A Note on Rough Fuzzy Ideal and Prime Ideal in Gamma Rings

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Abstract—**Rough Set (RS) is a relatively new technique for making imprecise and uncertain decisions. It is an emerging area of ambiguous mathematics associated with fuzzy set theory. In comparison with fuzzy sets and other theories, RS has more advantages. Unlike statistics, evidence theory, and fuzzy set theory, RS theory does not require any additional or preliminary information about data. RS theory deals with approximating arbitrary subsets of the universe using two subsets namely lower and upper approximation. The advantage of RS is that rough approximations can be expanded into fuzzy environments which aids in solving a wide variety of real-time problems. The goal of this study is to focus on Rough Fuzzy Prime Ideals (RFPI) in ring structure and explain specific aspects of its upper and lower approximations. Furthermore, Rough Fuzzy Ideal (RFI) and RFPI notions and characteristics are described.**

Index Terms—**Γ Ring, rough set, rough fuzzy set, rough fuzzy ideals, rough fuzzy prime ideal.**

I. Introduction

IN 1965 Zadeh introduced a fuzzy set theory to address uncertainty [1]. This theory proves effective in solving problems involving ambiguous, subjective, and incorrect assessments. It aids in describing the data using the fuzzy set theory. Numerous researchers have explored fuzzy versions of algebraic structures. In algebra, there are several types of structures, including gamma rings. Nobusawa invented the idea of the gamma ring [2]. This is rather prevalent compared to a ring. Barnes reduced the demands of Nobusawa's concept of gamma ring theory [3]. These two articles offered curious findings regarding gamma rings and were widely read by mathematicians after being released. In continuation of these studies, the researchers are interested in gamma rings with uniqueness. In 1992, Jun et al. applied the fuzzy set concept in the theory of gamma ring [4], [5]. Many significant conclusions about rings have been extended by using gamma rings. The gamma ring structure was used to examine the number of generalizations that are identical to their corresponding parts in ring theory [6]. Fuzzy primeness has received significant attention due to its significance in classical ring theory. Jun and Dutta discussed several characteristics of the fuzzy prime ideal [7], [8]. In 2019 Ardakari examined prime and semi prime gamma rings [9] and Kavikumar et al. primarily focused on fuzzy ideals in ring semirings [10]. To solve the issue of the non membership function in the fuzzy set, Atanassov invented the Intuitionistic Fuzzy Set (IFS). IFS are useful models for analyzing uncertainty and vagueness. Palaniappan et al. conducted a study on fuzzy prime ideals in gamma near ring structure [11], [12], [13]. Ezhilmaran et al. investigated artinian and notherian near-rings, and studied some of their properties [14]. Takallo et al. investigated the implicative MBJ- MBJ-neutrosophic ideal and proved its properties[15]. Pawlak proposed the RS theory to address vague and ambiguous knowledge [16]. Moreover, RS compared to a fuzzy set, employs ideas like approximation, dependence, reduction of attributes, etc. RS and fuzzy sets can be applied to all situations, however, RS remains superior to fuzzy sets. RS does not require assumptions about membership functions, but fuzzy sets do require an a priori membership function. Ali discussed about generalized rough sets [17]. Several new results of the rough algebraic structure were studied by Davvaz [18]. Kachanci et al. illustrated the rough prime (primary) ideals [19]. Davvaz discussed RS with algebraic structure [20]. The different algebraic structures are explained through RS by many researchers. In fuzzy environments, RS has been extended to Rough Fuzzy Sets (RFS) and fuzzy rough sets. RFS refers to a pair of fuzzy sets that are formed by approximating a fuzzy set in a crisp approximation space. RFS has recently gained popularity among researchers and has been used to study algebraic structures namely groups, rings, and near rings. The present study discusses RFPI in gamma rings. The RFS which utilized in many disciplines and it can help both with classification and improbability particularly when there is ambiguity. The various structures of algebra are explained through RS by many researchers [21] [22], [23]. The difference between a fuzzy rough set and an RFS is discussed by Dubois and Prade[24]. Many researchers have explored soft sets [25], [26]. Subha et al. and Bagirmaz stated rough prime ideals in semigroups [27], [28], [29] and Zhan and Marynirmala analyzed Rough Fuzzy Ideal (RFI) in hemirings and gamma rings [30], [31], [32]. Recently some researchers discussed RFI in gamma ring structure [33]. The aim of the present work is to discuss the RFPI in the gamma ring structure. In addition to these, we analyze some theorems related to RFI and RFPI.

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II. Preliminaries

Definition II.1. [3] Let $(M, +)$ and $(\Gamma, +)$ be additive abelian groups. If there exists a mapping $M \times \Gamma \times M \to M$ [the image of $x_1, x_2, x_3 \in M$ is denoted by $x_1 \alpha x_2$ for $x_1, x_2 \in M$, $\alpha \in \Gamma$] satisfying the following identities:

(1) $x_1 \alpha x_2 \in M$,

(2) $(x_1 + x_2)\alpha x_3 = x_1 \alpha x_3 + x_2 \alpha x_3$,
$x_1(\alpha + \beta)x_2 = x_1 \alpha x_2 + x_1 \beta x_2$,
$x_1 \alpha (x_2 + x_3) = x_1 \alpha x_2 + x_1 \alpha x_3$,

(3) $(x_1 \alpha x_2)\beta x_3 = x_1(\alpha x_2 \beta)x_3 = x_1 \alpha (x_2 \beta x_3)$,

for all $x_1, x_2, x_3 \in M$ implies $\alpha, \beta \in \Gamma$, then M is called a Γ Ring. If these axioms are strengthened to

(1') $x_1\alpha x_2 \in M$, $\alpha x_1\beta \in \Gamma$
(2') $(x_1 + x_2)\alpha x_3 = x_1\alpha x_3 + x_2\alpha x_3$,
$x_1(\alpha + \beta)x_2 = x_1\alpha x_2 + x_1\beta x_2$,
$x_1\alpha(x_2 + x_3) = x_1\alpha x_2 + x_1\alpha x_3$,
(3') $(x_1\alpha x_2)\beta x_3 = x_1(\alpha x_2\beta)x_3 = x_1\alpha(x_2\beta x_3)$,
(4') $x_1\alpha x_2 = 0$ for all $x_1, x_2 \in M$ implies $\alpha = 0$.

we then have a Γ Ring in the sense of Nobusawa.
Note that, it follows from (1) - (3) that $0\alpha x_2 = x_1 0 x_2 = x_1\alpha 0 = 0$ for all $x_1, x_2 \in$ M and $\alpha \in \Gamma$.

