Combinatorial Properties of Involutive Fibonacci Arrays

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***Abstract*—Various scholars have investigated Fibonacci arrays to uncover its combinatorial features and applications. As an extension of Involutive Fibonacci words, Involutive Fibonacci arrays were introduced. We will look at some of the combinatorial features of Involutive Fibonacci Arrays in this article.**

***Index Terms*—Fibonacci words, Involutive Fibonacci arrays, Two-dimensional languages, Morphic involution, Decomposition of arrays.**

I. Introduction

THE branch of mathematics known as 'Formal Languages' provides the fundamental machinery for developing compilers. It has helped a variety of other sciences, like computer networking, physics, biology, etc., expand during the past few decades. Over the years, researchers have offered a variety of models to extend formal languages to two dimensions, which are derived from concepts in the fields of pattern matching and image processing [1], [2]. Studies using combinatorics and other parallel computing models also contain instances of two-dimensional patterns.

Both combinatorics on words and formal languages explore different aspects of words and their combinatorial features, cutting across the borders of computer science and mathematics. Finding patterns within collections of symbols was the main emphasis of this field. Axel Thue, credited with starting the field of combinatorics on words in the early 1900s [3], [4], published articles on patterns and repetitions in words. In 1983, Lothaire published a book titled "Combinatorics on Words" [5].

Recent research has focused on many combinatorial properties of words, such as tandem repeats, square free property, primitive words, complexity, morphisms on words, borderness, periodicity, and so on. For example, Kari and Mahalingam examined involutively bordered words in [6], Czeizler et al. focused on primitivity properties based on the behaviour of molecules in DNA in [7], Yu studied the borderness properties of words in [8], and so on. Researchers have now expanded the study to include two-dimensional words and their combinatorial features. Amir and Benson [9] investigated periodicity in two-dimensional arrays. Various scholars explored the occurrence of palindromes in two-dimensional arrays [10], [11], [12], [13]. Studies related to partial words and partial arrays and their combinatorial properties have also been at their peak recently. To mention a few, Sasikala et al. [14] studied the partial languages; in [15] the authors discussed the tiling systems in partial array languages; Krishna Kumari and Arulprakasam [16] investigated the factors and subwords of rich partial words; and so on. Researchers have developed many algorithms to enhance the applications of these properties. To name a few, Knuth et al. [17] studied pattern matching in words, whereas Cole et al. [18] provided an algorithm for pattern matching in words and arrays, and Geizhals et al. [19] presented an approach for finding the maximal two-dimensional palindromes, and a few more. John Kaspar et al. [20] made a study relating to lattice automata.

Fibonacci numbers are defined as a sequence of numbers obtained from the recurrence relation

$$Fb(n) = Fb(n-1) + Fb(n-2)$$

for all $n \geq 2$ and $Fb(0) = 0, Fb(1) = 1$. Knuth identified the one-dimensional Fibonacci strings [21] as words corresponding to the Fibonacci numbers obtained by fixing the first two initial letters, say $f_0 = p$ and $f_1 = q$, and recursively obtaining

$$f_{n+2} = f_{n+1} \bullet f_n$$

for all $n \geq 0$, where $\bullet$ represents word concatenation. However, Allouche and Shalit [22] claim that it is difficult to pinpoint who first used the Fibonacci numbers. Researchers have conducted numerous studies over the years to reveal various combinatorial features. A. De. Luca's [23] work is one of its kind. Berstel conducted a study in 1986 in order to compile a list of all the proven results [24]. W.F. Chaun [25], [26], [27], [28] investigated several of their characteristics. G. Fici [29] discussed Fibonacci factorization and other infinite words. Mahalingam et al. [30] recently published a study on Watson-Crick palindromes in Watson-Crick conjugates.

Apostolico and Brimkov [31] came up with the following two-dimensional Fibonacci sequence: $f_{0,0} = a_1$, $f_{0,1} = b_1$, $f_{1,0} = c_1$, and $f_{1,1} = d_1$; a_1, b_1, c_1, and d_1 are letters from the finite alphabet $\mathcal{A}$; and

$$f_{s,(k+1)} = f_{s,k} \,\mathbin{\text{⦶}}\, f_{s,(k-1)}$$

and

$$f_{(j+1),s} = f_{j,s} \ominus f_{(j-1),s}$$

where ⦶ means vertical concatenation and $\ominus$ means horizontal concatenation. Mahalingam et al. [13] examined the nature of palindromes in Fibonacci arrays, and Kulkarni et al. [32] investigated some of the other combinatorial aspects of Fibonacci arrays. The application is found in [33], where the authors explored the near-field optical behaviour of two-dimensional Fibonacci plasmonic lattices created by electron-beam lithography over transparent quartz substrates. Since

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then, several studies on Fibonacci arrays' combinatorial features and applications have been published.

Lila Kari et al.[34] initially examined involutive Fibonacci words in 2021 as a result of inferred research on DNA computing[35]. They have also talked a lot about the properties of indexed ξ-Fibonacci words and bordered ξ-Fibonacci words, where ξ is an antimorphic or morphic involution over $\mathcal{A}^*$. Later in the year 2022, they studied primitivity in Fibonacci words[36].

Hannah Blasiyus and D.K. Sheena Christy [37] introduced the involutive Fibonacci arrays by classifying them as alternating, palindromic, and hairpin Fibonacci arrays, depending upon the nature of the concatenation applied.

In this paper, we study some of the other combinatorial properties of the involutive Fibonacci arrays.

This paper is organised as follows: Section II deals with recalling several definitions from the literature that are required for our study. In Section III, we define indexing of involutive Fibonacci arrays. In Section IV, we study some of the ways of decomposing the alternating Fibonacci arrays, and we prove that the languages of square alternating, palindromic, and hairpin Fibonacci arrays are two-dimensional codes. In Section V, we study the primitivity of involutive Fibonacci arrays.

II. Preliminaries

Here we recall a few definitions required for our study. For basic definitions we refer the reader to [32], [34], [37], [38] and [39].

Definition II.1. [38] A finite and non-empty collection of symbols called letters is known as *alphabet*, denoted by $\mathcal{A}$. $|\mathcal{A}|$ represents the total number of elements present in $\mathcal{A}$.

Definition II.2. [38] A finite or infinite sequence of letters from the alphabet is known as a *string* or a *word*.

If $\mathcal{A}$ is an alphabet, then $\mathcal{A}^*$ represents the set of strings created by joining zero or more letters from $\mathcal{A}$ together. A subset of $\mathcal{A}^*$ is usually known as *one-dimensional language*.

Definition II.3. [38] If w and x are two strings from $\mathcal{A}^*$, then *word concatenation* of w and x is got by joining the letters of x to the right-end of w. We denote this by $w \bullet x$ or simply wx. i.e., if $w = r_1 r_2 \cdots r_m$ and $x = s_1 s_2 \cdots s_n$, where $r_1, r_2, \cdots r_m$ and $s_1, s_2, \cdots s_n \in \mathcal{A}$, then $wx = w \bullet x = r_1 r_2 \cdots r_m s_1 s_2 \cdots s_n$

Definition II.4. [38] *Length* of a string w is the total number of letters present in it, denoted by $|w|$. If $|w| = 0$, we say that w is an *empty word*, usually denoted by λ. Note that $\mathcal{A}^* \setminus \{\lambda\} = \mathcal{A}^+$, where $\mathcal{A}^+$ is the collection of all non-empty strings over $\mathcal{A}$.

Definition II.5. [38] If $w \in \mathcal{A}^*$, then w^n is given by $w \bullet w \bullet w \bullet \cdots w$ (n-times), for all $n \geq 1$. If $n = 0$, then $w^n = w^0 = \lambda$, the empty string.

Definition II.6. [34] The *Fibonacci sequence* is a numerical sequence in which each term is obtained by summing its two preceding terms. i.e., $Fb(n) = Fb(n-1) + Fb(n-2)$ for all $n \geq 2$, fixing $Fb(0) = 0, Fb(1) = 1$. Each term of this sequence is termed as *Fibonacci number*.

Definition II.7. [34] The *sequence of standard and reverse Fibonacci words* $Fb(r,s)$ and $Fb'(r,s)$, where $r, s \in \mathcal{A}^*$ are defined as:

$$Fb(r,s) = \{f_m(r,s)\}_{m\geq 0} \text{ and } Fb'(r,s) = \{f'_m(r,s)\}_{m\geq 0},$$

where $f_m(r,s)$ and $f'_m(r,s)$ are defined recursively as:

$$f_0 = f'_0 = r \ ; \ f_1 = f'_1 = s;$$

and

$$f_m = f_{m-1} \bullet f_{m-2} \text{ and } f'_m = f'_{m-2} \bullet f'_{m-1} \text{ for } m \geq 2$$

Definition II.8. [34] A function that is its own inverse is called the *involution*, i.e., for any two non-empty sets X and Y, a function $\xi : X \to Y$ such that ξ^2 equals the identity is called the involution. i.e., $\xi(\xi(x)) = x, \forall x \in X$.

