On the Symmetry between State Estimation and Reference Tracking

Zahrotul Amalia Solakha, *Member, IAENG,* Salmah

***Abstract*—Both constrained state estimation and constrained reference tracking could be formulated as optimization problems, and the symmetry between these two problems could be shown. As variables in the stochastic model, both the mean of process noise and the mean of measurement noise do not have to be zero. The main contributions of this paper are that we take the mean of process noise and the mean of measurement noise, which do not have to be zero, and show the symmetry between constrained state estimation and constrained reference tracking for this case. In other words, we study the symmetry between state estimation and reference tracking with an additional constraint on the measurement noise and the process noise. By this symmetric relationship, the constrained state estimation problem can be solved as a constrained reference tracking problem, and vice versa, despite the presence of more general process and measurement noise.**

***Index Terms*—estimation, optimal control, constraints, tracking systems, noise.**

I. Introduction

CONTROL and estimation theories can be applied in many fields, along with the development of science and technology with an increasing variety and complexity of problems. Estimation problems occur in many fields, including chemistry, physics, geology, mining, and transportation, to name a few. Some applications of estimation are the estimation of malaria mortality in a developing country [1], the estimation for a Dothan model [2] and hyperbolic model [3], the estimation of a regression curve [4], and applications to hydrology assessment [5]. In these fields, optimal control can also be applied to solve several problems, such as applications of tracking control for mobile robots [6], agricultural vehicles [7], and vehicle slip ratio based on speed tracking [8]. Tracking control can also be applied for magnetic levitation systems [9], intelligent tracing cars [10], six-rotor UAVs [11], and moving ground targets [12].

Some research has been conducted on the relationship of these two problems, control and estimation problems. For linear systems and problems without constraints, Kailath et al. investigated the relationship between these problems [13]. A problem without constraints here means that there is no limit value for every variable in the problem.

Control and estimation problems are generally problems with constraints on the variables with respect to particular regions. The relationship between constrained estimation and control problems was investigated by Goodwin et al. in [14], [15], [16], in which each of the control and estimation problems is formulated as an optimization problem.

Manuscript received August 1, 2023; revised March 24, 2024.

Z.A. Solakha is an independent researcher in Yogyakarta, Indonesia (email: za.solakha@gmail.com)

Salmah is a Professor at Department of Mathematics, Universitas Gadjah Mada, Yogyakarta, Indonesia (corresponding author to provide phone: +62 8886 835 039; fax: +62 274 513 339; e-mail: syalmah@yahoo.com)

In [14] and [15], both means of measurement noise and process noise are zero, while in [16], they do not have to be zero on each time step. Moreover, in [14] and [16], they assumed that the process noise is limited in a particular region. Meanwhile, in [15], the limitation value is applied not only in the process noise but also in the initial state and the measurement noise. Using the Lagrangian duality theory [17], estimation problems can also result in control problems. In this case, the estimation problem is a constrained problem, whereas the control problem is unconstrained.

In [16], state estimation problem as a primal problem, while control problem as a dual problem. Then, it was shown that a duality relation between these two problems is written in a theorem. In [16], the symmetry between these two problems was also discussed, known as a connection of the LQR (linear quadratic regulator) and linear quadratic state estimation problems. The symmetry in [16] was illustrated in primal and dual problem configurations.

The duality between these problems (estimation and control) allows that if we have a control problem, we can solve that problem by solving a particular estimation problem with a duality relationship with that control problem, and vice versa. Moreover, the formulation of this duality results in the optimizers of both problems being the same, as mentioned in [14] and [15]. Furthermore, a constrained estimation problem is symmetric to an unconstrained control problem in which constraints are not preserved.

The existence of constraints on these problems was further investigated by Mare and Dona [18], who discussed reference tracking problems as control problems, which differs from the discussions by Goodwin et al. in [14], [15], [16], who considered them as linear quadratic regulator (LQR) problems. The constrained reference tracking problem discussed by Mare and Dona [18] is symmetric with the constrained estimation problem. The symmetry property in this problem preserves the existence of constraints.

By the symmetric relationship of both constrained reference tracking and constrained state estimation, solving one problem can also solve the other. If there is a constrained estimation problem, then the reference tracking problem that is symmetric with the constrained estimation problem can be solved, and vice versa. In addition, we can solve a constrained estimation problem by solving the constrained reference tracking problem that is symmetric with the original problem.

Furthermore, the symmetry between these problems (constrained reference tracking and constrained estimation) allows the algorithms and methods that apply to one problem to be used for other problems. It becomes interesting in terms of the development of control and estimation theories.

The duality between estimation and control problems and the symmetry between estimation and reference tracking problems is an introduction to further study of the theory, as reported in [19], [20], [21], [22]. Muller et al. in [19] investigated the duality and symmetry between constrained estimation and control problems with more general constraints on the initial state, process noise, and measurement noise. In [20], Zhang and Song proposed a new estimator for multiplicative noise systems and the duality of control for it. They then applied this theory to package throwing and multiple input delay systems. Song et al. in [21] studied the optimal linear estimation and its duality for multiplicative noise with time-delay systems. In [22], Song and Yan investigated the duality between state estimation and linear quadratic tracking for time-delay systems. Then, in 2018, using duality between estimation and control, Semushin et al. in [23] constructed further array designs of controller re-optimization algorithms. Gutierrez-Pachas and Costa in [24] studied the linear quadratic problem for time-reversed Markov jump parameters in the systems. The duality in this problem is with the filtering of Markov jump linear systems. A review of duality between linear quadratic regulator and linear estimation problems is well-written in [25], for discrete-time models.

Furthermore, this duality develops the similarity between model predictive control (MPC) and moving horizon estimation (MHE). One application of MHE is temperature distribution problems for fluid catalytic cracking units (See [26]). MHE and nonlinear MPC can also be applied in autonomous agricultural vehicles [27]. One application of optimal control and state estimation is in the periodic epidemic model as in [28] and in population dynamics as in [29], respectively. In [30], the combination of MHE and MPC can be applied to indoor air grade and power control in construction.

The primary contributions of this article are that we take both the mean of process noise and mean of measurement noise, which do not have to be zero, and we show the symmetry between constrained state estimation and constrained reference tracking for this case. In other words, we study the symmetry between state estimation and reference tracking with an additional constraint on the measurement and process noise. In this paper, we generalized results in [18] that take both the mean of process noise and the mean of measurement noise as zero.

Symmetry relation in this paper can help solve a problem with the existence of disturbances. It is essential to investigate because, as mentioned in [31], for many applications, the system is not only with zero mean Gaussian disturbances but also with non-zero mean non-Gaussian disturbances. An example of a system with these disturbances is the aerodynamic parameters measurement of aerial automobile problem [31].