Definition II.2. [16] Suppose the knowledge base $K = (U, R)$ with each subset X $\subseteq$ U and an equivalence relation R $\in$ IND(K) we associate two subsets
$\underline{apr}(X) = \bigcup\{Y \in U/R : Y \subseteq X\}$ and
$\overline{apr}(X) = \bigcup\{Y \in U/R : Y \cap X \neq \phi\}$,
called the apr-lower and apr-upper approximations of X respectivelty.

Definition II.3. [24] Let X $\subseteq$ U be a set, R be an equivalence relation on U and λ be a fuzzy set in U. Then upper and lower approximation of $\overline{apr}(\lambda)$ and $\underline{apr}(\lambda)$ of a fuzzy set λ by R are the fuzzy set of U/R with membership function is
$\mu_{\overline{apr}(\lambda)}(X_i) = sup\{\mu_\lambda(x)/w(X_i) = [x]_R\}$
$\mu_{\underline{apr}(\lambda)}(X_i) = inf\{\mu_\lambda(x)/w(X_i) = [x]_R\}$
where $\mu_{\overline{apr}(\lambda)}(X_i)$ (resp.$\mu_{\underline{apr}(\lambda)}(X_i)$ is the membership of X_i in $\overline{apr}(\lambda)$(resp. $\underline{apr}(\lambda)$). $(\overline{apr}(\lambda), \underline{apr}(\lambda))$ is called a RFS.

III. Rough Fuzzy Ideal in Γ Rings

Throughout the section M denotes Γ Ring.

Definition III.1. [33] An upper (resp. lower) RFS $\lambda = \langle \overline{apr}_\lambda, \underline{apr}_\lambda \rangle$ in M is said to be a Rough Fuzzy Left Ideal (RFLI) (resp. Rough Fuzzy Right Ideal (RFRI)) of a Γ Ring M if
(i) $\overline{apr}_\lambda(x_1 - x_2) \geq \{\overline{apr}_\lambda(x_1) \wedge \overline{apr}_\lambda(x_2)\}$ and
$\overline{apr}_\lambda(x_1\alpha x_2) \geq \overline{apr}_\lambda(x_2)$ [*resp.* $\overline{apr}_\lambda(x_1\alpha x_2) \geq \overline{apr}_\lambda(x_1)$]
(ii) $\underline{apr}_\lambda(x_1 - x_2) \leq \{\underline{apr}_\lambda(x_1) \vee \underline{apr}_\lambda(x_2)\}$ and
$\underline{apr}_\lambda(x_1\alpha x_2) \leq \underline{apr}_\lambda(x_2)$ [*resp.* $\underline{apr}_\lambda(x_1\alpha x_2) \leq \underline{apr}_\lambda(x_1)$]
for all $x_1, x_2 \in M$ and $\alpha \in \Gamma$.

Example III.2. *[33] Let M = {a, b, c, d} and α = {e, f, g, h} defined M and α as*

-	a	b	c	d
a	a	b	c	d
b	b	b	d	c
c	c	d	d	d
d	d	c	c	c

α	e	f	g	h
e	e	f	g	h
f	f	f	h	g
g	g	h	h	g
h	h	g	g	g

$$\overline{apr}_\lambda(x) = \begin{cases} 0.5, & \textit{if } x = a, e \\ 0.6, & \textit{if } x = b, f \\ 0.6, & \textit{if } x = c, d, g, h. \end{cases} \quad \textit{and}$$

$$\underline{apr}_\lambda(x) = \begin{cases} 0.5, & \textit{if } x = a, e \\ 0.3, & \textit{if } x = b, f \\ 0.2, & \textit{if } x = c, d, g, h. \end{cases}$$

By routine calculation, clearly M is a RFI.

Theorem III.3. *Let λ ($\neq$ 0) be a subset of M. If the RFS $\tilde{\lambda}$ = $\langle \overline{apr}_{\chi_\lambda}, \underline{apr}_{\chi_{\bar{\lambda}}} \rangle$ is a RFLI (resp. RFRI) of M, then λ is a ideal of M.*

Proof: Assume $\tilde{\lambda} = \langle \overline{apr}_{\chi_\lambda}, \underline{apr}_{\chi_{\bar{\lambda}}} \rangle$ be a RFLI (resp. RFRI) of M and let $x_1, x_2 \in \lambda$. Now $\overline{apr}_{\chi_\lambda}(x_1 - x_2) \geq \{\overline{apr}_{\chi_\lambda}(x_1) \wedge \overline{apr}_{\chi_\lambda}(x_2)\}$={$1 \wedge 1$} = 1, so that $\overline{apr}_{\chi_\lambda}(x_1 - x_2) = 1$. Hence $x_1 - x_2 \in \lambda$. Also $\overline{apr}_{\chi_\lambda}(x_1\alpha x_2) \geq \overline{apr}_{\chi_\lambda}(x_2)$ = 1 and so $\overline{apr}_{\chi_\lambda}(x_1\alpha x_2)$ = 1, that is $x_1\alpha x_2 \in \lambda$. And $\underline{apr}_{\chi_{\bar{\lambda}}}(x_1 - x_2) \leq \{\underline{apr}_{\chi_{\bar{\lambda}}}(x_1) \vee \underline{apr}_{\chi_{\bar{\lambda}}}(x_2)\}$={$(1 - \overline{apr}_{\chi_\lambda}(x_1)) \vee (1 - \overline{apr}_{\chi_\lambda}(x_2))$} {$0 \vee 0$} = 0, $\underline{apr}_{\chi_{\bar{\lambda}}}(x_1 - x_2)$ = $1 - \overline{apr}_{\chi_\lambda}(x_1 - x_2)$ = 0 implies that $\overline{apr}_{\chi_\lambda}(x_1 - x_2)$ = 1. Hence $x_1 - x_2 \in \lambda$. Also $\underline{apr}_{\chi_{\bar{\lambda}}}(x_1\alpha x_2) \leq \underline{apr}_{\chi_{\bar{\lambda}}}(x_2)$ = 0 and so $\underline{apr}_{\chi_{\bar{\lambda}}}(x_1\alpha x_2)$ = 0. Hence $\overline{apr}_{\chi_\lambda}(x_1\alpha x_2)$ = 1 and so $x_1\alpha x_2 \in \lambda$. Therefore λ is an ideal of M. ■

