Sufficient Conditions for Identical Synchronization in the Networks Constructed by Levels of n Reaction-Diffusion Systems of Hindmarsh-Rose Type with Linear Coupling

Phan Van Long Em

***Abstract*—This paper studies about the sufficient conditions for identical synchronization in the network constructed by levels consisting of n nodes. Each node is connected to other nodes in the upper level by linear coupling and represented by a reaction-diffusion system of Hindmarsh-Rose type. With this network topology, a sufficient condition on the coupling strength is identified to achieve the desired synchronization. The result shows that when the in-degree of the nodes grows, the network becomes easier to synchronize. In addition, the coupling strength needed to synchronize a chain network grows with its diameter.**

***Index Terms*—linear coupling, network constructed by levels, reaction-diffusion system of Hindmarsh-Rose type, identical synchronization.**

I. Introduction

SYNCHRONIZATION has been ubiquitously studied in many domains and many natural phenomena presenting the synchronization such as the movement of birds forming the cloud, the movement of fishes in the lake, the movement of the parade, the reception and transmission of a group of neurons [1], [4], [5], [6], [7], [8], and the identical synchronization in the neural network is investigated in this paper. In the human brain, there are a lot of neurons that connect together in order to form a network. A neural network is a community of neurons that are physiologically connected together. The exchange between cells is mainly based on electrochemical processes. In addition, this paper only considers the networks of n neurons coupled linearly, and each neuron is presented by a system of reaction-diffusion equations of Hindmarsh-Rose type (HR).

Recently, there have been a lot of research papers on synchronization of the neural network, but most of them only study cells stimulated by the equations of FitzHugh-Nagumo type [10], [11] or the system of ordinary differential equations of Hindmarsh-Rose type [2]. In [13], we have a published work about the system of reaction-diffusion equations of Hindmarsh-Rose type in complete networks with linear coupling, also with nonlinear coupling [14]. So, there is no study related to the system of reaction-diffusion equations of the Hindmarsh-Rose type in the networks constructed by level with linear coupling. Moreover, this type of network is more realistic than the complete networks. Hence, the research on this issue is literally meaningful and brings a practical application value to the currently applied mathematics, and this work is really an improvement compared to two previous published papers of the author [13].

In this article, we are interested in the identical synchronization in the network of n nodes constructed by levels. In order to understand such a type of network, we take one example of a network containing 42 nodes distributed in 9 levels that we can see in Fig. 1. Each node receives a signal only from nodes of the previous level. We know that a necessary condition for the synchronization of two nodes of a network is that either one of them must be influenced by the other one or both of them must be influenced by a third node. At network level, this implies the existence of at least one ”root” node from which all nodes can be reached. In this section, we use networks constructed by levels. Level 0 contains the root node. Nodes of level l (any integer) receive signal only from nodes of level $l-1, l \geq 1$. Thus the distance between the root and all the nodes of level l is exactly l. The simplest case of networks constructed by levels is a chain network in which each level contains only one node connected to the node of the previous level (see Fig. 2).

In this study, each node represents a neuron modeled by a system of reaction-diffusion equations on HR type and each edge represents a synaptical connection modeled by a coupling function. Hindmarsh-Rose model was actually obtained by simplifying the famous Hodgkin-Huxley model. In 1952, Hodgkin and Huxley introduced a four-dimensional mathematical system that could approximate many properties of neural membrane potential [2], [4], [7]. Based on this system, a lot of scientists published simpler models describing the neuron voltage dynamics. In 1982, Hindmarsh J. L. and Rose R. M. introduced a new simpler model called the Hindmarsh-Rose model [9]. This system was known as a simplified two-dimensional model from Hodgkin-Huxley's famous model [6]. Although this system is simpler, it has a lot of extraordinary analytical results and retains the energizing properties and biological significance of cells. It represents the equilibrium state, activity, and bursting of the neuron voltage. The system consists of two equations in the two variables u and v. The first variable is the fast one. It is excitatory and represents the transmembrane voltage. The second one is the slow recovery variable presenting some physical quantities, such as the electrical conductivity of ion currents across the membrane. The ordinary differential equations of Hindmarsh-Rose type are given by [2]:

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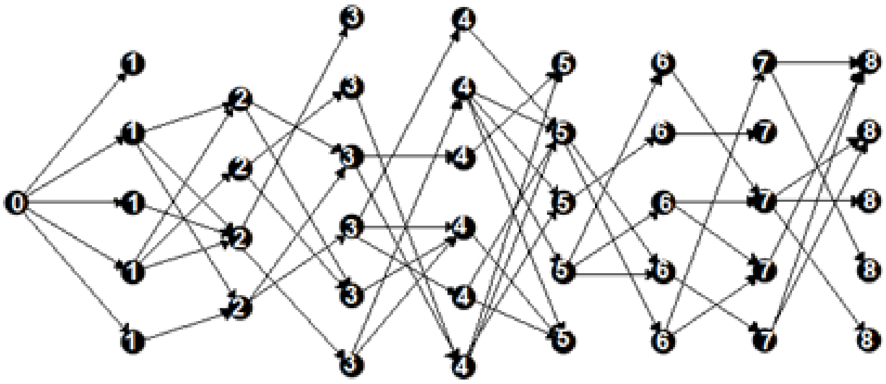