We arrange the structure of the paper as follows. In the second and third sections, we present state estimation and reference tracking along with formulations of their optimization problems. In the fourth section, we discuss the symmetry between state estimation and reference tracking with the presence of a more general process and measurement noise. We present numerical examples in the fifth section. Finally, we draw our conclusions in the sixth section.

II. STATE ESTIMATION

An estimation problem is a problem of estimating an unknown value based on available data. Estimation problems for deterministic systems are found in [32] and [33]. Estimation problems for stochastic systems can be done using the Kalman filter by minimizing the covariance of the estimation error [34]. This paper discusses an estimation problem for stochastic systems that contain process noise and measurement noise variables.

For a detailed explanation of the original constrained state estimation, we refer to the work of Mare and Dona [18] for the discrete linear state space equation given by:

$$x(k+1) = A_e x(k) + B_e w(k) \tag{1}$$

$$y(k) = C_e x(k) \tag{2}$$

$$v(k) = y(k)^{(d)} - C_e x(k) \tag{3}$$

where $x(k) \in \mathbb{R}^{n_x}$ is the state variable, $w(k) \in \mathbb{R}^{n_w}$ is the process noise, $y(k) \in \mathbb{R}^{n_y}$ is the true output, $y(k)^{(d)} \in \mathbb{R}^{n_y}$ is the measured output, and $v(k) \in \mathbb{R}^{n_y}$ is the measurement noise. Here A_e, B_e, and C_e are the constant matrixes with the sizes $n_x \times n_x$, $n_x \times n_w$, and $n_y \times n_x$, respectively.

In the Model (1)-(3), $x(0)$, $w(k)$, and $v(k)$ have probability density functions in Equation (4)-(6), respectively. They are independent and identically distributed with a truncated Gaussian distribution (See [35] for a detailed explanation of truncated Gaussian distribution).

$$p_W(w(k)) = \begin{cases} \dfrac{\exp\{-\frac{1}{2}(w(k)-\mu_w)^T Q_e^{-1}(w(k)-\mu_w)\}}{\int_{\Omega_w} \exp\{-\frac{1}{2}(\alpha-\mu_w)^T Q_e^{-1}(\alpha-\mu_w)\}d\alpha}, & w(k) \in \Omega_w \\ 0, & \text{others,} \end{cases} \tag{4}$$

$$p_V(v(k)) = \begin{cases} \dfrac{\exp\{-\frac{1}{2}(v(k)-\mu_v)^T R_e^{-1}(v(k)-\mu_v)\}}{\int_{\Omega_v} \exp\{-\frac{1}{2}(\alpha-\mu_v)^T R_e^{-1}(\alpha-\mu_v)\}d\alpha}, & v(k) \in \Omega_v \\ 0, & \text{others,} \end{cases} \tag{5}$$

$$p_{X_0}(x(0)) = \begin{cases} \dfrac{\exp\{-\frac{1}{2}(x(0)-\mu_0^{(e)})^T P_{e(0)}^{-1}(x(0)-\mu_0^{(e)})\}}{\int_{\Omega_{x(0)}} \exp\{-\frac{1}{2}(\alpha-\mu_0^{(e)})^T P_{e(0)}^{-1}(\alpha-\mu_0^{(e)})\}d\alpha}, x(0) \in \Omega_{x(0)} \\ 0, \text{others.} \end{cases} \tag{6}$$

Here, Q_e, R_e, and $P_{e(0)}$ are a positive definite matrix or positive constants; $\Omega_w \subseteq \mathbb{R}^{n_w}$, $\Omega_v \subseteq \mathbb{R}^{n_v}$, and $\Omega_{x(0)} \subseteq \mathbb{R}^{n_x}$ are nonempty sets. Parameters μ_w, μ_v, and $\mu_0^{(e)}$ above are the mean of process noise, measurement noise, and initial state, respectively.

Then, some vectors are defined by:

$$\mathbf{x} = \begin{bmatrix} x(1) \\ x(2) \\ \vdots \\ x(N) \end{bmatrix}, \quad \hat{\mathbf{x}} = \begin{bmatrix} \hat{x}(1) \\ \hat{x}(2) \\ \vdots \\ \hat{x}(N) \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} w(0) \\ w(1) \\ \vdots \\ w(N-1) \end{bmatrix},$$

$$\hat{\mathbf{w}} = \begin{bmatrix} \hat{w}(0) \\ \hat{w}(1) \\ \vdots \\ \hat{w}(N-1) \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} v(1) \\ v(2) \vdots \\ v(N) \end{bmatrix}, \quad \hat{\mathbf{v}} = \begin{bmatrix} \hat{v}(1) \\ \hat{v}(2) \\ \vdots \\ \hat{v}(N) \end{bmatrix},$$

$$\mathbf{y} = \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix}, \quad \mathbf{y^{(d)}} = \begin{bmatrix} y(1)^{(d)} \\ y(2)^{(d)} \\ \vdots \\ y(N)^{(d)} \end{bmatrix},$$

where $\hat{\mathbf{x}}$, $\hat{\mathbf{w}}$, and $\hat{\mathbf{v}}$, respectively, are vector estimations $\mathbf{x}$, $\mathbf{w}$, and $\mathbf{v}$.

Following the work presented in [14], [15], and [16], the state estimation problem can be written as the following minimization problem, as reported in [18]. Given the data $\mathbf{y^{(d)}}$, $\mu_0^{(e)}$, μ_w, and μ_v, solve the following problem:

$$\mathcal{P}_N^{e'} : V_N^{\text{OPT}}(\mu_0^{(e)}, \mu_w, \mu_v, \mathbf{y^{(d)}}) = \min_{\hat{x}(N)} V_N^*(\hat{x}(N), \mu_0^{(e)}, \mu_w, \mu_v, \mathbf{y^{(d)}}) \tag{7}$$

$$V_N^*(\hat{x}(N), \mu_0^{(e)}, \mu_w, \mu_v, \mathbf{y^{(d)}}) = \min_{\hat{\mathbf{w}}} V_N(\hat{x}(N), \mu_0^{(e)}, \mu_w, \mu_v, \mathbf{y^{(d)}}, \hat{\mathbf{w}}) \tag{8}$$

such that:

$$\hat{x}(k) = A_e^{-1}\hat{x}(k+1) - A_e^{-1}B_e\hat{w}(k) \quad \text{for } k = 0, 1, 2, \ldots, N-1, \tag{9}$$

$$\hat{w}(k) \in \Omega_w \quad \text{for } k = 0, 1, 2, \ldots, N-1, \tag{10}$$

$$\hat{v}(k) = y_k^d - C_e\hat{x}(k) \in \Omega_v \quad \text{for } k = 1, \ldots, N, \tag{11}$$

$$\hat{x}_0 \in \Omega_{x(0)}, \tag{12}$$

where

$$V_N(\hat{x}(N), \mu_0^{(e)}, \mu_w, \mu_v, \mathbf{y^{(d)}}, \hat{\mathbf{w}}) = \frac{1}{2}\sum_{k=0}^{N-1} \|\hat{w}(k) - \mu_w\|^2_{Q_e^{-1}} + \frac{1}{2}\sum_{k=1}^{N} \|\hat{v}(k) - \mu_v\|^2_{R_e^{-1}} + \frac{1}{2}\|\hat{x}(0) - \mu_0^{(e)}\|^2_{P_{e(0)}^{-1}}. \tag{13}$$