Theorem III.4. *If the RFS $\lambda = \langle \overline{apr}_\lambda, \underline{apr}_\lambda \rangle$ is a RFLI (resp. RFRI) of M, then we have $\overline{apr}_\lambda(0) \geq \overline{apr}_\lambda(x)$ and $\underline{apr}_\lambda(0) \leq \underline{apr}_\lambda(x)$, for all $x \in M$.*

Proof: Let λ be a RFLI (resp. RFRI) of M and for all $x \in M$. Now $\overline{apr}_\lambda(0) = \overline{apr}_\lambda(x - x) \geq \{\overline{apr}_\lambda(x) \wedge \overline{apr}_\lambda(x)\} = \overline{apr}_\lambda(x)$ and $\underline{apr}_\lambda(0) = \underline{apr}_\lambda(x - x) \leq \{\underline{apr}_\lambda(x) \vee \underline{apr}_\lambda(x)\} = \underline{apr}_\lambda(x)$. ■

Definition III.5. If $\lambda = \langle \overline{apr}_\lambda, \underline{apr}_\lambda \rangle$ be a RFS in M and $t \in$ [0,1]. Then U $(\overline{apr}_\lambda; t) = \{x \in M : \overline{apr}_\lambda(x) \geq t\}$ and $L(\underline{apr}_\lambda; t) = \{x \in M : \underline{apr}_\lambda(x) \leq t\}$ are said to be Upper Level Set (ULS) and Lower Level Set (LLS) of λ respectively.

Theorem III.6. *Consider $\lambda = \langle \overline{apr}_\lambda, \underline{apr}_\lambda \rangle$ is a RFLI (resp. RFRI) of M and t $\in$ [0,1], then*
(i) If t = 1, then the ULS $U(\overline{apr}_\lambda$; t) = 0 or λ of M.
(ii) If t = 0, then the LLS L $(apr_\lambda; t)$ = 0 or λ of M.

Proof: (i) Let t = 1 and let $x_1, x_2 \in U(\overline{apr}_\lambda$; t) Then $\overline{apr}_\lambda(x_1) \geq$ t = 1 and $\overline{apr}_\lambda(x_2) \geq$ t = 1. Accordingly $\overline{apr}_\lambda(x_1 - x_2) \geq \{\overline{apr}_\lambda(x_1) \wedge \{\overline{apr}_\lambda(x_2)\}$ ={$1 \wedge 1$} = 1 So that $x_1 - x_2 \in$ U $(\overline{apr}_\lambda; t)$. Now let $x_1 \in M$, $\alpha \in \Gamma$ and $x_2 \in$ U($\overline{apr}_\lambda$; t). Then $\overline{apr}_\lambda(x_1\alpha x_2) \geq \overline{apr}_\lambda(x_2) \geq$ t = 1 [resp. $\overline{apr}_\lambda(x_1\alpha x_2) \geq \overline{apr}_\lambda(x_1) \geq$ t = 1] and so $x_1\alpha x_2 \in U(\overline{apr}_\lambda$; t). Consequently $U(\overline{apr}_\lambda$; t) is an ideal of M.
(ii) Suppose that t = 0 and let $x_1, x_2 \in L(\underline{apr}_\lambda$; t) Then $\underline{apr}_\lambda(x_1) \leq$ t = 0 and $\underline{apr}_\lambda(x_2) \leq$ t = 0. Thus $\underline{apr}_\lambda$ $(x_1 - x_2)$ $\leq \{\underline{apr}_\lambda(x_1) \vee \underline{apr}_\lambda(x_2)\}$ = {$0 \vee 0$} = 0. So that $x_1 - x_2 \in$ L($\underline{apr}_\lambda$; t). Now let $x_1 \in M$, $\alpha \in \Gamma$ and $x_2 \in$ L($\underline{apr}_\lambda$; t). Then $\underline{apr}_\lambda$ $(x_1\alpha x_2) \leq \underline{apr}_\lambda(x_2) \leq$ t = 0. [resp.$\underline{apr}_\lambda(x_1\alpha x_2)$ $\leq \underline{apr}_\lambda(x_1) \leq$ t = 0]. So $x_1\alpha x_2 \in$ L($\underline{apr}_\lambda$; t). Therefore L($\underline{apr}_\lambda$; t) is an ideal of M. ■

Theorem III.7. *If the RFS λ=$\langle \overline{apr}_\lambda, \underline{apr}_\lambda \rangle$ is a RFLI (resp. RFRI) of M, then B = $\langle \overline{apr}_\lambda, 0 \rangle$ and C = $\langle$ 0, 1 - $\overline{apr}_\lambda \rangle$ are RFLI (resp. RFRI) Γ Ring M .*