Definition II.9. [34] A function $\xi : \mathcal{A}^* \to \mathcal{A}^*$ is termed as *morphism* over $\mathcal{A}^*$ if, $\xi(u \bullet v) = \xi(u) \bullet \xi(v)$, for all $u, v \in \mathcal{A}^*$ and $\xi(\lambda) = \lambda$.

(i.e.,) if $u = r_1 r_2 \cdots r_k$, then $\xi(u) = \xi(r_1 r_2 \cdots r_k) = \xi(r_1)\xi(r_2)\cdots\xi(r_k)$, for all $r_i \in \mathcal{A}, 1 \leq i \leq k$.

A function $\xi : \mathcal{A}^* \to \mathcal{A}^*$ is termed as *antimorphism* over $\mathcal{A}^*$ if, $\xi(u \bullet v) = \xi(v) \bullet \xi(u)$, for all $u, v \in \mathcal{A}^*$ and $\xi(\lambda) = \lambda$.

(i.e.,) if $u = r_1 r_2 \cdots r_k$, then $\xi(u) = \xi(r_1 r_2 \cdots r_k) = \xi(r_k)\cdots\xi(r_2)\xi(r_1)$, for all $r_i \in \mathcal{A}, 1 \leq i \leq k$.

Definition II.10. [34] A function $\xi : \mathcal{A}^* \to \mathcal{A}^*$ is termed as a *morphic involution* on $\mathcal{A}^*$ if it is an involution on $\mathcal{A}$ extended to be a morphism on $\mathcal{A}^*$.

A function $\xi : \mathcal{A}^* \to \mathcal{A}^*$ is termed as an *antimorphic involution* on $\mathcal{A}^*$ if it is an involution on $\mathcal{A}$ extended to be an antimorphism on $\mathcal{A}^*$.

Definition II.11. [34] The *sequence of atom standard and reverse alternating ξ-Fibonacci words* $G(\alpha,\beta)$ and $G'(\alpha,\beta)$ over $\mathcal{A} = \{\alpha, \beta\}$ are defined as:

$$G(\alpha,\beta) = \{g_m(\alpha,\beta)\}_{m\geq 0}$$

and

$$G'(\alpha,\beta) = \{g'_m(\alpha,\beta)\}_{m\geq 0},$$

where $g_m(\alpha,\beta)$ and $g'_m(\alpha,\beta)$ are defined recursively as: $g_0 = g'_0 = \alpha$; $g_1 = g'_1 = \beta$;
$g_m = \xi(g_{m-1}) \bullet g_{m-2}$ and $g'_m = g'_{m-2} \bullet \xi(g'_{m-1})$ for $m \geq 2$, where ξ represents a morphic involution over $\mathcal{A}^*$.

Definition II.12. [34] The *sequence of atom standard and reverse palindromic ξ-Fibonacci words* $W(\alpha,\beta)$ and $W'(\alpha,\beta)$ over $\mathcal{A} = \{\alpha, \beta\}$ are defined as:

$$W(\alpha,\beta) = \{w_m(\alpha,\beta)\}_{m\geq 0}$$

and

$$W'(\alpha,\beta) = \{w'_m(\alpha,\beta)\}_{m\geq 0},$$

where $w_m(\alpha,\beta)$ and $w'_m(\alpha,\beta)$ are defined recursively as: $w_0 = w'_0 = \alpha$; $w_1 = w'_1 = \beta$;
$w_m = \xi(w_{m-1}) \bullet \xi(w_{m-2})$ and $w'_m = \xi(w'_{m-2}) \bullet \xi(w'_{m-1})$ for $m \geq 2$, where ξ represents a morphic involution over $\mathcal{A}^*$.

Definition II.13. [34] The *sequence of atom standard and reverse hairpin ξ-Fibonacci words* $Z(\alpha,\beta)$ and $Z'(\alpha,\beta)$ over $\mathcal{A} = \{\alpha, \beta\}$ are defined as:

$$Z(\alpha,\beta) = \{z_m(\alpha,\beta)\}_{m\geq 0}$$

and

$$Z'(\alpha,\beta) = \{z'_m(\alpha,\beta)\}_{m\geq 0},$$

where $z_m(\alpha,\beta)$ and $z'_m(\alpha,\beta)$ are defined recursively as: $z_0 = z'_0 = \alpha$; $z_1 = z'_1 = \beta$;
$z_m = z_{m-1} \bullet \xi(z_{m-2})$ and $z'_m = \xi(z'_{m-2}) \bullet z'_{m-1}$, for $m \geq 2$, where ξ represents a morphic involution over $\mathcal{A}^*$.

Definition II.14. [39] A rectangular array of letters over a finite alphabet $\mathcal{A}$ is known as a *picture* or *matrix* over $\mathcal{A}$.

A collection of pictures is denoted by $\mathcal{A}^{**}$. A subset of $\mathcal{A}^{**}$ is usually known as *picture language (or 2D language)* over $\mathcal{A}$.

Definition II.15. [39] If

$$M_A = \begin{matrix} u_{11} & \dots & u_{1n} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ u_{m1} & \dots & u_{mn} \end{matrix}$$

and

$$M_B = \begin{matrix} v_{11} & \dots & v_{1n'} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ v_{m'1} & \dots & v_{m'n'} \end{matrix}$$

then the vertical concatenation of M_A and M_B is defined as

$$M_A \mathbin{\Phi} M_B = \begin{matrix} u_{11} & \dots & u_{1n} & v_{11} & \dots & v_{1n'} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ u_{m1} & \dots & u_{mn} & v_{m'1} & \dots & v_{m'n'} \end{matrix}$$

provided $m = m'$, and the horizontal concatenation of M_A and M_B is defined as

$$M_A \ominus M_B = \begin{matrix} u_{11} & \dots & u_{1n} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ u_{m1} & \dots & u_{mn} \\ v_{11} & \dots & v_{1n'} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ v_{m'1} & \dots & v_{m'n'} \end{matrix}, \text{ provided } n = n'.$$

Definition II.16. [32] The *size* or *order* of an array with r rows and s columns is denoted by $r \times s$, provided $r, s \geq 1$. The arrays of order $r \times 0$ or $0 \times s$ are undefined. An array of order 0×0 is known as *empty array*, denoted by Λ.

Note that $\mathcal{A}^{**} \setminus \{\Lambda\} = \mathcal{A}^{++}$, where $\mathcal{A}^{++}$ represents the collection of all non-empty arrays over $\mathcal{A}$.

Definition II.17. [37] For two-dimensional arrays, a function $\xi : \mathcal{A}^{**} \to \mathcal{A}^{**}$ is called as *morphism* if for any $\alpha, \beta \in \mathcal{A}^{**}$,

(i) $\xi(\alpha \mathbin{\Phi} \beta) = \xi(\alpha) \mathbin{\Phi} \xi(\beta)$, provided α and β have equal number of rows
(ii) $\xi(\alpha \ominus \beta) = \xi(\alpha) \ominus \xi(\beta)$, provided α and β have equal number of columns and
(iii) In particular, $\xi(\Lambda) = \Lambda$.

(i.e.,) If

$$\alpha = \begin{matrix} u_{11} & \dots & u_{1n} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ u_{m1} & \dots & u_{mn} \end{matrix}$$

and

$$\beta = \begin{matrix} v_{11} & \dots & v_{1n'} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ v_{m'1} & \dots & v_{m'n'} \end{matrix}$$

then

$$\xi(\alpha \mathbin{\Phi} \beta) = \begin{matrix} \xi(u_{11}) & \dots & \xi(u_{1n}) & \xi(v_{11}) & \dots & \xi(v_{1n'}) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \xi(u_{m1}) & \dots & \xi(u_{mn}) & \xi(v_{m'1}) & \dots & \xi(v_{m'n'}) \end{matrix}$$

provided $m = m'$, and

$$\xi(\alpha \ominus \beta) = \begin{matrix} \xi(u_{11}) & \dots & \xi(u_{1n}) \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ \xi(u_{m1}) & \dots & \xi(u_{mn}) \\ \xi(v_{11}) & \dots & \xi(v_{1n'}) \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ \xi(v_{m'1}) & \dots & \xi(v_{m'n'}) \end{matrix}$$

provided $n = n'$.

Definition II.18. [37] For two-dimensional arrays, a function $\xi : \mathcal{A}^{**} \to \mathcal{A}^{**}$ is called as *antimorphism* if for any $\alpha, \beta \in \mathcal{A}^{**}$,

(i) $\xi(\alpha \mathbin{\Phi} \beta) = \xi(\beta) \mathbin{\Phi} \xi(\alpha)$, provided α and β have equal number of rows
(ii) $\xi(\alpha \ominus \beta) = \xi(\beta) \ominus \xi(\alpha)$, provided α and β have equal number of columns and
(iii) In particular, $\xi(\Lambda) = \Lambda$.