Fig. 1. Example of a network constructed by levels containing 42 nodes distributed in 9 levels. Each node receives signal only from nodes of the previous level.

$$\begin{cases} \dfrac{du}{dt} = u_t = v - u^3 + au^2 + I, \\ \\ \dfrac{dv}{dt} = v_t = 1 - bu^2 - v, \end{cases} \quad (1)$$

where the parameters $a = 3, b = 5$ are constants determined by practical experience, I presents the external current.

However, the system (1) is not strong enough in order to describe the propagation of action potential. In order to solve this problem, the cable equation is investigated in this work. This mathematical system is obtained from a circuit model of the membrane and its intracellular and extracellular space to provide a quantitative description of current flow and voltage change both within and between neurons. It allows us to understand how cells function quantitatively and qualitatively. Hence, the reaction-diffusion system of Hindmarsh-Rose type (HR) is considered as follows:

$$\begin{cases} \dfrac{du}{dt} = u_t = v - u^3 + au^2 + I + d\Delta u, \\ \\ \dfrac{dv}{dt} = v_t = 1 - bu^2 - v, \end{cases} \quad (2)$$

where $u = u(x,t),\ v = v(x,t),\ (x,t) \in \Omega \times \mathbb{R}^+,\ d$ is a positive constant, Δu is the Laplace operator of u, $\Omega \subset \mathbb{R}^N$ is a regular bounded open set with Neumann zero flux boundary conditions, and N is a positive integer. This model allows the appearance of many patterns and relevant phenomena in physiology. This model consists of two nonlinear partial differential equations. The first one presents the action potential and the second one introduces the recovery variable describing some physical quantities, such as the electrical conductivity of ion currents across the membrane. Besides, the first equation is similar to the cable equation. It presents the distribution of the membrane potential along the axon of a single cell [6], [7]. Hereafter, system (2) is considered as a neural model, and a network of n coupled systems (2) is constructed as follows:

$$\begin{cases} u_{it} = v_i - u_i^3 + au_i^2 + I + d\Delta u_i - h(u_i, u_j), \\ v_{it} = 1 - bu_i^2 - v_i, \\ i, j = 1, ..., n, i \neq j, \end{cases} \quad (3)$$

where $(u_i, v_i), i = 1, 2, ..., n$ is defined by (2).

Function h presents the coupling function describing the type of connection between cell ith and jth. Neurons connect through synapses, then it leads to two types of connections between cells such as chemical connections and electrical ones. It is known that the chemical connection is more abundant than the electrical one. For easy research, this paper only focuses on electrical connection, then the coupling function is linear [2], [10], [11] and is given by the following formula:

$$h(u_i, u_j) = g_{syn} \sum_{j=1, j\neq i}^{n} c_{ij}(u_i - u_j), \quad i = 1, 2, ..., n.$$

Parameter g_{syn} represents the coupling strength. The coefficients c_{ij} are the elements of the connectivity matrix $C_n = (c_{ij})_{n\times n}$, defined by: $c_{ij} = 1$ if neuron ith and jth are coupled, $c_{ij} = 0$ if neuron ith and jth are not coupled, where $i, j = 1, 2, ..., n, i \neq j$.

II. Identical synchronization in the networks constructed by levels

Identical synchronization is defined as the coincidence of states of interacting systems [1], [10], [11]. Synchronization usually means having the same behavior at the same time [1]. Therefore, the synchronization of two dynamical systems could be understood that one system copies the behavior of the other. In other words, if the behaviors of some dynamical systems are synchronized, these systems are called synchronous. In the studies of Aziz-Alaoui [1] and Corson [2], it is said that a phenomenon of synchronization may appear in a network of many weakly coupled oscillators. The phenomenon of synchronization can be seen in a lot of different applications such as increasing the power of lasers, controlling oscillations in chemical reactions, encoding electronic messages for secure communications, or synchronizing the output of electric circuits [1], [3]. Mathematically, we have the following definition of identical synchronization:

Definition 1 (see [10]). *Let $S_i = (u_i, v_i),\ i = 1, 2, ..., n$ and $S = (S_1, S_2, ..., S_n)$ be a network. We say that S is identically synchronous if*

$$\lim_{t\to+\infty} \sum_{i=1}^{n-1} \left(\|u_i - u_{i+1}\|_{L^2(\Omega)} + \|v_i - v_{i+1}\|_{L^2(\Omega)} \right) = 0,$$

where $L^2(\Omega)$ is function space on Ω defined using a natural generalization of the 2-norm for finite-dimensional vector spaces.

