Impact of Vertex Duplication on Total Power Dominator Coloring in Graphs

J. Bala Samuvel, *Member, IAENG,* R. Stella Maragatham, Kezia Prem, and A. Divya Jebaseeli, *Member, IAENG*

***Abstract*—Graph theory, with its wide-ranging theoretical foundations and practical applications, continues to influence diverse scientific and engineering domains. Two fundamental concepts—graph domination and graph coloring—have given rise to several advanced variants, including power domination, dominator coloring, and, more recently, power dominator coloring. These extensions are especially relevant in applications such as network monitoring, fault detection, and communication systems. This study introduces and explores a further refinement known as total power dominator coloring, which integrates elements of both domination and coloring. For a given undirected, connected, finite, and simple graph $G = (V, E)$, a total power dominator coloring is defined as a proper vertex coloring in which each vertex power dominates all vertices in at least one other distinct color class. The minimum number of colors required to achieve such a coloring is called the total power dominator chromatic number, denoted by χ_{tpd}. The main objective of this work is to analyze the behavior of χ_{tpd} under vertex duplication, a graph operation that replicates a vertex along with its adjacency relations. We investigate this parameter across various classical graph families, including cycle graphs, path graphs, complete graphs, bipartite graphs, double fan graphs, octopus graphs, and the Venessa graph. The study examines this parameter across various classical graph families, with supporting diagrams provided to visually illustrate key definitions and examples.**

***Index Terms*—Total Power Dominator Coloring, Duplication, Fan Graph, Cycle, Complete Graph**

AMS Subject Classification: 05C15, 05C69

I. Introduction

GRAPH theory, with its rich historical background, stands as a fundamental pillar of discrete mathematics. One of its most influential subfields is graph coloring, which emerged in the mid-19$^{\text{th}}$ century. The origins of this concept are commonly attributed to Francis Guthrie, who, in 1852, while attempting to color a map of the counties of England, observed that no more than four distinct colors were necessary to ensure that adjacent regions—those sharing a common boundary—received different colors. Although this observation arose from a cartographic problem, it led to the formulation of the renowned Four Color Problem, which ultimately laid the foundation for the field of graph coloring. Since then, graph coloring has grown into a rich and well-established area of research, with significant theoretical developments and practical applications in areas such as scheduling, resource allocation, register assignment in compilers, and frequency assignment in wireless networks.

Another significant area within graph theory is *domination theory*, which began to emerge as a formal field of study in the 1960s. By 1998, Teresa W. Haynes et al. [11] compiled a comprehensive annotated bibliography that documented more than 1,200 publications, underscoring the rapid expansion and conceptual richness of the domain. Their survey identified and systematically classified over 75 distinct types of domination parameters, reflecting the depth and breadth of research activity in this area. Domination theory has since become an integral part of graph theory, with applications spanning network security, social network analysis, and optimization problems in distributed systems.

A notable advancement in domination theory was the introduction of total domination by Cockayne, Dawes, and Hedetniemi [3], which refined classical domination by requiring that every vertex in the graph be adjacent to at least one vertex in the dominating set. Building on this foundation, Gera [5], [6] proposed the concept of dominator coloring, a novel framework that integrates the principles of domination and proper vertex coloring. This innovation catalyzed further developments, including total dominator coloring [28], [29], global dominator coloring [8], and power-dominated coloring [18], each contributing to a deeper understanding of the structural interplay between domination and coloring within graph theory.

The concept of *power domination (PD)*, introduced by Haynes et al. [9], marked a significant advancement in the practical application of graph theory. Power domination was initially motivated by the need to optimize the placement of Phasor Measurement Units (PMUs) in electrical power networks. The objective is to achieve complete system observability while minimizing the number of PMUs deployed. This model incorporates both classical domination and propagation rules derived from electrical monitoring constraints. The power domination framework has profound implications for improving the efficiency, reliability, and cost-effectiveness of electrical grid operations, making it a vital tool in modern energy management and smart grid technologies.

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Building on this foundation, Kumar et al. [15] introduced the concept of *power dominator coloring (PDC)*, which elegantly integrates the principles of power domination with proper vertex coloring. This novel framework has since sparked significant interest and led to extensive investigations across a variety of graph classes. Uma Maheswari and Bala Samuvel J. [25] conducted a comprehensive study on PDC in specific families such as Bull, Flower, Helm, and Star graphs, substantiated with detailed illustrations and examples. Their research was further expanded in subsequent works [22] [24] to encompass more complex graph structures, including the Triangular Book with Bookmark, Jellyfish, Extended Jewel, Jewel, and Fan graphs. In a notable contribution, Uma Maheswari et al. [23] curated a catalogue of graphs characterized by a *PDCN*] equal to 3, offering valuable insights into the coloring properties of such graph classes.

Beyond traditional domains of domination and coloring, graph theory has increasingly permeated the field of cryptography, offering novel approaches to securing data and communication systems. Vani Shree and Dhanalakshmi [26] proposed a method that integrates graph labeling techniques with the RSA encryption algorithm to increase cryptographic complexity and enhance security. In a related study, Narayan et al. [17] examined graph-based encryption models, highlighting the importance of structural properties such as connectivity, vertex labeling, and graph topology in designing robust cryptographic protocols. More recently, research has shifted toward combining graph-theoretical insights with emerging computational paradigms such as deep learning. Samuvel et al. [20] introduced a hybrid framework that leverages graph structures alongside deep learning algorithms to strengthen data privacy within blockchain networks. Their contribution underscores the potential of graph theory as a foundational tool in the development of secure and intelligent distributed systems.

Recent developments in graph coloring have increasingly embraced structural insights and application-oriented frameworks. Haddadene and Issaadi [12] conducted a comprehensive study of *perfect graphs*, demonstrating that, in all induced subgraphs, the chromatic number is equal to the clique number. Vaidya and Isaac [27] investigated the concept of the *total chromatic number*, which unifies both edge and vertex coloring into a single framework. Li et al. [30] introduced the notion of the total dominator edge chromatic number, wherein every edge must be adjacent to edges in all other color classes. Further advancing this line of inquiry, Zhou et al. [32] proposed the adjacent vertex strongly distinguishing total coloring, with a focus on unicyclic graphs.

For planar graphs, Lou et al. [31] investigated 2-frugal coloring, a coloring scheme in which each color may appear at most twice in the neighborhood of any vertex. Their study, particularly focused on planar graphs with maximum degree six, highlights how structural constraints significantly influence the colorability of graphs. Collectively, such advances underscore the growing diversity of the field and its increasing relevance to practical applications, including fault-tolerant system design, identity labeling in networks, and efficient resource allocation.

An important structural operation in graph theory is *vertex duplication*, in which a new vertex is introduced that inherits all adjacency relations of an existing vertex. This concept has garnered significant attention due to its applicability in modeling redundancy within networks and control systems. Vertex duplication serves a critical function in the development of fault-tolerant architectures, where introducing redundancy enhances system reliability and reduces the risk of failure.

The research conducted by Kulli and Janakiram [16] explored the effects of vertex duplication on the domination number and its various extensions. Their foundational work established a basis for analyzing how structural modifications, such as vertex duplication, influence key domination parameters in graphs. In a related development, Dorfling and Hattingh [4] investigated duplication in the context of total domination, showing that the total domination number may either increase or remain unchanged depending on the underlying graph structure. Building upon these contributions, subsequent studies have focused on the impact of vertex duplication on coloring parameters linked to domination. This includes measures such as the *total dominator chromatic number* and related domination-based colorings [2], [21].

Inspired by recent advancements in domination-based graph coloring, J. Bala Samuvel introduced the novel concept of *Total Power Dominator Coloring (TPDC)*—a hybrid framework that synthesizes the principles of Total Dominator Coloring and Power Domination. The key parameter associated with this model, termed the *Total Power Dominator Chromatic Number*, denoted by χ_{tpd}, quantitatively captures the interplay between domination constraints and coloring strategies under vertex duplication [19]. This research primarily aims to explore the influence of vertex duplication on the value of χ_{tpd} across various classical graph families, thereby deepening the theoretical understanding of domination-coloring behavior under structural perturbations.

The findings of this research yield significant insights into the structural ramifications of vertex duplication within the framework of TPDC, thereby contributing to the broader discourse in graph-theoretic optimization. The derived results exhibit practical relevance in various applied domains such as network design, data monitoring, and control systems, where considerations of fault tolerance, redundancy, and operational efficiency are critically important.

II. Preliminaries

A complete and detailed list of the various terminologies, and the standard notations that are utilized throughout this study, can be seen in [5], [6], [1], [7], [14], [13], power dominator coloring [15], [22], [25], the definitions of a monitoring set [9], [10], and the vertex duplication in [4], [21] all play key roles in shaping the content of the findings expressed in this section. We have the chance to formally define in this section the notion of TPDC, and we also establish the corresponding notation that is used for the Total Power Dominator Chromatic Number (TPDCN) of a graphχ_{tpd}. To further and enrich the understanding of the topic under discussion, examples have been carefully provided to clearly show the practical application of TPDC in a variety of various graph structures one might encounter.

Definition 1. *Total Power Dominator Coloring (TPDC) The total power dominator coloring (TPDC) [19] is the coloring of the vertices in the graph (which is proper), so that each vertex ν_i of the G power dominates every vertex of some other color class (not the color class of the vertex ν_i. The total dominator chromatic number (TPDCN) $\chi_{tpd}(G)$, is the minimal number of colors that are necessary for the total power dominator coloring (TPDC) of the graph G.*

Observe that, in PDC, a color class with a single vertex have power-dominating ability, while the definition of TPDC calls for a more stringent requirement: every vertex must power dominate all the vertices in at least one color class other than its own. This guarantees domination beyond color groups, adding strength to the total nature of the coloring. For simplicity and uniformity, the abbreviations TPDC and TPDCN will from now on denote Total Power Dominator Coloring and Total Power Dominator Chromatic Number, respectively.

III. Main Results

Theorem 1. *For any $2 \leq n \leq 3$, the TPDCN for the graph $P_n^{'}$ created through the process of duplicating an arbitrary vertex υ in Path P_n is 2.*

Proof: Based on the number of vertices n in the original graph and the selection of the vertex ν to be duplicated by a new vertex υ, three distinct cases may arise.

Case (i): when $n = 2$ and the duplicated vertex is one of the end vertices.

Let P_2 the path have 2 nodes. Let ν_1, ν_2 be the vertices of Path P_2, where $\nu_1$and ν_2 are the pendent vertices. Let E(P_2) be the edges of the Path where E(P_2) = $\{\nu_1\nu_2\}$. Here the vertex set is defined as $|V(P_2)| = 2$. Now, on duplicate any one of pendent vertex ν_1 or ν_2.

To streamline the discussion while maintaining generality, assume that one of the pendant vertices, say ν_1, is selected for duplication. This duplication results in the creation of a new vertex υ, which inherits the exact neighborhood of ν_1. Since ν_1 is adjacent only to ν_2, the new vertex υ is also connected to ν_2. Consequently, the resulting graph P_2' has vertex set $V(P_2') = \{\nu_1, \nu_2, \upsilon\}$, and the edge set is defined as

$$E(P_2') = \{\nu_1\nu_2,\ \upsilon\nu_2\}.$$

This forms a star graph structure with center ν_2 and pendant vertices ν_1 and υ, isomorphic to the star graph $K_{1,2}$. The procedure outlined below is based on a systematic approach to coloring the vertices of the graph and guarantees a coloring that satisfies both the proper coloring and the TPDC requirements. Specifically, this method ensures that each color class is power dominated by at least one vertex from a distinct color class, thereby producing a valid TPDC configuration.

The graph P_2' is colored using two colors, c_1 and c_2, as follows:

- The central vertex ν_2 is assigned color c_1.
- The pendant vertices ν_1 and υ are assigned color c_2.

This coloring strategy ensures a proper vertex coloring of the graph, as no two adjacent vertices share the same color. It also satisfies the requirements for TPDC. The domination relationships under this coloring scheme are as follows:

- The central vertex ν_2, colored with c_1, is adjacent to both ν_1 and υ, thus power dominating the entire color class c_2.
- The pendant vertices ν_1 and υ, colored with c_2, are each adjacent to ν_2, thereby being power dominated by a vertex of the distinct color class c_1.

Therefore, each vertex in the graph P_2' is power dominated by at least one vertex belonging to a different color class. This satisfies all the conditions required for a valid TPDC.

To show that two colors are necessary, suppose that only one color is used for the TPDC of P_2'. In such a case, adjacent vertices must necessarily share the same color, which violates the condition of proper vertex coloring. Therefore, it is not possible to construct a valid TPDCof P_2' using fewer than two colors.

Hence, the color assignment described above is both valid and minimal. It ensures that each vertex in the graph power dominates all vertices in at least one color class different from its own, in accordance with the definition of TPDC . Consequently, the TPDCN for the graph P_2', created through the process of duplicating a pendant vertex in P_2, is 2. i.e., $\chi_{tpd}(P_2') = 2$.

Case (ii): When $n = 3$ and duplicating any pendant vertex

Let P_3 be the path graph with three vertices, denoted as ν_1, ν_2, ν_3, where ν_1 and ν_3 are the pendant (end) vertices. The edge set of the path is given by $E(P_3) = \{\nu_1\nu_2,\ \nu_2\nu_3\}$. Hence, the graph has $|V(P_3)| = 3$ and $|E(P_3)| = 2$.

To streamline the discussion while maintaining generality, assume that the pendant vertex ν_1 is selected for duplication. This duplication results in a new vertex υ, which inherits the exact neighborhood of ν_1. Since ν_1 is adjacent only to ν_2, the new vertex υ is also adjacent to ν_2. Thus, the resulting graph, denoted as P_3', has the vertex set $\{\upsilon, \nu_1, \nu_2, \nu_3\}$ and the edge set

$$E(P_3') = \{\nu_1\nu_2,\ \nu_2\nu_3,\ \upsilon\nu_2\}.$$

This forms a star-like configuration centered at ν_2, with three vertices—$\nu_1,\ \nu_3,\ \upsilon$—connected to it.

The procedure outlined below follows a systematic coloring strategy to ensure that the resulting coloring satisfies both proper coloring and TPDC requirements. Specifically, this method ensures that each color class is power dominated by at least one vertex from a different color class.

The graph P_3' is colored using two colors, c_1 and c_2, as follows:

- Assign color c_1 to the pendant vertices $\nu_1,\ \nu_3$, and the newly introduced vertex υ.
- Assign color c_2 to the central vertex ν_2.