(i.e.,) If

$$\alpha = \begin{matrix} u_{11} & \dots & u_{1n} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ u_{m1} & \dots & u_{mn} \end{matrix}$$

and

$$\beta = \begin{matrix} v_{11} & \dots & v_{1n'} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ v_{m'1} & \dots & v_{m'n'} \end{matrix}$$

then

$$\xi(\alpha \mathbin{\Phi} \beta) = \begin{matrix} \xi(v_{m'n'}) & \dots & \xi(v_{m'1}) & \xi(u_{mn}) & \dots & \xi(u_{m1}) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \xi(v_{1n'}) & \dots & \xi(v_{11}) & \xi(u_{1n}) & \dots & \xi(u_{11}) \end{matrix}$$

provided $m = m'$, and

$$\xi(\alpha \ominus \beta) = \begin{matrix} \xi(v_{m'n'}) & \dots & \xi(v_{m'1}) \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ \xi(v_{1n'}) & \dots & \xi(v_{11}) \\ \xi(u_{mn}) & \dots & \xi(u_{m1}) \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ \xi(u_{1n}) & \dots & \xi(u_{11}) \end{matrix}$$

provided $n = n'$.

Definition II.19. [37] A function $\xi : \mathcal{A}^{**} \to \mathcal{A}^{**}$ is known as a *morphic involution* over $\mathcal{A}^{**}$ if ξ is an involution on $\mathcal{A}$ extended to be a morphism over $\mathcal{A}^{**}$.

A function $\xi : \mathcal{A}^{**} \to \mathcal{A}^{**}$ is known as an *antimorphic involution* over $\mathcal{A}^{**}$ if ξ is an involution on $\mathcal{A}$ extended to be an antimorphism over $\mathcal{A}^{**}$.

Definition II.20. [37] Let $\mathcal{A}$ be an alphabet such that $|\mathcal{A}| \geq 2$. Then the *sequence of atom standard alternating ξ-Fibonacci arrays* $\{g_{m,n}\}_{m,n\geq 0}$ over $\mathcal{A}^{**}$ is defined recursively as:

(i) The initial arrays $\begin{matrix} g_{0,0} & g_{0,1} \\ g_{1,0} & g_{1,1} \end{matrix}$ are given by $\begin{matrix} a_1 & b_1 \\ c_1 & d_1 \end{matrix}$ where $a_1, b_1, c_1, d_1 \in \mathcal{A}$, with atleast one of a_1, b_1, c_1, d_1 being distinct from the rest.

(ii) For $s \geq 0$ and $j, k \geq 1$, $g_{s,k+1} = \xi(g_{s,k}) \oplus g_{s,k-1}$ and $g_{j+1,s} = \xi(g_{j,s}) \ominus g_{j-1,s}$, where ξ is a morphic involution over $\mathcal{A}^{**}$.

Definition II.21. [37] Let $\mathcal{A}$ be an alphabet such that $|\mathcal{A}| \geq 2$. Then the *sequence of atom reverse alternating ξ-Fibonacci arrays* $\{g'_{m,n}\}_{m,n\geq 0}$ over $\mathcal{A}^{**}$ is defined recursively as:

(i) The initial arrays $\begin{matrix} g'_{0,0} & g'_{0,1} \\ g'_{1,0} & g'_{1,1} \end{matrix}$ are given by $\begin{matrix} a_1 & b_1 \\ c_1 & d_1 \end{matrix}$, where $a_1, b_1, c_1, d_1 \in \mathcal{A}$, with atleast one of a_1, b_1, c_1, d_1 being distinct from the rest.

(ii) For $s \geq 0$ and $j, k \geq 1$, $g'_{s,k+1} = g'_{s,k-1} \oplus \xi(g'_{s,k})$ and $g'_{j+1,s} = g'_{j-1,s} \ominus \xi(g'_{j,s})$, where ξ is a morphic involution over $\mathcal{A}^{**}$.

Definition II.22. [37] Let $\mathcal{A}$ be an alphabet such that $|\mathcal{A}| \geq 2$. Then the *sequence of atom standard palindromic ξ-Fibonacci arrays* $\{w_{m,n}\}_{m,n\geq 0}$ over $\mathcal{A}^{**}$ is defined recursively as:

(i) The initial arrays $\begin{matrix} w_{0,0} & w_{0,1} \\ w_{1,0} & w_{1,1} \end{matrix}$ are given by $\begin{matrix} a_1 & b_1 \\ c_1 & d_1 \end{matrix}$ where $a_1, b_1, c_1, d_1 \in \mathcal{A}$, with atleast one of a_1, b_1, c_1, d_1 being distinct from the rest.

(ii) For $s \geq 0$ and $j, k \geq 1$, $w_{s,k+1} = \xi(w_{s,k}) \oplus \xi(w_{s,k-1})$ and $w_{j+1,s} = \xi(w_{j,s}) \ominus \xi(w_{j-1,s})$, where ξ is a morphic involution over $\mathcal{A}^{**}$

Definition II.23. [37] Let $\mathcal{A}$ be an alphabet such that $|\mathcal{A}| \geq 2$. Then the *sequence of atom reverse palindromic ξ-Fibonacci arrays* $\{w'_{m,n}\}_{m,n\geq 0}$ over $\mathcal{A}^{**}$ is defined recursively as:

(i) The initial arrays $\begin{matrix} w'_{0,0} & w'_{0,1} \\ w'_{1,0} & w'_{1,1} \end{matrix}$ are given by $\begin{matrix} a_1 & b_1 \\ c_1 & d_1 \end{matrix}$ where $a_1, b_1, c_1, d_1 \in \mathcal{A}$, with atleast one of a_1, b_1, c_1, d_1 being distinct from the rest.

(ii) For $s \geq 0$ and $j, k \geq 1$, $w'_{s,k+1} = \xi(w'_{s,k-1}) \oplus \xi(w'_{s,k})$ and $w'_{j+1,s} = \xi(w'_{j-1,s}) \ominus \xi(w'_{j,s})$, where ξ is a morphic involution over $\mathcal{A}^{**}$.

Definition II.24. [37] In two-dimension, the *sequence of atom standard hairpin ξ-Fibonacci arrays* $\{z_{m,n}\}_{m,n\geq 0}$ is defined as:

(i) The initial arrays $\begin{matrix} z_{0,0} & z_{0,1} \\ z_{1,0} & z_{1,1} \end{matrix}$ are given by $\begin{matrix} a_1 & b_1 \\ c_1 & d_1 \end{matrix}$ where $a_1, b_1, c_1, d_1 \in \mathcal{A}$, with atleast one of a_1, b_1, c_1, d_1 being distinct from the rest.

(ii) For $s \geq 0$ and $j, k \geq 1$, $z_{s,k+1} = z_{s,k} \oplus \xi(z_{s,k-1})$ and $z_{j+1,s} = z_{j,s} \ominus \xi(z_{j-1,s})$, where ξ is a morphic involution over $\mathcal{A}^{**}$.

Definition II.25. [37] In two-dimension, the *sequence of atom reverse hairpin ξ-Fibonacci arrays* $\{z'_{m,n}\}_{m,n\geq 0}$ is defined as:

(i) The initial arrays $\begin{matrix} z'_{0,0} & z'_{0,1} \\ z'_{1,0} & z'_{1,1} \end{matrix}$ are given by $\begin{matrix} a_1 & b_1 \\ c_1 & d_1 \end{matrix}$ where $a_1, b_1, c_1, d_1 \in \mathcal{A}$, with atleast one of a_1, b_1, c_1, d_1 being distinct from the rest.

(ii) For $s \geq 0$ and $j, k \geq 1$, $z'_{s,k+1} = \xi(z'_{s,k-1}) \oplus z'_{s,k}$ and $z'_{j+1,s} = \xi(z'_{j-1,s}) \ominus z'_{j,s}$, where ξ is a morphic involution over $\mathcal{A}^{**}$.

Definition II.26. [32] An array a from the alphabet $\mathcal{A}$ is said to be *primitive* if $a = (w^{k_1 \oplus})^{k_2 \ominus}$, then both k_1 and k_2 should be equal to 1.

III. Indexing the Involutive Fibonacci Arrays

We attempt to index the Involutive Fibonacci arrays by means of few known results in this section.

We recall from [37] about the size of the involutive Fibonacci arrays $g_{r,s}, g'_{r,s}, w_{r,s}, w'_{r,s}, z_{r,s}, z'_{r,s}$, which is given by $Fb(r) \times Fb(s)$.

Lemma III.1. If the initial array of the standard alternating Fibonacci array is $\begin{matrix} a_1 & b_1 \\ c_1 & d_1 \end{matrix}$ and if $a_1 \neq b_1$ and $c_1 \neq d_1$, then for $r, s \geq 1$, every row of the standard alternating Fibonacci array $g_{r,s}$, while written as a one-dimensional string is the standard alternating Fibonacci word g_s over $\{a_1, b_1, c_1, d_1\}$. Similarly, if $a_1 \neq c_1$ and $b_1 \neq d_1$ then for $r, s \geq 1$, every column of $g_{r,s}$, while written as a one-dimensional string is the alternating Fibonacci words g_r over $\{a_1, b_1, c_1, d_1\}$.