According to the description of the network constructed by levels, we can model such a network based on the HR system as follows:

$$\begin{cases} \varepsilon u_{1t} = v_1 - u_1^3 + au_1^2 + I + d\Delta u_1, \\ v_{1t} = 1 - bu_1^2 - v_1, \\ \varepsilon u_{it} = v_i - u_i^3 + au_i^2 + I + d\Delta u_i, \\ \qquad -g_j \sum_{u_k \in N(j-1)} \alpha_{ik}(u_i - u_k), \\ v_{it} = 1 - bu_i^2 - v_i, \\ i = 2, ..., n; j = 1, ..., l; 1 \le k < n, \end{cases} \tag{4}$$

where g_j is the coupling strength between neuron u_i of level j and neurons u_k of level $j-1$, $N(j)$ is the set of neurons of level j, and

$$\alpha_{ik} = \begin{cases} 0 & \text{if } u_i \text{ and } u_k \text{ are not connected}, \\ 1 & \text{if } u_i \text{ and } u_k \text{ are connected}. \end{cases}$$

Theorem 1. *If the coupling strength g_j verifies the condition:*

$$g_j \ge \max\left\{ \frac{a^2}{3N(\alpha_{icon}^{j-1})}, \frac{1}{4N(\alpha_{icon}^{j-1})\gamma} + \frac{(b-2a)^2}{4N(\alpha_{icon}^{j-1})(3-\gamma b^2)} \right\},$$

$j = 1, 2, ..., l$, where $0 < \gamma < \frac{3}{b^2}$, for all initial conditions $u_i(x, 0), v_i(x, 0), i = 1, 2, ..., n$, $N(\alpha_{icon}^{j-1})$ is the number of neurons of level $j-1$ connected to the neuron u_i of level j, the system (4) *will synchronize.*

Remark 1. Following this theorem, we find that more and more the number of neurons of level $j-1$ connected to the neuron u_i of level j is, less and less the threshold value of synchronization is.

Proof: We proceed by mathematical induction. We consider the following Lyapunov function:

$$\Phi_i^j(t) = \int_\Omega \left(\frac{1}{2} X^2 + \frac{\gamma}{2} Y^2 \right) dx,$$

where γ is a positive constant, $X = u_i - u_{icon}, Y = v_i - v_{icon}$ and $U = u_i + u_{icon}, i = 2, ..., n$, $u_i \in N(j), u_{icon} \in N(j-1)$ connected to u_i.

We show for all $j = 1, 2, ..., l$; $\frac{d\Phi_i^j(t)}{dt} \le 0$.

First of all, we consider $\Phi_i^1(t) = \int_\Omega \left(\frac{1}{2} X^2 + \frac{\gamma}{2} Y^2 \right) dx$, where $X = u_i - u_1, Y = v_i - v_1$ and $U = u_i + u_1, i = 2, ..., n$, $u_i \in N(1), u_{icon} = u_1$ connected to u_i.

We have then the system corresponding to the variables X, Y:

$$\begin{cases} \frac{dX}{dt} = Y - \frac{1}{4} X^3 + X(aU - \frac{3}{4} U^2 - g_1) + \Delta X, \\ \frac{dY}{dt} = -bXU - Y, \end{cases}$$

By deriving $\Phi_i^1(t)$ and using the Green's formula, we get:

$$\frac{d\Phi_i^1(t)}{dt} \le \int_\Omega \left(-\frac{X^4}{4} - \left(AX^2 - BXY + \gamma Y^2 \right) \right) dx,$$

where $A = \frac{3}{4} U^2 - aU + g_1, B = \gamma bU - 1$.

It can be seen that $AX^2 - BXY + \gamma Y^2 > 0$ if the following two conditions are verified:

(i) Since $A = \frac{3}{4} U^2 - aU + g_1$, the solutions of the equation $A = 0$ are $U_{1,2} = \frac{2\left(a \pm \sqrt{a^2 - 3g_1}\right)}{3}$ if $g_1 \le \frac{a^2}{3}$. Therefore, $A > 0$ if $g_1 > \frac{a^2}{3}$;

(ii) $\gamma A - \frac{B^2}{4} > 0 \Leftrightarrow (3 - \gamma b^2) U^2 - 2(a - 2b) U + 4g_1 - \frac{1}{\gamma} > 0$. This condition is satisfied if $g_1 > \frac{1}{4\gamma} + \frac{(b-2a)^2}{4(3-\gamma b^2)}$ and $\gamma < \frac{3}{b^2}$.

Then, if $g_1 \ge \max\left\{ \frac{a^2}{3}, \frac{1}{4\gamma} + \frac{(b-2a)^2}{4(3-\gamma b^2)} \right\}$, and $0 < \gamma < \frac{3}{b^2}$, we have $AX^2 - BXY + \gamma Y^2 > 0$.

Hence, $\frac{d\Phi_i^1(t)}{dt} \le 0$.