This vertex coloring satisfies the condition for a proper coloring, as adjacent vertices receive distinct colors. Additionally, the coloring meets the TPDC condition:

- The vertices $\nu_1,\ \nu_3,\ \upsilon$, all colored with c_1, are each adjacent to ν_2, which belongs to color class c_2; thus, color class c_2 is power dominated by vertices of color c_1.
- The vertex ν_2, colored with c_2, is adjacent to all vertices in color class c_1, and hence power dominates color class c_1.

Therefore, each color class is power dominated by at least one vertex from a different color class, satisfying the conditions of TPDC .

To establish the minimality of the coloring, suppose that only one color is used for the TPDC of P_3'. In that case, adjacent vertices must necessarily receive the same color, violating the condition of proper vertex coloring. Thus, a single color is insufficient for a valid TPDC of this graph.

Therefore, the color assignment described above is both valid and minimal, ensuring that each vertex power dominates at least one complete color class different from its own. Consequently, the TPDCN for the graph P_3', created through the process of a pendant vertex in the path graph P_3, is $\chi_{tpd}(P_3') = 2$.

Case (iii): When $n = 3$ and duplicating vertex ν_2

Let P_3 be the path graph with three vertices denoted as $\nu_1,\ \nu_2,\ \nu_3$, where ν_1 and ν_3 are the pendant (end) vertices. The edge set of the path is given by $E(P_3) = \{\nu_1\nu_2,\ \nu_2\nu_3\}$, and hence the graph has $|V(P_3)| = 3$ and $|E(P_3)| = 2$.

Now, the internal vertex ν_2 undergoes duplication. This process yields a new vertex υ, which inherits the exact neighborhood of ν_2. Since ν_2 is adjacent to both ν_1 and ν_3, the new vertex υ is also adjacent to both ν_1 and ν_3. Consequently, the resulting graph, denoted by P_3', has the vertex set $\{\upsilon,\ \nu_1,\ \nu_2,\ \nu_3\}$, and the edge set is defined as

$$E(P_3') = \{\nu_1\nu_2,\ \nu_2\nu_3,\ \upsilon\nu_1,\ \upsilon\nu_3\}.$$

This results in a quadrilateral-like structure in which the new vertex υ forms a mirror image of ν_2, maintaining adjacency to the same vertices.

The following procedure is employed to assign colors in order to determine the TPDCN and ensure that the coloring satisfies all conditions of TPDC.

The graph P_3' is colored using two colors, c_1 and c_2, as follows:

- Assign color c_1 to the two pendant vertices ν_1 and ν_3.
- Assign color c_2 to the internal vertex ν_2 and the newly introduced duplicate vertex υ.

This vertex coloring is proper because no two adjacent vertices share the same color. Furthermore, it satisfies the TPDC conditions. The domination relationships under this coloring scheme are as follows:

- The vertex ν_2, colored c_2, is adjacent to both ν_1 and ν_3, thereby power dominating the entire color class c_1.
- The duplicate vertex υ, also colored c_2, is adjacent to ν_1 and ν_3 as well, reinforcing the power domination of color class $\{c_1\}$.
- Conversely, the vertices ν_1 and ν_3, each assigned color c_1, are adjacent to both ν_2 and υ, thereby power dominating the color class $\{c_2\}$.

Therefore, every vertex in the graph P_3' is power dominated by a vertex from a different color class, and the conditions for a valid TPDC are fully satisfied.

To verify minimality, assume for contradiction that only one color is used. In that case, adjacent vertices such as ν_1 and ν_2 would receive the same color, violating the proper coloring requirement. Therefore, one color is insufficient for a valid TPDC.

Thus, the color assignment described is both valid and minimal, as it ensures that each vertex power dominates at least one complete color class distinct from its own. Consequently, the TPDCN for the graph P_3', *created through the process of duplicating a vertex ν_2* in the path graph P_3, is $\chi_{tpd}(P_3') = 2$.

In all three cases—whether the path graph has two or three vertices, and whether the duplication is performed on a pendant vertex or the internal vertex—a TPDCusing exactly two colors can be constructed. This coloring satisfies both the conditions of proper vertex coloring and power domination, thereby making it valid.

Furthermore, no valid TPDC exists using fewer than two colors, confirming the minimality of the coloring. Therefore, the TPDN of the resulting graph, created through the process of duplicating a vertex in the path graph, is

$$\chi_{tpd}(P_n') = 2; \quad for \quad 2 \leq n \leq 3\ .$$

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Example 1. *In figure 1, the TPDC of graph $P_3^{'}$ created through the process of duplicating a vertex ν_1 by υ in path P_3 is shown.*

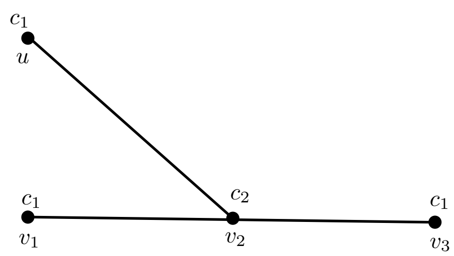

Fig. 1. Modified Path P_3', created through the process of duplicating a vertex ν_1, the color classes of the P_3' are $c_1 = \{\upsilon, \nu_1, \nu_3\}$, $c_2 = \{\nu_2\}$ Then $\chi_{tpd}(P_3') = 2$

Theorem 2. *For any $n \geq 4$, the TPDCN for the graph $P_n^{'}$ created through the process of duplicating a pendent vertex in Path P_n is 3.*

Proof: Let P_n be a path graph with $n \geq 4$ vertices. Denote the vertex set as $\{\nu_1, \nu_2, \nu_3, \nu_4, \dots, \nu_n\}$, where ν_1 and ν_n are the pendant (end) vertices. The edge set of the path is given by:

$$E(P_n) = \{\nu_i \nu_{i+1} \mid 1 \leq i \leq n-1\}.$$

Thus, $|V(P_n)| = n$.

Now, consider duplicating a pendant vertex, either ν_1 or ν_n. To streamline the discussion while maintaining generality, assume that the pendant vertex ν_1 is duplicated. Let υ be the newly introduced vertex that inherits the neighborhood of ν_1. Since ν_1 is adjacent only to ν_2, the vertex υ is also connected to ν_2. The resulting graph P_n' has the vertex set:

$$V(P_n') = \{\upsilon, \nu_1, \nu_2, \nu_3, \dots, \nu_n\},$$

and the edge set:

$$E(P_n') = \{\nu_i \nu_{i+1} \mid 1 \leq i \leq n-1\} \cup \{\upsilon \nu_2\}.$$

To determine the TPDCN of P_n', a vertex coloring strategy is employed that satisfies the conditions for a valid TPDC. The graph P_n' is colored using three colors: c_1, c_2, and c_3, as follows:

- Assign color c_1 to all vertices at odd-numbered positions, i.e., ν_{2i-1} for $1 \leq i \leq \left\lfloor \frac{n+1}{2} \right\rfloor$, and to the newly introduced vertex υ.
- Assign color c_2 to all even-numbered vertices ν_{2i} for $2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$, excluding ν_2.
- Assign color c_3 to the vertex ν_2, which now has degree 3.

This coloring is proper since adjacent vertices receive distinct colors. Furthermore, it satisfies the requirements of a TPDC . The domination relationships under this coloring are as follows:

- All vertices in color classes c_1 and c_2 are adjacent to ν_2, which belongs to color class c_3; thus, c_3 is power dominated.
- The vertex ν_2, colored c_3, is adjacent to all vertices in color classes c_1 and c_2, thereby power dominating both of them.

Hence, every vertex in the graph P_n' is power dominated by at least one vertex belonging to a distinct color class, satisfying all the conditions required for a valid TPDC.

To verify the minimality of this coloring, assume, for contradiction, that a valid TPDCof P_n' can be achieved using fewer than three colors. If only two colors are used, then vertex ν_2, which has degree 3 and is adjacent to three distinct vertices (ν_1, ν_3, and υ), must share its color with at least one of its neighbors. This violates the condition for proper vertex coloring, which requires that adjacent vertices receive distinct colors. Therefore, no proper TPDC exists with fewer than three colors.

Thus, the color assignment described is both valid and minimal, as it ensures that each vertex power dominates at least one complete color class distinct from its own. Consequently, the TPDCN for the graph P_n', *created through the process of duplicating a pendant vertex* ν_1 in the path graph P_n, is

$$\chi_{tpd}(P_n') = 3.$$

■

Example 2. *In figure 2, the TPDC of graph $P_5^{'}$ obtained by process of duplication of any pendent vertex ν_1 by υ in path P_5 is shown.*

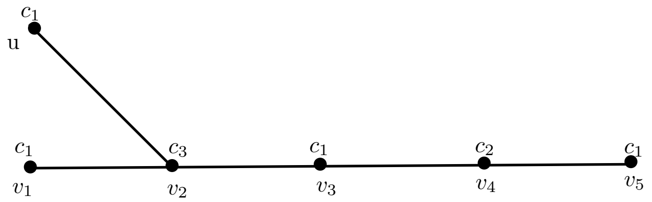

Fig. 2. Modified Path P_5' created through the process of duplicating a vertex ν_1, the color classes of the P_5' are $c_1 = \{\upsilon, \nu_1, \nu_3, \nu_5\}$, $c_2 = \{\nu_4\}, c_3 = \{\nu_2\}$, Then $\chi_{tpd}(P_6') = 3$.

Theorem 3. *For $n \geq 4$, the TPDCN for the graph $P_n^{'}$ created through the process of duplicating a vertex ν_i ($i = 2$ or $i = n-1$) in Path P_n is 3.*

Proof: Let P_n be a path graph with $n \geq 4$ vertices. Denote the vertex set as $\{\nu_1, \nu_2, \nu_3, \nu_4, \dots, \nu_n\}$, where ν_1 and ν_n are the pendant (end) vertices. The internal vertices ν_2 and ν_{n-1} each have degree 2. The edge set of the path graph is defined as:

$$E(P_n) = \{\nu_i \nu_{i+1} \mid 1 \leq i \leq n-1\}.$$

Hence, $|V(P_n)| = n$.

Now, consider duplicating one of the internal vertices ν_2 or ν_{n-1}. Without loss of generality, assume that the vertex ν_2 is selected for duplication. Let υ be the newly introduced vertex, which inherits the exact neighborhood of ν_2. Since ν_2 is adjacent to ν_1 and ν_3, the new vertex υ is also adjacent to both ν_1 and ν_3. As a result, the resulting graph P_n' has the vertex set:

$$V(P_n') = \{\upsilon, \nu_1, \nu_2, \nu_3, \nu_4, \dots, \nu_n\},$$

and the edge set:

$$E(P_n') = \{\nu_i \nu_{i+1} \mid 1 \leq i \leq n-1\} \cup \{\nu_1 \upsilon,\ \upsilon \nu_3\}.$$

Consequently, the degrees of vertices ν_1 and ν_3 become 3.

To determine the TPDCN of the graph P_n', a proper coloring is applied using three colors: c_1, c_2, and c_3, as follows:

- Assign color c_1 to vertex ν_1 and to all vertices at odd-numbered positions ν_{2i-1} for $3 \leq i \leq \left\lfloor \frac{n+1}{2} \right\rfloor$.
- Assign color c_2 to all vertices at even-numbered positions ν_{2i} for $2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$, and also to the newly introduced vertex υ.
- Assign color c_3 to vertex ν_3, which now has degree 3.

This coloring strategy satisfies the conditions for a proper vertex coloring, as no two adjacent vertices share the same color. Furthermore, it meets the criteria of a TPDC. The domination relationships under this coloring scheme are as follows:

- The vertices in color classes c_1 and c_2, which include ν_1, υ, ν_2, and all others except ν_3, are adjacent to ν_3, which belongs to color class c_3. Thus, the color class c_3 is power dominated.
- The vertex ν_3, colored c_3, is adjacent to multiple vertices in both color classes c_1 and c_2, thereby power dominating both classes.

Therefore, every vertex in the graph P_n' is power dominated by at least one vertex from a distinct color class, satisfying all the requirements of a valid TPDC.

To verify minimality, assume for contradiction that a valid TPDC of P_n' can be obtained using only two colors. Consider vertex ν_3, which is adjacent to three vertices: ν_2, ν_4, and υ. Since these are pairwise adjacent, assigning only two colors would inevitably cause a conflict where at least two adjacent vertices share the same color, violating the condition of proper coloring. Thus, a two-color TPDC is not possible.

Thus, the color assignment described above is both valid and minimal, as it ensures that each vertex power dominates at least one complete color class distinct from its own. Consequently, the TPDCN for the graph P_n', *created through the process of duplicating an internal vertex* ν_2 (or symmetrically ν_{n-1}) in the path graph P_n, is

$$\chi_{tpd}(P_n') = 3.$$

■

Example 3. *In figure 3, the TPDC of graph $P_6^{'}$ created through the process of duplicating a vertex ν_3 by υ in path P_6 is shown.*

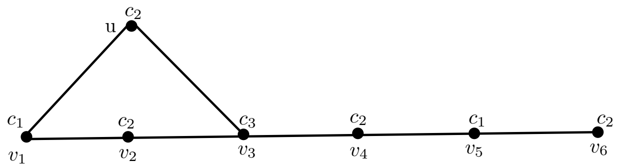

Fig. 3. Modified Path P_6' created through the process of duplicating a vertex ν_2, the color classes of the Path P_6' are $c_1 = \{\nu_1, \nu_5\}$, $c_2 = \{\upsilon, \nu_2, \nu_4, \nu_6\}, c_3 = \{\nu_3\}$, Then $\chi_{tpd}(P_6') = 3$.

Theorem 4. *For any $n \geq 4$, the TPDCN for the graph $P_n^{'}$ created through the process of duplicating a vertex with degree 2 (except ν_2 and ν_{n-1}) in Path P_n is 4.*

Proof: Let P_n be the path with $n \geq 4$ nodes. Let $\nu_1, \nu_2, \nu_3, \nu_4, \ldots, \nu_n$ be the vertices of path P_n, where ν_1 and ν_n are the pendent vertices. Let $E(P_n)$ be the edges of the path, where

$$E(P_n) = \{\nu_i\nu_{i+1} \mid 1 \leq i \leq n-1\}.$$

Here, $|V(P_n)| = n$ and $|E(P_n)| = n-1$. Now, duplicate any arbitrary vertex with degree 2, other than ν_2 and ν_{n-1}, to υ.