Proof: Let λ be a RFLI (resp. RFRI) of M. Then $\overline{apr}_\lambda(x_1 - x_2) \geq \{\overline{apr}_\lambda(x_1) \wedge \overline{apr}_\lambda(x_2)\}$ and $\overline{apr}_\lambda(x_1\alpha x_2) \geq \overline{apr}_\lambda(x_2)$ [resp. $\overline{apr}_\lambda(x_1\alpha x_2) \geq \overline{apr}_\lambda(x_1)$] and $\underline{apr}_\lambda(x_1 - x_2) \leq \{\underline{apr}_\lambda(x_1) \vee \underline{apr}_\lambda(x_2)\}$ and $\underline{apr}_\lambda(x_1\alpha x_2) \leq \underline{apr}_\lambda(x_2)$ [resp.$\underline{apr}_\lambda(x_1\alpha x_2) \leq \underline{apr}_\lambda(x_1)$], for all $x_1, x_2 \in$ M and $\alpha \in \Gamma$.
(i) Let B = $\langle \overline{apr}_\lambda, 0 \rangle$ then $\overline{apr}_B = \overline{apr}_\lambda$ and $\underline{apr}_B$= 0. Therefore $\overline{apr}_B(x_1 - x_2) = \overline{apr}_\lambda(x_1 - x_2) \geq \{\overline{apr}_\lambda(x_1) \wedge \overline{apr}_\lambda(x_2)\} = \{\overline{apr}_B(x_1) \wedge \overline{apr}_B(x_2)\}$ and $\overline{apr}_B(x_1\alpha x_2) = \overline{apr}_\lambda(x_1\alpha x_2) \geq \overline{apr}_\lambda(x_2) = \overline{apr}_B(x_2)$ [resp. $apr_B(x_1\alpha x_2)$ $\geq \overline{apr}_\lambda(x_1) = \overline{apr}_B(x_1)$]. Also $\underline{apr}_B(x_1 - x_2) = 0 \leq$ {$0 \vee 0$} = { $\underline{apr}_B(x_1) \vee \underline{apr}_B(x_2)$} and $\underline{apr}_B(x_1\alpha x_2)$ = 0 $\leq \underline{apr}_B(x_2)$ [resp. $\underline{apr}_B(x_1\alpha x_2)$ = 0 $\leq \underline{apr}_B(x_1)$]. Hence

B = $\langle \overline{\text{apr}}_\lambda, 0\rangle$ is a RFLI (resp. RFRI) of M.
(ii) Let C = $\langle$ 0, $1-\overline{apr}_\lambda\rangle$ Then $\overline{apr}_c$ = 0 and $\underline{apr}_c$ = $1-\overline{apr}_\lambda$. Therefore $\overline{apr}_c(x_1-x_2) \geq \{\overline{apr}_c(x_1)\wedge\overline{apr}_c(x_2\}$ and $\overline{apr}_c(x_1\ \alpha\ x_2)$ = 0 $\geq \overline{apr}_c(x_2)$[resp. $\overline{apr}_c(x_1\alpha x_2)=0$ $\geq \overline{apr}_c(x_1)$]. Also $\underline{apr}_c(x_1\alpha x_2)$ = 1- $\overline{apr}_\lambda(x_1\alpha x_2) \leq$ 1- $\overline{apr}_\lambda(x_2)$ = $\underline{apr}_c(x_2)$ [resp. $\underline{apr}_c(x_1\alpha x_2)$ = 1 -$\overline{apr}_\lambda(x_1\alpha x_2)$ $\leq$ 1- $\overline{apr}_\lambda(x_1)$ = $\underline{apr}_c(x_1)$]. Therefore C = $\langle$ 0, $1-\overline{apr}_\lambda\rangle$ is a RFLI (resp. RFRI) of M. ■

Definition III.8. Let $\{\lambda_\text{i}\}_{\text{i}\in J}$ be family of RFSs in X, where $\lambda_i = \left\langle \wedge\overline{\text{apr}}_{\lambda_\text{i}}, \vee\underline{apr}_{\lambda_\text{i}}\right\rangle$ for each $i \in J$.
(i) $\cap\lambda_\text{i} = \left\langle\wedge\overline{\text{apr}}_{\lambda_\text{i}}, \vee\ \underline{apr}_{\lambda_\text{i}}\right\rangle$, (ii) $\cup\lambda_\text{i} = \left\langle\vee\overline{\text{apr}}_{\lambda_\text{i}}, \wedge\underline{\text{apr}}_{\lambda_\text{i}}\right\rangle$.

Theorem III.9. *If $\{\lambda_i\}_{i\in J}$ is a family of RFLI (resp. RERI) of M, then λ_i=$\langle\vee\overline{apr}_{\lambda_i}, \wedge\underline{apr}_{\lambda_i}\rangle$ is a RFLI (resp. RFRI) of M.*