In Objective Function (13), there are some additional terms (μ_w and μ_v as the mean of process noises and mean of measurement noises, respectively) that do not have to be zero. Meanwhile, in [18], both the mean of process and measurement noises are zero.

III. Reference Tracking

A reference tracking problem is a control problem that brings the system output to a reference path. Reference tracking for a discrete system is discussed in [36]. This paper discusses constrained reference tracking from [18].

For a detailed explanation of the original constrained reference tracking, we refer to [18] for the following discrete linear state space model:

$$x(k+1) = A_c x(k) + B_c u(k) \tag{14}$$

$$y(k) = C_c x(k) \tag{15}$$

$$e(k) = y(k)^{(r)} - C_c x(k), \tag{16}$$

where $x(k) \in \mathbb{R}^{n_x}$ is the state variable, $u(k) \in \mathbb{R}^{n_u}$ is the control input, and $y(k) \in \mathbb{R}^{n_y}$ is the system output at time k. Here, $y(k)^{(r)} \in \mathbb{R}^{n_y}$ is the reference trajectory for the output and $e(k)$ is the tracking error. Matrixes A_c, B_c, and C_c are a constant matrixes with the sizes $n_x \times n_x$, $n_x \times n_u$, and $n_y \times n_x$, respectively. In this problem, the initial state $x(0)$ is given, and the assumption that the system is stabilizable is required [18]. To solve the constrained reference tracking problem, we must find the control input sequence that minimizes the objective function for the $N \in \{1, 2, 3, \ldots\}$ horizon such that:

$$u(k) \in \Omega_u \subseteq \mathbb{R}^{n_u}, \text{ for } k = 0, 1, 2, \ldots, N-1,$$
$$e(k) \in \Omega_e \subseteq \mathbb{R}^{n_y}, \text{ for } k = 0, 1, 2, \ldots, N-1,$$
$$x(N) \in \Omega_{x(N)},$$

where Ω_u, Ω_e, and $\Omega_{x(N)}$ are nonempty sets.

Then, define vectors $\mathbf{u}$ and $\mathbf{y^{(r)}}$ as following:

$$\mathbf{u} = \begin{bmatrix} u(0) \\ \vdots \\ u(N-1) \end{bmatrix}, \text{ and } \mathbf{y^{(r)}} = \begin{bmatrix} y(0)^{(r)} \\ \vdots \\ y(N-1)^{(r)} \end{bmatrix}.$$

Given the data $x(0)$, $\mu_N^{(c)}$ (mean of N-th state), μ_u (mean of control inputs), μ_e (mean of tracking errors) and $\mathbf{y^{(r)}}$. The following $\mathcal{P}_N^{c'}$ optimization problem is optimal control problem:

$$\mathcal{P}_N^{c'} : V_0^{\text{OPT}}(x(0), \mu_N^{(c)}, \mu_u, \mu_e, \mathbf{y^{(r)}}) = \min_{\mathbf{u}} V_0(x(0), \mu_N^{(c)}, \mu_u, \mu_e, \mathbf{y^{(r)}}, \mathbf{u}) \tag{17}$$

such that:

$$x(k+1) = A_c x(k) + B_c u(k) \quad \text{for } k = 0, 1, 2, \ldots, N-1, \tag{18}$$

$$u(k) \in \Omega_u \quad \text{for } k = 0, 1, 2, \ldots, N-1, \tag{19}$$

$$e(k) = y(k)^{(r)} - C_c x(k) \in \Omega_e \quad \text{for } k = 0, 1, 2, \ldots, N-1, \tag{20}$$

$$x(N) \in \Omega_{x(N)}, \tag{21}$$

where

$$V_0(x(0), \mu_N^{(c)}, \mu_u, \mu_e, \mathbf{y^{(r)}}, \mathbf{u}) = \frac{1}{2}\sum_{k=0}^{N-1} \left\{\|u(k) - \mu_u\|^2_{R_c} + \|e(k) - \mu_e\|^2_{Q_c}\right\} + \frac{1}{2}\|x(N) - \mu_N^{(c)}\|^2_{P_{c_N}}. \tag{22}$$

In Objective Function (22), there are some additional terms (μ_u and μ_e as the mean of control inputs and mean of tracking errors, respectively) that do not have to be zero. Meanwhile, in [18], both the mean of control inputs and tracking errors are zero. An analytical solution to constrained reference tracking problems can be found using dynamic programming as in [37] and [38].

IV. On the Symmetry between State Estimation and Reference Tracking

Here, we consider the constrained reference tracking problem $\mathcal{P}_N^{c'}$ given by Equations (17)-(22) and the constrained state estimation problem given by Equations (7)-(13). The relations between the variables and parameters in problems $\mathcal{P}_N^{c'}$ and $\mathcal{P}_N^{e'}$, taken from [18], are given in Table I and II, respectively. Table II also lists the additional parameters μ_w, μ_u, μ_v, and μ_e used in this paper. Here, μ_w and μ_v in the estimation problem can be translated into μ_u and μ_e in the reference tracking problem, respectively, such that the symmetry between these two problems can be preserved.