To streamline the discussion while maintaining generality, proceed under the assumption that the vertex ν_3, with degree 2, undergoes duplication. This process yields a new vertex υ, which inherits the exact neighborhood of ν_3, thus maintaining the structural properties of the original graph P_n. As a result, the resulting graph $P_n^{'}$ is generated, containing the vertices $\upsilon, \nu_1, \nu_2, \nu_3, \nu_4, \ldots, \nu_n$, and the edge set

$$E(P_n^{'}) = \{\nu_i\nu_{i+1} \mid 1 \leq i \leq n-1\} \cup \{\nu_2\upsilon,\ \upsilon\nu_4\}.$$

The degrees of both ν_2 and ν_4 are now 3.

To determine the TPDCN of the graph $P_n^{'}$, the following coloring strategy is applied, ensuring that all criteria of a valid TPDC are satisfied.

The graph $P_n^{'}$ is colored using four colors c_1, c_2, c_3, and c_4, based on vertex positions:

- Vertices at odd-numbered positions, i.e., $\{\nu_{2i-1} \mid 1 \leq i \leq \lfloor\frac{n+1}{2}\rfloor\}$, are colored with c_1.
- Vertices at even-numbered positions, i.e., $\{\nu_{2i} \mid 2 \leq i \leq \lfloor\frac{n}{2}\rfloor\}$, along with the newly added vertex υ, are colored with c_2.
- Vertex ν_2, which now has degree 3, is colored with c_3.
- Vertex ν_4, which also has degree 3, is colored with c_4.

The vertex coloring defined above satisfies the rules of a proper coloring, as no two adjacent vertices share the same color. The domination relationships under this coloring are as follows:

- All the vertices $\{\upsilon, \nu_1, \nu_2, \nu_4, \ldots, \nu_n\}$ power dominate either the color class $c_3 = \{\nu_2\}$ or the color class $c_4 = \{\nu_4\}$, since they are adjacent to at least one of these vertices.
- Conversely, the vertices ν_2 and ν_4, having degree 3 and colored with c_3 and c_4 respectively, are adjacent to multiple vertices from color classes c_1 and c_2, and thus power dominate them.

Therefore, each vertex in the graph P_n' is power dominated by at least one vertex belonging to a distinct color class, satisfying all the necessary conditions of a valid TPDC.

To verify minimality, assume for contradiction that the TPDC of P_n' can be achieved using fewer than four colors. If only three colors are used, then at least one of the high-degree vertices ν_2 or ν_4 must share a color with a neighboring vertex, violating the proper coloring condition. Alternatively, reducing the number of color classes would leave at least one color class not power dominated by a different one, violating the TPDC condition. Hence, a valid TPDC cannot exist with fewer than four colors.

Thus, the color assignment described is both valid and minimal, as it ensures that each vertex power dominates at least one complete color class distinct from its own. Consequently, the TPDCN for the graph P_n', *created through the process of duplicating an internal vertex of degree 2 other than ν_2 or ν_{n-1}* in the path graph P_n, is

$$\chi_{tpd}(P_n') = 4.$$

■

Example 4. *In figure 4,the TPDC of graph $P_6^{'}$ created through the process of duplicating a vertex ν_3 by υ in Path P_6 is shown.*

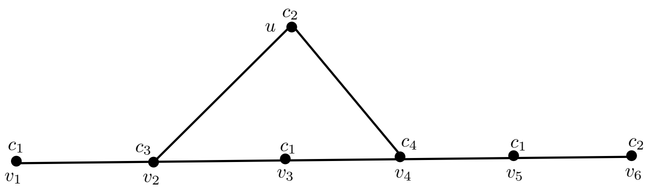

Fig. 4. Modified Path P_6' created through the process of duplicating a vertex ν_3, the color classes of the P_6', the color classes of the P_6' are $c_1 = \{\nu_1, \nu_3, \nu_5\}$, $c_2 = \{\upsilon, \nu_6\}$, $c_3 = \{\nu_2\}$, and $c_4 = \{\nu_4\}$ Then $\chi_{tpd}(P_6') = 4$

Theorem 5. *For any $n \geq 3$, the TPDCN for a graph $C_n^{'}$ created through the process of duplicating any arbitrary vertex with degree 2 in cycle $\chi_{tpd}(C_n) = 3,\ if\ n \geq 3$*

Proof: Let C_n denote the cycle graph with n vertices, where $n \geq 3$. Let the vertex set of C_n be $V(C_n) = \{\nu_1, \nu_2, \nu_3, \dots, \nu_n\}$, and the edge set be defined as

$$E(C_n) = \{\nu_i\nu_{i+1} \mid 1 \leq i \leq n-1\} \cup \{\nu_n\nu_1\}.$$

Thus, the graph forms a closed loop in which each vertex ν_i is connected to two neighbors: ν_{i-1} and ν_{i+1}, with indices taken modulo n. The number of vertices is $|V(C_n)| = n$, and the number of edges is $|E(C_n)| = n$.

The TPDCN for graphs obtained by duplicating a vertex in C_n is analyzed under three separate cases, depending on the selection of the vertex to be duplicated. The theorem is established by examining each of these cases in detail.

Case (i): When $n = 3$

Let C_3 be the cycle with $n = 3$ vertices. Let the vertex set of the cycle be $V(C_3) = \{\nu_1, \nu_2, \nu_3\}$, and the edge set be

$$E(C_3) = \{\nu_1\nu_2,\ \nu_2\nu_3,\ \nu_3\nu_1\}.$$

Thus, $|V(C_3)| = 3$ and $|E(C_3)| = 3$.

Now, consider the process of duplicating any arbitrary vertex of degree 2. To streamline the discussion while maintaining generality, assume that the vertex ν_3 is selected for duplication. This results in the introduction of a new vertex υ, which inherits the exact neighborhood of ν_3. Since ν_3 is adjacent to both ν_1 and ν_2, the new vertex υ will also be adjacent to ν_1 and ν_2. Therefore, the resulting graph, denoted by C_3', has the vertex set

$$V(C_3') = \{\nu_1, \nu_2, \nu_3, \upsilon\}$$

and the edge set

$$E(C_3') = \{\nu_1\nu_2,\ \nu_2\nu_3,\ \nu_3\nu_1,\ \nu_2\upsilon,\ \upsilon\nu_1\}.$$

In the modified graph C_3', the degrees of both ν_1 and ν_2 become 3, as they are now connected to three vertices each.

To determine the TPDCN, a systematic coloring strategy is applied that satisfies both proper coloring and power domination conditions. The graph C_3' is colored using three colors: c_1, c_2, and c_3, as follows:

- Assign color c_1 to vertex ν_1,
- Assign color c_2 to vertex ν_2,
- Assign color c_3 to vertices ν_3 and υ.

This coloring ensures that adjacent vertices receive distinct colors, thereby satisfying the proper coloring requirement. Furthermore, the power domination relationships under this coloring scheme are as follows:

- Vertex ν_1, colored c_1, is adjacent to both ν_2 (c_2) and ν_3 (c_3), and hence power dominates color classes c_2 and c_3.
- Vertex ν_2, colored c_2, is adjacent to ν_1 (c_1), ν_3 (c_3), and υ (c_3), thus power dominating color classes c_1 and c_3.
- Vertices ν_3 and υ, both colored c_3, are adjacent to ν_1 and ν_2, hence power dominating color classes c_1 and c_2.

Therefore, every vertex in the graph C_3' power dominates all vertices in at least one color class different from its own. This satisfies the condition of TPDC.

To demonstrate the minimality of this coloring, suppose that only two colors are used. Then, some adjacent vertices must share the same color, which violates the proper coloring condition. Alternatively, even if proper coloring is preserved, one of the color classes will not be power dominated by any vertex from a different color class, thereby violating the TPDC condition. Hence, a valid TPDC of C_3' is not possible with fewer than three colors.

Thus, the coloring assignment described above is both valid and minimal. It satisfies all the requirements for a TPDC . Therefore, the TPDCN of the graph C_3', created through the process of duplicating a vertex of degree 2 in the cycle C_3, is

$$\chi_{tpd}(C_3') = 3.$$

Case (ii): When $n = 4$

Let C_4 be the cycle graph with $n = 4$ vertices. Let the vertex set be $V(C_4) = \{\nu_1, \nu_2, \nu_3, \nu_4\}$, and the edge set be

$$E(C_4) = \{\nu_1\nu_2,\ \nu_2\nu_3,\ \nu_3\nu_4,\ \nu_4\nu_1\}.$$

Thus, $|V(C_4)| = 4$ and $|E(C_4)| = 4$.

Now, consider the process of duplicating any vertex of degree 2. To streamline the discussion while maintaining generality, assume that the vertex ν_3 is selected for duplication. This duplication results in the creation of a new vertex υ, which inherits the exact neighborhood of ν_3. Since ν_3 is adjacent to ν_2 and ν_4, the new vertex υ is also connected to both ν_2 and ν_4. Therefore, the resulting graph C_4' has vertex set

$$V(C_4') = \{\nu_1, \nu_2, \nu_3, \nu_4, \upsilon\}$$

and edge set

$$E(C_4') = \{\nu_1\nu_2,\ \nu_2\nu_3,\ \nu_3\nu_4,\ \nu_4\nu_1,\ \nu_2\upsilon,\ \upsilon\nu_4\}.$$

In the modified graph C_4', the degrees of both ν_2 and ν_4 become 3, while the other vertices remain of degree 2.

The following procedure is applied to assign colors in a manner that ensures a valid TPDC. The graph C_4' is colored using three colors: c_1, c_2, and c_3, as follows:

- Assign color c_1 to the vertices $\nu_1,\ \nu_3$, and the newly added vertex υ,
- Assign color c_2 to vertex ν_2,

- Assign color c_3 to vertex ν_4.

This coloring satisfies the proper coloring condition, as no two adjacent vertices share the same color. The power domination conditions are also satisfied, as detailed below:

- Vertices $\nu_1,\ \nu_3,\ \upsilon$, colored with c_1, are each adjacent to vertices of colors c_2 and c_3, thereby contributing to the power domination of both these color classes.
- Vertex ν_2, colored with c_2, is adjacent to ν_1 (c_1), ν_3 (c_1), and υ (c_1), thus power dominating color class c_1.
- Vertex ν_4, colored with c_3, is adjacent to ν_3, ν_1, and υ (all of which are in c_1), and hence power dominates color class c_1; additionally, it is adjacent to ν_2 (c_2), thus also power dominating c_2.

As each color class is power dominated by at least one vertex from a distinct color class, the coloring constitutes a valid TPDC of C'_4.

To demonstrate minimality, assume that only two colors are used. Then, it becomes impossible to maintain a proper coloring across all adjacent vertices in C'_4 while ensuring the power domination condition, especially for vertices with high degrees (such as ν_2 and ν_4). Hence, two colors are insufficient for a valid TPDC of C'_4.

Therefore, the coloring described is both valid and minimal. It satisfies all the requirements for TPDC . Consequently, the TPDCN for the graph C'_4, created through the process of duplicating a vertex of degree 2 in the cycle C_4, is

$$\chi_{tpd}(C'_4) = 3.$$

Case (iii): When $n \geq 5$

Let C_n be a cycle graph with $n \geq 5$ vertices, denoted as $V(C_n) = \{\nu_1, \nu_2, \ldots, \nu_n\}$, and edge set

$$E(C_n) = \{\nu_i\nu_{i+1} \mid 1 \leq i \leq n-1\} \cup \{\nu_n\nu_1\}.$$

Now, duplicate an arbitrary vertex of degree 2. Without loss of generality, assume the vertex ν_3 is selected for duplication. This process introduces a new vertex υ, which inherits the exact neighborhood of ν_3, i.e., it is connected to both ν_2 and ν_4. The resulting graph C'_n has the vertex set

$$V(C'_n) = \{\upsilon, \nu_1, \nu_2, \nu_3, \nu_4, \ldots, \nu_n\},$$

and the edge set

$$E(C'_n) = \{\nu_i\nu_{i+1} \mid 1 \leq i \leq n-1\} \cup \{\nu_n\nu_1\} \cup \{\nu_2\upsilon,\ \upsilon\nu_4\}.$$

In this modified graph, the degrees of vertices ν_2 and ν_4 increase to 3, while all other vertices (excluding υ) remain of degree 2.

The following procedure outlines a systematic coloring strategy that ensures the coloring satisfies both proper coloring and TPDC conditions. The graph C'_n is colored using three distinct colors: c_1, c_2, and c_3. The coloring scheme is defined as follows:

- Assign color c_1 to all vertices in odd positions, i.e., ν_{2i-1} for $1 \leq i \leq \frac{n+1}{2}$, and also to the newly added vertex υ,
- Assign color c_2 to all even-positioned vertices ν_{2i} for $2 \leq i \leq \frac{n}{2}$, except ν_2,
- Assign color c_3 to the vertex ν_2, which now has degree 3.

This coloring clearly satisfies the proper coloring condition since adjacent vertices receive different colors. It also satisfies the conditions for TPDC , as explained below:

- All vertices colored c_1 (including υ) are adjacent to ν_2, which belongs to c_3; hence, color class c_3 is power dominated.
- Vertex ν_4, which is adjacent to ν_3 (c_1), υ (c_1), and ν_5 (c_2), thereby power dominates both color classes c_1 and c_2.
- Vertex ν_2, colored c_3, is adjacent to ν_1 (c_1), ν_3 (c_1), and υ (c_1), thus also power dominating color class c_1.

Since each color class is power dominated by at least one vertex from a different color class, the coloring forms a valid TPDC .

To confirm minimality, assume that only two colors are used. In such a case, adjacent vertices would necessarily share the same color, violating the condition of proper coloring. Alternatively, it may become impossible to ensure that all color classes are power dominated by a different color class. Therefore, two colors are insufficient to form a valid TPDC.