Proof: Let $x_1, x_2 \in M$ and $\alpha \in \Gamma$. Then $(\bigcup_{i\in J} \overline{\text{apr}}_{\lambda_i})(x_1-x_2)$ = $\bigvee_{i\in J} \overline{apr}_{\lambda_i}(x_1-x_2) \geq \bigvee_{i\in J} (\overline{apr}_{\lambda_i}(x_1) \wedge \overline{apr}_{\lambda_i}(x_2))$= $(\bigvee_{i\in J} \overline{apr}_{\lambda_i}(x_1)) \wedge (\bigvee_{i\in J} \overline{apr}_{\lambda_i}(x_2))$=$(\bigcup_{i\in J} \overline{apr}_{\lambda_i})(x_1) \wedge (\bigcup_{i\in J} \overline{apr}_{\lambda_i})(x_2)$ and $(\bigcup_{i\in J} \underline{apr}_{\lambda_i})(x_1-x_2)$= $\bigwedge_{i\in J}\underline{apr}_{\lambda_i}(x_1-x_2) \leq \bigwedge_{i\in J} (\underline{apr}_{\lambda_i}(x_1) \vee \underline{apr}_{\lambda_i}(x_2))$= $(\bigwedge_{i\in J} \underline{apr}_{\lambda_i}(x_1)) \vee (\bigwedge_{i\in J} \overline{apr}_{\lambda_i}(x_2))$ = $(\bigcup_{i\in J} \underline{apr}_{\lambda_i})(x_1) \vee (\bigcup_{i\in J} \overline{apr}_{\lambda_i})(x_2)$. Also $(\bigcup_{i\in J} \overline{apr}_{\lambda_i})(x_1\alpha x_2)$= $\bigvee_{i\in J} \overline{apr}_{\lambda_i}(x_1\alpha x_2) \geq \bigvee_{i\in J} \overline{apr}_{\lambda_i}(x_2)$ = $(\bigcup_{i\in J} \overline{apr}_{\lambda_i})(x_2)$ and $(\bigcup_{i\in J} \underline{apr}_{\lambda_i})(x_1\alpha x_2)$= $\bigwedge_{i\in J} \underline{apr}_{\lambda_i}(x_1\alpha x_2) \leq \bigwedge_{i\in J} \underline{apr}_{\lambda_i}(x_2)$ =$(\bigcup_{i\in J} \underline{apr}_{\lambda_i})(x_2)$. Same for right ideals, $(\bigcup_{i\in J}\overline{apr}_{\lambda_i})(x_1\alpha x_2)$ = $\bigvee_{i\in J} \overline{apr}_{\lambda_i}(x_1\alpha x_2)$ $\geq \bigvee_{i\in J} \overline{apr}_{\lambda_i}(x_1)$ = $(\bigcup_{i\in J}\overline{apr}_{\lambda_i})(x_1)$, $(\bigcup_{i\in J} \underline{apr}_{\lambda_i})(x_1\alpha x_2)$ = $\bigwedge_{i\in J} \underline{apr}_{\lambda_i}(x_1\alpha x_2) \leq \bigwedge_{i\in J} \underline{apr}_{\lambda_i}(x_1)$ = $(\bigcup_{i\in J} \underline{apr}_{\lambda_i})(x_1)$. Therefore $(\bigcup_{i\in J}\lambda_i)$ is a RFLI(resp. RFRI) of M. ■

Definition III.10. Consider A = $\langle\overline{apr}_A, \underline{apr}_A\rangle$ and B = $\langle\overline{apr}_B, \underline{apr}_B\rangle$ be rough fuzzy subsets M, then AΓB is defined by

$$\overline{apr}_{A\Gamma B}(x_1) = \begin{cases} \bigvee_{x_1=x_2\alpha x_3} \{\overline{apr}_A(x_2)\wedge\overline{apr}_B(x_3)\}, & \\ & \text{if } x_1 = x_2\alpha x_3 \\ 0 & otherwise \end{cases}$$

$$\underline{apr}_{A\Gamma B}(x_1) = \begin{cases} \bigwedge_{x_1=x_2\alpha x_3} \{\underline{apr}_A(x_2)\vee\underline{apr}_B(x_3)\}, & \\ & \text{if } x_1 = x_2\alpha x_3 \\ 1 & otherwise \end{cases}$$

Theorem III.11. *If A=$\langle\overline{apr}_A, \underline{apr}_A\rangle$ and B=$\langle\overline{apr}_B, \underline{apr}_B\rangle$ be two RFLI (resp. RFRI) ideals of M, then A ∩ B is a RFLI (resp. RFRI) of M. If A is a RFRI and B is a RFLI A Γ B ⊆ A ∩ B.*

Proof: Consider A and B are RFLI (resp. RFRI) of M. Let $x_1, x_2 \in$ M and $\alpha \in \Gamma$. Then $\overline{apr}_{A\cap B}(x_1-x_2)$ = $\overline{apr}_A(x_1-x_2)\wedge\overline{apr}_B(x_1-x_2)\geq[\overline{apr}_A(x_1)\wedge\overline{apr}_A(x_2)]$ $\wedge$ $[\overline{apr}_B(x_1) \wedge \overline{apr}_B(x_2)]$ = $[\overline{apr}_A(x_1) \wedge \overline{apr}_B(x_1)]$ $\wedge$ $[\overline{apr}_A(x_2) \wedge \overline{apr}_B(x_2)]$ = $[\overline{apr}_{A\cap B}(x_1) \wedge \overline{apr}_{A\cap B}(x_2)]$, $\underline{apr}_{A\cap B}(x_1-x_2)$ = $\underline{apr}_A(x_1-x_2) \vee \underline{apr}_B(x_1-x_2)$ $\leq$ $[\underline{apr}_A(x_1) \vee \underline{apr}_A(x_2)]$ $\vee$ $[\underline{apr}_B(x_1) \vee \underline{apr}_B(x_2)]$ = $[\underline{apr}_A(x_1) \vee \underline{apr}_B(x_1)]$ $\vee$ $[\underline{apr}_A(x_2) \vee \underline{apr}_B(x_2)]$. =$[\underline{apr}_{A\cap B}(x_1) \vee \underline{apr}_{A\cap B}(x_2)]$