The symmetry between problems $\mathcal{P}_N^{c'}$ and $\mathcal{P}_N^{e'}$ is shown in Tables I and II. State variables at $k = 0, \ldots, N$ in the control problem are symmetric with state variables in the estimation

TABLE I
VARIABLE TRANSLATIONS [18]

$\mathcal{P}_N^{c'}$	$\mathcal{P}_N^{e'}$
$x(0)$	$\hat{x}(N)$
$x(1)$	$\hat{x}(N-1)$
$\vdots$	$\vdots$
$x(N)$	$\hat{x}(0)$
$u(0)$	$\hat{w}(N-1)$
$u(1)$	$\hat{w}(N-2)$
$\vdots$	$\vdots$
$u(N-1)$	$\hat{w}(0)$
$y(0)^{(r)}$	$y(N)^{(d)}$
$y(1)^{(r)}$	$y(N-1)^{(d)}$
$\vdots$	$\vdots$
$y(N-1)^{(r)}$	$y(1)^{(d)}$
$e(0)$	$\hat{v}(N)$
$e(1)$	$\hat{v}(N-1)$
$\vdots$	$\vdots$
$e(N-1)$	$\hat{v}(1)$

TABLE II
PARAMETER TRANSLATIONS

$\mathcal{P}_N^{c'}$	$\mathcal{P}_N^{e'}$
A_c	A_e^{-1}
B_c	$-A_e^{-1}B_e$
C_c	C_e
$\mu_N^{(c)}$	$\mu_0^{(e)}$
μ_w	μ_u
μ_v	μ_e
P_{c_N}	$P_{e(0)}^{-1}$
R_c	Q_e^{-1}
Q_c	R_e^{-1}
Ω_u	Ω_w
Ω_e	Ω_v
$\Omega_{x(N)}$	$\Omega_{x(0)}$

problem with reversed time steps, as shown in Table I. So do control input, reference trajectory, and tracking error in the control problem. Control input $u(k)$, reference trajectory $y(k)^{(r)}$, and tracking error $e(k)$ are symmetric with process noise $\hat{w}(k)$, data $y(k)^{(d)}$, and measurement noise $\hat{y}(k)$ in estimation problem, respectively, with reversed time steps.

Consequently, the parameter translations of both problems are shown in Table II. Here, parameter μ_w is symmetric with μ_u because process noises are symmetric with control inputs. Also, parameter μ_v is symmetric with μ_e because measurement noises are symmetric with tracking errors.

Using both tables, the optimization problem in Equations (8)-(13) can be written as problem $\mathcal{P}_N^{c'}$, as defined in Equations (17)-(22). Based on both tables, the state estimation problem $\mathcal{P}_N^{e'}$ can be solved using the five-step algorithm in [18] as follows:

1) Translate state estimation problem in Equations (8)-(13) into a control problem $\mathcal{P}_N^{c'}$ using Table I and II.
2) Solve the new problem $\mathcal{P}_N^{c'}$ such that the optimal control sequence $u(k)$ for $k = 0, 1, 2, \ldots, N-1$ and the optimal value of function $V_0^{\text{OPT}}(x(0), \mu_N^{(c)}, \mu_u, \mu_e, \mathbf{y^{(r)}})$ are found.
3) Solve the following problem:

$$x(0)^{\text{OPT}} = \arg\min_{x(0)} V_0^{\text{OPT}}(x(0), \mu_N^{(c)}, \mu_u, \mu_e, \mathbf{y^{(r)}}) \tag{23}$$

4) Obtain the all state x_k^{OPT} for $k = 0, 1, 2, \ldots, N$, from Equation (14), the value of $x(0)^{\text{OPT}}$ and u_k for $k = 0, 1, 2, \ldots, N-1$.
5) Translate $x(k)^{\text{OPT}}$ for $k = 0, 1, 2, \ldots, N$ into $\hat{x}(k)^{\text{OPT}}$ for $k = 0, 1, 2, \ldots, N$ as the original state estimation variables.

Based on Tables I and II, we can also solve the reference tracking problem using a similar algorithm.

V. NUMERICAL EXAMPLES

The following example is taken from [18] with the additional assumption. Consider the system in Equations (1)-(3) with the following matrix:

$$A_e = \begin{bmatrix} 0 & 1 \\ -1.4918 & 2.4428 \end{bmatrix}, \ B_e = \begin{bmatrix} 0 \\ 0.3730 \end{bmatrix},$$
$$C_e = \begin{bmatrix} 0.0701 & 0.0613 \end{bmatrix}.$$

We assume the measured output at $k = 1, 2, , 9$ is as follows:

$$\begin{bmatrix} y(1)^{(d)} \\ y(2)^{(d)} \\ y(3)^{(d)} \\ y(4)^{(d)} \\ y(5)^{(d)} \\ y(6)^{(d)} \\ y(7)^{(d)} \\ y(8)^{(d)} \\ y(9)^{(d)} \end{bmatrix} = \begin{bmatrix} 0{,}9 \\ 2 \\ -1 \\ 3 \\ 1 \\ 0.4 \\ 2.2 \\ -5 \\ -2.5 \end{bmatrix}. \tag{24}$$

We then formulate the constrained state estimation problem with some assumptions. In the system with Equations (1)-(3), we assume the measurement noise $\{v_k\}$ is independent with a normal distribution, a mean of 2.5 and covariance of 10. The process noise $\{w_k\}$ is also independent with a truncated normal distribution in the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$ with a mean of 0.25 and covariance of 5. We assumed that the initial state is normally distributed with the mean:

$$\mu_0^{(e)} = \begin{bmatrix} 5 & 1 \end{bmatrix}^T$$

and covariance matrix:

$$P_{e(0)} = \begin{bmatrix} 110.0753 & 161.4515 \\ 161.4515 & 257.8613 \end{bmatrix}.$$

In this case, $N = 9$, $\Omega_w = \left[-\frac{1}{2}, \frac{1}{2}\right]$, $\mu_v = 2.5$, $\mu_w = 0.25$, $Q_e = 10$, and $R_e = 5$. We solve the state estimation problem by transforming it into a reference tracking problem.

The first step, based on five-steps algorithm in Section IV, we obtain the following control problem:

$$\mathcal{P}_9^{c'} : V_0^{\text{OPT}}(x(0), \mu_9^{(c)}, \mu_u, \mu_e, \mathbf{y^{(r)}}) = \min_{\mathbf{u}} V_0(x(0), \mu_9^{(c)}, \mu_u, \mu_e, \mathbf{y^{(r)}}, \mathbf{u}) \tag{25}$$

such that:

$$x(k+1) = A_c x(k) + B_c u(k) \quad \text{for } k = 0, \ldots, 8, \tag{26}$$

$$u(k) \in \Omega_u \quad \text{for } k = 0, 1, \ldots, 8, \tag{27}$$

$$e(k) = y(k)^{(r)} - C_c x(k) \in \Omega_e \quad \text{for } k = 0, \ldots, 8, \tag{28}$$

$$x(9) \in \Omega_{x(9)}, \tag{29}$$

where

$$\begin{aligned}
&V_0(x(0), \mu_9^{(c)}, \mu_u, \mu_e, \mathbf{y^{(r)}}, \mathbf{u}) \\
&= \frac{1}{2}\sum_{k=0}^{8}\left\{\|u(k)-\mu_u\|_{R_c}^2 + \|e(k)-\mu_e\|_{Q_c}^2\right\} \\
&\quad + \frac{1}{2}\|x(9)-\mu_9^{(c)}\|_{P_{c_9}}^2,
\end{aligned}$$