Hence, the coloring described is both valid and minimal. It ensures that each vertex in the graph C'_n power dominates all vertices in at least one color class different from its own. Accordingly, the TPDCN of the graph C'_n, created through the process of duplicating a vertex of degree 2 in the cycle C_n, is

$$\chi_{tpd}(C'_n) = 3, \quad \text{for all } n \geq 5.$$

■

Example 5. *In figure 5, the TPDC of graph C'_4 created through the process of duplicating a vertex ν_3 by υ in cycle C_4 is shown.*

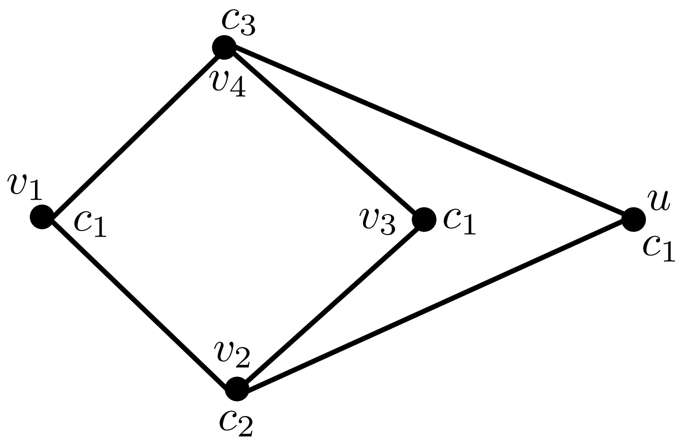

Fig. 5. Modified graph C'_4 created through the process of duplicating a vertex ν_3, the color classes of the C'_4 are $c_1 = \{\upsilon, \nu_1, \nu_3\}$, and $c_2 = \{\nu_2, \nu_2\}$. Then $\chi_{tpd}(C'_4) = 2$

Remark 1. *In the case of cycle graphs, it is noteworthy that notice that the case where $n = 1$ and $n = 2$ do not result in valid cycle graphs.*

For $n = 1$, the graph has one isolated a vertex with no edges, thus ruling out any cycle. Also, for $n = 2$, the graph can have at most one edge connecting the two vertices, and thus forming an elementary path rather than being in a cycle. Since neither of these configurations meets the definition of a cycle graph, the cases $n = 1$ & $n = 2$ are not considered in the theorem. Therefore, the theorem holds for cycle graphs with $n \geq 3$, where one can form a closed circuit with all vertices.

Theorem 6. *For any $n \geq 3$, the TPDCN for the graph $K_n^{'}$, created through the process of duplicating any arbitrary vertex in Complete graph K_n is n..*

Proof: Let K_n be the complete graph with n vertices. Denote the vertices as $\nu_1, \nu_2, \nu_3, \ldots, \nu_n$, where each vertex has degree $n-1$. The edge set of the complete graph is given by

$$E(K_n) = \{\nu_i\nu_j \mid 1 \leq i \leq n,\ 1 \leq j \leq n,\ i \neq j\}.$$

Here, $|V(K_n)| = n$ and $|E(K_n)| = \frac{n(n-1)}{2}$.

Now, create a duplicate of any arbitrarily chosen vertex ν_k with degree $n-1$, and denote the new vertex as υ. This duplication process generates a new vertex υ which inherits the exact neighborhood of ν_k, preserving the structural properties of the original graph K_n. The resulting graph, denoted K_n', contains the vertex set

$$V(K_n') = \{\upsilon,\ \nu_1,\ \nu_2,\ \ldots,\ \nu_n\},$$

and the edge set is defined as

$$E(K_n') = \{\nu_i\nu_j \mid 1 \leq i \leq n,\ 1 \leq j \leq n,\ i \neq j\} \\ \cup \{\upsilon\nu_j \mid 1 \leq j \leq n,\ j \neq k\}.$$

In this new graph, all vertices except ν_k have degree n, while ν_k and υ each retain degree $n-1$.

To determine the TPDCN, a systematic coloring strategy is adopted to satisfy both the proper coloring and TPDCrequirements. The graph K_n' is colored using n colors $c_1, c_2, \ldots, c_n$, such that:

- Each original vertex ν_i, for $1 \leq i \leq n$, is assigned a distinct color c_i,
- The duplicate vertex υ is assigned the same color c_k as the vertex ν_k it duplicates.

This coloring is proper because no two adjacent vertices share the same color, and υ is not adjacent to ν_k, the only vertex that shares its color. The coloring also satisfies the power domination condition:

- Each vertex ν_i is adjacent to all other vertices except itself, and hence power dominates all color classes c_j for $j \neq i$.
- The vertex υ, being adjacent to all vertices except ν_k, similarly power dominates all color classes other than c_k, and ν_k itself, being connected to all ν_j for $j \neq k$, also power dominates all required color classes.

Suppose, for contradiction, that fewer than n colors could be used for the TPDCof K_n'. In such a case, at least two adjacent vertices among the original vertices $\nu_1, \nu_2, \ldots, \nu_n$ would necessarily share the same color, violating the condition of proper vertex coloring. Alternatively, it may become impossible to ensure that each color class is power dominated by a different color class, as required by TPDC. Therefore, using fewer than n colors fails to meet one or both essential criteria. Hence, n colors are necessary.

Therefore, the coloring is both valid and minimal, and the TPDCN for the graph K_n', created through the process of duplicating a vertex in the complete graph K_n, is given by

$$\chi_{tpd}(K_n') = n.$$

■

Example 6. *In figure 6,the TPDC of graph $K_4^{'}$ created through the process of duplicating a vertex ν_3 by υ in Complete graph K_4 is shown.*

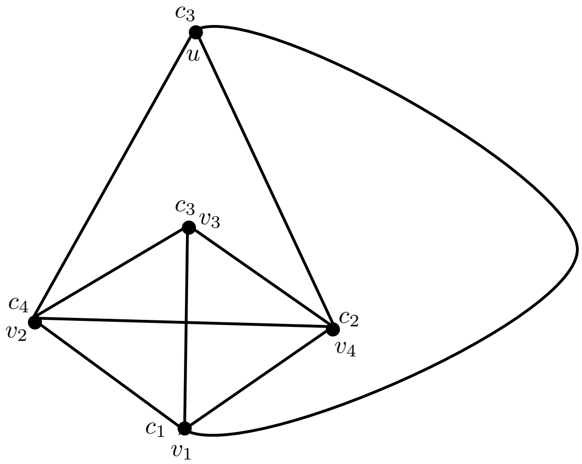

Fig. 6. Modified graph K_n' created through the process of duplicating a vertex ν_3, and the color classes of the K_4' are $c_1 = \{\nu_1\}$, $c_2 = \{\nu_4\}$, $c_3 = \{\nu_3, \upsilon\}$, and $c_4 = \{\nu_2\}$ Then $\chi_{tpd}(K_4') = 4$.

Remark 2. *In the case of complete graph K_n, it is noteworthy to notice that the case where $n = 1$ and $n = 2$ do result in valid complete graphs with trivial answers 1 and 2 respectively.*

In contrast to cycle graphs, it is especially interesting to note that complete graphs with When we examine the case of n being 1 and n being 2, we discover that both of these instances give us valid and well-defined graphs. Yet, it is significant to point out that these graphs have fairly trivial properties. For the case where n is 1, the entire graph, known as K_1, consists of a mere isolated vertex that is by itself with no edges to connect it to anything else. In spite of there being no connections whatsoever, this arrangement is still considered to be a complete graph because all potential edges that might be drawn between vertices are indeed present—although it should be pointed out that there are simply no other vertices to provide anything to connect to.

Likewise, for n=2, the entire graph K_2 has precisely two vertices that are linked by a single edge. As there is only one possible edge linking such two distinct vertices, and as this edge does exist, it is then true that the graph satisfies the conditions for completeness. Such small instances are easy examples wherein the combinatorial characteristics of a complete graph hold, but the resulting configurations are easy or are "trivial" in structure. Nevertheless, they have a way of being included in theoretical proofs or formal definitions, especially when trying to generalize properties that hold for all graphs.

Theorem 7. *For any $m \geq 1,\ n \geq 2$, the TPDCN for the graph $K'_{m,n}$ created through the process of duplicating any arbitrary vertex in bipartite graph $K_{m,n}$ is 2.*

Proof: Let $K_{m,n}$ be a complete bipartite graph with vertex sets V_1 and V_2, where

$$V_1 = \{\nu_1, \nu_2, \nu_3, \ldots, \nu_m\} \quad \text{and} \quad V_2 = \{\nu'_1, \nu'_2, \nu'_3, \ldots, \nu'_n\}.$$

Let $E(K_{m,n})$ denote the edge set of the bipartite graph, defined as

$$E(K_{m,n}) = \{\nu_i \nu'_j \mid 1 \leq i \leq m,\ 1 \leq j \leq n\}.$$

Here, the number of vertices in $K_{m,n}$ is $|V(K_{m,n})| = m+n$, and the number of edges is $|E(K_{m,n})| = mn$.

Based on the structural role of the duplicated vertex, two distinct cases arise in the bipartite graph $K_{m,n}$. The vertex selected for duplication may belong either to the partite set $V_1 = \{\nu_1, \nu_2, \ldots, \nu_m\}$ or to the partite set $V_2 = \{\nu'_1, \nu'_2, \ldots, \nu'_n\}$. Each case results in a structurally modified graph, with specific implications for TPDC, as detailed in the following sections.

Case (i): When $m = 1$, and duplicating any pendant vertex ν'_i in $K_{1,n}$.

When $m = 1$, the bipartite graph $K_{1,n}$ is isomorphic to the Star graph with apex vertex ν_1, and pendant vertices $\{\nu'_1, \nu'_2, \ldots, \nu'_n\}$ forming the second partite set. The edge set and vertex set of $K_{1,n}$ are defined as:

$$E(K_{1,n}) = \{\nu_1 \nu'_i \mid 1 \leq i \leq n\}$$

$$|V(K_{1,n})| = n+1,$$
$$|E(K_{1,n})| = n.$$

Now, create a duplicate of any arbitrary pendant vertex ν'_k in V_2. This duplication yields a new vertex υ, which inherits the exact neighborhood of ν'_k, i.e., it is adjacent to the apex vertex ν_1. The resulting graph $K'_{1,n}$ has:

$$V(K'_{1,n}) = \{\nu_1, \nu'_1, \nu'_2, \ldots, \nu'_n, \upsilon\}$$

$$E(K'_{1,n}) = \{\nu_1 \nu'_i \mid 1 \leq i \leq n\} \cup \{\nu_1 \upsilon\}$$

The procedure outlined below is based on a systematic approach to coloring the vertices of the graph and guarantees a coloring that satisfies both the proper coloring and the TPDC requirements. Specifically, this method ensures that every color class is power dominated by at least one vertex from a distinct color class, thereby producing a valid TPDC configuration.

The graph $K'_{1,n}$ is colored using two colors, c_1 and c_2, as follows:

- The apex vertex ν_1 is assigned color c_1.
- All pendant vertices $\{\nu'_1, \nu'_2, \ldots, \nu'_n\}$, as well as the duplicated vertex υ, are assigned color c_2.

This coloring is proper, since adjacent vertices receive different colors. The domination relationships under this coloring scheme are:

- The vertices $\{\nu'_1, \nu'_2, \ldots, \nu'_n, \upsilon\}$, being adjacent to ν_1, collectively power dominate the color class $c_1 = \{\nu_1\}$.
- The apex vertex ν_1, being adjacent to every vertex in c_2, power dominates the color class $c_2 = \{\nu'_1, \nu'_2, \ldots, \nu'_n, \upsilon\}$.

To show that two colors are necessary, assume only one color is used for a TPDC of $K'_{1,n}$. Then, at least one pair of adjacent vertices would share the same color, violating the requirement for proper coloring. Furthermore, if the coloring is proper but only uses one color, the power domination condition fails, as no vertex can power dominate a different color class. Hence, a valid TPDC cannot be achieved with fewer than two colors.

Therefore, the color assignment is both valid and minimal, as it ensures that each vertex in the graph power dominates all vertices in at least one color class distinct from its own, in accordance with the definition of TPDC. Thus, the TPDCN of the bipartite graph $K'_{1,n}$, created by duplicating any arbitrary pendant vertex of $K_{1,n}$, is:

$$\chi_{tpd}(K'_{1,n}) = 2.$$

Case (ii): When $m = 1$, and duplicating the apex vertex ν_1 in $K_{1,n}$.

When $m = 1$, the bipartite graph $K_{1,n}$ corresponds to the Star graph, with apex vertex ν_1 and pendant vertices $\{\nu'_1, \nu'_2, \ldots, \nu'_n\}$. The edge set and vertex set of $K_{1,n}$ are:

$$E(K_{1,n}) = \{\nu_1 \nu'_i \mid 1 \leq i \leq n\},$$

Now, consider duplicating the apex vertex ν_1. This process creates a new vertex υ, which inherits the exact neighborhood of ν_1, i.e., it is adjacent to all pendant vertices. The resulting graph $K'_{2,n}$ has:

$$V(K'_{2,n}) = \{\nu_1, \upsilon, \nu'_1, \nu'_2, \ldots, \nu'_n\}$$

$$E(K'_{2,n}) = \{\nu_1 \nu'_i \mid 1 \leq i \leq n\} \cup \{\upsilon \nu'_i \mid 1 \leq i \leq n\}$$

with

$$|V(K'_{2,n})| = n+2,$$
$$|E(K'_{2,n})| = 2n.$$

.

The procedure outlined below is based on a systematic approach to coloring the vertices of the graph and guarantees a coloring that satisfies both the proper coloring and the TPDC requirements. Specifically, this method ensures that every color class is power dominated by at least one vertex from a distinct color class, thereby producing a valid TPDC configuration.

The graph $K'_{2,n}$ is colored using two colors, c_1 and c_2, as follows:

- The apex vertex ν_1 and its duplicate υ are assigned color c_1.
- All pendant vertices $\{\nu'_1, \nu'_2, \ldots, \nu'_n\}$ are assigned color c_2.

This coloring satisfies the proper coloring condition, as all adjacent vertices receive distinct colors. The domination relationships under this coloring scheme are:

- The pendant vertices $\{\nu'_1, \nu'_2, \ldots, \nu'_n\}$, being adjacent to both ν_1 and υ, collectively power dominate the color class $c_1 = \{\nu_1, \upsilon\}$.
- The apex vertices ν_1 and υ, being adjacent to all pendant vertices, power dominate the color class $c_2 = \{\nu'_1, \nu'_2, \ldots, \nu'_n\}$.

To establish the minimality of this coloring, assume that only one color is used. Then, adjacent vertices must share the same color, violating the condition of proper vertex coloring. Furthermore, even if proper coloring is somehow preserved, the power domination condition fails, as a single color class cannot dominate itself. Therefore, at least two colors are necessary.

Hence, the color assignment described is both valid and minimal. Each vertex in the graph power dominates all vertices in at least one color class different from its own, fulfilling the requirements of TPDC. Therefore, the TPDCN of the graph $K'_{2,n}$, obtained by duplicating the apex vertex in the bipartite graph $K_{1,n}$, is:

$$\chi_{tpd}(K'_{2,n}) = 2.$$

Case (iii): When $m = 2$, and duplicating a vertex ν'_j from V_2 in $K_{2,n}$.