When we expand $g_{r,s}$ vertical-wise, we get

$$g_{r,s} = \xi(g_{r,s-1}) \oplus g_{r,s-2}$$

The above step clearly says that in the r^{th} row the s^{th} element is obtained by vertically concatenating $\xi(g_{r,s-1})$ i.e., the morphic involutive image of the preceding element of $g_{r,s}$, with $g_{r,s-2}$ i.e., the second preceding element of $g_{r,s}$, resembling the act of obtaining $g_s = \xi(g_{s-1}) \bullet g_{s-2}$. Similarly we can prove the other case using horizontal-wise expansion.

The upcoming results are similar to Lemma III.1.

Lemma III.2. If the initial array of the standard palindromic Fibonacci array is $\begin{matrix} a_1 & b_1 \\ c_1 & d_1 \end{matrix}$ and if $a_1 \neq b_1$ and $c_1 \neq d_1$, then for $r, s \geq 1$, every row of the standard palindromic Fibonacci array $w_{r,s}$, while written as a one-dimensional string is the standard palindromic Fibonacci word w_s over $\{a_1, b_1, c_1, d_1\}$. Similarly, if $a_1 \neq c_1$ and $b_1 \neq d_1$ then for $r, s \geq 1$, every column of $w_{r,s}$, while written as a one-dimensional string is the standard palindromic Fibonacci words w_r over $\{a_1, b_1, c_1, d_1\}$.

Lemma III.3. If the initial array of the standard hairpin Fibonacci array is $\begin{matrix} a_1 & b_1 \\ c_1 & d_1 \end{matrix}$ and if $a_1 \neq b_1$ and $c_1 \neq d_1$, then for $r, s \geq 1$, every row of the standard hairpin Fibonacci array $z_{r,s}$, while written as a one-dimensional string is the standard hairpin Fibonacci word z_s over $\{a_1, b_1, c_1, d_1\}$. Similarly, if $a_1 \neq c_1$ and $b_1 \neq d_1$ then for $r, s \geq 1$, every column of $z_{r,s}$, while written as a one-dimensional string is the standard hairpin Fibonacci words z_r over $\{a_1, b_1, c_1, d_1\}$.

Lemma III.4. If the initial array of the reverse alternating Fibonacci array is $\begin{matrix} a_1 & b_1 \\ c_1 & d_1 \end{matrix}$ and if $a_1 \neq b_1$ and $c_1 \neq d_1$, then for $r, s \geq 1$, every row of the reverse alternating Fibonacci array $g'_{r,s}$, while written as a one-dimensional string is the reverse alternating Fibonacci word g'_s over $\{a_1, b_1, c_1, d_1\}$. Similarly, if $a_1 \neq c_1$ and $b_1 \neq d_1$ then for $r, s \geq 1$, every column of $g'_{r,s}$, while written as a one-dimensional string is the reverse alternating Fibonacci words g'_r over $\{a_1, b_1, c_1, d_1\}$.

Lemma III.5. If the initial array of the reverse palindromic Fibonacci array is $\begin{matrix} a_1 & b_1 \\ c_1 & d_1 \end{matrix}$ and if $a_1 \neq b_1$ and $c_1 \neq d_1$, then for $r, s \geq 1$, every row of the reverse palindromic Fibonacci array $w'_{r,s}$, while written as a one-dimensional string is the reverse palindromic Fibonacci word w'_s over $\{a_1, b_1, c_1, d_1\}$. Similarly, if $a_1 \neq c_1$ and $b_1 \neq d_1$ then for $r, s \geq 1$, every column of $w'_{r,s}$, while written as a one-dimensional string is the reverse palindromic Fibonacci words w'_r over $\{a_1, b_1, c_1, d_1\}$.

Lemma III.6. If the initial array of the reverse hairpin Fibonacci array is $\begin{matrix} a_1 & b_1 \\ c_1 & d_1 \end{matrix}$ and if $a_1 \neq b_1$ and $c_1 \neq d_1$, then for $r, s \geq 1$, every row of the reverse hairpin Fibonacci array $z'_{r,s}$, while written as a one-dimensional string is the reverse hairpin Fibonacci word z'_s over $\{a_1, b_1, c_1, d_1\}$. Similarly, if $a_1 \neq c_1$ and $b_1 \neq d_1$ then for $r, s \geq 1$, every column of $z'_{r,s}$, while written as a one-dimensional string is the reverse hairpin Fibonacci words z'_r over $\{a_1, b_1, c_1, d_1\}$.

Now let us study about what reduced representation of an involutive Fibonacci word means.

Definition III.1. Given a standard alternating Fibonacci word g_m, for $m \geq 2$, if g_m is reduced using the recurrence relation $g_m = \xi(g_{m-1}) \bullet g_{m-2}$, until we get g_m expressed using only g_0, g_1, $\xi(g_0)$ or $\xi(g_1)$, then this way of representing any standard alternating Fibonacci word g_m is called the *reduced representation of* g_m.

(i.e.,) $g_m = g_{i_1} g_{i_2} \cdots g_{i_{Fb(n)}}$, where each $g_{i_j}, 1 \leq j \leq Fb(n)$ is either g_{i_j} itself or $\xi(g_{i_j})$, which we represent as (g_{i_j}) and $i_1, i_2, \cdots i_{Fb(n)} \in \{0, 1\}$.

The binary string $i_1 i_2 \cdots i_{Fb(n)}$ in which each $i_j, 1 \leq j \leq Fb(n)$ is either written as i_j or (i_j) to denote whether the index is of g_{i_j} or that of $\xi(g_{i_j})$ is termed as the *standard alternating Fibonacci reduced representation of the integer* n, denoted by $SAFR(n)$.

Example III.1. $SAFR(3) = 1(0)1, SAFR(4) = (1)0(1)(1)0$

Definition III.2. Given a standard palindromic Fibonacci word w_m, for $m \geq 2$, if w_m is reduced using the recurrence relation $w_m = \xi(w_{m-1}) \bullet \xi(w_{m-2})$, until we get w_m expressed using only w_0, w_1, $\xi(w_0)$ or $\xi(w_1)$, then this way of representing any standard palindromic Fibonacci word w_m is called the *reduced representation of* w_m.

(i.e.,) $w_m = w_{i_1} w_{i_2} \cdots w_{i_{Fb(n)}}$, where each $w_{i_j}, 1 \leq j \leq Fb(n)$ is either w_{i_j} itself or $\xi(w_{i_j})$, which we represent as (w_{i_j}) and $i_1, i_2, \cdots i_{Fb(n)} \in \{0, 1\}$.

The binary string $i_1 i_2 \cdots i_{Fb(n)}$ in which each $i_j, 1 \leq j \leq Fb(n)$ is either written as i_j or (i_j) to denote whether the index is of w_{i_j} or that of $\xi(w_{i_j})$ is termed as the *standard palindromic Fibonacci reduced representation of the integer* n, denoted by $SPFR(n)$.

Example III.2. $SPFR(3) = 10(1), SPFR(4) = (1)(0)110$

Definition III.3. Given a standard hairpin Fibonacci word z_m, for $m \geq 2$, if z_m is reduced using the recurrence relation $z_m = z_{m-1} \bullet \xi(z_{m-2})$, until we get z_m expressed using only z_0, z_1, $\xi(z_0)$ or $\xi(z_1)$, then this way of representing any standard hairpin Fibonacci word z_m is called the *reduced representation of* z_m.

(i.e.,) $z_m = z_{i_1} z_{i_2} \cdots z_{i_{Fb(n)}}$, where each $z_{i_j}, 1 \leq j \leq Fb(n)$ is either z_{i_j} itself or $\xi(z_{i_j})$, which we represent as (z_{i_j}) and $i_1, i_2, \cdots i_{Fb(n)} \in \{0, 1\}$.

The binary string $i_1 i_2 \cdots i_{Fb(n)}$ in which each $i_j, 1 \leq j \leq Fb(n)$ is either written as i_j or (i_j) to denote whether the index is of z_{i_j} or that of $\xi(z_{i_j})$ is termed as the *standard hairpin Fibonacci reduced representation of the integer* n, denoted by $SHFR(n)$.

Example III.3. $SHFR(3) = 1(0)(1), SHFR(4) = 1(0)(1)(1)0$

Definition III.4. Given a reverse alternating Fibonacci word g'_m, for $m \geq 2$, if g'_m is reduced using the recurrence relation $g'_m = g'_{m-2} \bullet \xi(g'_{m-1})$, until we get g'_m expressed using only g'_0, g'_1, $\xi(g'_0)$ or $\xi(g'_1)$, then this way of representing any reverse alternating Fibonacci word g'_m is called the *reduced representation of* g'_m.

(i.e.,) $g'_m = g'_{i_1} g'_{i_2} \cdots g'_{i_{Fb(n)}}$, where each $g'_{i_j}, 1 \leq j \leq Fb(n)$ is either g'_{i_j} itself or $\xi(g'_{i_j})$, which we represent as (g'_{i_j}) and $i_1, i_2, \cdots i_{Fb(n)} \in \{0, 1\}$.

