && B_4 := \{x, xu, uv, v\}, \\ &B_5 := \{y, xy, xu, u\}, && B_6 := \{y, xy, xu, uv, v\} \end{aligned}$$

which all belong to $\mathfrak{H}_{\mathrm{mdiag}}$ as can be verified easily. The collection of all minimal TDGs of $\mathcal{H}$ is $\{T_j : j \in [20]\}$:

$$\begin{aligned} &T_1 = \{u, x, y\}, && T_2 = \{v, x, y\}, \\ &T_3 = \{u, v, x\}, && T_4 = \{u, v, y\}, \\ &T_5 = \{v, x, xy\}, && T_6 = \{u, x, xu\}, \\ &T_7 = \{x, y, uv\}, && T_8 = \{u, x, xy\}, \\ &T_9 = \{u, v, xy\}, && T_{10} = \{u, x, uv\}, \\ &T_{11} = \{u, y, uv\}, && T_{12} = \{u, y, xu\}, \\ &T_{13} = \{v, x, xu\}, && T_{14} = \{v, y, xu\}, \\ &T_{15} = \{x, uv, xu\}, && T_{16} = \{u, uv, xy\}, \\ &T_{17} = \{x, xy, uv\}, && T_{18} = \{y, uv, xu\}, \\ &T_{19} = \{u, xy, xu\}, && T_{20} = \{v, xy, xu\} \end{aligned}$$

Observe that $T_6 = B_3$ of $\mathcal{H}_3 \in \mathfrak{G}(\mathcal{H})$, so a minimal TDG itself can be (the edge set of) a diagonal germ. In general, there also are disjoint TDGs, here e.g., $T_7 \cap T_9 = \emptyset$.

The following facts provide the connections to (minimal) diagonality.

Proposition 8: Let $\mathcal{H}, \mathcal{H}' \in \mathfrak{H}$ then:

(1) *$\mathcal{H} \in \mathfrak{H}_{\mathrm{diag}}$ iff $\mathfrak{G}(\mathcal{H}) \neq \emptyset$,*
(2) *$\mathcal{H} \in \mathfrak{H}_{\mathrm{mdiag}}$ iff $\mathfrak{G}(\mathcal{H}) = \{\mathcal{H}\}$,*
(3) *$\mathcal{H} \subseteq \mathcal{H}'$ implies $\mathfrak{G}(\mathcal{H}) \subseteq \mathfrak{G}(\mathcal{H}')$.*

PROOF. Let $\mathcal{H} \in \mathfrak{H}_{\mathrm{diag}}$ then it contains a minimal diagonal subhypergraph hence $\mathfrak{G}(\mathcal{H}) \neq \emptyset$. Reversely, let $\tilde{\mathcal{H}} \in \mathfrak{G}(\mathcal{H})$ then $\tilde{\mathcal{H}} \subseteq \mathcal{H}$ is diagonal so $\mathcal{H}$ is diagonal, so (1) is true. If $\mathcal{H} \in \mathfrak{H}_{\mathrm{mdiag}}$, by definition $\{\mathcal{H}\} \subseteq \mathfrak{G}(\mathcal{H})$. Let $\tilde{\mathcal{H}} \in \mathfrak{G}(\mathcal{H})$ then $\tilde{\mathcal{H}} \subseteq \mathcal{H}$ and $\tilde{\mathcal{H}} \in \mathfrak{H}_{\mathrm{mdiag}}$ hence $\tilde{\mathcal{H}} = \mathcal{H} \in \mathfrak{H}_{\mathrm{mdiag}}$. The converse direction is clear, so (2) is true; (3) is evident. □

Proposition 9: For $\mathcal{H} \in \mathfrak{H}_{\mathrm{diag}}$, one has $\delta(\mathcal{H}) \geq 2$ if $|\mathfrak{G}(\mathcal{H})| \geq 2$.

PROOF. Let $\mathcal{H}_j \in \mathfrak{G}(\mathcal{H})$, $j \in [2]$. If both are in one connected component $\mathcal{H}'$ of $\mathcal{H}$ then there is $b \in B(\mathcal{H}') \setminus B(\mathcal{H}_1)$ with $b \cap V(\mathcal{H}_1) \neq \emptyset$. Therefore $\delta(\mathcal{H}) \geq \delta(\mathcal{H}') \geq \delta(\{b\} \cup \mathcal{H}_1) \geq 2$ because at a common vertex of b and $V(\mathcal{H}_1)$ a bifurcation occurs. If both are in distinct components $\mathcal{H}'_j$, $j \in [2]$, then $\omega(\mathcal{H} \setminus \mathcal{H}'_1) \geq 2$. So by La. 1 (ii) in [16] $\delta(\mathcal{H}) = \delta(\mathcal{H} \setminus \mathcal{H}'_1)\omega(\mathcal{H}'_1) + \delta(\mathcal{H}'_1)\omega(\mathcal{H} \setminus \mathcal{H}'_1) - \delta(\mathcal{H}'_1)\delta(\mathcal{H} \setminus \mathcal{H}'_1) = \delta(\mathcal{H} \setminus \mathcal{H}'_1)(\omega(\mathcal{H}'_1) - \delta(\mathcal{H}'_1)) + \delta(\mathcal{H}'_1)\omega(\mathcal{H} \setminus \mathcal{H}'_1)$. Here the factor in the brackets equals $1 + \rho(\mathcal{H}'_1) \geq 1$, further $\delta(\mathcal{H}'_1) \geq 1$, and also $\delta(\mathcal{H} \setminus \mathcal{H}'_1) \geq \delta(\mathcal{H}'_2) \geq 1$. So $\delta(\mathcal{H}) \geq 3$. □

If $\mathcal{H} = (V, B)$ is diagonal, and T is one of its minimal TDGs, we set $\mathcal{H} \setminus T := (V, B \setminus T)$ in accordance with the next fact.

Lemma 7: Let $\mathcal{H} = (V, B) \in \mathfrak{H}_{\mathrm{diag}}$. If a minimal TDG of $\mathcal{H}$ is removed from B then the resulting BHG has the same vertex set V.

PROOF. First observe that since T is minimal, for every $b \in T$, there is $\tilde{\mathcal{H}} \in \mathfrak{G}(\mathcal{H})$ s.t. $T \cap B(\tilde{\mathcal{H}}) = \{b\}$. Let $\mathcal{H}' = (V', B') = \mathcal{H} \setminus T$ and suppose there is $x \in V \setminus V'$. Then there is $\tilde{b} \in T$ containing x, and $\tilde{\mathcal{H}} = (\tilde{V}, \tilde{B}) \in \mathfrak{G}(\mathcal{H})$ with $\tilde{B} \cap T = \{\tilde{b}\}$. Hence $\tilde{\mathcal{H}} \setminus \{\tilde{b}\} \subseteq \mathcal{H}'$ fails to be diagonal and $x \notin V(\tilde{\mathcal{H}} \setminus \{\tilde{b}\})$. So, extending a transversal in $\mathcal{F}(\tilde{\mathcal{H}} \setminus \{\tilde{b}\})$ over $\tilde{b}$ by an arbitrary clause $\tilde{c}$ yields a transversal in $\mathcal{F}(\tilde{\mathcal{H}})$ wherein $\tilde{c}$ can be solved independently via x. Therefore $\mathcal{F}_{\mathrm{diag}}(\tilde{\mathcal{H}}) = \emptyset$ implying a contradiction. □

Towards a characterization of maximal non-diagonality the next result relying on the previous one turns out to be useful.

Lemma 8: Let $\mathcal{H}' = (V, B') \in \mathfrak{H}_{\mathrm{diag}}$ and $\mathcal{H} = (V, B) \in \mathfrak{H}_{\mathrm{maxnd}}$ wrt. $\mathcal{H}'$ where $B \subseteq B'$ then $B' \setminus B$ is a minimal TDG of $\mathcal{H}'$.

PROOF. Let $T := B' \setminus B$ then by La. 7 $\mathcal{H} = \mathcal{H}' \setminus T$. First suppose that T fails to be a TDG of $\mathcal{H}'$. Then there is a minimal diagonal $\tilde{\mathcal{H}} = (\tilde{V}, \tilde{B}) \subseteq \mathcal{H}'$ s.t. $\tilde{B} \cap T = \emptyset$ implying $\tilde{\mathcal{H}} \in \mathfrak{G}(\mathcal{H}' \setminus T) \neq \emptyset$. According to Prop. 8 (1) $\mathcal{H} \in \mathfrak{H}_{\mathrm{diag}}$, yielding a contradiction, hence T is a TDG of $\mathcal{H}'$.

Next let T and $\tilde{T} \subsetneq T$ be TDGs of $\mathcal{H}'$. Then there is $b \in T \setminus \tilde{T}$ and it is claimed that $\mathcal{H} \cup \{b\}$ remains non-diagonal yielding a contradiction to $\mathcal{H} \in \mathfrak{H}_{\mathrm{maxnd}}$ implying the minimality of T. To verify the claim, using Prop. 8 (1), (3) suppose that $\mathcal{H}_b = (V_b, B_b) \in \mathfrak{G}(\mathcal{H} \cup \{b\}) \subseteq \mathfrak{G}(\mathcal{H}')$. Since $\mathcal{H} \cup \{b\} = \mathcal{H}' \setminus (T \setminus \{b\}) \subseteq \mathcal{H}' \setminus \tilde{T}$ on behalf of Prop. 8 (3) it is $\mathcal{H}_b \in \mathfrak{G}(\mathcal{H}' \setminus \tilde{T})$. So a contradiction is provided because by definition $\tilde{T} \cap B_b \neq \emptyset$. □

Now we are able to formulate a general criterion for maximal non-diagonality.

Theorem 7: Let $\mathcal{H}' = (V, B') \in \mathfrak{H}_{\mathrm{diag}}$. Then $\mathcal{H} = (V, B) \subseteq \mathcal{H}'$ is maximal non-diagonal iff $B' \setminus B$ is a minimal

TDG of $\mathcal{H}'$.

PROOF. The necessity directly is implied by Lemma 8. For the sufficiency let $T := B' \setminus B$ be a minimal TDG of $\mathcal{H}'$ and $\mathcal{H} := \mathcal{H}' \setminus T$. According to La. 7 $V(\mathcal{H}) = V$. First suppose that $\mathcal{H} \in \mathfrak{H}_{\mathrm{diag}}$, and let $\tilde{\mathcal{H}} \in \mathfrak{G}(\mathcal{H})$. Since $\mathcal{H} \subseteq \mathcal{H}'$ by Prop. 8 (3) $\tilde{\mathcal{H}} \in \mathfrak{G}(\mathcal{H}')$ yielding a contradiction because $T \cap B(\tilde{\mathcal{H}}) \neq \emptyset$, so $\mathcal{H}$ is non-diagonal. Next, suppose that there is $b \in T$ s.t. $\mathcal{H} \cup \{b\}$ remains non-diagonal, i.e., $\mathfrak{G}(\mathcal{H} \cup \{b\}) = \emptyset$. Let $\tilde{\mathcal{H}} \in \mathfrak{G}(\mathcal{H}')$ be arbitrary. Then $B(\tilde{\mathcal{H}}) \cap (T \setminus \{b\}) \neq \emptyset$, otherwise $\tilde{\mathcal{H}} \subseteq \mathcal{H} \cup \{b\}$, so $\tilde{\mathcal{H}} \in \mathfrak{G}(\mathcal{H} \cup \{b\})$ which is impossible. Thus $T \setminus \{b\}$ is a TDG of $\mathcal{H}'$ contradicting the minimality of T and settling $\mathcal{H} \in \mathfrak{H}_{\mathrm{maxnd}}$. □

As a direct consequence one obtains.