Suppose that the result is true until $l-1$. This implies that:

$$\frac{d\Phi_i^j(t)}{dt} \le 0, j = 1, 2, ..., l-1.$$

It implies that the origin is globally asymptotically stable for $\Phi_i^j(t), j = 1, 2, ..., l-1$.

We consider now the following function:

$$\Phi_i^l(t) = \int_\Omega \left(\frac{1}{2} X^2 + \frac{\gamma}{2} Y^2 \right) dx,$$

where γ is a positive constant, $X = u_i - u_{icon}, Y = v_i - v_{icon}$ and $U = u_i + u_{icon}, i = 2, ..., n$, neuron $u_i \in N(l), u_{icon} \in N(l-1)$ connected to u_i. By deriving $\Phi_i^l(t)$ and using the Green's formula, we obtain:

$$\begin{aligned} \frac{d\Phi_i^l(t)}{dt} \le \int_\Omega \Bigg[& -\frac{X^4}{4} - \left(AX^2 - BXY + \gamma Y^2 \right) \\ & - g_l \sum_{u_k \in N(l-1)} \alpha_{ik}(u_{icon} - u_k)(u_i - u_{icon}) \\ & + g_{l-1} \sum_{u_h \in N(l-2)} \alpha_{iconh}(u_{icon} - u_h)(u_i - u_{icon}) \Bigg] dx, \end{aligned}$$

where $A = \frac{3}{4} U^2 - aU + g_l N(\alpha_{icon}^{l-1}), B = \gamma bU - 1$, $N(\alpha_{icon}^{l-1})$ is the number of neurons on level $l-1$ connected to $u-i$ on level l. By applying the inequality of Young and Hölder, we can find the positive constants k_1 and k_2 such that:

$$\begin{aligned} \frac{d\Phi_i^l(t)}{dt} \le & \int_\Omega \left(-\frac{X^4}{4} - \left(AX^2 - BXY + \gamma Y^2 \right) \right) dx \\ & + k_1 \sum_{u_k, v_k \in N(l-1)} \alpha_{ik} \int_\Omega \Big[a\varepsilon (u_{icon} - u_k)^2 \\ & \quad + (v_{icon} - v_k)^2 \Big] dx \\ & + k_2 \sum_{u_h, v_h \in N(l-2)} \alpha_{iconh} \int_\Omega \Big[a\varepsilon (u_{icon} - u_h)^2 \\ & \quad + (v_{icon} - v_h)^2 \Big] dx \\ \le & \int_\Omega \left(-\frac{X^4}{4} - \left(AX^2 - BXY + \gamma Y^2 \right) \right) dx \\ & + \sum_{j=1,...,(l-1)} k_1 \Phi_i^j(t) + \sum_{j=1,...,(l-2)} k_2 \Phi_i^j(t). \end{aligned}$$

It can be seen that $AX^2 - BXY + \gamma Y^2 > 0$ if the following two conditions are verified:

(i) Since $A = \frac{3}{4}U^2 - aU + g_l N(\alpha_{icon}^{l-1})$, the solutions of the equation $A = 0$ are $U_{1,2} = \frac{2\left(a \pm \sqrt{a^2 - 3g_l N(\alpha_{icon}^{l-1})}\right)}{3}$ if $g_{syn} \le \frac{a^2}{3N(\alpha_{icon}^{l-1})}$. Therefore, $A > 0$ if $g_{syn} > \frac{a^2}{3N(\alpha_{icon}^{l-1})}$;

(ii) $\gamma A - \frac{B^2}{4} > 0 \Leftrightarrow (3 - \gamma b^2)U^2 - 2(a - 2b)U + 4g_l N(\alpha_{icon}^{l-1}) - \frac{1}{\gamma} > 0$. This condition is satisfied if

$$g_l > \frac{1}{4N(\alpha_{icon}^{l-1})\gamma} + \frac{(b-2a)^2}{4N(\alpha_{icon}^{l-1})(3-\gamma b^2)},$$

and $\gamma < \frac{3}{b^2}$.

Then, if

$$g_l \ge \max\left\{\frac{a^2}{3N(\alpha_{icon}^{l-1})}, \frac{1}{4N(\alpha_{icon}^{l-1})\gamma} + \frac{(b-2a)^2}{4N(\alpha_{icon}^{l-1})(3-\gamma b^2)}\right\}$$

and $0 < \gamma < \frac{3}{b^2}$, we have $AX^2 - BXY + \gamma Y^2 > 0$.

Hence, $\frac{d\Phi_i^l(t)}{dt} \le 0$. It implies that the origin is globally asymptotically stable for $\Phi_i^l(t)$. Hence, the neurons of network (4) is globally asymptotically synchronized. ■

Remark 2. As the result of Theorem 1, we can easily see that the coupling strength needs to reach certain threshold values in order to synchronize the network. Moreover, when the in-degree of the nodes grows, the network becomes easier to synchronize.

Following Theorem 1, we deduce the result into some particular networks, and we get some following corollaries.