The binary string $i_1 i_2 \cdots i_{Fb(n)}$ in which each $i_j, 1 \leq j \leq Fb(n)$ is either written as i_j or (i_j) to denote whether the index is of g'_{i_j} or that of $\xi(g'_{i_j})$ is termed as the *reverse alternating Fibonacci reduced representation of the integer* n, denoted by $RAFR(n)$.

Example III.4. $RAFR(3) = 1(0)1, RAFR(4) = 0(1)(1)0(1)$

Definition III.5. Given a reverse palindromic Fibonacci word w'_m, for $m \geq 2$, if w'_m is reduced using the recurrence relation $w'_m = \xi(w'_{m-2}) \bullet \xi(w'_{m-1})$, until we get w'_m expressed using only w'_0, w'_1, $\xi(w_0)$ or $\xi(w_1)$, then this way of representing any reverse palindromic Fibonacci word w'_m is called the *reduced representation of* w'_m.

(i.e.,) $w'_m = w'_{i_1} w'_{i_2} \cdots w'_{i_{Fb(n)}}$, where each $w'_{i_j}, 1 \leq j \leq Fb(n)$ is either w'_{i_j} itself or $\xi(w'_{i_j})$, which we represent as (w'_{i_j}) and $i_1, i_2, \cdots i_{Fb(n)} \in \{0, 1\}$.

The binary string $i_1 i_2 \cdots i_{Fb(n)}$ in which each $i_j, 1 \leq j \leq Fb(n)$ is either written as i_j or (i_j) to denote whether the index is of w'_{i_j} or that of $\xi(w'_{i_j})$ is termed as the *reverse palindromic Fibonacci reduced representation of the integer* n, denoted by $RPFR(n)$.

Example III.5. $RPFR(3) = (1)01, RPFR(4) = 011(0)(1)$

Definition III.6. Given a reverse hairpin Fibonacci word z'_m, for $m \geq 2$, if z'_m is reduced using the recurrence relation $z'_m = \xi(z'_{m-2}) \bullet z'_{m-1}$, until we get z'_m expressed using only z'_0, z_1, $\xi(z_0)$ or $\xi(z_1)$, then this way of representing any reverse hairpin Fibonacci word z'_m is called the *reduced representation of* z'_m.

(i.e.,) $z'_m = z'_{i_1} z'_{i_2} \cdots z'_{i_{Fb(n)}}$, where each $z'_{i_j}, 1 \leq j \leq Fb(n)$ is either z'_{i_j} itself or $\xi(z'_{i_j})$, which we represent as (z'_{i_j}) and $i_1, i_2, \cdots i_{Fb(n)} \in \{0, 1\}$.

The binary string $i_1 i_2 \cdots i_{Fb(n)}$ in which each $i_j, 1 \leq j \leq Fb(n)$ is either written as i_j or (i_j) to denote whether the index is of z'_{i_j} or that of $\xi(z'_{i_j})$ is termed as the *reverse hairpin Fibonacci reduced representation of the integer* n, denoted by $RHFR(n)$.

Example III.6. $RHFR(3) = (1)(0)1, RHFR(4) = 0(1)(1)(0)1$

Note: $SAFR(n)$ is the reversal of $RAFR(n)$, $SPFR(n)$ is the reversal of $RPFR(n)$ and $SHFR(n)$ is the reversal of $RHFR(n)$.

Theorem III.1. *Let* $SAFR(m) = i_1 i_2 \cdots i_{Fb(m)}$ *and* $SAFR(n) = j_1 j_2 \cdots j_{Fb(n)}$. *Then the elements in* $g_{m,n}$ *are indexed such that they are ordered pairs of the Cartesian product* $\{i_1, i_2, \cdots i_{Fb(m)}\} \times \{j_1, j_2, \cdots j_{Fb(n)}\}$.

Proof: We know that $g_{m,n}$ is of the size $\big(Fb(m), Fb(n)\big)$. Therefore we expand $g_{m,n}$ vertical-wise till the vertical index of every element turns to be either $0, 1, (0)$ or (1). The concatenation of the vertical indices thus results in the $SAFR(n) = j_1 j_2 \cdots j_{Fb(n)}$, where $j'_k s$ can be either $0, 1, (0)$ or (1).

Further when the horizontal-wise expansion is done till the horizontal-index of every element turns to be either $0, 1, (0)$ or (1), the concatenation of the horizontal-indices, from top to bottom of any column results in the $SAFR(m) = i_1 i_2 \cdots i_{Fb(m)}$, where $i'_k s$ can be either $0, 1, (0)$ or (1).

(i.e.,) $g_{m,n} = \big(g_{i_1,j_1} \oplus g_{i_1,j_2} \oplus \cdots \oplus g_{i_1,j_{Fb(n)}}\big) \ominus \cdots \ominus \big(g_{i_{Fb(m)},j_1} \oplus g_{i_{Fb(m)},j_2} \oplus \cdots g_{i_{Fb(m)},j_{Fb(n)}}\big)$, where every g_{i_k,j_k} is either g_{i_k,j_k} or $\xi(g_{i_k}, g_{j_k})$.

Hence, the indices of elements in $g_{m,n}$ are the ordered pairs of the Cartesian product

$\{i_1, i_2, \cdots i_{Fb(m)}\} \times \{j_1, j_2, \cdots j_{Fb(n)}\}$. ■

Note:

1) This result holds good for the standard palindromic, standard hairpin, reverse alternating, reverse palindromic and reverse hairpin Fibonacci arrays as well.
2) The Cartesian product of (a) and b is (a, b), a and (b) is (a, b), (a) and (b) is a, b and a and b is a, b, where $a, b \in \mathcal{A}$.

Example III.7. The index of elements in $g_{3,5}$ are indexed such that they are the ordered pairs of the Cartesian product $\{1, (0), 1\} \times \{1, (0), 1, 1, (0), 1, (0), 1\}$, the index of elements in $w_{3,5}$ are indexed such that they are the ordered pairs of the Cartesian product $\{1, 0, (1)\} \times \{1, 0, (1), (1), (0), (1), (0), 1\}$, the index of elements in $z_{3,5}$ are indexed such that they are the ordered pairs of the Cartesian product $\{1, (0), (1)\} \times \{1, (0), (1), (1), 0, (1), 0, 1\}$, the index of elements in $g'_{3,5}$ are indexed such that they are the ordered pairs of the Cartesian product $\{1, (0), 1\} \times \{1, (0), 1, (0), 1, 1, (0), 1\}$, the index of elements in $w'_{3,5}$ are indexed such that they are the ordered pairs of the Cartesian product $\{(1), 0, 1\} \times \{1, (0), (1), (0), (1), (1), 0, 1\}$ and the index of elements in $z'_{3,5}$ are indexed such that they are the ordered pairs of the Cartesian product $\{(1), (0), 1\} \times \{1, 0, (1), 0, (1), (1), (0), 1\}$.

IV. Decomposition of Involutive Fibonacci Arrays

In this section we see few properties on decomposing or breaking the involutive Fibonacci arrays into sub-arrays. In view of this we now recall a result from the literature.

Lemma IV.1. *[34] For* ξ *being a morphic involution over* $\mathcal{A}^*$, *let* $g_0 = a_1$ *and* $g_1 = b_1$, *where* $a_1, b_1 \in \mathcal{A}$. *Then for* $n \geq 2$,

$$g_n = \begin{cases} s_n xy, & \text{if } n \text{ is odd} \\ s_n pq, & \text{if } n \text{ is even} \end{cases}$$

where $s_2 = s'_2 = \lambda, x = \xi(a_1), y = b_1, p = \xi(b_1)$ *and* $q = a_1$.

We now use the above result to prove the theorem below.