Corollary 2: For every diagonal BHG there is a non-diagonal sub-BHG which is maximal non-diagonal wrt. it: Let $\mathcal{H}' = (V, B') \in \mathfrak{H}_{\mathrm{diag}}$ then for every minimal TDG T of $\mathcal{H}'$ one has $\mathcal{H}' \setminus T \in \mathfrak{H}_{\mathrm{maxnd}}$. □

Also further dense maximal non-diagonal BHGs directly arise in general terms:

Corollary 3: For every $n \in \mathbb{N}$, $n \geq 2$, there is a dense maximal non-diagonal BHG wrt. $\mathcal{K}_n$, namely $\mathcal{K}_n \setminus T$, for every minimal TDG T of $\mathcal{K}_n$. □

In view of Thm. 2 so one abstractly also obtains further non-diagonal BHGs that fail to be maximal non-diagonal: For V with $|V| \geq 3$, let T, T' be disjoint TDGs of $\mathcal{K}_V$ then evidently $\mathcal{H} := \mathcal{K}_V \setminus (T \cup T') \in \mathfrak{H}_0 \setminus \mathfrak{H}_{\mathrm{maxnd}}$. A more concrete and also constructive result is stated next regarding (dense) maximal non-diagonal instances, of arbitrary size.

Theorem 8: For fixed $n \in \mathbb{N}$ one has:

(1) *There are connected 2-regular BHGs of size n, density 1, which are dense maximal non-diagonal wrt. $\mathcal{K}_n$, where $n \geq 2$.*
(2) *There are BHGs of size $3n$, density 1, which are maximal non-diagonal wrt. the disjoint union of n $\mathcal{K}_3$-instances.*
(3) *There are connected, except for one vertex 2-regular, BHGs of size $3n+1$, density 1, which are dense maximal non-diagonal wrt. $\mathcal{K}_{3n+1}$.*

PROOF. Let $n \in \mathbb{N}$, $n \geq 2$, be fixed. Set $\mathcal{H}_n = (V_n, B_n)$ with $V_n = \{x_j : j \in [n]\}$ and $B_n = \{x_1\} \cup \{x_j x_{j+1} : j \in [n-1]\}$. Then $\mathcal{H}_n$ is 2-regular, has n vertices and size n, hence is of density 1. Moreover $\mathcal{H}_n$ is non-diagonal because either of its transversals can be solved by the independent assignment of x_1 for the clause over the unique loop, and by x_{j+1} for the clause over $x_j x_{j+1}$, for all $j \in [n-1]$. Considering $F_0 := \{x_1\} \cup \{\bar{x}_j x_{j+1} : j \in [n-1]\} \in \mathcal{F}(\mathcal{H})$ obviously has $x_1 = 1 \Rightarrow x_2 = 1 \Rightarrow \cdots \Rightarrow x_n = 1$ as its unique model. Thus by Thm. 5 (2), $\mathcal{H}_n$ is dense maximal non-diagonal wrt. $\mathcal{K}_n$, so (1) is true.

Set $\mathcal{H}_j = (V_j, B_j)$ with $V_j = \{x_j, u_j, y_j\}$ and $B_j = \{y_j, u_j x_j, x_j y_j\}$. Then $\mathcal{H}_j \in \mathfrak{H}_0^{\mathrm{con}}$ because every transversal of it has exactly three clauses, each of which can be solved independently by one of the three variables. Moreover for $F_j := \{y_j, u_j \bar{x}_j, x_j \bar{y}_j\} \in \mathcal{F}(\mathcal{H}_j)$ one has $\mathcal{M}(F_j) = \{w_j\}$ where $w_j = u_j x_j y_j$, as can be verified easily. So, by Thm. 5 (2), $\mathcal{H}_j$ is dense maximal non-diagonal wrt. $\mathcal{K}_{V_j}$. Moreover it is assumed that the V_j are vertex-disjoint, for all $j \in [n]$. According to Prop. 7 (1) then it follows that the disjoint union $\mathcal{H}'_{(n)} := \bigcup_{j \in [n]} \mathcal{H}_j$ having size $3n$ and therefore density 1, is maximal non-diagonal wrt. $\bigcup_{j \in [n]} \mathcal{K}_{V_j}$, so (2) is true.

Let $\mathcal{H}'_{(n)}$, $\mathcal{H}_j$, F_j and w_j, $j \in [n]$, as defined previously. Set $\mathcal{H}_{(n)} = (V_{(n)}, B_{(n)}) := \mathcal{H}'_{(n)} \cup \{b_{(n)}\}$ where $b_{(n)} := u_1 \ldots u_n u$, and $u \notin V_j$, $j \in [n]$. Obviously $|V_{(n)}| = |B_{(n)}| = 3n+1$, so $\mathcal{H}_{(n)}$ has density 1, and it is claimed that $\mathcal{H}_{(n)} \in \mathfrak{H}_0^{\mathrm{con}}$ where the connectedness is ensured because of $b_{(n)}$. For $F' \in \mathcal{F}(\mathcal{H}_{(n)})$ arbitrary there are unique $F'_j \in \mathcal{F}(\mathcal{H}_j)$, $j \in [n]$, and $c \in W_{b_{(n)}}$ s.t. $F' = \{c\} \cup \bigcup_{j \in [n]} F'_j$. Let $w'_j \in \mathcal{M}(F'_j)$ then $w' := \bigcup_{j \in [n]} w'_j$ as disjoint union solves $\bigcup_{j \in [n]} F'_j$; c can be solved via u independently, so $\mathcal{H}_{(n)} \in \mathfrak{H}_0$. Finally, consider $F := \{\bar{u}_1 \ldots \bar{u}_n u\} \cup \bigcup_{j \in [n]} F_j$. Then obviously $\mathcal{M}(F) = \{\{u\} \cup \bigcup_{j \in [n]} w_j\}$ implying the dense maximal non-diagonality of $\mathcal{H}_{(n)}$ wrt. $\mathcal{K}_{3n+1}$ by Thm. 5 (2). Finally, every vertex of $\mathcal{H}_{(n)}$, except for u, occurs in exactly two edges. □

For strict maximal i-non-diagonal instances, cf. Def. 4 (2) and Remark 2, relying on loops one obtains:

Corollary 4: Let $n \in \mathbb{N}$. There are strict maximal i-non-diagonal BHGs

(1) *$\mathcal{H}_n$ wrt. $\mathcal{H}_n \cup \{x\}$, where $i = 2^{n-2}$, and $\{x\} \notin B(\mathcal{H}_n)$, $n \geq 2$,*
(2) *$\mathcal{H}_{(n)}$ wrt. $\mathcal{H}_{(n)} \cup \{\{x_j\} : j \in [n]\}$, where $i = 2^{3n-1}$, and $\{x_j\} \notin B(\mathcal{H}_{(n)})$, $j \in [n]$.*

PROOF. For verifying (1), assume $n \geq 2$ and refer to $\mathcal{H}_n$ as defined in the proof of Thm. 8 (1), setting $x := x_2$. By its dense maximal non-diagonality $\mathcal{H}_n \cup \{x_2\} \in \mathfrak{H}_{\mathrm{diag}}$ containing only one diagonal, even simple germ isomorphic to the $\mathcal{K}_2$ with the edge set $\{x_1, x_2, x_1 x_2\}$. Either of the further edges $x_k x_{k+1}$, $k \in [n-1] \setminus \{1\}$, provides two independent bifurcations for a fixed representative of the diagonal orbit of the germ implying $\delta(\mathcal{H}_n \cup \{x_2\}) = 2^{n-2}$.

Regarding (2) fix $n \in \mathbb{N}$ arbitrarily, and refer to $\mathcal{H}_j$, $j \in [n]$, $\mathcal{H}'_{(n)}$, $\mathcal{H}_{(n)}$, and $b_{(n)}$ as defined in the proof of Thm. 8 (2), (3). For fixed $j' \in [n]$, first observe that $\hat{\mathcal{H}}_{j'} := \mathcal{H}_{j'} \cup \{x_{j'}\}$ with the edge set $\{x_{j'}, y_{j'}, u_{j'} x_{j'}, x_{j'} y_{j'}\}$ has only one diagonal germ $\tilde{\mathcal{H}}_{j'}$ belonging to $\mathfrak{H}_{\mathrm{simp}}^{\mathrm{con}}$ where $B(\tilde{\mathcal{H}}_{j'}) := \{x_{j'}, y_{j'}, x_{j'} y_{j'}\}$. It is claimed that $\delta(\hat{\mathcal{H}}_{j'}) = 2$. Indeed, there are exactly two possible bifurcations for a fixed representative of the single diagonal orbit of $\tilde{\mathcal{H}}_{j'}$, namely only one for each of the two literals over $x_{j'}$ in the clause over the additional edge $\{u_{j'} x_{j'}\}$, because $u_{j'} \notin V(\tilde{\mathcal{H}}_{j'})$. Since a clause over this edge thus is satisfiable by $u_{j'}$ independently, there is no satisfiable orbit-representative of $\tilde{\mathcal{H}}_{j'}$ that can become diagonal when enlarged over $\{u_{j'} x_{j'}\}$, so the claim is true.

Further, $\mathcal{H}'_{(n)} \cup \{x_{j'}\} = \hat{\mathcal{H}}_{j'} \cup \bigcup_{j \in [n] \setminus \{j'\}} \mathcal{H}_j$, where the components $\mathcal{H}_j \in \mathfrak{H}_0$, $j \in [n] \setminus \{j'\}$, and $\hat{\mathcal{H}}_{j'} \in \mathfrak{H}_{\mathrm{diag}}$ are mutually vertex-disjoint; also $\omega(\mathcal{H}_j) = 4$, $j \in [n]$. Again applying La. 1 (ii) in [16] straightforwardly yields $\delta(\mathcal{H}'_{(n)} \cup \{x_{j'}\}) = \delta(\hat{\mathcal{H}}_{j'}) \prod_{j \in [n] \setminus \{j'\}} \omega(\mathcal{H}_j) = 2^{2n-1}$, for all $j' \in [n]$. By definition $\mathcal{H}_{(n)} = \mathcal{H}'_{(n)} \cup \{u_1 \ldots u_n u\}$, so

$$\delta(\mathcal{H}_{(n)} \cup \{x_{j'}\}) = \delta(\mathcal{H}'_{(n)} \cup \{x_{j'}\}) 2^n = 2^{3n-1}$$

for all $j' \in [n]$. Here it is used that no satisfiable transversal over $\mathcal{H}'_{(n)} \cup \{x_{j'}\}$ can become unsatisfiable by its extension over $b_{(n)}$ having the unique vertex u. Further, for a fixed representative of a diagonal orbit over $\mathcal{H}'_{(n)} \cup \{x_{j'}\}$ there are exactly 2^n bifurcations for the literals over $u_j \in b_{(n)}$, $j \in [n]$. □

Definition 7: For $\mathcal{H} = (V, B)$ and $x \notin V$ let $\mathcal{H}^{\uparrow x} = (V^{\uparrow x}, B^{\uparrow x})$ where $V^{\uparrow x} := V \cup \{x\}$ and $B^{\uparrow x}$ is obtained *from B by adding the loop $\{x\}$ and enlarging every $b \in B$ about x. Let $\mathcal{H}^{\uparrow x}$ be refered to as the* x-lift *of $\mathcal{H}$. Similarly, given $F \in \mathcal{F}(\mathcal{H})$ let every $F' \in \mathcal{F}(\mathcal{H}^{\uparrow x})$ s.t. $F'[V] = F$ be called an* x-lift *of F.*

Observe that the x-lift of $\mathcal{H}$ contains a single loop, exactly one more edge and exactly one more vertex than $\mathcal{H}$. Obviously $\mathcal{H}^{\uparrow x} \in \mathfrak{H}^{\text{con}}$, $\mathcal{H}^{\uparrow x}[V] = \mathcal{H}$.