Corollary 1. *Consider the chain network (for example, Fig. 2), suppose that:*

$$g_j \ge \max\left\{\frac{a^2}{3}, \frac{1}{4\gamma} + \frac{(b-2a)^2}{4(3-\gamma b^2)}\right\}, j = 1, 2, ..., l,$$

where $0 < \gamma < \frac{3}{b^2}$, *for all initial conditions* $u_i(x,0), v_i(x,0), i = 1, 2, ..., n$, *the system* (4) *will synchronize.*

In order to see clearly the influence of in-degrees which is equal to the number of edges with that node as the target, we consider the modified chain network (for example, see Fig. 3(a)) in which on level γ (any integer bigger than 1), we introduce k nodes, each node receives the signal of node on level $\gamma - 1$ and send the signal to node on level $\gamma + 1$. Thus, the in-degree of the root is 0, the in-degree of nodes on level $\gamma + 1$ is k and the in-degree of the others is 1. In this case, we have the following corollary.

Corollary 2. *Consider the modified chain network (for example, Fig. 3(a)) where the level* $\gamma - 1$ *contains* k *nodes. Suppose that:*

$$g_{\gamma+1} > \frac{M}{k} \quad \text{and } g_j > M, \ \ j = \{1, 2, ..., l\} \backslash (\gamma + 1).$$

Then, the network corresponding to Fig. 3(a) synchronizes.

Remark 3. As the result of Corollary 2, when the in-degree of nodes grows, the network becomes easier to synchronize.

We consider now a regular network, for example, Fig. 3(b). In such a network, all levels (except level 0) contain the same number of nodes k and each node is connected to all nodes on the previous level. To ensure the same in-degree for all nodes, the nodes on level 1 are connected to the root with k edges.

Corollary 3. *Consider the regular level network (for example, Fig. 3(b)) where each level contains k nodes connected to all the nodes from the previous level. Suppose that:*

$$g_j > \frac{M}{k}, \ \ j = 1, 2, ..., l.$$

Then, the network corresponding to Fig. 3(b) synchronizes.

III. Numerical results and discussion

This section focuses on finding numerically the minimal values of coupling strength to observe the synchronization between n subsystems modeling the function of neuron networks, and to verify the effectiveness of the theoretical results above. The integration is realized by using C++ and the results are represented by Gnuplot.

A. Example 1.

In this example, the paper shows the numerical results obtained by integrating the modified chain network (see Fig. 4) of linearly coupled reaction-diffusion systems of HR type, where

$$n = 4, a = 3, b = 5, I = 0, d = 1, i = 1, 2,$$
$$[0; T] \times \Omega = [0; 200] \times [0; 100] \times [0; 100].$$

Fig. 4 presents a modified chain network containing 4 nodes. Level 0 contains one node called root node, level 1 contains 2 nodes, and level 2 contains 1 node. Hence, the in-degree of 2 nodes on level 1 is 1, and the in-degree of the node on level 2 is 2.

In this section, we would like to compare the coupling strengths between different levels in order to check the effectiveness of Corollary 2. Then, we label the coupling strengths between node 1 and nodes on level 1 (node 2 and 3) as g_1, and the coupling strengths between nodes on level 1 (node 2 and 3) and node 4 as g_2.

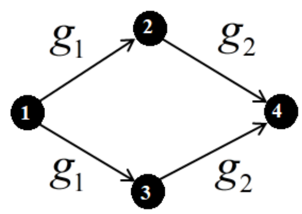

Fig. 4. A modified chain network where level 1 contains two nodes and level 2 contains one node.

Fig. 5 below presents the synchronization errors between nodes with respect to different values of coupling strength. Specifically, Fig. 5(a), 5(b), 5(c), 5(d) represent, respectively, the synchronization errors of the coupled solutions $(u_1(x_1, x_2, t), u_2(x_1, x_2, t))$, $(u_1(x_1, x_2, t), u_3(x_1, x_2, t))$,

Fig. 2. Chain network containing 10 neurons. In this network, neuron i ($i > 1$) receives only the signal from neuron $i - 1$.