Theorem IV.1. *Let* $\{g_{r,s}\}_{r,s \geq 0}$ *represent the sequence of standard alternating Fibonacci Arrays over* $\mathcal{A} = \{a_1, b_1, c_1, d_1\}$, *with initial array* $\begin{matrix} a_1 & b_1 \\ c_1 & d_1 \end{matrix}$. *For* $r, s \geq 3$,

$$g_{r,s} = \big(D_1 \oplus D_2\big) \ominus \big(D_3 \oplus D_4\big)$$

with $|D_1|_{horizontal} = |D_2|_{horizontal}$ *and* $|D_1|_{vertical} = |D_3|_{vertical}$, *such that*

1) D_1 *is a 2D palindrome of size*
$$(Fb(r) - 2) \times (Fb(s) - 2)$$
2) D_2 *and* D_3 *are arrays of sizes* $(Fb(r) - 2) \times 2$ *and* $2 \times (Fb(s) - 2)$ *respectively.*
3) D_4 *is an array of size* 2×2, *such that*
   a) $(d_1 \oplus \xi(c_1)) \ominus (\xi(b_1) \oplus a_1)$, *if* r *and* s *are even*
   b) $(a_1 \oplus \xi(b_1)) \ominus (\xi(c_1) \oplus d_1)$, *if* r *and* s *are odd*
   c) $(c_1 \oplus \xi(d_1)) \ominus (\xi(a_1) \oplus b_1)$, *if* r *is even and* s *is odd*
   d) $(b_1 \oplus \xi(a_1)) \ominus (\xi(d_1) \oplus c_1)$, *if* r *is odd and* s *is even*

Proof:

1) By Lemma III.1, every row (each column) of the alternating Fibonacci array $g_{r,s}$ while written as a one dimensional string is the alternating Fibonacci word g_s (or g_r) respectively. And by Lemma IV.1 every rows (and columns) will be having a palindromic prefix of size $(Fb(r) - 2) \times (Fb(s) - 2)$ of $g_{r,s}$ and thus the prefix array D_1 is a palindrome of size $(Fb(r) - 2) \times (Fb(s) - 2)$.
2) The sizes of D_2 and D_3 are obvious since $|D_1|_{\text{ horizontal}} = |D_2|_{\text{ horizontal}}$ and $|D_1|_{\text{ vertical}} = |D_3|_{\text{ vertical}}$.
3) Consider the case when r and s are even. From Lemma IV.1, the alternating Fibonacci reduced representation of both r and s ends with (1) and 0. Therefore, the members of the suffix of size 2×2 of $g_{r,s}$ have the indices which seems to be the ordered pairs of the Cartesian product $\{(1), 0\} \times \{(1), 0\}$. Hence the suffix is $(d_1 \oplus \xi(c_1) \ominus (\xi(b_1) \oplus a_1)$.

■

Let us now recall a result from [34].

Lemma IV.2. *[34] Let* ξ *be a morphic involution on* $\mathcal{A}^*$ *and let* $g_0 = a_1$ *and* $g_1 = b_1$. *Then* $g_n = x_n y_n$ *such that* x_n *and* y_n *are palindromes for all* $n \geq 2$, *such that* $|x_n| = (Fb(n-1) - 2)$ *and* $|y_n| = (Fb(n-2) + 2)$.

Theorem IV.2. *For $r, s \geq 4$, we have*

$$g_{r,s} = (D_1 \oplus D_2) \ominus (D_3 \oplus D_4)$$

where D_1, D_2, D_3 and D_4 are all palindromic sub-arrays such that

1) $size(D_1) = (Fb(r-1)-2) \times (Fb(s-1)-2)$
2) $size(D_2) = (Fb(r-1)-2) \times (Fb(s-2)+2)$
3) $size(D_3) = (Fb(r-2)+2) \times (Fb(s-1)-2)$
4) $size(D_4) = (Fb(r-2)+2) \times (Fb(s-2)+2)$

This decomposition is unique.

Proof: We first prove the validation of decomposition given in the statement and then prove the uniqueness. If $a_1 \neq b_1$ and $c_1 \neq d_1$. Then by Lemma III.1, each row of an alternating Fibonacci array, written as a 1D word is an alternating Fibonacci word g_s. By Lemma IV.2, every g_s can be written as product of two palindromes of lengths $Fb(s-1)-2$ and $Fb(s-2)+2$. Thus, in an alternating Fibonacci array $g_{r,s}$, each row can be represented as a vertical concatenation of two palindromes of sizes $1 \times (Fb(s-1)-2)$ and $1 \times (Fb(s-2)+2)$. Similarly we can discuss when $a_1 \neq c_1$ and $b_1 \neq d_1$. Since every row and every column is a 1D palindrome we say that the result holds. To prove the uniqueness, if a_1, b_1, c_1 and d_1 are all unique then its trivial. In all the other cases, at-least one row or column will be a 1D alternating Fibonacci word and so the representation will be unique. ■

Thus we have brought out two different ways of decomposing the involutive Fibonacci arrays.

Now we recall the following definitions from [40].

Definition IV.1. [40] The *domain of a picture* p is the set of coordinates

$$dom(p) = \{1, 2, \cdots, |p|_{horizontal}\} \times \{1, 2, \cdots, |p|_{vertical}\}$$

where $|p|_{horizontal}$ and $|p|_{vertical}$ represents the number of rows and columns in p.

We denote by $p(i,j)$ the symbol in p at (i,j) coordinate. We need to note that the positions in $dom(p)$ follow lexicographic order. (i.e.,) $(i,j) < (i_0, j_0)$ if either $i < i_0$ or $i = i_0$ and $j < j_0$.

Definition IV.2. [40] The *subdomain of* $dom(p)$ is a set d of the form

$$\{i, i+1, \cdots, i_0\} \times \{j, j+1, \cdots, j_0\}$$

where $1 \leq i \leq i_0 \leq |p|_{horizontal}, 1 \leq j \leq j_0 \leq |p|_{vertical}$, denoted usually as $[(i,j),(i_0,j_0)]$.

Definition IV.3. [40] The *subpicture of* p associated to the subdomain $[(i,j),(i_0,j_0)]$ is the portion of p corresponding to positions in the subdomain and is denoted by $p[(i,j),(i_0,j_0)]$.

Example IV.1. Let $\mathcal{A} = \{a_1, b_1, c_1, d_1\}$. Then the picture $p = \begin{pmatrix} a_1 & b_1 & a_1 & b_1 \\ b_1 & c_1 & c_1 & d_1 \\ d_1 & a_1 & b_1 & c_1 \end{pmatrix}$ is over $\mathcal{A}$, whose domain is $\{1,2,3\} \times \{1,2,3,4\}$. One of the subdomains of this domain is $\{2,3\} \times \{2,3\}$, also denoted as $[(2,2),(3,3)]$. Subpicture corresponding to $[(2,2),(3,3)]$ is $\begin{pmatrix} c_1 & c_1 \\ a_1 & b_1 \end{pmatrix}$

Definition IV.4. [40] *Tiling star of* X, denoted by X^{**}, is the set of pictures p, whose domain can be partitioned into disjoint subdomains $\{d_1, d_2, \cdots, d_m\}$, such that, for any $c = 1, 2, \cdots m$, the subpicture $p(c)$ of p corresponding to the subdomain $d_c \in X$.

Example IV.2. Let

$$X = \left\{ \begin{pmatrix} a_1 & b_1 \end{pmatrix}, \begin{pmatrix} c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_1 \\ c_1 \end{pmatrix}, \begin{pmatrix} b_1 \\ d_1 \end{pmatrix}, \begin{pmatrix} a_1 & a_1 \end{pmatrix}, \begin{pmatrix} a_1 & a_1 \\ c_1 & c_1 \end{pmatrix} \right\}$$

Let us consider a picture $p = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ over $\mathcal{A} = \{a_1, b_1, c_1, d_1\}$ and let the disjoint subdomains of p be $d_1 = [(1,1),(1,2)]$ and $d_2 = [(2,1),(2,2)]$. Then $p(1)$ of p is $p(1) = \begin{pmatrix} a_1 & b_1 \end{pmatrix}$ and $p(2)$ of p is $p(2) = \begin{pmatrix} c_1 & d_1 \end{pmatrix} \in X$. Hence the picture $p \in X^{**}$.

Definition IV.5. [40] Language X^{**} is called the *set of all tilings by* X.

If $p \in X^{**}$, then partition $d = \{d_1, d_2, \cdots, d_m\}$ of $dom(p)$ together with $\{p_1, p_2, \cdots, p_m\}$ is called *tiling decomposition of* p *in* X.

$X \subset \mathcal{A}^{**}$ is said to be a *Two-Dimensional code* if any $p \in \mathcal{A}^{**}$ has atmost one tiling decomposition in X.

Example IV.3. $X = \left\{ \begin{pmatrix} p & q \end{pmatrix}, \begin{pmatrix} p \\ q \end{pmatrix}, \begin{pmatrix} p & p \\ p & p \end{pmatrix} \right\}$ is a code as all the pictures made out of them can be decomposed in exactly one way or not.

Example IV.4. $X = \left\{ \begin{pmatrix} p & q \end{pmatrix}, \begin{pmatrix} q & p \end{pmatrix}, \begin{pmatrix} p \\ p \end{pmatrix} \right\}$ is not a code since the picture $\begin{pmatrix} p & q & p \\ p & q & p \end{pmatrix}$ can be decomposed in X in two different ways as

p q	**p**
p q	**p**

p	**q p**
p	**q p**

Now we prove that the language of square involutive Fibonacci arrays is a 2D code.

Theorem IV.3. *Let $\{g_{m,n}\}_{m,n \geq 0}$ be the sequence of alternating Fibonacci arrays defined on the morphic involution ξ over $\mathcal{A}^{**}$, where $\mathcal{A} = \{a_1, b_1, c_1, d_1\}$. If none of these letters a_1, b_1, c_1 or d_1 are identical, then the language $L = \{g_{n,n} : n \geq 0\}$ is a two-dimensional code.*

Proof: We prove this by the method of contradiction by assuming that L is not a code. This implies that there exist a word $w \in \mathcal{A}^{++}$ such that it has two distinct decomposition in L, say d_1 and d_2.