$$\begin{aligned}
A_c &= A_e^{-1} = \begin{bmatrix} 1{,}6375 & -0{,}6703 \\ 1 & 0 \end{bmatrix}, \\
B_c &= -A_e^{-1}B_e = \begin{bmatrix} 0{,}25 \\ 0 \end{bmatrix}, \\
C_c &= C_e = \begin{bmatrix} 0{,}0701 & 0{,}0613 \end{bmatrix}, \\
\mu_9^{(c)} &= \mu_{e_0} = \begin{bmatrix} 5 & 1 \end{bmatrix}^T, \\
\mu_e &= \mu_v = 2.5, \\
\mu_u &= \mu_w = 0.25, \\
P_{c_9} &= P_{e_0}^{-1} = \begin{bmatrix} 0{,}1113 & -0{,}0697 \\ -0{,}0697 & 0{,}0475 \end{bmatrix}, \\
R_c &= Q_e^{-1} = 0{,}1, \\
Q_c &= R_e^{-1} = 0{,}2, \\
\Omega_u &= \Omega_w = [-\frac{1}{2}, \frac{1}{2}].
\end{aligned}$$

Subsequently, we solve the new problem $\mathcal{P}_9^{c'}$. We obtain the optimal control sequence for $k = 0, \ldots, N-1$ as follows:

$$u(k) = \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(8) \end{bmatrix} = \begin{bmatrix} 0.3005 \\ 0.2899 \\ 0.2394 \\ 0.3138 \\ 0.4196 \\ 0.5000 \\ 0.5000 \\ 0.5000 \\ 0.5000 \end{bmatrix}.$$

Hence, the optimal cost function is:

$$\begin{aligned}
&V_0(x(0), \mu_9^{(c)}, \mu_u, \mu_e, \mathbf{y^{(r)}}) \qquad (30) \\
&= \frac{1}{2}\sum_{k=0}^{8}\left\{\|u(k)-\mu_u\|_{R_c}^2 + \|e(k)-\mu_e\|_{Q_c}^2\right\} \\
&\quad + \frac{1}{2}\|x(9)-\mu_9^{(c)}\|_{P_{c_9}}^2 \\
&= 0.1690 + \frac{1}{2}\sum_{k=0}^{8}\|e(k)-\mu_e\|_{Q_c}^2 \\
&\quad + \frac{1}{2}\|x_9-\mu_{c_9}\|_{P_{c_9}}^2.
\end{aligned}$$

In the third step, we solve Equation (30) then we obtain $x(0)$ that minimize Equation (30) as follows:

$$x(0)^{\text{OPT}} = \begin{bmatrix} -18.6583 \\ -33.7207 \end{bmatrix}.$$

The fourth step is obtaining all state $x(k)^{\text{OPT}}$ for $k = 0, 1, \ldots, N$. In this case, $x(0)^{\text{OPT}}$ was found, such that we require to find $x(k)^{\text{OPT}}$ for $k = 1, \ldots, N$. Based on Equation (14), the value of $x(0)^{\text{OPT}}$ and the optimal $u(k)$ for $k = 0, 1, \ldots, N-1$, then $\{x(0)^{\text{OPT}}, x(1)^{\text{OPT}}, \ldots, x(9)^{\text{OPT}}\}$ is equal to $\left\{\begin{bmatrix} -18.6583 \\ -33.7207 \end{bmatrix}, \begin{bmatrix} -7.8735 \\ -18.6583 \end{bmatrix}, \begin{bmatrix} -0.3130 \\ -7.8735 \end{bmatrix}, \begin{bmatrix} 4.8251 \\ -0.3130 \end{bmatrix}, \begin{bmatrix} 8.1894 \\ 4.8251 \end{bmatrix}, \begin{bmatrix} 10.2804 \\ 8.1894 \end{bmatrix}, \begin{bmatrix} 11.4695 \\ 10.2804 \end{bmatrix}, \begin{bmatrix} 12.0148 \\ 11.4695 \end{bmatrix}, \begin{bmatrix} 12.1107 \\ 12.0148 \end{bmatrix}, \begin{bmatrix} 11.9023 \\ 12.1107 \end{bmatrix}\right\}$.

In the last step, we translate $x(k)^{\text{OPT}}$ for $k = 0, 1, 2, \ldots, N$ into $\hat{x}(k)^{\text{OPT}}$ for $k = 0, 1, 2 \ldots, N$ which are the original state estimation variables. Using Table I, then

$$\begin{aligned}
\{\hat{x}(0)^{\text{OPT}}, \ldots, \hat{x}(9)^{\text{OPT}}\} &= \{x(9)^{\text{OPT}}, \ldots, x(0)^{\text{OPT}}\} \\
&= \left\{\begin{bmatrix} 11.9023 \\ 12.1107 \end{bmatrix}, \begin{bmatrix} 12.1107 \\ 12.0148 \end{bmatrix}, \begin{bmatrix} 12.0148 \\ 11.4695 \end{bmatrix}, \right. \\
&\quad \begin{bmatrix} 11.4695 \\ 10.2804 \end{bmatrix}, \begin{bmatrix} 10.2804 \\ 8.1894 \end{bmatrix}, \begin{bmatrix} 8.1894 \\ 4.8251 \end{bmatrix}, \\
&\quad \begin{bmatrix} 4.8251 \\ -0.3130 \end{bmatrix}, \begin{bmatrix} -0.3130 \\ -7.8735 \end{bmatrix}, \begin{bmatrix} -7.8735 \\ -18.6583 \end{bmatrix}, \\
&\quad \left.\begin{bmatrix} -18.6583 \\ -33.7207 \end{bmatrix}\right\}.
\end{aligned}$$

Thus, the solution of above estimation problem is found, i.e. the first until the last state estimation, respectively are $\begin{bmatrix} 12.1107 \\ 12.0148 \end{bmatrix}$, $\begin{bmatrix} 12.0148 \\ 11.4695 \end{bmatrix}$, $\begin{bmatrix} 11.4695 \\ 10.2804 \end{bmatrix}$, $\begin{bmatrix} 10.2804 \\ 8.1894 \end{bmatrix}$, $\begin{bmatrix} 8.1894 \\ 4.8251 \end{bmatrix}$, $\begin{bmatrix} 4.8251 \\ -0.3130 \end{bmatrix}$, $\begin{bmatrix} -0.3130 \\ -7.8735 \end{bmatrix}$, $\begin{bmatrix} -7.8735 \\ -18.6583 \end{bmatrix}$, $\begin{bmatrix} -18.6583 \\ -33.7207 \end{bmatrix}$.