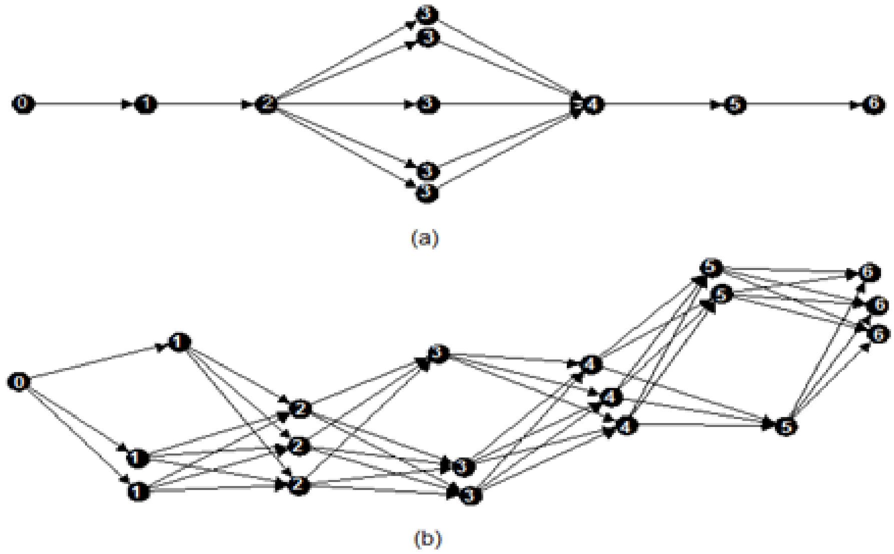

Fig. 3. Figure (a) is a modified chain network where level 3 contains five nodes. Figure (b) is a regular-level network where each level contains 3 nodes connected to all the nodes from the previous level.

$(u_2(x_1, x_2, t), u_4(x_1, x_2, t))$, $(u_3(x_1, x_2, t), u_4(x_1, x_2, t))$, where $g_1 = 1.3, g_2 = 1$, $t \in [0; T]$ and for all $(x_1, x_2) \in \Omega$. The simulations show that all synchronization errors reach zero, which means:

$$u_1(x_1, x_2, t) \approx u_2(x_1, x_2, t),$$

$$u_1(x_1, x_2, t) \approx u_3(x_1, x_2, t),$$

$$u_2(x_1, x_2, t) \approx u_4(x_1, x_2, t),$$

$$u_3(x_1, x_2, t) \approx u_4(x_1, x_2, t),$$

for all $(x_1, x_2) \in \Omega$. In other words, the network synchronizes.

Fig. 5(e), 5(f), 5(g), 5(h) represent, respectively, the synchronization errors of the coupled solutions $(u_1(x_1, x_2, t), u_2(x_1, x_2, t))$, $(u_1(x_1, x_2, t), u_3(x_1, x_2, t))$, $(u_2(x_1, x_2, t), u_4(x_1, x_2, t))$, $(u_3(x_1, x_2, t), u_4(x_1, x_2, t))$, where $g_1 = 1, g_2 = 1.3$, $t \in [0; T]$ and for all $(x_1, x_2) \in \Omega$. It is easy to see that all synchronization errors do not reach zero, which means the network does not synchronize in this case.

Similarly, we take more different values of coupling strengths g_1 and g_2 to get the synchronization. The numerical results are presented in Table I. After observation, in order to get the synchronization in such a network, the minimal values necessary of coupling strengths g_1 and g_2 should be 1.3 and 1, respectively.

When $g_1 = 1.3$ and g_2 go up to 1.2 or 1.3, the synchronization also happens, since the minimal value necessary of g_2 is equal to 1. In other words, the values 1.2 or 1.3 have already passed the threshold value of g_2. It implies that to get the synchronization of node 4 is easier than nodes 2 and 3. Remind that the in-degree of node 4 is 2, and the in-degree of nodes 2 or 3 is 1. It really matches with the result of Corollary 2.

When $g_2 = 1.3$ and g_1 goes down to 1 or 1.2, the synchronization does not occur. It means that the synchronization of nodes on level 1 is not easier than node 4 on level 2. It also shows that this result matches with Corollary 2.

When $g_1 = g_2 = 1.2$, the synchronization does not happen, since the value 1.2 is lower than the threshold value of g_1 to get the synchronization. And when $g_1 = g_2 = 1.5$ (or bigger), the synchronization certainly occurs, since this value is bigger than the threshold values of g_1 and g_2.

TABLE I
VALUES OF COUPLING STRENGTHS g_1 AND g_2 TO OBTAIN THE SYNCHRONIZATION

g_1	g_2	Synchronization
1.3	1.3	yes
1.3	1.2	yes
1.3	1	yes
1	1.3	no
1.2	1.3	no
1.2	1.2	no
1.5	1.5	yes

B. *Example 2.*

In the following, we label the coupling strength as g_n according to n which is the number of nodes of the network. The paper shows the numerical results obtained by integrating the chain network (see Fig. 2) of linearly coupled reaction-diffusion systems of HR type, where