Lemma 9: For $\mathcal{H} = (V, B)$, $x \notin V$, one has:

(i) *$\mathcal{H} \in \mathfrak{H}_0$ iff $\mathcal{H}^{\uparrow x} \in \mathfrak{H}_0$.*
(ii) *$\mathcal{H} \in \mathfrak{H}_{\text{mdiag}}$ iff $\mathcal{H}^{\uparrow x} \in \mathfrak{H}_{\text{mdiag}}$.*
(iii) *$\delta(\mathcal{H}^{\uparrow x}) \geq \delta(\mathcal{H})$.*
(iv) *Let $\mathcal{H} \in \mathfrak{H}_{\text{diag}}$. Then $\delta(\mathcal{H}^{\uparrow x}) = \delta(\mathcal{H})$ iff $\mathcal{H} \in \mathfrak{H}_{\text{mdiag}}$.*

PROOF. If $\mathcal{H} \in \mathfrak{H}_0$ meaning $\mathcal{H}^{\uparrow x}[V] \in \mathfrak{H}_0$, so $\mathcal{H}^{\uparrow x} \in \mathfrak{H}_0$ because the unit clause of every $F' \in \mathcal{F}(\mathcal{H}^{\uparrow x})$ can be solved independently. Reversely, let $\mathcal{H}^{\uparrow x} \in \mathfrak{H}_0$, and suppose there is $F \in \mathcal{F}_{\text{diag}}(\mathcal{H})$. Then there is an x-lift $F' \in \mathcal{F}_{\text{diag}}(\mathcal{H}^{\uparrow x})$ by enlarging each clause of F about $\bar{x}$, and adding the unit clause $\{x\}$. So a contradiction occurs.

Addressing (ii) let $\mathcal{H} \in \mathfrak{H}_{\text{mdiag}}$ then $\mathcal{H}^{\uparrow x} \in \mathfrak{H}_{\text{diag}}$, because of (i). For arbitrary $F' \in \mathcal{F}_{\text{diag}}(\mathcal{H}^{\uparrow x})$ one has $F'[V] \in \mathcal{I}$ because $\mathcal{H}^{\uparrow x}[V] = \mathcal{H} \in \mathfrak{H}_{\text{mdiag}}$. For fixed $b' \in B^{\uparrow x}$ set $F := F' \setminus F'_{b'}$. Consider two cases. (1) $b' = \{x\}$: Then $F[V] = F'[V] \in \mathcal{F}(\mathcal{H}) \cap \mathcal{I}$, so choosing an arbitrary $c \in F[V]$, the partial assignment $w(V(c)) = c^{\gamma}$ can be extended to a model $w \in W_V$ of $F[V] \setminus \{c\}$ due to La. 1 (v). Thus $F \setminus \{c'\} \in \text{SAT}$, where $c'[V] = c$. Finally, $c' \in F$ can be satisfied via x.

(2) $b' \neq \{x\}$: Let $b := b' \setminus \{x\}$, then $F[V] \in \mathcal{F}(\mathcal{H} \setminus \{b\})$ is satisfiable by the minimal diagonality, hence every model $w \in W_V$ of $F[V]$ also satisfies $F(V)$. Finally, the remaining unit clause can be satisfied via x, so $F \in \text{SAT}$, implying $\mathcal{H}^{\uparrow x} \in \mathfrak{H}_{\text{mdiag}}$.

Reversely assume $\mathcal{H}^{\uparrow x} \in \mathfrak{H}_{\text{mdiag}}$ then $\mathcal{H} = \mathcal{H}^{\uparrow x}[V] \in \mathfrak{H}_{\text{diag}}$ by (i). Next let $b \in B$, $b' := b \cup \{x\}$, $F \in \mathcal{F}_{\text{diag}}(\mathcal{H})$ and suppose $\hat{F} := F \setminus F_b \in \text{UNSAT}$. Then there is a diagonal x-lift $\hat{F}' \in \mathcal{F}_{\text{diag}}(\mathcal{H}^{\uparrow x} \setminus \{b'\})$ of $\hat{F}$ yielding a contradiction, and verifying (ii).

In view of (i) assume $\mathcal{F}_{\text{diag}}(\mathcal{H}) \neq \emptyset$ then $\mathcal{F}_{\text{diag}}(\mathcal{H}^{\uparrow x}) \neq \emptyset$. Let $F_j \in \mathcal{F}_{\text{diag}}(\mathcal{H})$, and F'_j be a diagonal x-lift of F_j, $j \in [2]$. Suppose F_1, F_2 live in distinct G_V-orbits, but there is $X' \in G_{V^{\uparrow x}}$ s.t. $F_1'^{X'} = F'_2$. Since $x \notin V$ one has that $G_{V^{\uparrow x}}$ is the direct sum of (the abelian groups) G_V and $G_{\{x\}}$, so there are unique $X \in G_V, Y \in G_{\{x\}}$ with $X' = X \oplus Y$. Therefore one obtains $F_1^X = F'_1[V]^X = F'_2[V] = F_2$, yielding a contradiction and implying $\delta(\mathcal{H}^{\uparrow x}) \geq \delta(\mathcal{H})$.

Regarding (iv) assume $\mathcal{H} \in \mathfrak{H}_{\text{mdiag}}$ then also $\mathcal{H}^{\uparrow x} \in \mathfrak{H}_{\text{mdiag}}$ by (ii). Let $F'_j \in \mathcal{F}_{\text{diag}}(\mathcal{H}^{\uparrow x})$, $j \in [2]$, live in distinct $G_{V^{\uparrow x}}$-orbits, but suppose there is $X \in G_V$ s.t. $F'_1[V]^X = F'_2[V]$. If there is a non-unit $c \in F'_1$ containing the same x-literal as its unit clause then both clauses can be solved via x. So setting all remaining literals in c to 0 yields a partial assignment that can be extended to a model of $F'_1 \in \mathcal{I}$ due to La. 1 (v) yielding a contradiction.

Hence all non-unit clauses of F'_1 have the same literal over x which is the complement of that in its unit clause; the same holds for F'_2. In consequence there is a unique $Y \in G_{\{x\}}$ s.t. $Z := X \oplus Y \in G_{V^{\uparrow x}}$ with $F_1'^{Z} = F'_2$ providing a contradiction. Thus $\delta(\mathcal{H}^{\uparrow x}) \leq \delta(\mathcal{H})$ meaning $\delta(\mathcal{H}^{\uparrow x}) = \delta(\mathcal{H})$ relying on (iii).

Finally assume $\mathcal{H} \notin \mathfrak{H}_{\text{mdiag}}$ then with (ii) $\mathcal{H}^{\uparrow x} \notin \mathfrak{H}_{\text{mdiag}}$. Let $F \in \mathcal{F}_{\text{diag}}(\mathcal{H})$ be arbitrary, and F' be a diagonal x-lift of F. So there is $\tilde{\mathcal{H}} \in \mathfrak{G}(\mathcal{H}^{\uparrow x})$ and according to Prop. 8 (2) also $b' \in B^{\uparrow x} \setminus B(\tilde{\mathcal{H}})$. W.l.o.g. one can assume that the restriction of F' to $\tilde{\mathcal{H}}$ is unsatisfiable. Let F'' be obtained from F' via substituting $l(x) \in F'_{b'}$ by $\overline{l(x)}$. Then obviously F', F'' live in two distinct $G_{V^{\uparrow x}}$-orbits and also $F'' \in \text{UNSAT}$ because it has the same restriction to $\tilde{\mathcal{H}}$ as F'. Therefore $\delta(\mathcal{H}^{\uparrow x}) > \delta(\mathcal{H})$ finishing the proof by contraposition. □

Consider the 2-uniform $\mathcal{H} \in \mathfrak{H}_{\text{mdiag}}$ defined by x-connecting two FPP(1)-BHGs which is minimal diagonal due to La. 4 (i). Further, $\mathcal{H}$ has size 6 and 5 vertices. Removing a single edge provides a maximal non-diagonal BHG of size 5 and density 1. Iterating the lifting process starting with $\mathcal{H}$ and using in each step a new vertex x_j by La. 9 (ii), the minimal diagonality of every intermediate instance is maintained, for all $j \in [m-5]$.

Thus finally one obtains a minimal diagonal BHG of size $m+1$ that has exactly m vertices. Removing exactly one edge provides a maximal non-diagonal instance of size m and density 1 which in particular is loopless if the loop was removed.

Moreover since $\delta(\mathcal{H}) = 1$, by La. 9 (ii), (iv) also the x_j-lift in each iteration remains simple, for $j \in [m-5]$. In summary we constructively proved:

Proposition 10: For every $m \in \mathbb{N}$, $m \geq 5$, there is a (loopless) BHG of size m and density 1, that is maximal non-diagonal wrt. a member of the class $\mathfrak{H}^{\text{con}}_{\text{simp}}$. □

Adapting the previous proof using the 2-uniform instance of size 6 provided in the proof of La. 4 (ii) as the initial BHG, one concludes on basis of La. 9 (iv):

Corollary 5: There is a (loopless) BHG of size m, $m \geq 5$, which is maximal non-diagonal wrt. a member of $\hat{\mathfrak{H}}_3$.

Maximal non-diagonality becomes diagonality by increasing the BHG about exactly one edge. A retraction non-diagonal instance becomes diagonal by a retraction involving exactly one edge that fails to be equal to the vertex set. The inclusion-relation between those classes is stated in the next result.

Proposition 11: Maximal non-diagonality and retraction non-diagonality provide distinct, yet overlapping subclasses of $\mathfrak{H}_0$, more precisely:

(1) *There are maximal non-diagonal BHGs that are not retraction non-diagonal.*
(2) *There are loopless, maximal non-diagonal BHGs that are retraction non-diagonal. Moreover, diagonality resulting from a retraction in general does not mean minimal diagonality.*
(3) *There are loopless, retraction non-diagonal BHGs that fail to be maximal non-diagonal.*

PROOF. For (1) consider $\mathcal{K}_V \in \mathfrak{H}_{\text{mdiag}}$ with $V = \{x, y\}$ and $b := \{x\}$ then in view of La. 4, $\mathcal{H} := \mathcal{K}_V \setminus \{b\} \in \mathfrak{H}_{\text{maxnd}}$. $\mathcal{H}[V \setminus \{y\}]$ consists of the loop $\{x\}$ only hence is non-diagonal. Moreover, since $V \setminus \{x, y\} = \emptyset$, $\mathcal{H} \notin \mathfrak{H}_{\text{retnd}}$.