Fig. 1 shows the simulation results. The data $y(k)^{(d)}$ for $k = 1, 2, , 9$ in (24) are merged with $y(0)^{(d)} = 0.4118$, so that we obtain the "∇" curve ($-\nabla-$) in Fig. 1a. Based on Table I, we obtain data $y(k)^{(r)}$ for $k = 0, 1, \ldots, 9$ and the "$\times$" curve ($-\times-$) in Fig 1b.

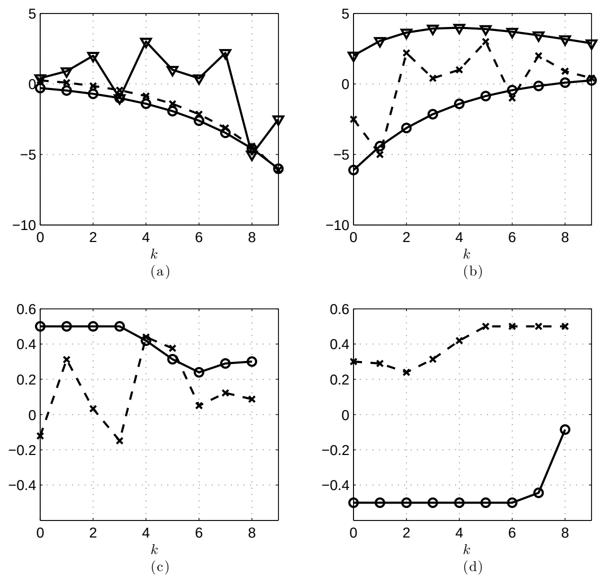

Fig. 1. Simulation result for $\mu_w = 0.25$ and $\mu_v = 2.5$: (a) $y(k)$ $(-o-)$, $y(k)^{(d)}$ $(-\nabla-)$, $\hat{y}_{k|N}$ $(-\times-)$; (b) $y(k)^{(r)}$ $(-\times-)$, $\tilde{y}_k$ $(-o-)$, $y(k)^{\text{OPT}}$ $(-\nabla-)$; (c) $w(k)$ $(-\times-)$, $\hat{w}(k)^{\text{OPT}}$ $(-o-)$; (d) $\tilde{u}(k)$ $(-\times-)$, $u(k)^{\text{OPT}}$ $(-o-)$

The true values of the process noise, $w(k)$ for $k = 0, 1, , 8$, are shown in the "$\times$" curve ($-\times-$) in Fig. 1c. These data are generated based on a normal distribution with a mean of 0.25 and covariance of Q_e, which are truncated in the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

The true value of the output, $y(k)$ for $k = 0, 1, , 9$, is depicted in the "o" curve ($-o-$) in Fig. 1a. These data are obtained based on the value of the initial state $x(0)$, process noise $w(k)$, and Equations (1)-(2). Here, the value of $x(0)$ is generated based on a normal distribution with a mean $\mu_0^{(e)}$ and covariance $P_{e(0)}$.

The solution to the original state estimation provides the optimal state and process noise to obtain the optimal output. The optimal state is $\hat{x}(k)^{\text{OPT}}$, where $k = 0, 1, \ldots, 9$, which is shown in Fig. 2c, while the process noise is $\hat{w}(k)^{\text{OPT}}$ where $k = 0, 1, \ldots, 8$, which is depicted in the "o" curve ($-o-$) in Fig. 1c. As shown in Fig. 2c, there are two curves because each state has two elements. The first element of $\hat{x}(k)^{\text{OPT}}$ is denoted by $\hat{x}_1(k)^{\text{OPT}}$, where $k = 0, 1, \ldots, 9$, depicted in the "$\times$" curve ($- \times -$). Meanwhile, the second element is $\hat{x}_2(k)^{\text{OPT}}$, where $k = 0, 1, \ldots, 9$, depicted in the "o" curve ($-o-$) in Fig. 2c. Then, the output obtained from the optimization result of the state estimation is $\hat{y}_{k|N}$, which is depicted in Fig. 1a in the "$\times$" curve ($- \times -$). The value $\hat{y}_{k|N}$ is obtained based on the value of $\hat{x}(k)^{\text{OPT}}$.

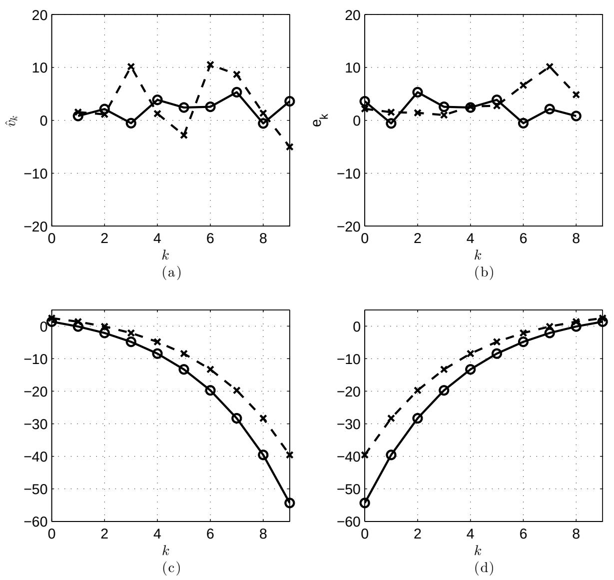

Fig. 2. Simulation result for $\mu_w = 0.25$ and $\mu_v = 2.5$: (a) $\hat{v}(k)$ as the optimization result ($-o-$) and the true $\hat{v}(k)$ ($- \times -$) ; (b) $e(k)$ as the optimization result ($-o-$) and the true $e(k)$ ($- \times -$); (c) $\hat{x}_1(k)^{\text{OPT}}$ ($- \times -$) and $\hat{x}_2(k)^{\text{OPT}}$ ($-o-$) as the elements of optimal state $\hat{x}(k)^{\text{OPT}}$ from the original state estimation problem (d) $\tilde{x}_1(k)$ ($- \times -$) and $\tilde{x}_2(k)$ ($-o-$) as the elements of optimal state $\tilde{x}(k)$ from the symmetric control problem

A reference tracking problem that is symmetric with the original state estimation results in optimal control and state, so we obtain the optimal output. Here, optimal control is denoted by $\tilde{u}(k)$ and is depicted in the "$\times$" curve ($- \times -$) in Fig 1d. Meanwhile, the optimal state is $\tilde{x}(k)$, where $k = 0, 1, \ldots, 9$, which is shown in Fig. 2d. As shown in Fig. 2d, there are two curves because each state has two elements. The first element of $\tilde{x}(k)$ is denoted by $\tilde{x}_1(k)$, where $k = 0, 1, \ldots, 9$, depicted in the "$\times$" curve ($- \times -$). Meanwhile, the second element is $\tilde{x}_2(k)$, where $k = 0, 1, \ldots, 9$, depicted in the "o" curve ($-o-$) in Fig. 2d. Then, the outputs $\tilde{y}(k)$ are obtained based on the value of the state and optimal control. Output $\tilde{y}(k)$ is depicted by the "o" curve ($-o-$) in Fig. 1b.