$$n = 2, a = 3, b = 5, I = 0, d = 1, i = 1, 2,$$

$$[0; T] \times \Omega = [0; 200] \times [0; 100] \times [0; 100] .$$

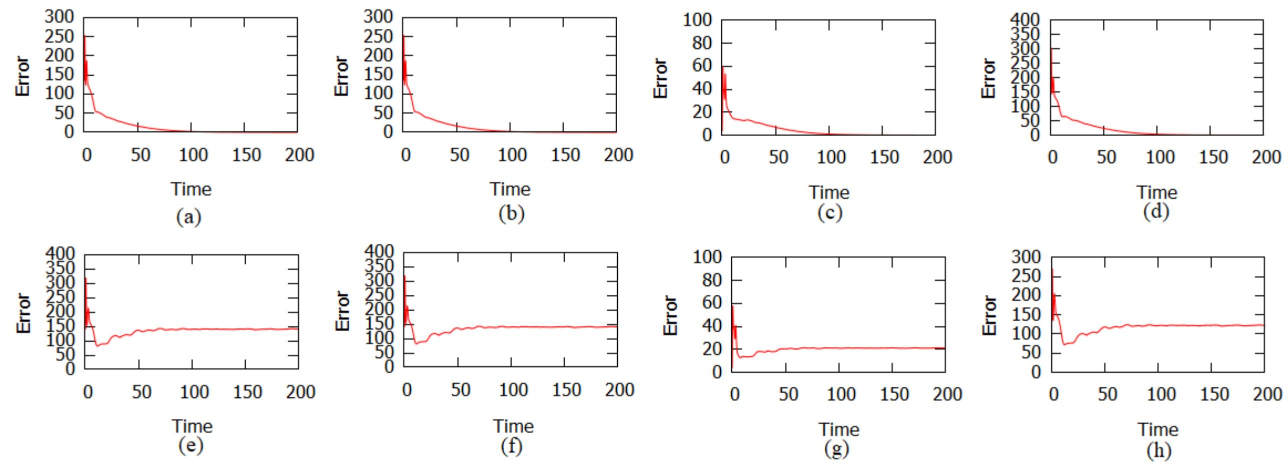

Fig. 5. Synchronization errors between nodes of modified chain network where level 1 contains two nodes and level 2 contains one node.

Fig. 6 below illustrates the identical synchronization of the chain network of 2 systems of reaction-diffusion equations of Hindmarsh-Rose type with linear coupling (see Fig. 2). The simulations show that the system synchronizes from the value $g_2 = 0.75$.

Fig. 6(a), 6(d), 6(g), 6(j) represent the synchronization errors of the coupled solutions $(u_1(x_1, x_2, t), u_2(x_1, x_2, t))$ where $t \in [0; T]$ and for all $(x_1, x_2) \in \Omega$.

In Fig. 6(j) with $g_2 = 0.75$, the simulation shows that the synchronization errors reach zero, which means:

$$u_1(x_1, x_2, t) \approx u_2(x_1, x_2, t),$$

for all $(x_1, x_2) \in \Omega$.

Fig. 6(b), 6(e), 6(h), 6(k) represent the solutions $u_1(x_1, x_2, 190)$, and Fig. 6(c), 6(f), 6(i), 6(l) represent the solutions $u_2(x_1, x_2, 190)$ of the chain network of 2 neurons from the moment when no synchronization has occurred until they have the same shape, i.e., the synchronization is performed.

Before synchronization with $g_2 = 0.1$, Fig. 6(a) represents the synchronization error between u_1 and u_2, for all $(x_1, x_2) \in \Omega$; Fig. 6(b) and 6(c) represent the solutions $u_1(x_1, x_2, 190)$ and $u_2(x_1, x_2, 190)$, respectively, when they are coupled together in the chain network; the results are similarly done for $g_2 = 0.3$ (Fig. 6(d), 6(e), 6(f)) and $g_2 = 0.5$ (Fig. 6(g), 6(h), 6(i)). For $g_2 = 0.75$ (Fig. 6(j), 6(k), 6(l)), the synchronization occurs. Since it is easy to see that the synchronization errors in Fig. 6(j) reach zero, and all patterns in Fig. 6(k), 6(l) are the same.

In other words, the coupling strength is over or equal to $g_2 = 0.75$, two linearly coupled neurons in the chain network have synchronous properties. By doing similarly for the chain networks in which the number of nodes gradually increases, the values of coupling strength with respect to the number of neurons n are reported in Table II. In Table II, for each value of n, we seek one necessary value of coupling strength to get the synchronization in the chain networks with respect to n from 2 to 20.

Based on these numerical experiments, it is clear to see that the coupling strength required to obtain the synchronization of n neurons depends on the number of neurons. Indeed, the points in Fig. 7 represent the coupling strength of synchronization with respect to the number of neurons in the chain networks from Table II, and we would like to find a relationship between the number of neurons n and the coupling strength reported in Table II. This relationship is presented by the following function:

$$g_n = 0.1n + 0.55. \quad (5)$$

In Fig. 7, the function (5) is represented by a line where the points corresponding to the coupling strengths are almost on. It means the coupling strength necessary to obtain the synchronization in the chain networks follows the law given by (5). These simulations show that the bigger the number of neurons is, the bigger the coupling strength is. In other words, the synchronization is more difficult to take place when the diameter of chain networks increases.