Let us take the bottom right blocks of d_1 and d_2, say g_{r_1,r_1} and g_{r_2,r_2}, respectively, such that $r_1 \neq r_2$. Assume that $r_1 \leq r_2$. Since both g_{r_1,r_1} and g_{r_2,r_2} are square arrays and $r_1 \leq r_2$, g_{r_1,r_1} must be a suffix of g_{r_2,r_2}. Assume that r_2 is odd. Then by Theorem IV.1, suffix of g_{r_2,r_2} is $(a_1 \oplus \xi(b_1)) \ominus (\xi(c_1) \oplus d_1)$. But we have g_{r_1,r_1} to be a suffix of g_{r_2,r_2} and so suffix of g_{r_1,r_1} is also $(a_1 \oplus \xi(b_1)) \ominus (\xi(c_1) \oplus d_1)$ and thus we have r_1 also to be odd.

Case (i): If r_1 is the immediate odd predecessor of r_2, then $r_2 = r_1 + 2$ and there exists a r_3 such that $r_3 = r_1 + 1$ and $r_3 = r_2 - 1$. Then we have $Fb(r_2) = Fb(r_1) + Fb(r_3)$.

Since r_1 and r_2 are odd, the $SAFR(r_1)$ and $SAFR(r_2)$ ends with $(0)1$. Therefore the last columns of g_{r_1,r_1} and g_{r_2,r_2} has $\xi(b_1)$ and d_1.

Since r_3 is even, the $SAFR(r_3)$ ends with $(1)0$ and then we find that the members in the last column of g_{r_2,r_2} which lies above the last column of g_{r_1,r_1} should consist of $\xi(c_1)$ and a_1, which is a contradiction.

Case (ii): If $r_2 \geq r_1 + 2$, then g_{r_1,r_1} is a 2D suffix of g_{r_2-2,r_2-2}. To arrive at contradiction, we can utilise the same logic as in Case(i).

From both the cases we conclude that $r_1 = r_2$. In the same way we can argue for all the blocks of d_1 and d_2 and hence conclude that $d_1 = d_2$. The case when r_1 is even can be studied similarly.

Thus, w has a unique decomposition in L and therefore L is a 2D code. ■

Corollary IV.1. *The languages $L = \{w_{n,n} : n \geq 0\}$ and $L = \{z_{n,n} : n \geq 0\}$ are also two-dimensional codes. This can be proved just as Theorem IV.3.*

V. Primitivity and Non-recognizability in Involutive Fibonacci Arrays

In this section we study the primitive nature of the involutive Fibonacci arrays and we establish that the languages of involutive Fibonacci arrays are not tiling recognizable. First, let us recall a theorem from the literature.

Theorem V.1. *[32] The set of Fibonacci arrays $f_{m,n}$ for $m, n \geq 2$ is 2D primitive except if $a_1 = c_1$ & $b_1 = d_1$ and $a_1 = b_1$ & $c_1 = d_1$. And if $b_1 \neq d_1$ & $c_1 \neq d_1$, then the initial arrays $f_{0,0}, f_{0,1}, f_{1,0}$ & $f_{1,1}$ and $f_{1,n}$ for $n \geq 2$ & $f_{m,1}$ for $m \geq 2$ are also 2D primitive arrays.*

Now we prove a similar result for the standard palindromic Fibonacci arrays.

Theorem V.2. *The set of standard palindromic Fibonacci arrays $w_{m,n}$, for $m, n \geq 2$, where $w_{m,n}$ is defined over morphic involution ξ on $\mathcal{A}^{**}$ with initial array $\begin{matrix} a_1 & b_1 \\ c_1 & d_1 \end{matrix}$, where $a \neq b, c \neq d$ and $a \neq c, b \neq d$ is 2D primitive.*

Proof: We prove this by classifying the morphic involution on which $w_{m,n}$ are defined as identity and non-identity.

Case (i): When ξ is identity

In this case the palindromic Fibonacci array $w_{m,n}$ turns out to be a standard Fibonacci array $f_{m,n}$. In that case by Theorem V.1 the result holds.

Case (ii): When ξ is non-identity

Assume the contrary that the set of $w_{m,n}$ is non-primitive. This implies that $w_{m,n}$ can be written as $w_{m,n} = (a^{r_1 \oplus})^{r_2 \ominus}$, with atleast one of r_1 or r_2 is strictly greater than 1.

Without loss of generality, assume that $r_1 > 1$ and $r_2 = 1$. This implies that $w_{m,n} = a^{r_1 \oplus}$ for some $a \in \mathcal{A}^{**}$. This means that for every $i, 1 \leq i \leq Fb(m)$, the i^{th} row of a, say $a^{(i)}$, will yield the i^{th} row of $w_{m,n}$, say $w^{(i)}_{m,n}$, as $w^{(i)}_{m,n} = (a^{(i)})^{r_1 \oplus}$. This leads to a contradiction as we assume that ξ is a non-identity morphic involution and hence atleast one of the rows of $w_{m,n}$ will not be 1D primitive. Therefore we conclude that $w_{m,n}$ is 2D primitive. ■

The below theorems can be proved in a similar way.

Theorem V.3. *The set of standard alternating Fibonacci arrays $g_{m,n}$, for $m, n \geq 2$, where $g_{m,n}$ is defined over morphic involution ξ on $\mathcal{A}^{**}$ with initial array $\begin{matrix} a_1 & b_1 \\ c_1 & d_1 \end{matrix}$, where $a_1 \neq b_1, c_1 \neq d_1$ and $a_1 \neq c_1, b_1 \neq d_1$ is 2D primitive.*

Theorem V.4. *The set of standard hairpin Fibonacci arrays $z_{m,n}$, for $m, n \geq 2$, where $z_{m,n}$ is defined over morphic involution ξ on $\mathcal{A}^{**}$ with initial array $\begin{matrix} a_1 & b_1 \\ c_1 & d_1 \end{matrix}$, where $a_1 \neq b_1, c_1 \neq d_1$ and $a_1 \neq c_1, b_1 \neq d_1$ is 2D primitive*

Theorem V.5. *The set of reverse alternating Fibonacci arrays $g'_{m,n}$, for $m, n \geq 2$, where $g'_{m,n}$ is defined over morphic involution ξ on $\mathcal{A}^{**}$ with initial array $\begin{matrix} a_1 & b_1 \\ c_1 & d_1 \end{matrix}$, where $a_1 \neq b_1, c_1 \neq d_1$ and $a_1 \neq c_1, b_1 \neq d_1$ is 2D primitive.*

Theorem V.6. *The set of reverse palindromic Fibonacci arrays $w'_{m,n}$, for $m, n \geq 2$, where $w'_{m,n}$ is defined over morphic involution ξ on $\mathcal{A}^{**}$ with initial array $\begin{matrix} a_1 & b_1 \\ c_1 & d_1 \end{matrix}$, where $a \neq b, c \neq d$ and $a \neq c, b \neq d$ is 2D primitive.*

Theorem V.7. *The set of reverse hairpin Fibonacci arrays $z'_{m,n}$, for $m, n \geq 2$, where $z'_{m,n}$ is defined over morphic involution ξ on $\mathcal{A}^{**}$ with initial array $\begin{matrix} a_1 & b_1 \\ c_1 & d_1 \end{matrix}$, where $a_1 \neq b_1, c_1 \neq d_1$ and $a_1 \neq c_1, b_1 \neq d_1$ is 2D primitive*

Thus we find that 2D Involutive Fibonacci arrays are primitive under certain conditions.

Now let us recall few definitions from [2]:

Let $\mathcal{A}$ and $\mathcal{B}$ be finite alphabets. For an array $a = [a_{i,j}] \in \mathcal{A}^{**}$ of order $m \times n$, the *projection by mapping* $\pi : \mathcal{A} \to \mathcal{B}$ of a is an array $a' = [a'_{i,j}] \in \mathcal{A}^{**}$ such that $a'_{i,j} = \pi(a_{i,j})$, for all $1 \leq i \leq m, 1 \leq j \leq n$. Similarly, the *projection by mapping* π of a two-dimensional language L is the language $L' = \{a' : a' = \pi(a), \forall a \in L\} \subseteq \mathcal{A}^{**}$. A *tiling system* τ is a quadruple $\tau = (\mathcal{B}, \mathcal{A}, \Theta, \pi)$, where $\mathcal{A}$ and $\mathcal{B}$ are two finite alphabets, Θ is a finite set of tiles over the alphabet $\mathcal{A} \cup \{\#\}$ and $\pi : \mathcal{A} \to \mathcal{B}$ is a projection. A language $L' \subseteq \mathcal{B}$ is said to be *recognizable* if there exists a tiling system $\tau = (\mathcal{B}, \mathcal{A}, \Theta, \pi)$ such that $L' = L'(\tau)$.

In [32], Kulkarni et al. proved that the language of Fibonacci arrays is not a tiling recognizable two-dimensional language. Similarly, we find that the languages of standard and reverse alternating Fibonacci arrays, standard and reverse palindromic Fibonacci arrays and standard and reverse hairpin Fibonacci arrays are also not tiling recognizable language. Hence we state this as a result in Theorem V.8.