Let $K_{2,n}$ be a bipartite graph with two partite sets:

$$V_1 = \{\nu_1, \nu_2\}, \quad V_2 = \{\nu'_1, \nu'_2, \nu'_3, \ldots, \nu'_n\},$$

where each vertex in V_1 is connected to every vertex in V_2. The edge set and graph order are defined as:

$$E(K_{2,n}) = \{\nu_i\nu'_j \mid 1 \le i \le 2,\ 1 \le j \le n\},$$

$$|V(K_{2,n})| = n + 2,$$

$$|E(K_{2,n})| = 2n.$$

Now, assume that an arbitrary vertex ν'_2 from the set V_2 undergoes duplication. The new vertex υ inherits the exact neighborhood of ν'_2, that is, it becomes adjacent to both ν_1 and ν_2. The resulting graph, denoted $K'_{2,n}$, has vertex set:

$$V(K'_{2,n}) = \{\nu_1, \nu_2, \nu'_1, \nu'_2, \ldots, \nu'_n, \upsilon\}$$

and edge set:

$$E(K'_{2,n}) = \{\nu_i\nu'_j \mid 1 \le i \le 2,\ 1 \le j \le n\} \cup \{\nu_1\upsilon,\ \nu_2\upsilon\}.$$

Thus,

$$|V(K'_{2,n})| = n + 3,$$
$$|E(K'_{2,n})| = 2n + 2.$$

The procedure outlined below is based on a systematic approach to coloring the vertices of the graph and guarantees a coloring that satisfies both proper coloring and TPDC requirements. Specifically, this method ensures that every color class is power dominated by at least one vertex from a distinct color class, thereby producing a valid TPDC configuration.

The graph $K'_{2,n}$ is colored using two colors, c_1 and c_2, as follows:

- Vertices in $V_1 = \{\nu_1, \nu_2\}$ are assigned color c_1.
- Vertices in $V_2 = \{\nu'_1, \nu'_2, \ldots, \nu'_n\}$, along with the duplicate vertex υ, are assigned color c_2.

This coloring satisfies the proper coloring condition, as no two adjacent vertices receive the same color. The domination relationships under this coloring scheme are:

- Vertices in $V_1 = \{\nu_1, \nu_2\}$, being adjacent to all of V_2 and υ, collectively power dominate the color class $c_2 = \{\nu'_1, \nu'_2, \ldots, \nu'_n, \upsilon\}$.
- Vertices in $V_2 \cup \{\upsilon\}$, being adjacent to both ν_1 and ν_2, collectively power dominate the color class $c_1 = \{\nu_1, \nu_2\}$.

To prove the minimality of the coloring, assume that only one color is used. Then, adjacent vertices would share the same color, violating the proper coloring condition. Even if proper coloring were somehow maintained, the power domination requirement would fail, since a single color class cannot power dominate another. Hence, two colors are both necessary and sufficient.

Therefore, the color assignment is both valid and minimal, as it ensures that each vertex in the graph power dominates all vertices in at least one color class different from its own, in accordance with the definition of TPDC. The TPDCN of the graph $K'_{2,n}$, obtained by duplicating an arbitrary vertex from V_2 in the bipartite graph $K_{2,n}$, is:

$$\chi_{tpd}(K'_{2,n}) = 2.$$

Case (iv): When $m \ge 3$, and duplicating a vertex ν'_i from V_2 in $K_{m,n}$.

Let $K_{m,n}$ be a bipartite graph with two partite sets:

$$V_1 = \{\nu_1, \nu_2, \ldots, \nu_m\}, \quad V_2 = \{\nu'_1, \nu'_2, \ldots, \nu'_n\},$$

where each vertex in V_1 is adjacent to every vertex in V_2. The edge set and order of the graph are given by:

$$E(K_{m,n}) = \{\nu_i\nu'_j \mid 1 \le i \le m,\ 1 \le j \le n\},$$
$$|V(K_{m,n})| = m + n,$$
$$|E(K_{m,n})| = mn.$$

Assume that a vertex ν'_i from V_2 undergoes duplication. The new vertex υ inherits the exact neighborhood of ν'_i, i.e., it is adjacent to all vertices in V_1. The resulting graph, denoted $K'_{m,n}$, has vertex set:

$$V(K'_{m,n}) = \{\nu_1, \nu_2, \ldots, \nu_m,\ \nu'_1, \nu'_2, \ldots, \nu'_n,\ \upsilon\}$$

and edge set:

$$E(K'_{m,n}) = \{\nu_i\nu'_j \mid 1 \le i \le m,\ 1 \le j \le n\} \cup \{\nu_l\upsilon \mid 1 \le l \le m\}.$$

Thus,

$$|V(K'_{m,n})| = m + n + 1,$$
$$|E(K'_{m,n})| = mn + m.$$

The procedure outlined below is based on a systematic approach to coloring the vertices of the graph and guarantees a coloring that satisfies both proper coloring and TPDC requirements. Specifically, this method ensures that every color class is power dominated by at least one vertex from a distinct color class, thereby producing a valid TPDC configuration.

The graph $K'_{m,n}$ is colored using two colors, c_1 and c_2, as follows:

- Vertices in $V_1 = \{\nu_1, \nu_2, \ldots, \nu_m\}$ are assigned color c_1.
- Vertices in $V_2 = \{\nu'_1, \nu'_2, \ldots, \nu'_n\}$, along with the duplicated vertex υ, are assigned color c_2.

This coloring satisfies the proper coloring condition, as adjacent vertices receive distinct colors. The domination relationships under this coloring scheme are as follows:

- The vertices in $V_1 = \{\nu_1, \nu_2, \ldots, \nu_m\}$, being adjacent to all vertices in V_2 and to υ, collectively power dominate the color class
$$c_2 = \{\nu'_1, \nu'_2, \ldots, \nu'_n, \upsilon\}.$$
- The vertices in $V_2 \cup \{\upsilon\} = \{\nu'_1, \nu'_2, \ldots, \nu'_n, \upsilon\}$, being adjacent to all vertices in V_1, collectively power dominate the color class
$$c_1 = \{\nu_1, \nu_2, \ldots, \nu_m\}.$$

To prove the minimality of this coloring, assume that only one color is used. Then, adjacent vertices must share the same color, which contradicts the condition of proper vertex coloring. Even if such a coloring avoids direct conflicts, the power domination condition will be violated, as no color class will be dominated by a different one. Hence, two colors are both necessary and sufficient.

Therefore, the color assignment is both valid and minimal, as it ensures that each vertex in the graph power dominates all vertices in at least one color class different from its own, in accordance with the definition of TPDC. Consequently, the TPDCN for the bipartite graph $K'_{m,n}$, obtained by duplicating an arbitrary vertex from either V_1 or V_2, is:

$$\chi_{tpd}(K'_{m,n}) = 2.$$

In all four structural scenarios examined—whether duplicating a pendant vertex in $K_{1,n}$, duplicating the apex vertex in $K_{1,n}$, duplicating a vertex in $K_{2,n}$, or duplicating any vertex in $K_{m,n}$ for $m \geq 3$—a TPDCusing exactly two colors can be successfully constructed. In each case, the coloring satisfies both the proper vertex coloring condition and the power domination requirement, thereby making it both valid and minimal. Therefore, the TPDCN for the modified bipartite graph is

$$\chi_{tpd}(K'_{m,n}) = 2.$$

■

Example 7. *In figure 7, the TPDC for the bipartite graph graph $K^{'}_{1,11}$ created through the process of duplicating a pendent vertex ν_6 by υ in bipartite graph $K_{1,11}$ is shown.*

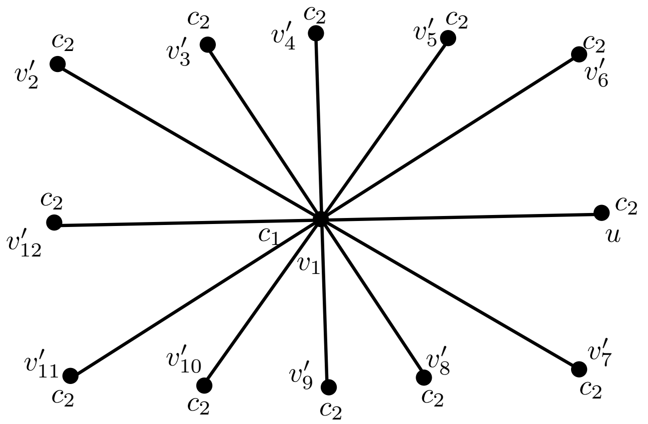

Fig. 7. Modified graph $K^{'}_{1,11}$, created through the process of duplicating a pendent vertex ν_6, and the color classes of the $K'_{1,11}$ are $c_1 = \{\upsilon, \nu_1, \nu_3\}$, $c_2 = \{\nu_2, \nu_4\}$ Then $\chi_{tpd}(\ K^{'}_{1,11})$=2.

Remark 3. *In the case of bipartite graph $K_{m,n}$, it is noteworthy to notice that the case where $n = 1$ and $m = 1$ do result in a valid Path graph, which was proved in Theorem 1.*

In the context of bipartite graphs, specifically the complete bipartite graph denoted as $K_{m,n}$, it is noteworthy that the case where $m = 2$ and $n = 1$ results in a graph that is structurally equivalent to a simple path graph. In this configuration, the graph consists of two vertices in one partition and a single vertex in the other. Each of the two vertices in the $m-$partition is connected to the single vertex in the $n-$ partition, forming a Y-shaped structure that, when viewed as an undirected graph, is isomorphic to a path of length two.

This observation confirms that $K_{2,1}$ represents a valid and minimal example of a path graph, a result that is formally established in Theorem 1. It highlights how certain complete bipartite graphs, even with small values of m and n, can correspond to well-known graph classes under specific conditions. This case also illustrates how bipartite graphs can encompass a diverse range of structures, including paths, cycles (under proper configurations), and more complex networks.

Theorem 8. *For any $n \geq 3$, the TPDCN for the graph $F^{'}_n$, created through the process of duplicating any arbitrary vertex of Fan graph F_n is 3.*

Proof: Let F_n be the Fan graph with vertex set $\{\nu_0,\ \nu_1, \nu_2, \ldots, \nu_n\}$, where ν_0 is the apex vertex and the remaining vertices form a path P_n. Let $E(F_n)$ denote the edge set of the Fan graph, where

$$E(F_n) = \{\nu_0\nu_i \mid 1 \leq i \leq n\} \cup \{\nu_i\nu_{i+1} \mid 1 \leq i \leq n-1\}.$$

Here, $|V(F_n)| = n + 1$, where n is any positive integer.

Based on the structural role of the duplicated vertex, two distinct cases arise in the Fan graph F_n. The vertex chosen for duplication may either be the apex vertex ν_0 or any arbitrary vertex from the path P_n. Each case results in a structurally distinct graph with corresponding implications for TPDC , as detailed below.

Case (i): Duplication of the apex vertex

Assume that the apex vertex ν_0 undergoes duplication. This process yields a new vertex υ, which inherits the exact neighborhood of ν_0, thereby preserving the structure of the original graph. The resulting graph F'_n has vertex set

$$\{\upsilon,\ \nu_0,\ \nu_1, \nu_2, \ldots, \nu_n\}$$

and edge set

$$\begin{aligned} E(F'_n) = &\{\nu_0\nu_i \mid 1 \leq i \leq n\} \\ &\cup \{\nu_i\nu_{i+1} \mid 1 \leq i \leq n-1\} \\ &\cup \{\upsilon\nu_i \mid 1 \leq i \leq n\}. \end{aligned}$$

The procedure outlined below is based on a systematic approach to coloring the vertices of the graph and guarantees a coloring that satisfies both proper coloring and TPDC requirements. Specifically, this method ensures that every color class is power dominated by at least one vertex from a distinct color class, thereby producing a valid TPDC configuration.

The graph F'_n is colored using three colors, c_1, c_2, and c_3, as follows:

- The apex vertex ν_0 and the duplicate vertex υ are assigned color c_1.
- Vertices at even-numbered positions along the path, i.e., $\{\nu_{2i} \mid 1 \leq i \leq \frac{n}{2}\}$, are assigned color c_2.
- Vertices at odd-numbered positions along the path, i.e., $\{\nu_{2i-1} \mid 1 \leq i \leq \frac{n+1}{2}\}$, are assigned color c_3.

The domination relationships under this coloring scheme are as follows:

- The path vertices $\{\nu_1, \nu_2, \ldots, \nu_n\}$ are adjacent to both apex vertices and therefore collectively power dominate the color class $c_1 = \{\nu_0, \upsilon\}$.
- The apex vertices ν_0 and υ are adjacent to all path vertices and hence together power dominate the color classes:
  - $\{c_2\}$, which contains all even-indexed path vertices, and
  - $\{c_3\}$, which contains all odd-indexed path vertices.

To demonstrate the necessity of three colors for a valid TPDC of F'_n, suppose, for the sake of contradiction, that only two colors are used. Under this assumption, it becomes inevitable that some adjacent vertices must share the same color, thereby violating the condition of proper vertex coloring. Alternatively, even if a proper coloring is somehow preserved, the power domination condition would fail, as at least one color class would not be monitored or dominated by any vertex from a different color class. Consequently, it is impossible to construct a valid TPDC for F'_n using fewer than three colors.

Therefore, the color assignment described is both valid and minimal, as it guarantees that each vertex in the graph power dominates all vertices in at least one color class different from its own, in accordance with the definition of TPDC. Consequently, the TPDCN for the graph F'_n, obtained by duplicating the apex vertex in the Fan graph F_n, is

$$\chi_{tpd}(F'_n) = 3.$$

Case (ii): Duplication of an arbitrary path vertex

Assume that an arbitrary vertex ν_k from the path P_n in the Fan graph F_n undergoes duplication. This process yields a new vertex υ, which inherits the exact neighborhood of ν_k, i.e., it connects to ν_0, ν_{k-1}, and ν_{k+1}. The resulting graph F'_n has vertex set

$$\{\upsilon,\ \nu_0,\ \nu_1, \nu_2, \ldots, \nu_n\}$$

and edge set

$$\begin{aligned} E(F'_n) = &\{\nu_0\nu_i \mid 1 \leq i \leq n\} \cup \{\nu_i\nu_{i+1} \mid 1 \leq i \leq n-1\} \\ &\cup \{\upsilon\nu_{k-1},\ \upsilon\nu_{k+1},\ \upsilon\nu_0\}. \end{aligned}$$

The coloring procedure uses the same strategy to ensure both proper coloring and TPDC:

- The apex vertex ν_0 is assigned color c_1.
- Vertices at even-numbered positions along the path, $\{\nu_{2i} \mid 1 \leq i \leq \frac{n}{2}\}$, are assigned color c_2.
- Vertices at odd-numbered positions, $\{\nu_{2i-1} \mid 1 \leq i \leq \frac{n+1}{2}\}$, are assigned color c_3.
- The duplicated vertex υ is assigned the same color as the original vertex ν_k.