Also $\overline{apr}_A(x_1\alpha x_2) \geq \overline{apr}_A(x_2)$ and $\underline{apr}_A(x_1\alpha x_2) \leq \underline{apr}_A(x_2)$, $\overline{apr}_B(x_1\alpha x_2) \geq \overline{apr}_B(x_2)$ and $\underline{apr}_B(x_1\alpha x_2)$ $\leq \underline{apr}_B(x_2)$ [resp. $\overline{apr}_A(x_1\alpha x_2) \geq \overline{apr}_A(x_1)$ and $\underline{apr}_A(x_1\alpha x_2) \leq \underline{apr}_A(x_1)$, $\overline{apr}_B(x_1\alpha x_2) \geq \overline{apr}_B(x_1)$ and $\underline{apr}_B(x_1\alpha x_2) \leq \underline{apr}_B(x_1)$]. Now $\overline{apr}_{A\cap B}(x_1\alpha x_2)$ = $\overline{apr}_A(x_1\alpha x_2) \wedge \overline{apr}_B(x_1\alpha x_2) \geq [\overline{apr}_A(x_2) \wedge \overline{apr}_B(x_2)]$ = $\overline{apr}_{A\cap B}(x_2)$. $\underline{apr}_{A\cap B}(x_1\alpha x_2)$ = $\underline{apr}_A(x_1\alpha x_2) \vee \underline{apr}_B(x_1\alpha x_2) \leq [\underline{apr}_A(x_2) \vee \underline{apr}_A(x_2)]$ = $\underline{apr}_{A\cap B}(x_2)$. Hence $A\cap B$ is a RFLI (resp. RFRI) of M. To prove second part, if $\overline{apr}_{A\Gamma B}(x) = 0$ and $\underline{apr}_{A\Gamma B}(x) = 1$. Suppose that $A\Gamma B \neq$ (0,1). By the definition of $A\Gamma B$, $\overline{apr}_A(x_1)$ = $\overline{apr}_A(x_2\alpha x_3) \geq \overline{apr}_A(x_2)$ and $\underline{apr}_A(x_1)$ = $\underline{apr}_A(x_2\alpha x_3) \leq \underline{apr}_A(x_2)$, $\overline{apr}_B(x_1)$ = $\overline{apr}_B(x_2\alpha x_3) \geq$ $\overline{\text{apr}}_B(x_3)$ and $\underline{apr}_B(x_1)$ = $\underline{apr}_B(x_2\alpha x_3) \leq \underline{apr}_B(x_3)$. Since A is a RFRI and B is a RFLI, we have $\overline{apr}_A(x_1)$ = $\overline{apr}_A(x_2\alpha x_3) \geq \overline{apr}_A(x_2)$ and $\underline{apr}_A(x_1)$ = $\underline{apr}_A(x_2\alpha x_3)$ $\leq \underline{apr}_A(x_2)$, $\overline{apr}_B(x)$ = $\overline{apr}_B(x_2\alpha x_3) \geq \overline{apr}_B(x_3)$ and $\underline{apr}_B(x_1)$ = $\underline{apr}_B(x_2\alpha x_3) \leq \underline{apr}_B(x_3)$. Hence by definition III.10., $\overline{apr}_{A\Gamma B}(x_1)$ = $\bigvee_{x_1=x_2\alpha x_3} \{\overline{apr}_A(x_2) \wedge \overline{apr}_B(x_3)\} \leq$ $\overline{apr}_A(x_1) \wedge \overline{apr}_B(x_1)$ = $\overline{apr}_{A\cap B}\ (x_1)$, $\underline{apr}_{A\Gamma B}(x_1)$ = $\bigwedge_{x_1=x_2\alpha x_3} \{\underline{apr}_A(x_2)\vee\underline{apr}_B(x_3)\} \geq \underline{apr}_A(x_1) \vee \underline{apr}_B(x_1)$ = $\underline{apr}_{A\cap B}\ (x_1)$. Hence AΓB ⊆ A∩B. ■

Corollary III.12. *If A = $\langle\ \overline{apr}_A, \underline{apr}_A\rangle$ and B=$\langle\overline{apr}_B, \underline{apr}_B\rangle$ are two RFLI (resp. RFRI) of M, then A∪B is a RFLI (resp. RFRI) of M.*

Definition III.13. Suppose M is called regular if for each a ∈ M there exists $x_1 \in$ M and $\alpha, \beta \in \Gamma$ such that a = $a\alpha x_1\beta a$.

Result III.14. *A Γ Ring M is called regular iff $I\Gamma J = I \cap J$, for each right ideal I and for each left ideal J of M.*

Theorem III.15. *If M is regular if for each RFRI A and for each RFLI B of M, AΓB = A∩B.*

Proof: Let M is regular. By Theorem III.11., $A\Gamma B \subseteq A \cap B$. To prove A ∩B ⊆ $A\Gamma B$. Let $a \in M$ and $\alpha, \beta \in \Gamma$. Then, by the definition there exists $x_1 \in$ M such that $a = a\alpha x_1\beta a$. Thus $\overline{apr}_A(a)$ = $\overline{apr}_A(a\alpha x_1\beta a) \geq \overline{apr}_A(a\alpha x_1) \geq \underline{apr}(a)$, $\underline{apr}_A(a)$ = $\underline{apr}_A(a\alpha x_1\beta a) \leq \underline{apr}_A(a\alpha x_1) \leq \underline{apr}(a)$. So $\overline{apr}_A(a\alpha x_1)$ $\geq \overline{apr}_A(a)$ and $\underline{apr}_A(a\alpha x_1) \leq \underline{apr}_A(a)$. On the other hand, $\overline{apr}_{A\Gamma B}$(a) = $\bigvee_{a=a\alpha x_1\beta a} \{\overline{apr}_A(a\alpha x_1) \wedge \overline{apr}_B(a)\} \geq$ $\{\overline{apr}_A(a) \wedge \overline{apr}_B(a)\} = \overline{apr}_{A\cap B}(a)$, $\underline{apr}_{A\Gamma B}$(a) = $\bigwedge_{a=a\alpha x_1\beta a}$ $\{\underline{apr}_A(a\alpha x_1) \vee \underline{apr}_B(a)\} \leq \{\underline{apr}_A(a) \vee \overline{apr}_B(a)\}$ = $\underline{apr}_{A\cap B}(a)$. Thus $A\cap B \subseteq A\Gamma B$. Hence $A\Gamma B = A\cap B$. ■

Definition III.16. *[3]* An ideal P of the Γ Ring M is called prime if any ideals A and B of M, AΓB ⊆ P implies A⊆P or B⊆P.

Definition III.17. If P be a RFI of M. Then P is said to be prime if P is not a constant mapping and for any RFIs A, B of a Γ Ring M, AΓB ⊆ P ⇒ A⊆ P or B⊆P.