Regarding (2) consider $\mathcal{H} = (V, B) \in \mathfrak{H}_{\text{diag}}$, and $\mathcal{H}' = \mathcal{H} \setminus \{xy_2\} = (V, B') \in \mathfrak{H}_{\text{maxnd}}$ as defined in Example 2. Then for $b' = xy_1 \in B'$ the retraction $\mathcal{H}'' := \mathcal{H}'[V \setminus b'] \in \mathfrak{H}_{\text{diag}}$ because $\{y_3, y_4, y_3y_4\} \subsetneq B(\mathcal{H}'')$. Thus also $\mathcal{H}' \in \mathfrak{H}_{\text{retnd}}$. The second assertion of (2) is true because $\mathcal{H}''$ obviously is no minimal diagonal BHG.

For (3) let $\mathcal{H} \in \mathfrak{H}_{\text{xlin}}$ be isomorphic to FPP(2), hence $\mathcal{H} \in \mathfrak{H}_0$. According to La. 4 (ii) one has $\mathcal{H} \in \mathfrak{H}_{\text{retnd}}$, and due to Thm. 2, $\mathcal{H} \notin \mathfrak{H}_{\text{maxnd}}$. □

VI. A Connection To Maximal Satisfiable Formulas

Recall that for $C' \in \mathrm{UNSAT}$ a subformula $C \in \mathrm{SAT}$ is C'-maximal satisfiable if $C \cup \{c\} \in \mathrm{UNSAT}$, for every $c \in C' \setminus C$, as defined in [14]. As a non-trivial example consider the total clause set $K_{\mathcal{H}} \in \mathrm{UNSAT}$ over an arbitrary (not necessarily diagonal) $\mathcal{H} = (V, B)$. For every $w \in W_V$ with $K_{\mathcal{H}} \setminus w(B)$, a $K_{\mathcal{H}}$-maximal satisfiable formula is provided [14]. Note that the BHG of $K_{\mathcal{H}} \setminus w(B)$ equals $\mathcal{H}$, and more generally, there can arise maximal non-diagonal BHGs from a C'-maximal satisfiable formula C only in case $\mathcal{H}(C') \neq \mathcal{H}(C)$.

As another example, let $I \in \mathcal{I}$. For every $c \in I$, $I \setminus \{c\}$ is an I-maximal satisfiable formula [14]. So for $\mathcal{H} \in \mathfrak{H}_{\mathrm{mdiag}}$, every $F \in \mathcal{F}_{\mathrm{diag}}(\mathcal{H})$ provides an F-maximal satisfiable formula $F \setminus \{c\}$, for each $c \in F$.

Whether there are deeper connections between these concepts shall be addressed next. A useful concept here for the instances in UNSAT is the parameter $\mu(C) := \min\{|w(B(C)) \cap C| : w \in W_{V(C)}\} > 0$, together with $W^{\mu(C)}_{V(C)} := \{w \in W_{V(C)} : |w(B(C)) \cap C| = \mu(C)\}$ [14]. In case $\mu(C) = 0$ there is a compatible transversal in the complement formula which by Thm. 1 implies $C \in \mathrm{SAT}$. The next result regards transversals only:

Theorem 9: Let $C \in \mathrm{UNSAT}$ s.t. $|C_b| = 1$, for all $b \in B(C)$. If there is $w \in W^{\mu(C)}_{V(C)}$ s.t. $\mathcal{H}_0 := \mathcal{H}(C \setminus w(B(C)))$ is non-diagonal then $\mathcal{H}_0 \subsetneq \mathcal{H}(C)$ is maximal non-diagonal.

PROOF. The fibre-condition on C ensures that it is a transversal of its BHG, hence $\mathcal{H} := \mathcal{H}(C) =: (V, B) \in \mathfrak{H}_{\mathrm{diag}}$. Let $w \in W^{\mu(C)}_V$ s.t. $\mathcal{H}_0 := \mathcal{H}(C_0)$, where $C_0 := C \setminus w(B)$, is non-diagonal, hence $\mathcal{H}_0$ is a proper sub-BHG. According to the proof of Thm. 7 in [14] C_0 then is a C-maximal satisfiable formula.

Suppose there is $x \in V \setminus V_0$ then there also is $c \in C \setminus C_0$ containing a literal over x which can be satisfied independently of C_0. Thus $C_0 \cup \{c\} \in \mathrm{SAT}$ providing a contradiction hence $V = V_0$. By assumption $\mathcal{H}_0$ is non-diagonal. Let $b \in B \setminus B(\mathcal{H}_0)$ be arbitrary then there is $c \in C_b \cap W_b$ which is not in C_0. Because of the C-maximal satisfiability, $C_0 \cup \{c\} \in \mathrm{UNSAT}$ which according to the condition on C also is a transversal of $\mathcal{H}_0 \cup \{b\}$. Thus $\delta(\mathcal{H}_0 \cup \{b\}) > 0$, for every $b \in B \setminus B_0$, implying that $\mathcal{H}_0$ is maximal non-diagonal w.r.t. $\mathcal{H}$. ☐

Restricting μ to diagonal transversals induces the following parameters on diagonal BHGs.

Definition 8: Let $\mathcal{H} = (V, B)$ be diagonal. Let $\lambda(\mathcal{H}) := \min\{|w(B) \cap F| : w \in W_V, F \in \mathcal{F}_{\mathrm{diag}}(\mathcal{H})\}$ be the lower intersection index *of $\mathcal{H}$. Similarly the* upper intersection index *of $\mathcal{H}$ is defined by $\nu(\mathcal{H}) := \max\{|w(B) \cap F| : w \in W_V, F \in \mathcal{F}_{\mathrm{diag}}(\mathcal{H})\}$. Moreover let $\mathcal{W}^{\tau(\mathcal{H})} := \{(w, F) \in W_V \times \mathcal{F}_{\mathrm{diag}}(\mathcal{H}) : |w(B) \cap F| = \tau(\mathcal{H})\}$, for $\tau \in \{\lambda, \nu\}$.*

Regarding λ, ν as integer-valued mappings on $\mathfrak{H}_{\mathrm{diag}}$ one has.

Lemma 10: There is no upper bound for the values of ν on $\mathfrak{H}^{\mathrm{con}}_{\mathrm{diag}}$, and the lower bound of λ on $\mathfrak{H}_{\mathrm{diag}}$ is 1. Moreover, restricted to $\mathfrak{H}_{\mathrm{mdiag}}$, λ equals the constant 1.

PROOF. Regarding the first claim consider the minimal diagonal and even simple $\mathcal{K}_{V_0}$ with $V_0 := \{u, v\}$, so $B_0 := \{u, v, uv\}$. For the diagonal transversal $F_0 := \{u, v, \bar{u}\bar{v}\}$ and $w_0 := \{u, v\} \in W_{V_0}$ hence $w_0(B_0) = B_0$, one has $|w_0(B_0) \cap F_0| = 2 = \lceil |B_0|/2 \rceil = |B_0| - 1$ which also coincides with $\nu(\mathcal{K}_{V_0})$ as can be verified easily.

Let $s \in \mathbb{N}$ and $\mathcal{H}_j = (V_j, B_j) \in \mathfrak{H}$, $j \in [s]$, be arbitrary, mutually vertex-disjoint BHGs s.t. also $V_j \cap V_0 = \emptyset$, for all $j \in [s]$. Set $V := \bigcup_{j \in [s]} V_j$, $B := \bigcup_{j \in [s]} B_j$ and let $w \in W_V$, hence $w(B)$ yields a compatible transversal over B. Then $w'(B') := w_0(B_0) \cup w(B)$ provides $w' \in W_{V'}$ where $V' := V_0 \cup V$, $B' := B_0 \cup B$. Moreover $F' := F_0 \cup w(B) \in \mathcal{F}_{\mathrm{diag}}(\mathcal{H}')$ where $\mathcal{H}' := (V', B')$, and $|w'(B') \cap F'| = |B'| - 1$. Introduce a new variable u and select a single variable $u_j \in V_j$, for all $j \in [s]$. Defining a new edge $\hat{b} := uu_1 \cdots u_s$ and setting $\hat{B} := B' \cup \{\hat{b}\}$ yields a connected diagonal BHG $\hat{\mathcal{H}} = (\hat{V}, \hat{B})$. Finally, setting $\hat{w}(\hat{B}) := w'(B') \cup \{\hat{b}\}$ and $\hat{F} := F' \cup \{\hat{b}\}$ implies $\nu(\hat{\mathcal{H}}) \geq |\hat{w}(\hat{B}) \cap \hat{F}| = |\hat{B}| - 1 > s$, which is arbitrarily fixed.

The lower bound statement for λ directly follows from the last assertion. For its verification observe that $\lambda(\mathcal{H}) = \min\{\mu(F) : F \in \mathcal{F}_{\mathrm{diag}}(\mathcal{H})\}$. La. 1 (ii) directly implies $\mu(C) = 1$ for every $C \in \mathcal{I}$. Since $\mathcal{F}_{\mathrm{diag}}(\mathcal{H}) \subseteq \mathcal{I}$, for every minimal diagonal BHG, one obtains $\lambda | \mathfrak{H}_{\mathrm{mdiag}} = 1$. ☐

Proposition 12: For arbitrary $\mathcal{H}' \in \mathfrak{H}_{\mathrm{diag}}$ and every $(w, F) \in \mathcal{W}^{\nu(\mathcal{H}')}$ one has $\mathcal{H}(F \setminus w(B(\mathcal{H}))) \in \mathfrak{H}_0$.

PROOF. Let $\mathcal{H}' = (V, B)$ and $\mathcal{H}_0 = (V_0, B_0) := \mathcal{H}(F \setminus w(B))$ and suppose there is $F_0 \in \mathcal{F}_{\mathrm{diag}}(\mathcal{H}_0)$. Then by the definition of its diagonality there is $b \in B_0 \subsetneq B$ s.t. $F_{0b} = w(b)$ because $w(B_0)$ can be identified with a compatible transversal over B_0. Obviously $w(b) \neq F_b$. Hence extending F_0 to $\hat{F}_0$ over B by setting $\hat{F}_{0b'} := F_{b'}$, for all $b' \in B \setminus B_0$, yields a transversal in $\mathcal{F}_{\mathrm{diag}}(\mathcal{H}')$ with $|w(B) \cap \hat{F}_0| \geq |w(B) \cap F| + 1 > \nu(\mathcal{H}')$. Thus it appears a contradiction providing the claim. ☐

Theorem 10: For every $\mathcal{H} \in \mathfrak{H}_{\mathrm{diag}}$ one has:

1. *$\lambda(\mathcal{H}) = 1$,*
2. *$\nu(\mathcal{H}) > \lambda(\mathcal{H})$, if $\mathcal{H} \notin \mathfrak{H}_{\mathrm{mdiag}}$.*
3. *$\nu(\mathcal{H}) = \lambda(\mathcal{H})$ implies $\mathcal{H} \in \mathfrak{H}_{\mathrm{mdiag}}$ s.t. there is no pair $b, b' \in B(\mathcal{H})$ with $b \cap b' = \emptyset$.*

PROOF. Let $\mathcal{H} = (V, B)$ and $\tilde{\mathcal{H}} = (\tilde{V}, \tilde{B}) \in \mathfrak{G}(\mathcal{H})$ be fixed due to Prop. 8 (1). If $\mathcal{H} = \tilde{\mathcal{H}}$ we are done because of La. 10. Otherwise set $\mathcal{H}_0 := \mathcal{H} \setminus \tilde{\mathcal{H}} = (V_0, B_0)$ where $B_0 = B \setminus \tilde{B}$ and $V_0 = V(B_0)$ then $\tilde{B} \cap B_0 = \emptyset$ but, in general, $\tilde{V} \cap V_0 \neq \emptyset$.