Now, we explain the optimal control and system output obtained by giving a certain value for the initial state. Following [18], we take $x(0)^* = [20, 10]^T$. At this initial state, we have another optimization problem. The solution to this problem results in the optimal control sequence $u(k)^{\text{OPT}}$ depicted in the "o" curve ($-o-$) in Fig. 1d. Then, based on this optimal control, we obtain the output denoted by $y(k)^{\text{OPT}}$, which is depicted in Fig. 1b as the "∇" curve ($-\nabla-$).

Fig. 2a shows two curves of measurement noise ($\hat{v}(k)$). The value of $\hat{v}(k)$ obtained by solving the initial problem (state estimation problem) is given by the "o" curve ($-o-$). The true values of $\hat{v}(k)$, given by "$\times$" curve ($- \times -$), are chosen randomly based on their distribution.

Fig. 2b shows two curves of tracking error ($e(k)$). The value of $e(k)$ obtained by solving the dual problem (reference tracking problem) is given by the "o" curve ($-o-$). The true values of $e(k)$, given by "$\times$" curve ($- \times -$), are chosen randomly based on their distribution.

In this case, the symmetric relationship between reference tracking and state estimation can be seen in the following points:

1) The symmetry between the output curve $\hat{y}_{k|N}$ obtained by solving the original state estimation problem in Fig 1a and the output curve $\tilde{y}_k$ obtained by solving the reference tracking problem that is symmetric with the original problem in Fig. 1b.
2) The symmetry between the optimal control curve $\tilde{u}(k)$ in Fig. 1d and the process noise curve $\hat{w}(k)^{\text{OPT}}$ in Fig. 1c.
3) The symmetry between the measurement noise curve $\hat{v}(k)$ in Fig. 2a ("o" curve) and the tracking error curve $e(k)$ in Fig. 2b ("o" curve). We draw these curves by solving the minimization problem.
4) The symmetry between the optimal state $\hat{x}(k)^{\text{OPT}}$ from the original state estimation problem in Fig. 2c and $\tilde{x}(k)$ from the symmetric control problem in Fig. 2d. Here, $\hat{x}_1(k)^{\text{OPT}}$ curve is symmetric with $\tilde{x}_1(k)$ curve, and $\hat{x}_2(k)^{\text{OPT}}$ curve is symmetric with $\tilde{x}_2(k)$ curve.

In addition, it is clear that the curve $y(k)^{(d)}$ in Fig. 1a is also symmetric with the curve $y(k)^{(r)}$ in Fig. 1b because of the data translation.

The optimal control curve $\tilde{u}(k)$ in Fig. 1d and the process noise curve $\hat{w}(k)^{\text{OPT}}$ in Fig. 1c have the most significant difference compared with the optimal control and process noise curves on [18] that uses $\mu_w = \mu_v = 0$. Other than that, the output values depicted in Fig. 1a and Fig. 1b are slightly different from those obtained by [18]. Furthermore, this paper presents the optimal state (both from the original state estimation and the symmetric control problems), measurement noise, and tracking error curves in Fig. 2 that are not presented in [18].

VI. Conclusion

In this paper, we demonstrated that the symmetry between constrained reference tracking and constrained state estimation can be shown in the case of a state estimation problem where the mean of process noises and mean of measurement noise do not have to be zero. Both the mean of process and measurement noise affect the measurement in the state estimation problem and the optimal control for reference tracking that is symmetric with the state estimation problem.

References

[1] S. L. Braide, and D. C. Idoniboyeobu, "Determinants of Estimated Measurement for Malaria Mortality using Modified State Estimation