The domination relationships under this coloring scheme are as follows:

- The path vertices $\{\nu_1, \nu_2, \ldots, \nu_n\}$ and the duplicated vertex υ, all of which are adjacent to the apex vertex ν_0, collectively power dominate the color class $c_1 = \{\nu_0\}$.
- The apex vertex ν_0, being adjacent to every path vertex, power dominates the following:
  - The color class $\{c_2\}$, consisting of even-indexed path vertices.
  - The color class $\{c_3\}$, consisting of odd-indexed path vertices.

To demonstrate the necessity of three colors for a valid TPDC of F'_n, suppose, for the sake of contradiction, that only two colors are used. Under this assumption, there would inevitably exist adjacent vertices sharing the same color, thereby violating the proper vertex coloring condition. Alternatively, even if such a coloring avoids direct conflicts, it would fail the TPDC condition, as at least one color class would remain unmonitored by any vertex from another color class. Hence, it is impossible to construct a valid TPDC for F'_n using fewer than three colors.

Therefore, the color assignment described is both valid and minimal, as it guarantees that each vertex in the graph power dominates all vertices in at least one color class different from its own, in accordance with the definition of TPDC. Consequently, the TPDCN for the graph F_n', obtained by duplicating an arbitrary path vertex in the Fan graph F_n, is

$$\chi_{tpd}(F_n') = 3.$$

In both scenarios considered in the modified Fan graph F_n'—namely, duplicating the apex vertex or duplicating any arbitrary path vertex—a TPDC using exactly three colors can be successfully constructed. In each case, the coloring satisfies both the proper vertex coloring condition and the power domination requirement, making it both valid and minimal. Therefore, the TPDCN of the modified Fan graph is

$$\chi_{tpd}(F_n') = 3.$$

■

Example 8. *In figure 8, the TPDC for the graph $F_n^{'}$ created through the process of duplicating a ν_2 by υ in Fan graph $F_n^{'}$ is shown.*

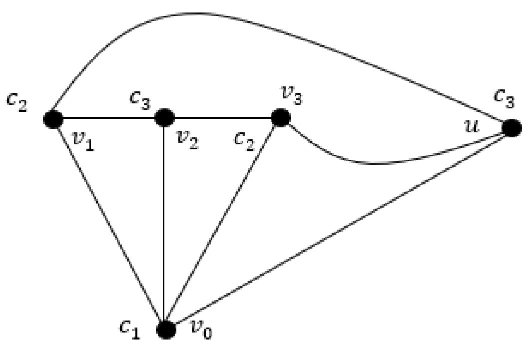

Fig. 8. Modified Fan graph $F_n^{'}$ created through the process of duplicating a vertex ν_2 the color classes of the F_4' are $c_1 = \{\nu_0\}$, $c_2 = \{\nu_1, \nu_3\}$ and $c_3 = \{\nu_2, \upsilon\}$ Then $\chi_{tpd}(F_4^{'})$=3.

Theorem 9. *For any $n \geq 3$, the TPDCN for the graph $DF_n^{'}$, created through the process of duplicating any arbitrary vertex of Double Fan graph DF_n is 3.*

Proof:
Let DF_n be the double fan graph with vertex set $\{\nu_0, \nu_1, \nu_2, \ldots, \nu_n, \nu_0^{'}\}$, where ν_0 and $\nu_0^{'}$ are the apex vertices, and all the vertices of the path P_n, namely $\{\nu_1, \nu_2, \ldots, \nu_n\}$, are shared with both apex vertices ν_0 and $\nu_0^{'}$. Let $E(DF_n)$ denote the edge set of the double fan graph, where $E(DF_n) = \{\nu_0\nu_i \mid 1 \leq i \leq n\} \cup \{\nu_i\nu_{i+1} \mid 1 \leq i < n\}$ $\cup\{\nu_0^{'}\nu_i \mid 1 \leq i \leq n\}$. Here, $|V(DF_n)| = n + 2$, where n is any positive integer. Based on the selection of the vertex ν to be duplicated by a new vertex υ, two cases may arise.

Case (i): Duplicating any one of the apex vertex
To streamline the discussion while maintaining generality, we assume that one of the apex vertices, namely ν_0, undergoes duplication. This process yields a new vertex υ, which inherits the exact neighborhood of ν_0, thereby preserving the structural properties of the original graph DF_n. As a result, the duplicated graph $DF_n^{'}$ is obtained, with the vertex set $\{\upsilon, \nu_1, \nu_2, \nu_3, \ldots, \nu_n, \nu_0, \nu_0^{'}\}$ and the edge set defined as, $E(DF_n^{'}) = \{\nu_0\nu_i \mid 1 \leq i \leq n\} \cup \{\nu_i\nu_{i+1} \mid 1 \leq i < n\}$ $\cup\{\nu_0^{'}\nu_i \mid 1 \leq i \leq n\} \cup \{\upsilon\nu_i \mid 1 \leq i \leq n\}$.

The procedure outlined below is based on a systematic approach to coloring the vertices of the graph and guarantees a coloring that satisfies both proper coloring and the TPDC requirements. Specifically, this method ensures that every color class is power dominated by at least one vertex from a distinct color class, thereby producing a valid TPDC configuration.

The graph DF_n' is colored using three colors, c_1, c_2, and c_3, according to the positions of the vertices:

- Vertices $\nu_0, \nu_0^{'}$, which are the apex vertices, along with the newly introduced duplicate vertex υ, are all assigned color c_1.
- Vertices at even-numbered positions along the path, that is, $\{\nu_{2i} \mid 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$, are assigned color c_2.
- Vertices at odd-numbered positions along the path, that is, $\{\nu_{2i-1} \mid 1 \leq i \leq \lceil \frac{n}{2} \rceil\}$, are assigned color c_3.

This procedure guarantees a proper vertex coloring of the graph, ensuring that no two adjacent vertices receive the same color. Hence, it satisfies the fundamental requirement of proper coloring needed for TPDC. Additionally, the assignment of colors ensures that each color class is power dominated by a vertex of a different color class, thereby fulfilling the TPDC condition.

The domination relationships under this coloring scheme are as follows:

- All the path vertices $\{\nu_1, \nu_2, \ldots, \nu_n\}$ power dominate the apex color class $c_1 = \{\upsilon, \nu_0, \nu_0^{'}\}$, since each apex vertex and its duplicate are adjacent to all vertices on the path.
- All apex vertices and the duplicate vertex $\{\nu_0, \nu_0^{'}, \upsilon\}$ power dominate the path vertices in color class c_2 (even-indexed) and c_3 (odd-indexed), as they are adjacent to every path vertex.

To show that three colors are necessary, suppose that only two colors are used for the TPDC of DF_n'. In such a case, some adjacent vertices must necessarily share the same color, thereby violating the condition of proper vertex coloring. Alternatively, even if proper coloring is somehow maintained, the power domination condition will be violated, as at least one color class will not be power dominated by any vertex from another color class. Therefore, it is not possible to construct a valid TPDC of DF_n' using fewer than three colors.

Therefore, the color assignment is both valid and minimal, as it ensures that each vertex in the graph power dominates all vertices in at least one color class different from its own, in accordance with the definition of TPDC. Therefore, every vertex in the graph $DF_n^{'}$ power-dominates every vertex from at least one distinct color class. The TPDCN for the graph $DF_n^{'}$ created through the process of duplicating one of the apex vertex ν_0 in double fan graph DF_n is 3. i.e.,$\chi_{tpd}\left(DF_n^{'}\right) = 3$.

Case (ii): Duplicating any arbitrary vertex of path P_n in Double fan graph DF_n

To streamline the discussion while maintaining generality, let us assume that an arbitrary vertex ν_k of the path P_n in the double fan graph DF_n undergoes duplication. This duplication results in the creation of a new vertex υ, which inherits the exact neighborhood of ν_k. As a result, the structural properties of the original graph DF_n are preserved. The resulting graph, denoted by DF_n', has the vertex set $\{\upsilon, \nu_1, \nu_2, \nu_3, \nu_4, \ldots, \nu_n, \nu_0, \nu_0'\}$, and the edge set is given by: $E(DF_n') = \{\nu_0\nu_i \mid 1 \leq i \leq n\} \cup \{\nu_i\nu_{i+1} \mid 1 \leq i < n\}$ $\cup\{\nu_0'\nu_i \mid 1 \leq i \leq n\} \cup \{\upsilon\nu_{k-1},\ \upsilon\nu_{k+1},\ \upsilon\nu_0, \upsilon\nu_0'\}$.

The procedure outlined below follows a systematic approach to coloring the vertices of the graph, ensuring that the resulting coloring satisfies both the proper coloring condition and the TPDC requirements. Furthermore, this procedure guarantees that each color class is power dominated by at least one vertex from a different color class, thereby achieving a valid TPDC configuration.

The graph $DF_n^{'}$ is colored using three colors, c_1, c_2, and c_3, based on the positions of the vertices:

- The apex vertices ν_0 and $\nu_0^{'}$ are assigned the color c_1.
- The vertices at even-numbered positions along the path, i.e., $\{\nu_{2i} \mid 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$, are assigned the color c_2.
- The vertices at odd-numbered positions along the path, i.e., $\{\nu_{2i-1} \mid 1 \leq i \leq \lceil \frac{n+1}{2} \rceil\}$, are assigned the color c_3.
- The newly introduced vertex υ is assigned the same color as the duplicated vertex ν_k.

The proposed coloring ensures that adjacent vertices are assigned distinct colors, thereby fulfilling the condition for proper vertex coloring, which is essential for TPDC. The domination relationships under this coloring scheme are as follows:

- The set of path vertices along with the duplicated vertex υ, that is, $\{\nu_1,\ \nu_2, \ldots, \nu_n,\ \upsilon\}$, collectively power dominate the apex vertices ν_0 and $\nu_0^{'}$, which are colored with c_1.
- In turn, the apex vertices ν_0 and $\nu_0^{'}$ power dominate the path vertices, thereby covering the color classes c_2 and c_3.

This vertex domination among different color classes confirms that the coloring satisfies all the conditions required for a valid TPDC.

To demonstrate the necessity of three colors for a valid TPDC of DF_n', suppose, for the sake of contradiction, that only two colors are used. Under this assumption, there would inevitably exist adjacent vertices sharing the same color, thereby violating the condition of proper vertex coloring. Alternatively, even if a proper coloring is somehow achieved with two colors, the power domination condition would fail, as at least one color class would not be monitored or dominated by any vertex from a different color class. Consequently, it is impossible to construct a TPDC DF_n' using fewer than three colors.

Therefore, the color assignment described is both valid and minimal, as it guarantees that each vertex in the graph power dominates all vertices in at least one color class different from its own, in accordance with the definition of TPDC. Consequently, the TPDCN for the graph DF_n', obtained by duplicating an arbitrary vertex in the double fan graph DF_n, is 3. That is, $\chi_{tpd}(DF_n') = 3$.

In both scenarios—whether duplicating an apex vertex or a path vertex—a TPDC with exactly three colors can be constructed. This coloring satisfies both the proper vertex coloring condition and the power domination requirement, making it both valid and minimal. Therefore, the TPDCN for the modified double fan graph is

$$\chi_{tpd}(DF_n') = 3.$$

■

Example 9. *In figure 9, the TPDC for the graph $DF_n^{'}$ created through the process of duplicating a ν_2 by υ in Double fan graph $DF_n^{'}$ is shown.*

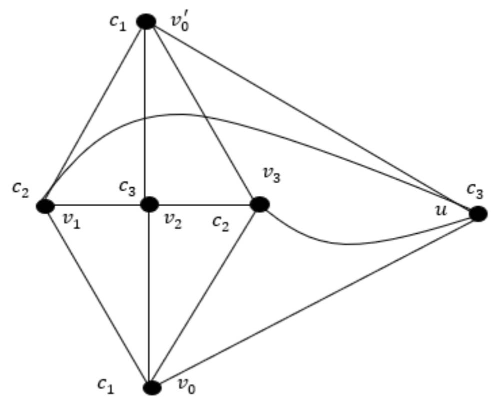

Fig. 9. Modified Double Fan graph $DF_n^{'}$ created through the process of duplicating a vertex ν_2 the color classes of the DF_4' are $c_1 = \{\nu_0, \nu_0^{'}\}$, $c_2 = \{\nu_1, \nu_3\}$ and $c_3 = \{\nu_2, \upsilon\}$ Then $\chi_{tpd}(\ DF_4^{'})$=3.

Theorem 10. *For any $n \geq 3$, the TPDCN for graph $O_n^{'}$, created through the process of duplicating any arbitrary vertex of the Octopus graph O_n is 3.*

Proof: Let O_n be the Octopus graph with vertex set $V(O_n) = \{\nu_1, \nu_2, \nu_3, \ldots, \nu_{2n+1}\}$, where ν_1 is designated as the apex vertex. The vertex subset $\{\nu_2, \nu_3, \ldots, \nu_{n+1}\}$ forms a Fan graph F_n connected sequentially, while $\{\nu_{n+2}, \nu_{n+3}, \ldots, \nu_{2n+1}\}$ forms a Star graph $K_{1,n}$, where each vertex is connected only to the apex vertex ν_1.

Let $E(O_n)$ denote the edge set of the Octopus graph, defined as:

$$E(O_n) = \{\nu_1\nu_i \mid 2 \leq i \leq 2n+1\} \cup \{\nu_i\nu_{i+1} \mid 2 \leq i \leq n\}.$$

Here, $|V(O_n)| = 2n + 1$, where n is a positive integer. The graph consists of a single apex vertex ν_1 that connects to all other vertices. The path subgraph formed by ν_2 through ν_{n+1} constitutes a Fan, and ν_{n+2} through ν_{2n+1} are pendant vertices forming the Star component.

To determine the TPDCN $\chi_{tpd}(O_n)$ under vertex duplication, we will analyze different duplication scenarios with corresponding proper TPDC s.

Case (i): Duplicating an apex vertex.