Choose an arbitrary $(\tilde{w}_1, \tilde{F}_1) \in \mathcal{W}^{\lambda(\tilde{\mathcal{H}})}$. Evidently $G_{\tilde{V}}$ operates transitive on $W_{\tilde{V}}$, so there is $U \subset \tilde{V}$ s.t. $\tilde{w} := \tilde{w}_1^U \in W_{\tilde{V}}$ contains only negative literals. Moreover $\tilde{F} := \tilde{F}_2^U \in \mathcal{F}_{\mathrm{diag}}(\tilde{\mathcal{H}})$ because both lie in the same $G_{\tilde{V}}$-orbit; also one has $(\tilde{w}, \tilde{F}) \in \mathcal{W}^{\lambda(\tilde{\mathcal{H}})}$. Since $B_0 \in \mathcal{F}_{\mathrm{comp}}(\mathcal{H}_0)$ contains positive literals only, setting $w(B) := \tilde{w}(\tilde{B}) \cup B_0^\gamma$ yields a well-defined $w \in W_V$. Further, one has $F \in \mathcal{F}_{\mathrm{diag}}(\mathcal{H})$, for $F := \tilde{F} \cup B_0$, because $\tilde{F} \in \mathrm{UNSAT}$. It is claimed that $|w(B) \cap F| = 1$ proving $\lambda(\mathcal{H}) = 1$. To verify the claim, note that $|\tilde{w}(\tilde{B}) \cap \tilde{F}| = 1$ by La. 10. Further, by construction $B_0^\gamma \cap B_0 = \emptyset$, $\tilde{w}(\tilde{B}) \cap B_0 = \emptyset$ and $B_0^\gamma \cap \tilde{F} = \emptyset$ finishing the verification of (1).

Assume $\mathcal{H} \notin \mathfrak{H}_{\mathrm{mdiag}}$ then there are $\tilde{\mathcal{H}} = (\tilde{V}, \tilde{B}) \in \mathfrak{G}(\mathcal{H})$ and $\mathcal{H}_0 := \mathcal{H} \setminus \tilde{\mathcal{H}} \neq \emptyset$. Fix $F_0 \in \mathcal{F}_{\mathrm{comp}}(\mathcal{H}_0)$, $(\tilde{w}, \tilde{F}) \in \mathcal{W}^{\lambda(\tilde{\mathcal{H}})}$ s.t. analogous to the previous construction $w \in W_V$ is well-defined via $w(B) := \tilde{w}(\tilde{B}) \cup F_0$, and $F \in \mathcal{F}_{\mathrm{diag}}(\mathcal{H})$ via $F := \tilde{F} \cup F_0$. Since $|F_0| \geq 1$ one has $|w(B) \cap F| = |\tilde{w}(\tilde{B}) \cap \tilde{F}| + |F_0| \geq 2$ according to La. 10, hence $\nu(\mathcal{H}) > 1 = \lambda(\mathcal{H})$.

Regarding (3) let $\mathcal{H} \in \mathfrak{H}_{\mathrm{mdiag}}$ then $|B| > 1$. Assume there are $b_0, b_0' \in B$ with $b_0 \cap b_0' = \emptyset$ and let $F \in \mathcal{F}_{\mathrm{diag}}(\mathcal{H})$ be arbitrary. Define $w \in W_V$ via $w(b_0) = F_{b_0}$, $w(b_0') = F_{b_0'}$ and for all $b \in B \setminus \{b_0, b_0'\}$ by setting $w(b)$ s.t. $V(w(b_0) \cap w(b)) = b_0 \cap b$ and $V(w(b_0') \cap w(b)) = b_0' \cap b$. Hence $|F \cap w| \geq 2$. □

From the proof of (2) above one especially extracts:

Corollary 6: For $\mathcal{H} \in \mathfrak{H}_{\text{diag}}$ one has:
$\lambda(\mathcal{H}) = \min\{|B(\mathcal{H}) \cap F| : F \in \mathcal{F}_{\text{diag}}(\mathcal{H})\}$,
$\nu(\mathcal{H}) = \max\{|B(\mathcal{H}) \cap F| : F \in \mathcal{F}_{\text{diag}}(\mathcal{H})\}$. □

VII. A Non-Commutative Join-Operation

The simple instances provided in the proof of Prop. 10 contain loops. Thus its maximal non-diagonal derivatives are loopless only in very rare cases. In this section we are able to construct simple BHGs of arbitrary size and density (almost) 1 that are k-uniform hence especially loopless, for every fixed $k \geq 2$. To that end the following definition turns out to be useful. It provides the junction of two formulas resp. two BHGs below the level of set union, namely on the clause, respectively edge level:

Definition 9: For $m \in \mathbb{N}$, let $\text{CNF}(m) := \{C \in \text{CNF} : |C| = m\}$, and $\mathfrak{H}(m) := \{\mathcal{H} \in \mathfrak{H} : |\mathcal{H}| = m\}$.

(a) *For $C, D \in \text{CNF}(m)$ s.t. $V(C) \cap V(D) = \emptyset$, and $\sigma \in \text{Bij}(C, D)$, the σ-join of C and D is defined by*
$$C \otimes_\sigma D := \bigcup_{c \in C} [\{c \cup \{l\} : l \in \sigma(c)\} \cup \{\sigma(c)^\gamma\}] \in \text{CNF}$$

(b) *For vertex-disjoint $\mathcal{H}_j = (V_j, B_j) \in \mathfrak{H}(m)$, $j \in [2]$, and $\sigma \in \text{Bij}(B_1, B_2)$ let $\mathcal{H} = (V_1 \cup V_2, B_1 \otimes_\sigma B_2) =: \mathcal{H}_1 \otimes_\sigma \mathcal{H}_2 \in \mathfrak{H}$ be the σ-join of $\mathcal{H}_j$, $j \in [2]$.*

Part (b) of the definition directly relies on (a) by observing that the hyperedge set of a BHG can be identified with a formula (containing positive literals only). The operation $\otimes_\sigma$ neither is commutative nor it is associative on $\text{CNF}(m)$, respectively on $\mathfrak{H}(m)$ as can be verified easily.

Assuming that $C, D \in \text{CNF}(m)$ are transversals and that its clauses are uniquely labeled via the index set $[m]$, then the same labeling transfers to the members of $B_1 := B(C), B_2 := B(D)$, respectively. Hence $\sigma \in \text{Bij}(C, D)$ can be identified with a unique member of the symmetric group S_m, which directly yields the corresponding bijection in $\text{Bij}(B_1, B_2)$, and vice versa. Thus from Def. 9 directly one concludes:

Lemma 11: If $C, D \in \text{CNF}(m)$ with $V(C) \cap V(D) = \emptyset$ are labeled transversals over the corresponding equally labeled $B(C), B(D)$ then $\mathcal{H}(C \otimes_\sigma D) = \mathcal{H}(C) \otimes_\sigma \mathcal{H}(D)$, for every fixed $\sigma \in S_m$. □

Lemma 12: Let $m \in \mathbb{N}$, $\mathcal{H}_j = (V_j, B_j) \in \mathfrak{H}(m)$, $F_j \in \mathcal{F}(\mathcal{H}_j)$, $j \in [2]$, $\sigma \in \text{Bij}(B_1, B_2)$, $V := V_1 \cup V_2$, $V_1 \cup V_2 = \emptyset$ and $F := F_1 \otimes_\sigma F_2$. Then for $X \in G_{V_1}, Y \in G_{V_2}$, $Z \in G_V$ one has $F^Z = F_1^X \otimes_\sigma F_2^Y$ iff $Z = X \oplus Y$.

Proof. Since $V_j \subset V$ one has $G_{V_j} \leq G_V$ as a subgroup, $j \in [2]$. Since $G_{V_1} \cap G_{V_2} = \{\emptyset\}$, which is the neutral element G_V is the direct sum of G_{V_j}, $j \in [2]$, in the sense of (abelean) group theory. Thus for $Z \in G_V$ there are unique (disjoint) $X \in G_{V_1}, Y \in G_{V_2}$ s.t. $Z = X \oplus Y$ and vice versa. The claim therefore follows directly from the definition of the σ-join and recalling that $U \in G_V$ acts on a clause c via $c^U := c^{U \cap V(c)}$ meaning the complementation of exactly its literals over the variables in U, if existing. □

Recalling that a linear formula by definition is a transversal hence is required to be free of complementary unit clauses, several properties of the σ-join are collected next.

Proposition 13: For $m \in \mathbb{N}$, $C, D \in \text{CNF}(m)$ with $V(C) \cap V(D) = \emptyset$, $\sigma \in \text{Bij}(C, D)$, $J := C \otimes_\sigma D$ one has:

1. *$\mathcal{H}(J) \in \mathfrak{H}^{\text{con}}$ if $\mathcal{H}(C) \in \mathfrak{H}^{\text{con}}$ or $\mathcal{H}(D) \in \mathfrak{H}^{\text{con}}$.*
2. *$|V(J)| = |V(C)| + |V(D)|$, $|J| = \|D\| + m$, $\|J\| = 2\|D\| + \sum_{c \in C} |c| \cdot |\sigma(c)|$.*
3. *$J \in \text{UNSAT}$ iff $C \in \text{UNSAT}$ or $D \in \text{UNSAT}$.*
4. *$J \in \mathcal{I}$ iff $C \in \mathcal{I}$, and $D \in \text{SAT}$ s.t. for every $d \in D$, either of the partial assignments $d^\gamma, (d \setminus \{l\})^\gamma \cup \{l\} \in W_{V(d)}$, for every $l \in d$, can be extended to a model of $D \setminus \{d\}$.*
5. *J is a transversal iff D is a transversal and for every $b \in B(C)$ with $|C_b| > 1$ one has $V(\sigma(c)) \cap V(\sigma(c')) = \emptyset$, for all distinct $c, c' \in C_b$.*
6. *J is linear iff D is linear, s.t. for every non-unit clause $c \in C$ one has $|\sigma(c)| = 1$, moreover $\mathcal{H}(C) \in \mathfrak{H}_{\text{lin}}$, and C is linear up to possible pairs of complementary unit clauses, and for each such pair $c, c' \in C$ one has $V(\sigma(c)) \cap V(\sigma(c')) = \emptyset$.*
7. *J is exact linear iff $m = 1$ and the clause in C is unit or the clause in D is unit.*
8. *$\mathcal{H}(J)$ is k-uniform iff $\mathcal{H}(C)$ is $(k-1)$-uniform and $\mathcal{H}(D)$ is k-uniform, for fixed $k \in \mathbb{N}, k \geq 2$.*
9. *Let C, D be transversals. Then $\mathcal{H}(J)$ is k-uniform and r-regular iff $r, k \in \mathbb{N} \setminus \{1\}$, $r \equiv 0 \bmod 2$, $k|r$, moreover $\mathcal{H}(C)$ is $(k-1)$-uniform, $\frac{r}{k}$-regular, and $\mathcal{H}(D)$ is k-uniform, $\frac{r}{2}$-regular. Furthermore then $|V(C)| = \frac{k-1}{2}|V(D)|$, $k \cong 1 \bmod 2$, $k > 1$.*

Proof. For $c \in C$ with $d := \sigma(c) \in D$, the subformula $J^c := \{c \cup \{l\} : l \in d\} \cup \{d^\gamma\}$ of J is determined via copying c exactly $|d|$ times and enlarging each copy by exactly one distinct literal of d, finally adding the clause d^γ. Thus $\mathcal{H}(J^c) \in \mathfrak{H}^{\text{con}}$, and $\mathcal{H}(D) \subseteq \mathcal{H}(J)$from which (1) can be concluded easily. The first assertion of (2) is clear. Evidently $|J^c| = 1 + |\sigma(c)|$, for $c \in C$. Since σ is a bijection $J^{c_1} \cap J^{c_2} = \emptyset$ whenever $c_1, c_2 \in C$ are distinct. So $J = \bigcup_{c \in C} J^c$ is a disjoint union providing the second assertion of (2). Finally, $\|J^c\| = |d| \cdot |c| + 2|d|$ yielding the last statement.