Theorem V.8. *The languages $L_1 = \{g_{r,s} : r, s \geq 2\}$, $L_2 = \{w_{r,s} : r, s \geq 2\}$, $L_3 = \{z_{r,s} : r, s \geq 2\}$, $L_4 = \{g'_{r,s} : r, s \geq 2\}$, $L_5 = \{w'_{r,s} : r, s \geq 2\}$ and $L_6 = \{z'_{r,s} : r, s \geq 2\}$ are not tiling recognizable two-dimensional languages.*

VI. Conclusion

This paper establishes some of the combinatorial properties of the involutive Fibonacci arrays such as indexing, decomposition into palindromic sub-arrays, primitivity and non-recognizability. We have also proved that the language of the Involutive square Fibonacci arrays is a code. Further topics of research includes writing algorithms for checking arrays to be involutive or not, and so on.

References

[1] Giammarresi, D., Restivo, A. "Recognizable picture languages," International Journal of Pattern Recognition and Artificial Intelligence, vol. 6, pp. 241-256, 1992.

[2] Giammarresi, D., Restivo, A. "Two-Dimensional languages," In: Rozenberg, G., Salomaa, A. (eds) Handbook of Formal Languages. Springer, Berlin, Heidelberg, 1997.

[3] Thue, A. "Über Unendliche Zeichenreihen," Norske Vid Selsk. Skr. I Mat-Nat Kl.(Christiana), vol. 7, pp. 1-22, 1906.

[4] Trygve Nagell, Atle Selberg, Sigmund Selberg, Knut Thalberg "Selected Mathematical Papers of Axel Thue," Universitetsforlaget, Oslo, 1977.

[5] Lothaire, M. "Combinatorics on Words," Encyclopedia of Mathematics and its Applications, Addison - Wesley, United States, 1983.

[6] Kari, L., Mahalingam, K. "Involutively Bordered words," International Journal of Foundations of Computer Science, vol. 18, no. 5, pp. 1089-1106, 2022.

[7] Czeizler, E., Kari, L., Seki, S. "On a special class of primitive words," Theoretical Computer Science, vol. 411, pp. 617-630, 2010.

[8] S.S.Yu "Languages and Codes," National Chung-Hsing University, Taichung, Taiwan 402, Lecture Notes, Department of Computer Science, 2005.

[9] Amir, A., Benson, G. "Two-dimensional periodicity in rectangular arrays," SIAM Journal on Computing, vol. 27, no. 1, pp. 90-106, 1998.

[10] Kulkarni, M.S., Mahalingam, K. "Two-Dimensional Palindromes and their Properties," In: Drewes, F., Martín-Vide, C., Truthe, B. (eds) Language and Automata Theory and Applications. LATA 2017. Lecture Notes in Computer Science, 10168, Springer, Cham, 2017.

[11] Mahalingam, K., Pandoh, P. "On the maximum number of distinct palindromic sub-arrays," In: Martín-Vide, C., Okhotin, A., Shapira, D. (eds) Language and Automata Theory and Applications. LATA 2019. Lecture Notes in Computer Science, 11417, Springer, Cham, 2019.

[12] Mahalingam, K., Pandoh, P., Krithivasan, K. "On the least number of palindromes in two-dimensional words," Theoretical Computer Science, vol. 807, pp. 246-256, 2020.

[13] Mahalingam, K., Sivasankar, M., Krithivasan, K. "Palindromic Properties of Two-Dimensional Fibonacci Words," Romanian Journal of Information Science and Technology, vol. 21, pp. 267-277, 2018.

[14] Sasikala, K., Rajkumar Dare, V., D.G. Thomas "Learning of Partial Languages," Engineering Letters, vol. 14 no. 2, pp. 72-80, 2007.

[15] Janaki, K., Arulprakasam, R. "Tiling systems and Domino systems for partial array languages," IAENG International Journal of Applied Mathematics, vol. 52, no. 2, pp. 418-425, 2022.

[16] Krishna Kumari, R., Arulprakasam, R. "Factors and Subwords of Rich Partial Words," IAENG International Journal of Applied Mathematics, vol. 53, no.2, pp. 534-539, 2023.

[17] Knuth, D. E., J. H. Morris, Jr, Pratt, V.R. "Fast pattern matching in strings," SIAM Journal on Computing, vol. 6, no. 2, pp. 323-350, 1977.

[18] Cole, R., Crochemore, M., Galil, Z., Gasieniec, L., Hariharan, R., Muthukrishnan, S., Park, K.,Rytter, W. "Optimally Fast parallel algorithms for Preprocessing and Pattern Matching in One and Two Dimensions," IEEE 34th Annual Foundations of Computer Science, 1993.

[19] Geizhals, S.H., Sokol, D. "Finding maximal 2-dimensional palindromes," Information and Computation, vol. 266, pp. 161-172, 2019.

[20] A. John Kaspar, D.K. Sheena Christy, D.G. Thomas "Lattice Regular Grammar-Automata," IAENG International Journal of Applied Mathematics, vol. 52, no.4, pp. 1123-1129, 2022.

[21] Knuth, D. E. "The Art of Computer Programming," Volume 1: Fundamental Algorithms, 1^{st} edition, Addison - Wesley, United States, 1968.

[22] Allouche, Jean-Paul., Shallit, Jeffrey "Automatic Sequences: Theory, Applications, Generalizations," Cambridge University Press, 2003.

[23] Aldo de Luca. "A Combinatorial property of the Fibonacci words," Information Processing Letters, vol.12, no. 4, pp. 193-195, 1981.

[24] Jean Berstel "Fibonacci Words- A Survey," In: The Book of L, Springer, Berlin, Heidelberg, 1986.

[25] Chuan, W-F. "Fibonacci words," The Fibonacci Quarterly, vol. 30, no. 1, pp. 68-76, 1992.

[26] Chuan, W-F. "Symmetric Fibonacci words," The Fibonacci Quarterly, vol. 31, no. 3, pp. 251-255, 1993.

[27] Chuan, W-F. "Generating Fibonacci words," The Fibonacci Quarterly, vol. 33, no. 2, pp. 104-112, 1995.

[28] Chuan, W-F., "Subwords of the Golden Sequence and the Fibonacci words," In: Bergum, G.E., Philippou, A.N., Horadam, A.F. (eds) Applications of Fibonacci Numbers. Springer, Dordrecht, 1996.

[29] Fici, Gabriele. "Factorizations of the Fibonacci infinite word," Journal of Integer Sequences, vol. 18, no. 9, pp. 1-4, 2015.

[30] Mahalingam, K., Maity, A., Pandoh, P. "Counting (Watson-Crick) palindromes in Watson-Crick conjugates," Information and Computation, vol. 285, no. B, 104863, 2022.

[31] Apostolico, A., Brimkov, V.E."Fibonacci arrays and their two – dimensional repetitions," Theoretical Computer Science, vol. 237, pp. 263-273, 2000.

[32] Kulkarni, M. S., Mahalingam, K., Mohankumar, S. "Combinatorial Properties of Fibonacci Arrays," In: Gopal, T., Watada, J. (eds) Theory and Applications of Models of Computation. TAMC 2019. Lecture Notes in Computer Science, 11436, Springer, Cham, 2019.

[33] Dallapiccola, R., Gopinath, A., Stellacci, F., Negro, L. D. "Quasi-Periodic distribution of Plasmon modes in Two-dimensional Fibonacci arrays of Metal Nanoparticles," Optics Express, vol.16, no. 8, pp. 5544-5555, 2008.

[34] Kari, L., Kulkarni, M. S., Mahalingam, K., Wang, Z. "Involutive Fibonacci Words," Journal of Automata, Languages and Combinatorics, vol. 26, pp. 255-280, 2021.

[35] Kari, L., Kitto, R., G. Thierrin "Codes, Involutions, and DNA encodings," In: Brauer, W., Ehrig, H., Karhumäki, J., Salomaa, A. (eds) Formal and Natural Computing, Lecture Notes in Computer Science, 2300, Springer, Berlin, Heidelberg, 2002.

[36] Kari, L., Mahalingam, K., Pandoh, P., Wang, Z., "Primitivity of Atom Watson-Crick Fibonacci words," Journal of Automata, Languages and Combinatorics, vol. 27, pp. 151-178, 2022.

[37] Hannah Blasiyus, D.K. Sheena Christy"On Involutive co-Fibonacci words and Involutive Fibonacci Arrays," Communicated.

[38] Linz, P., Rodger, S. H. "An Introduction to Formal languages and Automata," 7^{th} Edition, Sudbury, Massachusetts, Jones & Bartlett, 2023.

[39] Siromoney, G. and Siromoney, R., Krithivasan, K. "Picture Languages with Array Rewriting Rules," Information and Control, vol. 20, pp. 447-470, 1973.

[40] Anselmo, M., Giammarresi, D., Madonia, M. "Prefix picture codes: A Decidable class of Two-dimensional codes," International Journal of Foundations of Computer Science, vol. 25, no. 8, pp. 1017-1031, 2014.