Example III.18. *Let M = {e, f, g, h} and α ∈ Γ. Let us defined the Cayley table using the binary operations of a Γ Ring as follows*

-	e	f	g	h
e	e	f	g	h
f	f	g	h	e
g	g	g	e	g
h	h	e	f	g

α	e	f	g	h
e	e	e	e	e
f	e	g	e	g
g	e	e	e	e
h	e	g	e	g

Then $\overline{apr}_\lambda(x)$ and $\underline{apr}_\lambda(x)$ as stated below

$\overline{apr}_\lambda(x) = \begin{cases} 0.5, & \text{if } x = e, g \\ 0.3, & \text{if otherwise} \end{cases}$ *and*

$\underline{apr}_\lambda(x) = \begin{cases} 0.2, & \text{if } x = e, g \\ 0.7, & \text{if otherwise} \end{cases}$

Clearly (M, Γ) is a RFI.
Consider the subsets of M are $\{\{e\}, \{f, g, h\}\}$ and $\lambda = \{e, g\} \subseteq$ M.
By the Cayley table $\overline{apr}_\lambda(x)$={$e, f, g, h$}, and $\underline{apr}_\lambda(x)$={$e$} are RFPI of M.
But λ is not a RFPI of M.
Since hΓf = g $\subseteq \lambda$ then h $\nsubseteq \lambda$ and f $\nsubseteq \lambda$.

Theorem III.19. *Let J be an ideal of M such that $J \neq M$, Then J is a prime ideal of M iff $(\chi_J, \overline{\chi_J})$ is a RFPI of M.*

Proof: Necessary Part: If J is a prime ideal of M and let P = $(\chi_J, \overline{\chi_J})$. Since J $\neq$ M. P is not a constant mapping on M. Let A and B be two RFI of M such that AΓB $\subseteq$P and A $\not\subset$ P or B $\not\subset$ P there exists $x_1, x_2 \in M$ such that $\overline{apr}_A(x_1) > \overline{apr}_P(x_1)$ = $\chi_J(x_1)$, $\underline{apr}_A(x_1) < \underline{apr}_P(x_1)$ = $\overline{\chi_J}(x_1)$ and $\overline{apr}_B(x_2) > \overline{apr}_P(x_2) = \chi_J(x_2)$, $\underline{apr}_B(x_2) < \underline{apr}_P(x_2)$ = $\overline{\chi_J}(x_2)$. So $\overline{apr}_A(x_1) \neq 0$, $\underline{apr}_A(x_1) \neq 1$ and $\overline{apr}_B(x_2) \neq 0$, $\underline{apr}_B(x_2) \neq 1$. But $\chi_J(x_1) = 0$ and $\chi_J(x_2)$ = 0. So $x_1 \notin J, x_2 \notin J$. Since J is a prime ideal of M, by theorem 5[3] that there exist $x_3 \in$ M and $\alpha, \beta \in \Gamma$ such that $x_1\alpha x_3\beta x_2 \notin$ J. Let $a = x_1\alpha x_3\ \beta\ x_2$. Then $\chi_J(a)$ = 0 and $\overline{\chi_J}$(a)=1. Thus AΓB(a) = (0,1). But $\overline{apr}_{A\Gamma B}(a)$ = $\bigvee_{a=c\gamma d}[\overline{apr}_A(c) \wedge \overline{apr}_B(d)] \geq \overline{apr}_A(x_1\alpha x_3) \wedge \overline{apr}_B(x_2)$ (since a = $x_1\alpha x_3\beta x_2) \geq \overline{apr}_A(x_1) \wedge \overline{apr}_B(x_2) > 0$ (Since $\overline{apr}_A(x_1) \neq 0$ and $\overline{apr}_B(x_2) \neq 0$) and $\underline{apr}_{A\Gamma B}(a)$ = $\bigwedge_{a=c\gamma d}$ $[\underline{apr}_A(c) \vee \underline{apr}_B(d)] \leq \underline{apr}_A(x_1\alpha x_3) \vee \underline{apr}_B(x_2)$ $\leq \underline{apr}_A(x_1) \vee \overline{apr}_B(x_2) < 1$. (Since $\underline{apr}_A(x_1) \neq 1$ and $\underline{apr}_B(x_1) \neq 1$) Then $A\Gamma B(a) \neq$ (0,1). This is a contradiction. Hence for any RFI A and B, $A\Gamma B \subseteq$ P implies A$\subseteq$P or B$\subseteq$P. Hence P is a RFPI of M.

Sufficient Part:
Suppose P = $(\chi_J, \overline{\chi_J})$ is a RFPI of M. Since P is not a constant mapping on M, J $\neq$ M. Let A, B be two ideals of M such that $A\Gamma B \subseteq$ J. Let $\overline{A} = (\chi_A, \overline{\chi_A})$ and $\overline{B}$=$(\chi_B, \overline{\chi_B})$ be two RFI of M. Consider the product $\overline{A}\Gamma\overline{B}$. Let $x_1 \in$ M. If $\overline{A}\Gamma\overline{B}(x)$ = (0,1). Clearly $\overline{A}\Gamma\overline{B}\subseteq$ P. Suppose $\overline{A}\Gamma\overline{B}(x) \neq$ (0,1) Then $\overline{apr}_{\bar{A}\Gamma\bar{B}}(x_1)$ = $\bigvee_{x_1=x_2\alpha x_3}$ $[\chi_A(x_2) \wedge \chi_B(x_3)] \neq$ 0. $\underline{apr}_{\bar{A}\Gamma\bar{B}}(x_1)$ = $\bigwedge_{x_1=x_2\alpha x_3}$ $[\chi_B(x_2) \vee \overline{\chi_B}(x_3)] \neq 1$. Thus there exist $x_2, x_3 \in$ M with $x_1 = x_2\alpha x_3$ such that $\chi_A(x_2) \neq$ 0, $\overline{\chi_J}(x_2) \neq 1$ and $\chi_B(x_3) \neq 0$, $\overline{\chi_B}(x_3) \neq 1$. So $\chi_A(x_2)$=1, $\overline{\chi_A}(x_2)$=0 and $\overline{\chi_B}(x_3)$ = 1, $\overline{\chi_A}(x_3)$ = 0. This implies $x_2 \in A$ and $x_3 \in B$. Thus $x_1 = x_2\alpha x_3 \in A\Gamma B \subseteq J$, so $\chi_J(x_1)$ = 1, $\overline{\chi_J}(x_1)$ = 0. It follows that $\bar{A}\Gamma\bar{B}(x) \subseteq$P. Since P is a RFPI of M either $\bar{A} \subseteq$ P or $\bar{B} \subseteq$ P either A $\subseteq$ J or B $\subseteq$ J. Hence J is a prime ideal of M. ■