Regarding (3) because of $D^\gamma \subseteq J$ one has $J \in \text{UNSAT}$ if $D \in \text{UNSAT}$. Now let $C \in \text{UNSAT}$ and suppose there is a model $w \in W_{V(J)}$ of J. Then there is $c \in C$ with $\sigma(c) =: d \in D$ s.t. $w(V(c)) = c^\gamma$ otherwise $C \in \text{SAT}$. Therefore either of $c \cup \{l\} \in J^c$ can be satisfied only via setting l to 1, for every $l \in d$, meaning $w(V(d)) = d$. Hence $d^\gamma \in J^c$ is unsatisfied yielding a contradiction. Reversely assume that $C, D \in \text{SAT}$ with (disjoint) partial assignments $w_C \in W_{V(C)}$, $w_D \in W_{V(D)}$ satisfying C, respectively $D^\gamma \subseteq J$. Obviously the retraction $J'[V(C)]$ of the remaining subformula $J' := J \setminus D^\gamma$ also is satisfied by w_C which therefore satisfies J', too. Thus $w_C \cup w_D \in W_{V(J)}$ is a model of J.

For the if-direction of (4), let $C \in \mathcal{I}$ and $D \in \text{SAT}$ then $J \in \text{UNSAT}$ according to (3). For arbitrary $t \in J$ there are unique $c \in C$, $d := \sigma(c) \in D$ s.t. $t \in J^c$; let $\tilde{J} := J \setminus J^c, \hat{J} := J \setminus \{t\}$. Evidently $C \setminus \{c\} \in \text{SAT}$, so let $w_C \in W_{V(C)}$ be a corresponding (partial) model, hence also satisfying the retraction $\tilde{J}[V(C)] \in \text{SAT}$, and therefore $\tilde{J} \setminus D^\gamma$, too. According to La. 1 (v) one has $w_C(V(c)) = c^\gamma$ and either there is $l \in d$ with $t = c \cup \{l\}$ or $t = d^\gamma \in D^\gamma$. In the latter case each clause in $J^c \setminus \{t\}$ must be solved by its unique literal over $V(d)$, i.e., forcing the partial assignment

$d = t^\gamma$. By assumption it can be extended to a (partial) model $w_D \in W_{V(D)}$ satisfying also the remaining formula $D^\gamma \setminus \{t\}$ yielding $w_C \cup w_D \in \mathcal{M}(\hat{J})$. Finally, if $t = c \cup \{l\}$, all clauses in $J^c \setminus \{t\}$, especially d^γ have to be satisfied via $(d \setminus \{l\}) \cup \{\bar{l}\} \in W_{V(d)}$. By assumption this partial assignment can be extended to a (partial) model $w_D \in W_{V(D)}$ for $D^\gamma \setminus \{d^\gamma\}$. Hence $w_C \cup w_D \in \mathcal{M}(\hat{J})$ implying $J \in \mathcal{I}$.

Reversely assume $C \notin \mathcal{I}$ then either (i): $C \in \mathrm{SAT}$ which in case of $D \in \mathrm{SAT}$ means $J \in \mathrm{SAT}$ due to (3). If $D \in \mathrm{UNSAT}$ then $D^\gamma \in \mathrm{UNSAT}$, so removing a clause from $J \setminus D^\gamma$ remains the resulting formula unsatisfiable, therefore $J \notin \mathcal{I}$. Or (ii): $C \in \mathrm{UNSAT}$ and there is $c \in C$ with $C \setminus \{c\} \in \mathrm{UNSAT}$ then $J \in \mathrm{UNSAT}$ via (3). Suppose there is $w \in \mathcal{M}(\tilde{J})$ where $\tilde{J} := J \setminus \{c \cup \{l\}\}$, for fixed $l \in \sigma(c)$. Then there is $\hat{c} \in C \setminus \{c\}$ s.t. $w(V(\hat{c})) = \hat{c}^\gamma$ otherwise $C \setminus \{c\} \in \mathrm{SAT}$. Therefore $\hat{c} \cup \{\hat{l}\} \in \tilde{J}^{\hat{c}}$ can be satisfied only via setting $\hat{l}$ to 1, for every $\hat{l} \in \sigma(\hat{c})$ meaning $w(V(\sigma(\hat{c}))) = \sigma(\hat{c})$. Hence $\sigma(\hat{c})^\gamma \in \tilde{J}^{\hat{c}}$ is unsatisfied establishing $J \notin \mathcal{I}$ by contradiction and yielding (4).

Since $D^\gamma \subseteq J$, J can be a transversal only if D is a transversal. Therefore and because of $V(C) \cap V(D) = \emptyset$, J is a transversal iff $(*)$: $\forall t, \hat{t} \in J \setminus D^\gamma$, $t \neq \hat{t}$: $V(t) \neq V(\hat{t})$. So, for every fixed $c \in C$ and distinct $t, \hat{t} \in J^c \setminus D^\gamma$, obviously $(*)$ is guaranteed because $|V(\sigma(c))| = |\sigma(c)|$. For distinct $c, \hat{c} \in C$ s.t. $V(c) \neq V(\hat{c})$ also $(*)$ is true for the corresponding $t = c \cup \{l\} \in J^c, \hat{t} = \hat{c} \cup \{\hat{l}\} \in J^{\hat{c}}$, where $l \in \sigma(c), \hat{l} \in \sigma(\hat{c})$. Finally, for distinct $c, \hat{c} \in C$ with $V(c) = V(\hat{c})$, $(*)$ is true iff $V(\sigma(c)) \cap V(\sigma(\hat{c})) = \emptyset$ proving (5).

Evidently J is linear iff D and $J \setminus D^\gamma$ are linear. The latter condition is equivalent with (i): $|C_b| = 1$, say $C_b = \{c\}$, and then also $|\sigma(c)| = 1$, for all $b \in B(C)$ with $|b| > 1$, meaning $\mathcal{H}(C) \in \mathfrak{H}_{\mathrm{lin}}$ and also that C is linear, except for possible pairs of complementary unit clauses. Since D is linear all those unit clauses corresponding to $|\sigma(c)| = 1$ are pairwise variable disjoint. And (ii): For every $b \in B(C)$ with $|b| = 1$ and $|C_b| = 2$, hence $C_b = \{c, c^\gamma\}$, it must be $V(\sigma(c)) \cap V(\sigma(c^\gamma)) = \emptyset$.

Regarding (7), J is exact linear iff J is linear and $V(t) \cap V(\hat{t}) \neq \emptyset$, for all $t, \hat{t} \in J$. According to the linearity of J, $|c| > 1 \Rightarrow |\sigma(c)| = 1$ equivalent with $|\sigma(c)| > 1 \Rightarrow |c| = 1$, for every $c \in C$. So a subformula J^c can be of the following forms only. (i): $\{c \cup \{l\}\} \cup \{\bar{l}\}$, with $\sigma(c) = \{l\}$. (ii): $\{c \cup \{l_j\} : j \in [|\sigma(c)|]\} \cup \{\sigma(c)^\gamma\}$ with $|c| = 1$, and $\sigma(c) = \{l_j : j \in [|\sigma(c)|]\}$ including the case $|\sigma(c)| = 1$. Suppose there are $J^c, J^{\hat{c}} \subseteq J$, for distinct $c, \hat{c} \in C$. If both are of type (i) then D is forced to have complementary unit clauses contradicting its linearity. If both are of type (ii) then one must have $c = \hat{c}^\gamma$ implying $V(\sigma(c)) \cap V(\sigma(\hat{c})) = \emptyset$ due to (6) contradicting the exact linearity. Finally, if J^c is of type (i), and $J^{\hat{c}}$ of type (ii) then $\sigma(\hat{c})$ cannot be a unit clause as above. But then l must occur in every clause of $J^{\hat{c}} \setminus \{\sigma(\hat{c})^\gamma\}$ of which there are at least 2 meaning that $\sigma(\hat{c})$ contains $l, \bar{l}$, so in summary one must have $m = 1$.

For (8), observe that since $\mathcal{H}(D) \subset \mathcal{H}(J)$ the k-uniformity of $\mathcal{H}(D)$ is necessary for that of $\mathcal{H}(J)$. Moreover, by definition every edge in $B(J) \setminus B(D)$ contains exactly one vertex of $V(D)$, thus $\mathcal{H}(C)$ must be $(k-1)$-uniform and $k \geq 2$. The reverse direction also is immediately implied by the definition of the σ-join.

Addressing (9) the uniformity conditions follow from (8). Since C, D are transversals, due to (5) J is a transversal, too. According to La. 3 in the regular case one has $o_J(x) = |J(x)| = \deg(\mathcal{H}(J))$, for every $x \in V(J)$. Then either $x \in V(D)$ or $x \in V(C)$. Since $D^\gamma \subseteq J$ and each clause of D occurs as $\sigma(c)$ in J, for every $x \in V(D)$ one has $o_J(x) = 2o_D(x) = 2\deg(\mathcal{H}(D)) = \deg(\mathcal{H}(J))$ due to the same lemma. That is equivalent with $\deg(\mathcal{H}(D)) = r/2$ if $\mathcal{H}(J)$ is r-regular, so r must be even. For $x \in V(C)$ one has in the regular and uniform cases $\deg(\mathcal{H}(J)) = \sum_{c \in C(x)} |\sigma(c)| = o_C(x)k = k\deg(\mathcal{H}(C))$ using La. 3. This is equivalent with $\deg(\mathcal{H}(C)) = \frac{r}{k}$, meaning that k must divide r. Moreover $\|C\| = \sum_{c \in C} |c| = (k-1)m = \sum_{x \in V(C)} o_C(x) = \frac{r}{k}|V(C)|$, and analogously $km = \frac{r}{2}|V(D)|$. Hence $|V(C)| = \frac{k-1}{2}|V(D)|$, so $k \cong 1 \mod 2$ finishing the proof. $\square$

The next statement provides BHGs ensuring the conditions of Prop. 13 (4) at least for transversals, and also the conditions of (5).