Assume that the apex vertex ν_1 undergoes duplication. This operation yields a new vertex υ, which inherits the exact neighborhood of ν_1, thereby preserving the structural integrity of the original graph O_n. Consequently, the resulting graph, denoted as O_n', is constructed with the vertex set

$$V(O_n') = \{\upsilon, \nu_1, \nu_2, \nu_3, \ldots, \nu_{n+1}, \nu_{n+2}, \nu_{n+3}, \ldots, \nu_{2n+1}\}$$

and the edge set given by $E(O_n') = \{\nu_1\nu_i \mid 2 \leq i \leq 2n + 1\} \cup \{\nu_i\nu_{i+1} \mid 2 \leq i \leq n\} \cup \{\upsilon\nu_i \mid 2 \leq i \leq 2n + 1\}$.

To determine the TPDCN of the graph O_n', we apply a systematic vertex coloring strategy that satisfies both the proper vertex coloring condition and the power domination constraint.

The graph O_n' is colored using three colors: c_1, c_2, and c_3, as follows:

- Vertices ν_1 and its duplicate υ (the apex vertices) are assigned color c_1.
- Vertices at even-numbered positions in the fan path, i.e., $\{\nu_{2i} \mid 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$, are assigned color c_2.
- Vertices at odd-numbered positions in the fan path, i.e., $\{\nu_{2i-1} \mid 1 \leq i \leq \lceil \frac{n+1}{2} \rceil\}$, are assigned color c_3.
- All pendant vertices of the star, i.e., $\{\nu_{n+2}, \nu_{n+3}, \ldots, \nu_{2n+1}\}$, are assigned color c_2.

This coloring configuration guarantees a proper vertex coloring, since no two adjacent vertices share the same color. Furthermore, the power domination requirement is satisfied under the following domination relationships:

- All vertices in the fan path $\{\nu_2, \nu_3, \ldots, \nu_{n+1}\}$, being adjacent to ν_1 and υ, collectively power dominate the color class $c_1 = \{\nu_1, \upsilon\}$.
- All pendant vertices $\{\nu_{n+2}, \ldots, \nu_{2n+1}\}$ are also adjacent to ν_1 and υ, hence they too contribute to the power domination of c_1.
- In turn, the apex vertices ν_1 and υ are adjacent to every fan and star vertex, thus power dominating both c_2 and c_3.

To demonstrate the necessity of three colors, suppose only two colors are used. In such a case, adjacent vertices would inevitably share the same color, thereby violating the requirement of proper coloring. Alternatively, if the coloring is somehow proper with two colors, the power domination condition fails because at least one color class would not be dominated by any vertex of another color class. Therefore, three colors are necessary.

Therefore, the color assignment is both valid and minimal, as it ensures that each vertex in the graph O'_n power dominates all vertices in at least one color class different from its own, in accordance with the definition of TPDC. The TPDCN for the graph O'_n created through the process of duplicating apex vertex in Octopus graph O_n is 3. i.e., $\chi_{tpd}\left(O_n'\right) = 3$.

Case (ii): Duplicating an arbitrary vertex of the Fan graph in Octopus graph O_n.

Assume that the vertex ν_3, which lies on the Fan graph F_n, undergoes duplication. This process results in the creation of a new vertex υ, which inherits the exact neighborhood of ν_3. Consequently, the structural properties of the original Octopus graph O_n are preserved. The resulting graph is denoted by O_n', and has the vertex set:

$$V(O_n') = \{\upsilon, \nu_1, \nu_2, \nu_3, \nu_4, \ldots, \nu_{n+1}, \nu_{n+2}, \ldots, \nu_{2n+1}\}$$

and the edge set: $E(O_n') = \{\nu_1\nu_i \mid 2 \leq i \leq 2n + 1\} \cup \{\nu_i\nu_{i+1} \mid 2 \leq i \leq n\} \cup \{\upsilon\nu_2, \upsilon\nu_4, \upsilon\nu_1\}$.

The following coloring strategy guarantees a TPDC of O_n' that is both valid and minimal. The graph is colored using three distinct colors: c_1, c_2, and c_3, assigned as follows:

- The apex vertex ν_1 is assigned color c_1.
- Vertices at even-numbered positions along the Fan path, i.e., $\{\nu_{2i} \mid 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$, are assigned color c_2.
- Vertices at odd-numbered positions along the Fan path, i.e., $\{\nu_{2i-1} \mid 1 \leq i \leq \lceil \frac{n+1}{2} \rceil\}$, including the duplicated vertex υ, are assigned color c_3.
- All the pendant vertices of the star graph, i.e., $\{\nu_{n+2}, \nu_{n+3}, \ldots, \nu_{2n+1}\}$, are assigned color c_2.

This coloring satisfies the condition of a proper coloring because no two adjacent vertices share the same color. Moreover, the coloring satisfies the TPDC requirements as described below:

- The vertices of the Fan graph F_n, including the duplicated vertex υ, collectively power dominate the color class $c_1 = \{\nu_1\}$, since each is adjacent to the apex vertex.
- All pendant vertices of the star graph $K_{1,n}$, colored c_2, are adjacent to ν_1 (colored c_1), and hence power dominate the class c_1.
- The apex vertex ν_1 is adjacent to every vertex of both color classes c_2 and c_3, and thus power dominates those classes.

To demonstrate the minimality of the coloring, suppose that only two colors are used. Under this assumption, it is impossible to assign distinct colors to all adjacent vertices while simultaneously ensuring that every color class is power dominated by a vertex from a different color class. Therefore, at least three colors are necessary to achieve a valid TPDC for the graph O_n'.

Thus, the coloring presented is both valid and minimal, fulfilling all the criteria for TPDC . The TPDCN of the graph O_n', resulting from the duplication of a Fan graph vertex in the Octopus graph O_n, is

$$\chi_{tpd}(O'_n) = 3.$$

Case (iii): Duplicating an arbitrary pendent vertex of Star graph in Octopus graph O_n.

To maintain generality, consider the case in which a pendant vertex ν_{n+k} of the star subgraph $K_{1,n}$ in the Octopus graph O_n undergoes duplication. This process produces a new vertex υ, which inherits the exact neighborhood of ν_{n+k}, thereby preserving the structural properties of the original graph. The resulting graph, denoted as O'_n, contains the following vertex set:

$$V(O'_n) = \{\nu_1, \nu_2, \ldots, \nu_n, \nu_{n+1}, \nu_{n+2}, , \ldots, \nu_{2n+1}, \upsilon\}.$$

The corresponding edge set is defined as:

$$\begin{aligned} E(O'_n) = &\{\nu_1\nu_i \mid 2 \leq i \leq 2n+1\} \\ &\cup \{\nu_i\nu_{i+1} \mid 2 \leq i \leq n\} \\ &\cup \{\upsilon\nu_1\}. \end{aligned}$$

To determine the TPDCN of O'_n, a valid coloring strategy employing three colors c_1, c_2, and c_3 is applied. The colors are assigned as follows:

- Color c_1 is assigned to the apex vertex ν_1 and to all odd-indexed vertices of the Fan graph:
$$\{\nu_1\} \cup \{\nu_{2i-1} \mid 1 \leq i \leq \lceil \tfrac{n+1}{2} \rceil\}.$$
- Color c_2 is assigned to the even-indexed vertices of the Fan graph:
$$\{\nu_{2i} \mid 1 \leq i \leq \lfloor \tfrac{n}{2} \rfloor\}.$$
- Color c_3 is assigned to all pendant vertices of the star subgraph, including the newly introduced vertex υ:
$$\{\nu_{n+2}, \nu_{n+3}, \ldots, \nu_{2n+1}, \upsilon\}.$$

This coloring satisfies the conditions for a proper vertex coloring, ensuring that no two adjacent vertices receive the same color. Furthermore, the TPDC requirements are fulfilled as described below:

- The vertices of the Fan graph F_n, namely $\{\nu_2, \nu_3, \ldots, \nu_{n+1}\}$, are adjacent to the apex vertex ν_1 and therefore power dominate the color class c_1.
- All pendant vertices of the star subgraph, including υ, are adjacent to ν_1 and thus power dominate the color class c_1.
- The apex vertex ν_1 is adjacent to all vertices in both c_2 and c_3, and therefore power dominates those color classes.

To establish the minimality of the coloring, suppose that only two colors are used. Under such an assumption, it becomes impossible to assign distinct colors to all adjacent vertices while ensuring that each color class is power dominated by a vertex of a different color class. Thus, at least three colors are necessary to achieve a valid TPDC.

Hence, the coloring described above is both valid and minimal, and satisfies all the criteria of TPDC . Therefore, the TPDCN of the graph O'_n, obtained by duplicating an arbitrary pendant vertex of the star subgraph in the Octopus graph O_n, is: $\chi_{tpd}(O'_n) = 3$.

Therefore, the color assignment is both valid and minimal, as it ensures that each vertex in the graph power dominates all vertices in at least one color class different from its own, in accordance with the definition of TPDC. The TPDCN for the graph O'_n created through the process of duplicating any arbitrary pendent vertex in a Star graph of Octopus graph O_n is 3..i.e., $\chi_{tpd}\left(O'_n\right) = 3$.

In all the three cases whether duplicating an apex vertex of a Fan graph or a path vertex or pendedent vertex of the star graph —a TPDC with exactly three colors can be constructed. This coloring satisfies both the proper vertex coloring condition and the power domination requirement, making it both valid and minimal. Therefore, the TPDCN for the modified Octopus graph is

$$\chi_{tpd}(O'_n) = 3.$$

■

Example 10. *In figure 10, the TPDC for the graph O'_5 created through the process of duplicating a pendent vertex ν_{11} by υ in Octopus graph O_5 is shown*

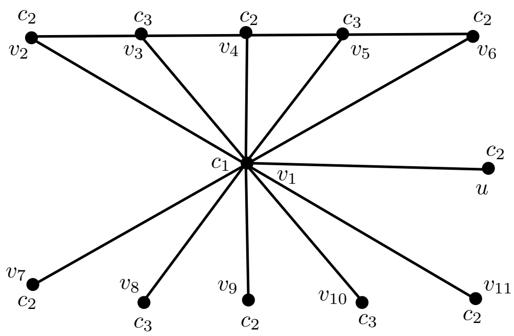

Fig. 10. A modified octopus graph O'_5 created through the process of duplicating a pendent vertex ν_{11}, the color classes of the O'_4 are $c_1 = \{\nu_1\}$, $c_2 = \{\upsilon, \nu_2, \nu_4, \nu_6, \nu_7, \nu_9, \nu_{11}\}$ and $c_3 = \{\nu_3, \nu_5, \nu_8, \nu_{10}$ Then $\chi_{tpd}(\ O'_5)$=3.

Theorem 11. *For any $n \geq 3$, the TPDCN for graph FP'_n, created through the process of duplicating any arbitrary vertex of the Flower pot graph FP_n is 3.*

Proof: Let FP_n denote a flower pot graph defined on $2n+1$ vertices, where n is any positive integer. The vertex set of FP_n consists of three components: an apex vertex ν_1; a cycle of n vertices $\{\nu_2, \nu_3, \ldots, \nu_{n+1}\}$ that form the cycle graph C_n; and a set of n pendant vertices $\{\nu_{n+2}, \nu_{n+3}, \ldots, \nu_{2n+1}\}$ that are connected to the apex ν_1, forming a star graph $K_{1,n}$. The edge set of FP_n is constructed as follows: the cycle part consists of the edges $\{\nu_i\nu_{i+1} \mid 2 \leq i \leq n\} \cup \{\nu_{n+1}\nu_2\}$; the star part consists of the edges $\{\nu_1\nu_j \mid n+2 \leq j \leq 2n+1\}$; and two additional edges $\nu_1\nu_2$ and $\nu_1\nu_{n+1}$ connect the apex vertex to two vertices of the cycle, linking the cycle and the star components. Thus, the graph FP_n has $|V(FP_n)| = 2n+1$ vertices and $|E(FP_n)| = 2n+2$ edges. The overall structure resembles a flower pot, with the cycle graph forming the rim of the pot, the star graph representing the flower or leaves, and the apex vertex ν_1 acting as the central hub connecting both structures.

Case (i): Duplicating the Apex Vertex.

Assume that the apex vertex ν_1 of the Flower Pot Graph FP_n undergoes duplication. This operation introduces a new vertex υ, which inherits the exact neighborhood of ν_1. That is, the vertex υ is connected to all the vertices to which ν_1 is originally adjacent. As a result, the structure and connectivity of the graph are preserved, and the new graph, denoted FP'_n, is formed. The vertex set of FP'_n is given by $\{\upsilon,\ \nu_1,\ \nu_2,\ \nu_3,\ \ldots,\ \nu_{n+1},\ \nu_{n+2},\ \ldots,\ \nu_{2n+1}\}$, which includes the duplicated apex vertex along with the original vertices of FP_n. The corresponding edge set of FP'_n is defined as follows:

$$\begin{aligned} E(FP'_n) = &\{\nu_1\nu_i \mid n+2 \le i \le 2n+1\} \cup \{\nu_1\nu_2, \nu_1\nu_{n+1}\} \\ &\cup \{\nu_i\nu_{i+1} \mid 2 \le i \le n\} \cup \{\nu_{n+1}\nu_2\} \\ &\cup \{\upsilon\nu_i \mid n+2 \le i \le 2n+1\} \cup \{\upsilon\nu_2, \upsilon\nu_{n+1}\}. \end{aligned}$$

Here, the duplicated vertex υ mirrors the role of the apex ν_1 by connecting to all vertices of the star component as well as to two specific vertices of the cycle. This duplication preserves the graph's original properties while creating an extended structure suitable for further analysis, particularly in contexts such as power domination coloring or structural resilience.

The TPDC of the graph FP'_n is defined using three colors c_1, c_2, c_3, with the following vertex assignments:

- The apex vertex ν_1, its duplicate ν'_1, and the newly introduced vertex υ are assigned color c_1.
- Vertices at even-numbered positions along the cycle, i.e., $\{\nu_{2i} \mid 1 \le i \le \lfloor \frac{n}{2} \rfloor\}$, are assigned color c_2.
- Vertices at odd-numbered positions along the cycle, i.e., $\{\nu_{2i-1} \mid 1 \le i \le \lceil \frac{n+1}{2} \rceil\}$, are assigned color c_3.
- All the pendant vertices of the star graph, i.e., $\{\nu_{n+2}, \nu_{n+3}, \ldots, \nu_{2n+1}\}$, are assigned color c_2.

This coloring guarantees a proper vertex coloring and ensures that each color class is power dominated by vertices from a different color class, thereby satisfying the conditions for a valid TPDC.