TABLE II
MINIMAL COUPLING STRENGTH NECESSARY TO OBSERVE THE SYNCHRONIZATION

n	2	3	4	5	6
g_n	0.75	0.85	0.94	1.045	1.14
n	7	8	9	10	11
g_n	1.23	1.35	1.44	1.54	1.65
n		12	13	14	15
g_n		1.74	1.845	1.94	2.04
n	16	17	18	19	20
g_n	2.14	2.245	2.34	2.445	2.55

IV. CONCLUSION

This study gave sufficient conditions on the coupling strength to achieve the synchronization in the networks constructed by levels of n linearly coupled systems of reaction-diffusion equations of Hindmarsh-Rose type. The result shows that when the in-degree of nodes grows, the network becomes easier to synchronize. Numerically, it displays that the synchronization is stable when the coupling strength exceeds a certain threshold and a compromise between the theoretical and numerical results can be reached. Especially, in the chain network, the bigger the number of neurons is, the more difficult the phenomenon of synchronization will be obtained. In other words, the value of coupling strength grows with its diameter. In addition, it is necessary to conduct further studies on the different synchronization regimes in free networks coupled with chemical synapses.

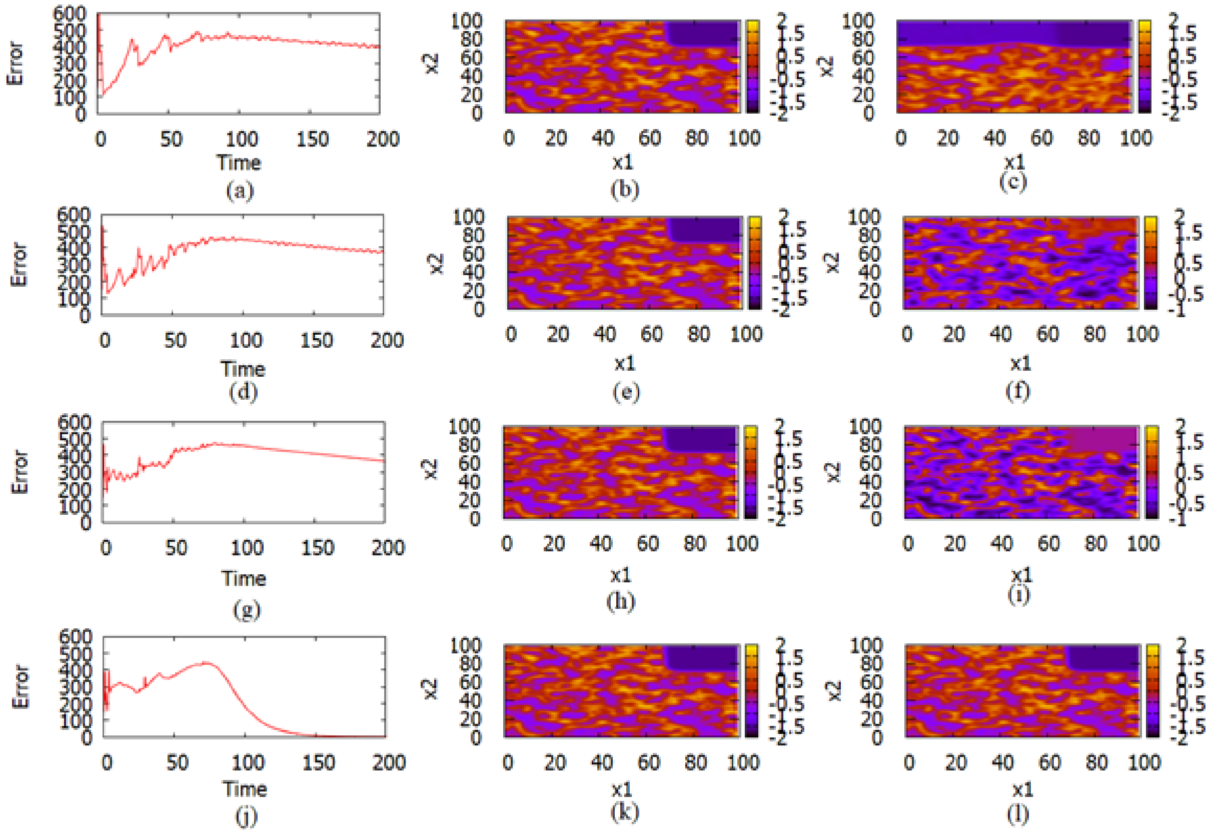

Fig. 6. Synchronization in the chain network of 2 electrically connected cells.

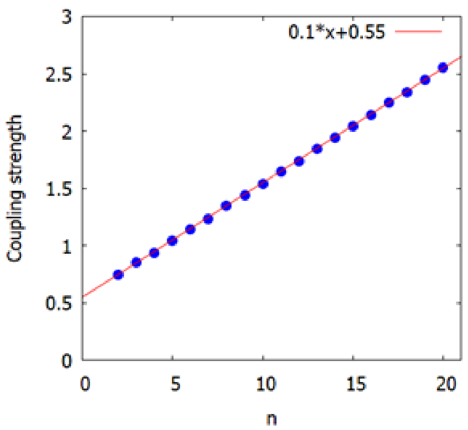

Fig. 7. The evolution of the coupling strength with respect to the number of neurons.

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