Corollary 7: Let C, D, J as in Prop. 13.

(1) *If C, D are transversals or $\mathcal{H}(D)$ is trivial, then J is a transversal.*
(2) *$J \in \mathcal{F}_{\mathrm{diag}}(\mathcal{H}(J)) \cap \mathcal{I}$ if $C \in \mathcal{F}_{\mathrm{diag}}(\mathcal{H}(C)) \cap \mathcal{I}$, $D \in \mathcal{F}(\mathcal{H}(D))$ and $\mathcal{H}(D)$ is an all-unique-vertex BHG.*

PROOF. The assertion (1) directly follows from Prop. 13 (5). Every $D \in \mathcal{F}(\mathcal{H}(D))$ is satisfiable because $\mathcal{H}(D)$ is an all-unique-vertex BHG. For the same reason the requirements of Prop. 13 (4) are fulfilled obviously, especially in the case that $\mathcal{H}(D)$ contains loops only, so (2) follows with Prop. 13 (4), and assertion (1). $\square$

Corollary 8: For $m \in \mathbb{N}$, let $\mathcal{H}_j = (V_j, B_j) \in \mathfrak{H}(m)$, $j \in [2]$, be vertex-disjoint, $\sigma \in \mathrm{Bij}(B_1, B_2)$ then for $\mathcal{H} = \mathcal{H}_1 \otimes_\sigma \mathcal{H}_2$ one has:

(1) *$\mathcal{H} \in \mathfrak{H}_{\mathrm{lin}}$ iff $\mathcal{H}_j \in \mathfrak{H}_{\mathrm{lin}}$, $j \in [2]$, s.t. for every non-loop $b \in B_1$ its image $\sigma(b) \in B_2$ is a loop.*
(2) *For every integer $k \geq 2$, $\mathcal{H}$ is k-uniform iff $\mathcal{H}_1$ is $(k-1)$-uniform and $\mathcal{H}_2$ is k-uniform.*

PROOF. In Prop. 13 (6), respectively (8), set $C := B_1$, $D := B_2$ evidently being (labeled) transversals implying $\mathcal{H}(J) = \mathcal{H}$ according to La. 11. So Prop. 13 (6), (8) implies (1), (2), respectively. $\square$

As shown above if both $C, D \in \mathrm{SAT}$ then also their σ-join is satisfiable. In general, that does not transfer to arbitrary transversals of σ-joined BHGs:

Remark 5: Let $\mathcal{H} = \mathcal{H}_1 \otimes_\sigma \mathcal{H}_2$, for vertex-disjoint $\mathcal{H}_j = (V_j, B_j) \in \mathfrak{H}_0(m)$, $j \in [2]$, where m and $\sigma \in \mathrm{Bij}(B_1, B_2)$ are fixed. Then, in general, $\mathcal{F}_{\mathrm{diag}}(\mathcal{H})$ fails to be empty which is already the case for $m = 2$, and $\mathcal{H}_1$, $\mathcal{H}_2$ are unique-vertex BHGs (cf. Def. 1): Let $\mathcal{H}_1$ with $V_1 = \{u, v\}$ consist of loops only, hence it is 1-uniform and even an all-unique-vertex, hence trivial BHG. Let $B_2 = \{xy, yz\}$ and $\sigma(u) = xy, \sigma(v) = yz$. Since $\mathcal{H}_2$ is 2-uniform, also $\mathcal{H}$ is 2-uniform with $B(\mathcal{H}) = \{ux, uy, xy, vy, vz, yz\}$. Evidently $\mathcal{H}$ is isomorphic to two FPP(1)-components that are y-connected implying $\mathcal{H} \in \mathfrak{H}^{\mathrm{con}}_{\mathrm{simp}} \subset \mathfrak{H}_{\mathrm{mdiag}}$ by La. 4 (i).

Theorem 11: For $m \in \mathbb{N}$, let $\mathcal{H}_j = (V_j, B_j) \in \mathfrak{H}(m)$, $j \in [2]$, be vertex-disjoint and $\sigma \in \mathrm{Bij}(B_1, B_2)$ be arbitrary. Moreover let $\mathcal{H}_1 \in \mathfrak{H}_{\mathrm{diag}}$, $\mathcal{H}_2$ be trivial, and $\mathcal{H} := \mathcal{H}_1 \otimes_\sigma \mathcal{H}_2$. Then $\mathcal{H} \in \mathfrak{H}_{\mathrm{mdiag}}$ iff $\mathcal{H}_1 \in \mathfrak{H}_{\mathrm{mdiag}}$. Moreover in this case $\delta(\mathcal{H}) = \delta(\mathcal{H}_1)$.

PROOF. W.l.o.g. let B_j, and $F_j \in \mathcal{F}(\mathcal{H}_j)$ be equally labeled over $[m]$, $j \in [2]$. Hence $\sigma \in S_m$ and $\mathcal{H} = \mathcal{H}(F_1 \otimes_\sigma F_2) =: (V, B)$ via La. 11.

First let $\mathcal{H}_1 \in \mathfrak{H}_{\rm mdiag}$ and $F_1 \in \mathcal{F}_{\rm diag}(\mathcal{H}_1) \subset \mathcal{I}$ then $F_1 \otimes_\sigma F_2 \in \mathcal{F}_{\rm diag}(\mathcal{H}) \cap \mathcal{I}$ by Cor. 7 (2). Hence $\mathcal{H} \in \mathfrak{H}_{\rm diag}$ and also $\mathcal{H} \in \mathfrak{H}^{\rm con}$ due to Prop. 13 (1) because $\mathcal{H}_1 \in \mathfrak{H}^{\rm con}$.

It is claimed that for $F \in \mathcal{F}(\mathcal{H})$ one has $F \in \mathcal{F}_{\rm diag}(\mathcal{H})$ iff there are $F_j, j \in [2]$, as above s.t. $F = F_1 \otimes_\sigma F_2$. From this claim one derives $\mathcal{F}_{\rm diag}(\mathcal{H}) \subset \mathcal{I}$ thus $\mathcal{H} \in \mathfrak{H}_{\rm mdiag}$.

Since the if-direction of the claim is clear by the previous argument, let $F \in \mathcal{F}_{\rm diag}(\mathcal{H})$ be arbitrary. For every $b \in B_2 \subset B$ considering its subformula $F(b)$, one has $F = \bigcup_{b \in B_2} F(b)$ as disjoint union, because $\mathcal{H}_2$ is an all-unique-vertex BHG. Further, let $\hat{F} = F \setminus F(V_1) = \bigcup_{b \in B_2} F_b \in \mathcal{F}(\mathcal{H}_2)$. Suppose there is $\tilde{b} \in B_2$ with $b' = \sigma^{-1}(\tilde{b}) \in B_1$ s.t. $|(F(\tilde{b}))[b']| > 1$, and let $\tilde{T} \in$ UNSAT be a transversal of the retraction $F[V_1] = \bigcup_{b \in B_2} (F(b))[\sigma^{-1}(b)]$ then $\tilde{T} \in \mathcal{F}_{\rm diag}(\mathcal{H}_1)$ because $F[V_1]$ is $\mathcal{H}_1$-based. Hence $\tilde{T} \in \mathcal{I}$ because $\mathcal{H}_1$ is minimal diagonal.

Substituting the clause $\tilde{T}_{b'}$ by another one in $(F(\tilde{b}))[b']$ yields another transversal T of $F[V_1]$ which however is satisfiable, by La. 6 in [16]. Let $w \in W_{V_1}$ be a model of T. Since $T = \bigcup_{b \in B_2} T_{\sigma^{-1}(b)}$, for every $b \in B_2$ there is a clause c_b of $F(b)$ containing $T_{\sigma^{-1}(b)}$ which is satisfied by w. Thus the unique literal over V_2 in c_b can be assigned to solve F_b of $\hat{F}$, for all $b \in B_2$, so satisfying $\hat{F}$. Finally the remaining literals over $b \in B_2$ exactly one in each of the clauses in $F(b) \setminus \{c_b, F_b\}$ can be assigned s.t. all these clauses are satisfied, so for all $b \in B_2$, because $\mathcal{H}_2$ is trivial. Hence $F \in$ SAT providing a contradiction and implying $|(F(b))[\sigma^{-1}(b)]| = 1$, for all $b \in B_2$, therefore $F[V_1] \in \mathcal{F}(\mathcal{H}_1)$. Also $F[V_1] \in$ UNSAT otherwise $F \in$ SAT because $\hat{F} \in$ SAT.

Next let $\tilde{F} := F(V_1)$ and suppose there is $\hat{b} \in B_2$ s.t. the clause $F_{\hat{b}}$ and a clause of $\tilde{F}(\hat{b})$, say $c_{\hat{b}}$, have a common literal l satisfying $c_{\hat{b}}, F_{\hat{b}}$. The remaining literals of $F_{\hat{b}}$ then can be assigned for solving all further clauses of $F(\hat{b})$. Let $w \in W_{\sigma^{-1}(\hat{b})}$ be a partial assignment setting all literals over V_1 in $c_{\hat{b}} \setminus \{l\}$ to 0. Since $F[V_1] \in \mathcal{I}$, w can be extended to a model of $F[V_1] \setminus \tilde{F}(\hat{b})[\sigma^{-1}(\hat{b})]$ due to La. 1 (v). Hence $F \setminus \hat{F}$ and F_b are satisfied but all literals in $\hat{F} \setminus F_b$ are unassigned. These can solve the remaining clauses of F because $\mathcal{H}_2$ is trivial meaning $F \in$ SAT. So by contradiction $\tilde{F}[b]$ and F_b fail to have a literal in common, for all $b \in B_2$. Thus $F = F[V_1] \otimes_\sigma \hat{F}$, where $F[V_1] \in \mathcal{F}_{\rm diag}(\mathcal{H}_1), \hat{F} \in \mathcal{F}(\mathcal{H}_2)$, and the claim is verified.

Conversely assume $\mathcal{H} \in \mathfrak{H}_{\rm mdiag}$ then $\mathcal{F}_{\rm diag}(\mathcal{H}) \subseteq \mathcal{I}$ but suppose $\mathcal{H}_1 \in \mathfrak{H}_{\rm diag} \setminus \mathfrak{H}_{\rm mdiag}$, especially meaning that $m > 1$. So there is $F_1 \in \mathcal{F}_{\rm diag}(\mathcal{H}_1) \setminus \mathcal{I}$ and let $F_2 \in \mathcal{F}(\mathcal{H}_2)$ be arbitrary then $F := F_1 \otimes_\sigma F_2 \in \mathcal{F}_{\rm diag}(\mathcal{H})$ by Prop. 13 (3). There is $c \in F_1$ s.t. $F_1' := F_1 \setminus \{c\} \in {\rm UNSAT} \cap \mathcal{F}(\mathcal{H}_1 \setminus \{V(c)\})$. Let $\sigma(c) =: d \in F_2$, $F_2' := F_2 \setminus \{d\} \in \mathcal{F}(\mathcal{H}_2 \setminus \{V(d)\})$ and $F^c := \{d^\gamma\} \cup \{c \cup \{l\} : l \in d\} \subset F$.