Model in a Case of a Developing Country," Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering and Computer Science 2019, 22-24 October, 2019, San Francisco, USA, pp376-389
[2] Huiping Jiao, Piaopiao Zhou, Heling Li, and Chao Wei, "Parameter Estimation for Dothan Model Driven by Small Symmetrical Noise Based on Discrete Observation," Engineering Letters, vol. 30, no.4, pp1210-1216, 2022
[3] C. Wei, "Parameter estimation for Hyperbolic model with small noises based on discrete observations," Engineering Letters, vol. 30, no. 1, pp. 243–249, 2022.
[4] Made Ayu Dwi Octavanny, I Nyoman Budiantara, Heri Kuswanto, and Dyah Putri Rahmawati, "The Estimation of a Regression Curve by Using Mixed Truncated Spline and Fourier Series Models for Longitudinal Data," Engineering Letters, vol. 30, no.1, pp335-343, 2022
[5] Jutamas Boonradsamee, Winai Bodhisuwan, and Uraiwan Jaroengeratikun, "An Estimation Approach of Classical Tail Index and an Application to Hydrology Assessment," Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering 2021, 7-9 July, 2021, London, U.K., pp19-24
[6] X. Liu, R. Zhou, and G. Cai, "New Non-Time Based Tracking Control for Mobile Robot," International Journal of Robotic Engineering, vol. 5, no. 25, 2020.
[7] Z. Liu, W. Zheng, N. Wang, Z. Lyu, and W. Zhang, "Trajectory tracking control of agricultural vehicles based on disturbance test," International Journal of Agricultural and Biological Engineering, vol. 13, no. 2, pp. 138-145, 2020.
[8] X. Zhao and G. Guo, "Model Reference Adaptive Control of Vehicle Slip Ratio Based on Speed Tracking," Applied Sciences, vol. 10, no. 10, 2020.
[9] E. Giraldo, "Real-time control of a magnetic levitation system for time-varying reference tracking," IAENG International Journal of Applied Mathematics, vol. 51, no. 3, pp. 792–798, 2021.
[10] Lie Yu, Lei Ding, and Yukang Tian, "Tracking Control for Intelligent Tracing Car based on Novel Path Tracking Strategy," IAENG International Journal of Applied Mathematics, vol. 53, no.2, pp664-669, 2023
[11] Wangyue Di, Xin Tong, and Zhigang Li, "Tracking Control of the Six-rotor UAV with Input Dead-zone and Saturation," IAENG International Journal of Applied Mathematics, vol. 52, no.4, pp928-934, 2022
[12] Dongqiao Bai, Jikun Tian, Ding Li, and Shouzhi Li, "Multiple UAVs Tracking for Moving Ground Target," Engineering Letters, vol. 30, no.2, pp829-834, 2022
[13] T. Kailath, A. H. Sayed, and B. Hassibi, *Linear Estimation*, New Jersey: Prentice-Hall, Inc., 2000.
[14] G. C. Goodwin, J. A. De Dona, M. M. Seron, and X. W. Zhuo, "On the duality of constrained estimation and control," Proceedings of the 2004 American Control Conference, pp. 2148-2153, 2004.
[15] G. C. Goodwin, M. M. Seron, and J. A. De Dona, *Constrained Control and Estimation: An Optimisation Approach*, London: Springer, 2005.
[16] G. C. Goodwin, J. A. De Dona, M. M. Seron, and X. W. Zhuo, "Lagrangian duality between constrained estimation and control", Automatica, vol. 41, no. 6, pp. 935-944, 2005.
[17] M. S. Bazaraa, H. D. Sherali, and C. M. Shetty, *Nonlinear Programming: Theory and Algorithms (3rd ed.)*, New Jersey: John Wiley & Sons, 2006.
[18] J. B. Mare and J. A. De Dona, "Symmetry between constrained reference tracking and constrained state estimation," Automatica, vol. 45, no. 1, pp. 207-211, 2009.
[19] C. Muller, X. W. Zhuo, and J. A. De Dona, "Duality and symmetry in constrained estimation and control problems," Automatica, vol. 42, no. 12, pp. 2183-2188, 2006.
[20] H. Zhang and X. Song, "New estimator and duality of control for multiplicative noise systems with applications to packet dropping and multiple input delay systems," Proceedings of the IEEE International Conference on Control and Automation, pp. 75–80, 2009.
[21] X. Song, X. Yan, and D. Yuan, "Optimal linear estimation for multiplicative noise systems with time delay and its duality," Proceedings of the 31st Chinese Control Conference, pp. 1614-1619, 2012.
[22] X. Song and X. Yan, "Duality between state estimation and linear quadratic tracking for time-delay systems," Applied Mathematical Sciences (Ruse), vol. 6, no. 85, pp. 4239-4248, 2012.
[23] I. V. Semushin, J. V. Tsyganova, and A. V. Tsyganov, "Array Patterns of Regulator Re-optimization Algorithms via Duality in Estimation and Control," IFAC Papers-Online, vol. 51, no. 32, pp. 776-781, 2018.
[24] D. A. Gutierrez-Pachas and E. F. Costa, "On the Linear Quadratic Problem for Systems With Time-Reversed Markov Jump Parameters and the Duality With Filtering of Markov Jump Linear Systems," IEEE Transactions on Automatic Control, vol. 63, no. 9, pp. 3040–3045, 2018.
[25] X. Song, X. Yan, and X. Li, "Survey of duality between linear quadratic regulation and linear estimation problems for discrete-time systems," Advances in Difference Equations, no. 90, 2019.
[26] M. Li, Y. Zheng, and S. Li, "Moving horizon optimal estimation for temperature distribution of fccu riser reactor," Industrial & Engineering Chemistry Research, vol. 57, no. 36, pp. 12136-12148, 2018.
[27] T. Kraus, H. J. Ferreau, E. Kayacan, H. Ramon, J. De Baerdemaeker, M. Diehl, and W. Saeys, "Moving horizon estimation and nonlinear model predictive control for autonomous agricultural vehicles," Computers and Electronics in Agriculture, vol. 98, pp. 25–33, 2013.
[28] S. Rosa and D. F. M. Torres, "Parameter Estimation, Sensitivity Analysis and Optimal Control of a Periodic Epidemic Model with Application to HRSV in Florida," Statistics, Optimization, Information Computing, vol. 6, pp. 139–149, 2018.
[29] S. Infante, L. Sanchez, and A. Hernandez, "Stochastic Models to Estimate Population Dynamics", Statistics, Optimization, Information Computing, vol. 7, pp. 311–328, 2019.
[30] H. S. Ganesh, K. Seo, H. E. Fritz, T. F. Edgar, A. Novoselac, and M. Baldea, "Indoor air quality and energy management in buildings using combined moving horizon estimation and model predictive control," Journal of Building Engineering, vol. 33, 2021.
[31] H. Bolandhemmat, C. Clark, and F. Golnaraghi, "A Solution to the State Estimation Problem of Systems with Unknown Inputs," Recent Patents on Mechanical Engenieering, vol. 5, no. 2, pp. 1-11, 2012.
[32] G. J. Olsder, *Mathematical Systems Theory (1st ed.)*, Netherlands: Delftse Uitgevers Maatschappij, 1994.
[33] F. L. Lewis, L. Xie, and D. Popa, *Optimal and Robust Estimation (2nd ed.)*, New York: CRC Press, 2008.
[34] R. E. Kalman, "A New Approach to Linear Filtering and Prediction Problems," Transactions on ASME - Journal of Basic Engineering, vol. 82, pp. 35–45, 1960.
[35] J. Burkardt, *The truncated normal distribution*, 2014, Available: http://people.sc.fsu.edu/ jburkardt/presentations/truncated_normal.pdf.
[36] F. L. Lewis and V. L. Syrmos, *Optimal Control (2nd ed.)*, New York: John Wiley & Sons, Inc., 1995.
[37] J. B. Mare and J. A. De Dona, "Analytical solution of constrained reference tracking problems by Dynamic Programming," Proceedings of the IEEE conference on decision and control and european control conference, 2005.
[38] J. B. Mare and J. A. De Dona, "Solution of the input-constrained LQR problem using Dynamic Programming," Systems & Control Letters, vol. 56, no. 5, pp. 342–348, 2007.

Zahrotul Amalia Solakha received a bachelor's and master's degree in mathematics from Universitas Gadjah Mada, Yogyakarta, Indonesia, in 2017 and 2019, respectively. In 2019, she joined the Sekolah Tinggi Agama Islam Al-Anwar Sarang, Rembang, Indonesia, as a lecturer. Currently, she is an independent researcher. Her research interests include control theory, mathematics modeling, and optimization.

Salmah holds the title of Professor at the Department of Mathematics at Universitas Gadjah Mada, Yogyakarta, Indonesia. She completed her undergraduate and doctoral studies at the same department in 1989 and 2006. In 1995, she obtained her master's degree from the Department of Mathematics at Institut Teknologi Bandung in Bandung. Her research focuses on dynamic games, control theory, optimization, and its applications.