Theorem III.20. *Let P be a RFPI of M and let $M_P = \{x_1 \in M : P(x_1) = p(0)\}$. Then M_P is a prime ideal of M.*

Proof: Let $x_1, x_2 \in M_P$. Then $P(x_1)$ = P(0) and $p(x_2)$ = P(0). Thus $\overline{apr}_P(x_1 - x_2) \geq \overline{apr}_P(x_1) \wedge \overline{apr}_P(x_2)$ = $\overline{apr}_P(0)$ and $\underline{apr}_P(x_1 - x_2) \leq \underline{apr}_P(x_1) \vee \underline{apr}_P(x_2)$ = $\underline{apr}_P(0)$. Since P is a RFI, $\overline{apr}_P(0) = \overline{apr}_P(0\alpha(x_1 - x_2)$ $\geq \overline{apr}_p(x_1 - x_2)$ and $\underline{apr}_P(0)$ = $\underline{apr}_p(0\alpha(x_1 - x_2) \leq$ $\underline{apr}_P(x_1 - x_2)$. $x_1 - x_2 \in M_p$ and let $x_2 \in$ M and let $x_1 \in M_P$. Then $\overline{apr}_p(x_2\alpha x_1) \geq \overline{apr}_p(x_1)$ = $\overline{apr}_p$ (0) and $\underline{apr}_P(x_2\alpha x_1) \leq \underline{apr}_P(x_1)$ = $\underline{apr}_P(0)$. Therefore $x_2\alpha x_1 \in$ M_P Hence M_P is an ideal of M. Let J and K are ideals of M such that JΓK $\subseteq M_P$.
Define A = $P(0)(\chi_J, \overline{\chi_J})$ and B=$P(0)(\chi_K, \overline{\chi_K})$, Where $P(0)(\chi_J, \overline{\chi_J})$ = $(\overline{apr}_p(0)\ \chi_J, \underline{apr}_p(0)\ \overline{\chi_J})$ and P(0)$(\chi_K, \overline{\chi_K})$ = $(\overline{apr}_P(0)\ \chi_K, \underline{apr}_P(0)\ \overline{\chi_K})$. Then to prove A and B are RFI of M. Let $x_1 \in$ M, Suppose that AΓB (x) = (0,1) then AΓB $\subseteq$ P. Also let AΓB (x) $\neq$ (0,1) $\overline{apr}_{A\Gamma B}(x)$ = $\bigvee_{x_1=x_2\alpha x_3}$ $[\overline{apr}_A(x_2) \wedge \overline{apr}_B(x_3)]$ = $\bigvee_{x_1=x_2\alpha x_3}$ $[\overline{apr}_P(0)\chi_J(x_2) \wedge \overline{apr}_P(0)\ \chi_K(x_3)] \neq 0$ and $\underline{apr}_{A\Gamma B}$(x) = $\bigwedge_{x_1=x_2\alpha x_3}$ $[\underline{apr}_A(x_2) \vee \underline{apr}_B(x_3)]$ = $\bigwedge_{x_1=x_2\alpha x_3}[\underline{apr}_P(0)\overline{\chi_J}(x_2) \vee \underline{apr}_P(0)\ \overline{\chi_K}(x_3)] \neq 1$. Thus there exist $x_2, x_3 \in M$ with $x_1 = x_2\alpha x_3$ such that $\overline{apr}_p(0)$ $\chi_J(x_2) \wedge \overline{apr}_p(0)\ \chi_K(x_3) \neq 0$ and $\underline{apr}_P(0)\ \overline{\chi_J}(x_2) \vee$ $\underline{apr}_P(0)\ \overline{\chi_K}(x_3) \neq 1$. Thus $x_2 \in J$ and $x_3 \in K$, that is, $x_1 = x_2\alpha x_3 \in$ JK $\subseteq M_P$. So P(x) = P(0) that is, AΓB $\subseteq$ P. Since P is a RFPI and A, B are RFI, either A $\subseteq$ P or B $\subseteq$ P. Suppose A $\subseteq$ P. Then $P(0)(\chi_J, \overline{\chi_J}) \subseteq$ P. Assume J$\subseteq$ M_P. Then there exists a $\in J$ such that a $\notin M_P$. Thus P(a) $\neq$ P(0) and $\overline{apr}_p(a) < \overline{apr}_p(0)$ and $\underline{apr}_p(a) > \underline{apr}_p(0)$. Then $\overline{apr}_A(a) = \overline{apr}_p(0)\chi_J(a)$= $\overline{apr}_p(0) > \overline{apr}_p(a)$ and $\underline{apr}_A(a) = \underline{apr}_p(0)\overline{\chi_J} = 0 \leq \underline{apr}_p(0) < \underline{apr}_p(a)$. This is the contradict the assumption $A \subseteq$ P. So $J \subseteq M_P$. By similarly to show if $B \subseteq$P then $K \subseteq M_P$. Hence M_P is a prime ideal. ■

IV. Conclusion

RS theory has a wide range of application potential. Designed with an innovative approach RS theory is a useful information-processing tool that has found extensive applications in various fields. In addition to addressing new uncertain information systems, RS is also useful in optimizing many existing soft computing techniques. Various structures are being visualized in a new way by researchers. In this article, we discussed the RFPI in the Γ Ring structure. In the future, rough fuzzy concepts can be applied to various algebraic structures like Γ semirings, Γ modules, Γ fields, etc.

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