Then one has $\mathcal{H}_1 \setminus \{V(c)\} \in \mathfrak{H}_{\rm diag}$, $\mathcal{H}_2 \setminus \{V(d)\}$ is trivial, and both are members of $\mathfrak{H}(m-1)$ with $m-1 > 0$. Therefore $F_1' \otimes_{\sigma'} F_2' = F \setminus F^c \in$ UNSAT again due to Prop. 13 (3). Here $\sigma' \in {\rm Bij}(B_1 \setminus \{V(c)\}, B_2 \setminus \{V(d)\})$ denotes the restriction of σ to $B_1 \setminus \{V(c)\}$. So, $F \notin \mathcal{I}$ providing a contradiction and completing the proof of the first assertion.

Finally assume there are $F, F' \in \mathcal{F}_{\rm diag}(\mathcal{H})$ living in distinct G_V-orbits then as proven before there are $F_1, F_1' \in \mathcal{F}_{\rm diag}(\mathcal{H}_1)$, $F_2, F_2' \in \mathcal{F}(\mathcal{H}_2)$ s.t. $F = F_1 \otimes_\sigma F_2$, $F' = F_1' \otimes_\sigma F_2'$. Suppose there is $X \in G_{V_1}$ unique s.t. $F_1' = F_1^X$. Prop. 1 (4) implies $\omega(\mathcal{H}_2) = 1$, so there is $Y \in G_{V_2}$ with $F_2' = F_2^Y$ which by La. 12 yields $Z := X \oplus Y \in G_V$ with $F' = F^Z$, providing a contradiction. Hence $\delta(\mathcal{H}) \le \delta(\mathcal{H}_1)$. Conversely let $F_1, F_1' \in \mathcal{F}_{\rm diag}(\mathcal{H}_1)$ be members of distinct G_{V_1}-orbits, and assume there are $F_2, F_2' \in \mathcal{F}(\mathcal{H}_2)$ s.t. there is $Z \in G_V$ with $F' = F^Z$ where $F = F_1 \otimes_\sigma F_2$, $F' = F_1' \otimes_\sigma F_2'$. By La. 12 then there are $X \in G_{V_1}$ and $Y \in G_{V_2}$ with $Z = X \oplus Y$ implying $F_1' = F_1^X$ so $\delta(\mathcal{H}) = \delta(\mathcal{H}_1)$. $\square$

Corollary 9: For every $k \in \mathbb{N}$, $k \ge 2$, there is a k-uniform $\mathcal{H}_k \in \mathfrak{H}^{\rm con}_{\rm simp}$ of size $(k+1)!$, with $(k+1)! - 1$ vertices, so of density 1 asymptotically; further there is a k-uniform $\hat{\mathcal{H}}_k \in \mathfrak{H}_{\rm mdiag}$ of size $(k+1)!$, s.t. $\delta(\hat{\mathcal{H}}_k) = 3$.

PROOF. Proceeding by induction on k, the base is already provided e.g., by $\mathcal{H}_2$ of size 6 and 5 variables as defined in La. 4 (i). Next, let $\mathcal{H}_k = (V_k, B_k) \in \mathfrak{H}^{\rm con}_{\rm simp}$ be k-uniform with $k \ge 2$, $m_k := |\mathcal{H}_k| = (k+1)!$ edges and $n_k := |V_k| = m_k - 1$ vertices. Further, let $\mathcal{H}' = (V', B') \in \mathfrak{H}_0(m_k)$ s.t. $V_k \cap V' = \emptyset$ be a $(k+1)$-uniform, trivial BHG. Fixing $\sigma \in {\rm Bij}(B_k, B')$, according to Cor. 8 (2), yields the $(k+1)$-uniform $\mathcal{H}_{k+1} := \mathcal{H}_k \otimes_\sigma \mathcal{H}'$. Due to Thm. 11 one directly concludes $\mathcal{H}_{k+1} \in \mathfrak{H}^{\rm con}_{\rm simp}$. Moreover by Prop. 13 (2) and the induction hypothesis one has $m_{k+1} = \|\mathcal{H}'\| + m_k = (k+2)m_k = (k+2)!$, and also $n_{k+1} = n_k + (k+1)m_k = (k+2)! - 1$.

The second assertion is achieved analogously using as induction base the minimal diagonal, non-simple BHG with $\delta = 3$ as provided by La. 4 (ii). That BHG is 2-uniform, and of size 6 as above. Due to Thm. 11 the value $\delta = 3$ and also the minimal diagonality are inductive invariants. $\square$

The previous result in combination with La. 5 yields:

Corollary 10: For every $k \ge 2$, there are k-uniform BHGs of size $(k+1)! - 1$ in $\mathfrak{H}_{\rm maxnd}$ and also in $\hat{\mathfrak{H}}^{(3)}_{\rm maxnd}$. $\square$

VIII. Conclusions and Open Problems

The class of maximal non-diagonal BHGs has been investigated, containing all dense maximal non-diagonal instances as a proper subclass. Several structural properties could be revealed, although numerous questions remain open so far. So, observe that the BHG in the proof of the second statement of Thm. 4 contains a loop which one should substitute by a loopless instance, if possible.

Refering to La 4 (ii) the existence of a strict retraction 3-non-diagonal BHG is explicitly established. And in view of Prop. 3 for infinitely many i there are strict retraction i-non-diagonal BHGs relying on the results in [18], cf. e.g. Cor. 6,7, resp. Thm. 10. However clarifying the general case is still open. Regarding the dense maximal non-diagonality refering to Remark 4 we provided the existence of dense maximal non-diagonal BHGs wrt. $\mathcal{K}_n$, for all $n > 1$, relying on loops however. Due to Prop. 10 there are loopless dense maximal (i-)non-diagonal BHGs wrt. $\mathcal{K}_n$, for every $n > 1$, too. Refering to Def. 4 establishing $\hat{\mathfrak{H}}^{(i)}_{\rm maxnd} \ne \emptyset$ is open, for arbitrary $i \in \mathbb{N}$, whereas via Cor. 4 it is provided, for every $i = 2^{n-1}$, $n \in \mathbb{N}$, relying on instances containing loops.

Next, there arise several computational problems along with their complexities: First of all, given a BHG, determine the complexity to decide whether it is diagonal. This problem is closely related to the problem: Given $\mathcal{H}$ and $F \in \mathcal{F}(\mathcal{H})$, decide whether F is diagonal. Observe that testing whether F is compatible can be performed in linear time in the size of the formula relying on appropriate data structures. Suppose it could be efficiently decided whether a non-compatible F is satisfiable. Then also the decision whether a linear formula is satisfiable was easy. However, the latter problem is well known to be NP-complete [19]. Thus the decision whether a transversal is diagonal, in general is at least NP-complete, too. We conclude that a test for the diagonality of a BHG should not rely on SAT-testing of its transversals. And it arises the question whether there is another approach.

Also the complexity for deciding the minimal diagonality of an instance $\mathcal{H}$ is unknown. Here it might be helpful to clarify whether the criterion $\mathcal{F}_{\text{diag}}(\mathcal{H}) \subset \mathcal{I}$ for minimal diagonality of $\mathcal{H}$ could be relaxed to $\mathcal{F}_{\text{diag}}(\mathcal{H}) \cap \mathcal{I} \neq \emptyset$.

Next suppose it could be tested fast whether a transversal of a linear BHG is minimal unsatisfiable. Then the same was true for an arbitrary linear formula, implying that also SAT for linear formulas was decidable easily. So the question how minimal diagonality or at least simplicity could be decided efficiently remains open, especially in view of the fact that there even are arbitrary large simple BHGs by Cor. 9.

Due to Cor. 2 every diagonal BHG admits a maximal non-diagonal subhypergraph. Thus, given a diagonal $\mathcal{H}'$ and $\mathcal{H} \subset \mathcal{H}'$, the decision whether $\mathcal{H}$ is maximal non-diagonal could be performed by testing whether $B(\mathcal{H}') \setminus B(\mathcal{H})$ is a minimal TDG of $\mathcal{H}'$. Hence the complexity for deciding whether a given set of hyperedges forms a minimal TDG has to be investigated.

In view of Cor. 3 an investigation of the structure of the space of all minimal TDGs of $\mathcal{K}_n$, $n \in \mathbb{N}$, could be of interest. Refering to Prop. 11 (1), constructing a loopless example instead, would be of higher value.

Further, it is unclear whether $\nu(\mathcal{H})$ restricted to minimal diagonal BHGs has an upper bound depending on the size of the corresponding $B(\mathcal{H})$. Here we conjecture that this bound is given by $\lceil |B(\mathcal{H})|/2 \rceil$. Note that $\nu(\mathcal{K}_2)$ exactly equals this bound, cf. the proof of La. 10. In this context also Prop. 12 needs to be sharpened providing a condition for maximal non-diagonality. Given $\mathcal{H} = (V, B) \in \mathfrak{H}_{\text{diag}}$ refering to Def. 8, one also might ask whether there is a deeper structural relationship between instances $\mathcal{H}_j := \mathcal{H} \setminus w_j(B)$, for different pairs $(w_j, F_j) \in \mathcal{W}^{\nu(\mathcal{H})}$, $j \in [2]$.

The assumption $C \in \text{UNSAT}$, in general, fails to imply $\mathcal{H}(C) \in \mathfrak{H}_{\text{diag}}$. Moreover, μ does not equal 1 on all of UNSAT: As e.g., the total clause set $K_{\mathcal{H}}$ even of a non-diagonal BHG $\mathcal{H} = (V, B)$ fulfills $\mu(K_{\mathcal{H}}) = |B|$.

Additionally defining a function $\psi : \text{UNSAT} \to \mathbb{N}$ via $\psi(C) := \max\{|w \cap C| : w \in W_{V(C)}\}$ yields the relationship $\nu(\mathcal{H}) = \max\{\psi(F) : F \in \mathcal{F}_{\text{diag}}(\mathcal{H})\}$ to the upper intersection index of a diagonal BHG. Observe that also $\psi(K_{\mathcal{H}}) = |B|$, so the question whether there is an instance, for which both mappings, μ, ψ become equal, must be answered positive.

However it remains open whether there is a minimal diagonal $\mathcal{H}$ s.t. $\nu(\mathcal{H}) = \lambda(\mathcal{H}) = 1$. Recall that the lifted version $\mathcal{H}^{\uparrow x}$ of a minimal diagonal $\mathcal{H}$ remains minimal diagonal; it also fulfills the condition stated in Thm. 10 (3). But one easily verifies that $\nu(\mathcal{H}^{\uparrow x}) \geq \nu(\mathcal{H})$. E.g. concretely one has $\nu(\mathcal{K}_2^{\uparrow x}) = 2 = \nu(\mathcal{K}_2)$.

Refering to La. 11 on the one hand it could be interesting to investigate the structural aspects of the classes $\{\mathcal{H}_1 \otimes_\sigma \mathcal{H}_2 : \sigma \in S_m\}$, for fixed $m \in \mathbb{N}$, $\mathcal{H}_j \in \mathfrak{H}(m)$, $j \in [2]$. On the other hand, one can provide examples of non-transversals $C, D \in \text{CNF}(m)$ such that $C \otimes_\sigma D$ and therefore $\mathcal{H}(C \otimes_\sigma D)$ are well-defined, for every $\sigma \in \text{Bij}(C, D)$, but $\mathcal{H}(C) \otimes_\sigma \mathcal{H}(D)$ fails to exist for all $\sigma \in \text{Bij}(B(C), B(D))$.

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