The domination relationships under this coloring scheme are as follows:

- The vertices of the cycle graph, $\{\nu_2, \nu_3, \nu_4, \ldots, \nu_{n+1}\}$, power dominate the color class $c_1 = \{\nu_1, \upsilon\}$, since both apex vertices are adjacent to all cycle vertices.
- Every pendant vertex of the star graph $K_{1,n}$, i.e., $\{\nu_i \mid n+2 \le i \le 2n+1\}$, also power dominates the color class $c_1 = \{\nu_1, \upsilon\}$, as each of these vertices is connected to both apex vertices.
- The apex vertex ν_1 power dominates the cycle vertices colored with c_2 (even-indexed) and c_3 (odd-indexed), since it is adjacent to all vertices in the cycle graph.

To show that three colors are necessary, suppose that only two colors are used for the TPDC of the graph FP'_n. In such a case, some adjacent vertices must necessarily share the same color, thereby violating the condition of proper vertex coloring. Alternatively, even if proper coloring is somehow preserved, the power domination condition will be violated, as at least one color class will not be power dominated by any vertex from another color class. Therefore, it is not possible to construct a valid TPDC of FP'_n using fewer than three colors.

Hence, the color assignment is both valid and minimal, as it ensures that every vertex in the graph power dominates all vertices in at least one color class different from its own, in accordance with the definition of TPDC. That is, each vertex in the graph FP'_n power dominates all the vertices of at least one distinct color class.

Therefore, the TPDCN of the graph FP'_n, created through the duplication of the apex vertex ν_1 in the Flower Pot graph FP_n, is:

$$\chi_{tpd}\,(FP'_n) = 3.$$

Case (ii): Duplicating any arbitrary vertex of Cycle graph in Flower pot graph FP_n.

To streamline the discussion while maintaining generality, let us assume that an arbitrary vertex ν_k of the cycle C_n in the Flower Pot graph FP_n undergoes duplication. This duplication results in the creation of a new vertex υ, which inherits the exact neighborhood of ν_k. As a result, the structural properties of the original graph FP_n are preserved.

The resulting graph, denoted by FP'_n, has the vertex set:

$$\{\upsilon,\ \nu_1,\ \nu_2,\ \nu_3,\ \ldots,\ \nu_{n+1},\ \nu_{n+2},\ \ldots,\ \nu_{2n+1}\},$$

and the edge set is given by:

$$\begin{aligned} E(FP'_n) = &\{\nu_1\nu_i \mid n+2 \le i \le 2n+1\} \cup \{\nu_1\nu_2,\ \nu_1\nu_{n+1}\} \\ &\cup \{\nu_i\nu_{i+1} \mid 2 \le i \le n\} \cup \{\nu_{n+1}\nu_2\} \\ &\cup \{\upsilon\nu_{k-1},\ \upsilon\nu_{k+1}\}. \end{aligned}$$

Here, υ replicates the role of ν_k, maintaining the connections to its neighboring vertices in the cycle C_n, thereby preserving the cycle structure and the overall Flower Pot graph configuration.

The procedure provided below, based on a systematic approach to coloring the graph's vertices, guarantees a resulting coloring that satisfies both the proper coloring and TPDC requirements. Moreover, applying this procedure ensures that every color class is power dominated by at least one vertex from a different color class, thereby establishing a valid TPDC configuration.

The graph FP'_n is colored using three colors c_1, c_2, and c_3, based on the positions and roles of the vertices in the structure:

- The apex vertex ν_1 is assigned color c_1.
- Vertices at even-numbered positions in the cycle (path portion of the fan), i.e., $\{\nu_{2i} \mid 1 \le i \le \frac{n}{2}\}$, are assigned color c_2.

- Vertices at odd-numbered positions in the cycle, i.e., $\{\nu_{2i-1} \mid 1 \leq i \leq \frac{n+1}{2}\}$, along with the newly introduced vertex υ, are assigned color c_3.
- All pendant vertices of the star graph $K_{1,n}$, i.e., $\{\nu_{n+2}, \nu_{n+3}, \ldots, \nu_{2n+1}\}$, are also assigned color c_2.

This procedure guarantees a proper vertex coloring of the graph, ensuring that no two adjacent vertices receive the same color, thereby maintaining the fundamental condition required for a TPDC.

The domination relationships under this coloring scheme are as follows:

- The cycle vertices $\{\nu_2, \nu_3, \nu_4, \ldots, \nu_{n+1}\}$, along with the duplicated vertex υ, collectively power dominate the color class $c_1 = \{\nu_1\}$, as each of these vertices is adjacent to the apex vertex ν_1.
- Every pendant vertex of the star graph $K_{1,n}$, i.e., $\{\nu_i \mid n+2 \leq i \leq 2n+1\}$, is also adjacent to ν_1, and thus contributes to the power domination of the color class c_1.
- The apex vertex ν_1, being adjacent to all cycle and star vertices, power dominates both color classes c_2 and c_3, which include the cycle path vertices and the pendant vertices of the star graph.

To demonstrate the necessity of three colors for a valid TPDC of FP_n', suppose, for contradiction, that only two colors are used. Under this assumption, some adjacent vertices would inevitably share the same color, violating the condition of proper vertex coloring. Even if proper coloring is somehow maintained, the power domination requirement would fail, as at least one color class would not be monitored by any vertex of another color class. Hence, it is impossible to construct a valid TPDC of FP_n' using fewer than three colors.

Therefore, the color assignment described is both valid and minimal, as it ensures that each vertex in the graph power dominates all vertices in at least one color class different from its own, fully satisfying the definition of TPDC .

Consequently, the TPDCN for the graph FP_n', formed by duplicating an apex or cycle vertex in the Flower Pot graph FP_n, is:

$$\chi_{tpd}(FP_n') = 3.$$

Case (iii): Duplicating any arbitrary pendent vertex of Star graph in Flower Pot graph FP_n.

To streamline the discussion while maintaining generality, let us assume that the pendant vertex ν_{n+k} in the Flower Pot graph FP_n undergoes duplication. This process introduces a new vertex υ, which inherits the exact neighborhood of ν_{n+k}, i.e., it is adjacent to the apex vertex ν_1. Thus, the structural properties of the original graph FP_n are preserved.

The resulting graph, denoted by FP_n', has the vertex set:

$$\{\nu_1,\ \nu_2,\ \nu_3,\ \ldots,\ \nu_{n+1},\ \nu_{n+2},\ \ldots,\ \nu_{2n+1},\ \upsilon\},$$

and the edge set:

$$\begin{aligned} E(FP_n') = &\{\nu_1\nu_i \mid n+2 \leq i \leq 2n+1\} \cup \{\nu_1\nu_2,\ \nu_1\nu_{n+1}\} \\ &\cup \{\nu_i\nu_{i+1} \mid 2 \leq i \leq n\} \cup \{\nu_{n+1}\nu_2\} \\ &\cup \{\upsilon\nu_1\}. \end{aligned}$$

Here, the duplicated vertex υ acts as an additional pendant vertex adjacent to the apex ν_1, preserving the star-like structure attached to ν_1 while keeping the original topology intact.

To determine the TPDCN of the graph FP_n', the following coloring strategy is applied, ensuring that all conditions of a valid TPDC are satisfied.

The graph FP_n' is colored using three colors, c_1, c_2, and c_3, based on vertex positions:

- The apex vertex ν_1, along with the vertices at odd-numbered positions on the cycle C_n, i.e., $\{\nu_{2i-1} \mid 1 \leq i \leq \frac{n+1}{2}\}$, are assigned color c_1.
- The vertices at even-numbered positions on the cycle, i.e., $\{\nu_{2i} \mid 1 \leq i \leq \frac{n}{2}\}$, are assigned color c_2.
- All pendant vertices of the star $K_{1,n}$, i.e., $\{\nu_{n+2}, \nu_{n+3}, \ldots, \nu_{2n+1}\}$, together with the newly introduced duplicated vertex υ, are also assigned color c_2.

This procedure guarantees a proper vertex coloring of the graph, ensuring that no two adjacent vertices receive the same color, thereby satisfying the fundamental requirement of a TPDC.

The domination relationships under this coloring scheme are as follows:

- The vertices of the cycle C_n, i.e., $\{\nu_2, \nu_3, \ldots, \nu_{n+1}\}$, collectively power dominate the color class $c_1 = \{\nu_1\}$, as they are adjacent to the apex vertex.
- All pendant vertices of the star $K_{1,n}$, i.e., $\{\nu_{n+2}, \nu_{n+3}, \ldots, \nu_{2n+1}\}$, together with the duplicated pendant vertex υ, also power dominate the color class $c_1 = \{\nu_1\}$, since each is adjacent to ν_1.
- The apex vertex ν_1 power dominates both color classes c_2 and c_3, as it is adjacent to all cycle vertices and pendant vertices of the star.

To demonstrate the necessity of three colors for a valid TPDC of FP_n', suppose, for contradiction, that only two colors are used. Under this assumption, some adjacent vertices would inevitably share the same color, violating the condition of proper vertex coloring. Even if proper coloring is somehow maintained, the power domination requirement would fail, as at least one color class would not be monitored by any vertex of another color class. Hence, it is impossible to construct a valid TPDC of FP_n' using fewer than three colors.

Therefore, the color assignment described is both valid and minimal, as it ensures that each vertex in the graph power dominates all vertices in at least one color class different from its own, fully satisfying the definition of TPDC. The TPDCN for the graph FP'_n created through the process of duplicating any arbitrary pendent vertex in a Star graph of Flower Pot FP_n is 3..i.e.,

$$\chi_{tpd}\left(FP'_n\right) = 3.$$

In all three scenarios—whether duplicating an apex vertex, a vertex on the cycle, or a pendant vertex—a TPDC with exactly three colors can be successfully constructed. This coloring satisfies the constraints of being a proper vertex coloring, wherein adjacent vertices receive different colors, and also meets the power domination condition, wherein all vertices are eventually observed starting from a dominating set composed of colored vertices.

Since no coloring with fewer than three colors can simultaneously satisfy both the proper coloring and the power domination requirements for the modified graph, the constructed coloring is both valid and minimal. Therefore, the TPDCN of the modified double fan graph FP'_n is given by

$$\chi_{\text{tpd}}(FP'_n) = 3.$$

■

Example 11. *In figure 11, the TPDC for the graph FP'_5 created through the process of duplicating a pendent vertex ν_{11} by υ in Flower Pot graph FP_5 is shown*

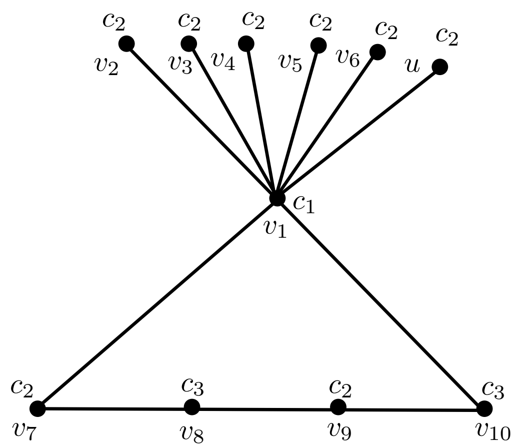

Fig. 11. Modified flower Pot graph FP'_5 created through the process of duplicating a pendent vertex ν_{11}, the color classes of the FP'_4 are $c_1 = \{\nu_1\}$, $c_2 = \{\upsilon, \nu_2, \nu_4, \nu_6, \nu_7, \nu_9, \nu_{11}\}$ and $c_3 = \{\nu_3, \nu_5, \nu_8, \nu_{10}$ Then $\chi_{tpd}(FP'_5)$=3.

Theorem 12. *For any $n \geq 3$, the TPDCN for ν'_n, created through the process of duplicating any arbitrary vertex of Vanessa graph, ν_n is 3.*

Proof: Let V_n be the Vanessa graph with vertex ν_0 as the apex vertex. Let $\{\nu_1, \nu_2, \nu_3, \ldots, \nu_n\}$ be the vertices of the first fan graph F_n, and let $\{\nu'_1, \nu'_2, \nu'_3, \ldots, \nu'_n\}$ be the vertices of the second fan graph F_n. Further, let $\{\nu_{n+1}, \nu_{n+2}, \nu_{n+3}, \ldots, \nu_{2n}\}$ be the pendant vertices of a star graph $K_{1,n}$, all of which are adjacent to the apex vertex ν_0. Let $E(\nu_n)$ denote the edge set of the Vanessa graph, where

$$\begin{aligned} E(V_n) = \{&\nu_0\nu_i \mid 1 \leq i \leq 2n\} \\ &\cup \{\nu_i\nu_{i+1} \mid 1 \leq i \leq n-1\} \\ &\cup \{\nu_0\nu'_i \mid 1 \leq i \leq n\} \\ &\cup \{\nu'_i\nu'_{i+1} \mid 1 \leq i \leq n-1\}. \end{aligned}$$

Here, $|V(\nu_n)| = 3n + 1$, where n is any positive integer.

The graph thus consists of a central apex vertex ν_0, two fan graphs F_n formed by paths of length $n-1$ each connected to ν_0, and a star graph $K_{1,n}$ formed by n pendant vertices also adjacent to ν_0.

Case (i): Duplicating an apex vertex.

Assume that the apex vertex ν_1 undergoes duplication. This process yields a new vertex υ, which inherits the exact neighborhood of ν_1, thus maintaining the structural properties of the original graph V_n. As a result, the modified graph V'_n is generated, having the vertex set,

$$\begin{aligned} E(V_n) = \{&\upsilon,\ \nu_0,\ \nu_1, \nu_2, \nu_3, \nu_4, \ldots, \nu_n,\ \nu_{n+1}, \nu_{n+2}, \nu_{n+3}, \\ &\ldots, \nu_{2n},\ \nu'_1, \nu'_2, \nu'_3, \ldots, \nu'_n\}. \end{aligned}$$

The edge set of the modified graph V'_n is given by:

$$\begin{aligned} E(V'_n) = \{&\nu_0\nu_i \mid 1 \leq i \leq 2n\} \\ &\cup \{\nu_i\nu_{i+1} \mid 1 \leq i \leq n-1\} \\ &\cup \{\nu_0\nu'_i \mid 1 \leq i \leq n\} \\ &\cup \{\nu'_i\nu'_{i+1} \mid 1 \leq i \leq n-1\} \\ &\cup \{\upsilon\nu_i \mid 1 \leq i \leq 2n\} \\ &\cup \{\upsilon\nu'_i \mid 1 \leq i \leq n\}. \end{aligned}$$

The procedure provided below, based on a systematic approach to coloring the graph's vertices, guarantees a resulting coloring that satisfies both the proper coloring condition and the TPDC requirements. Moreover, this coloring ensures that every color class is power dominated by at least one vertex from a different color class, thus achieving a valid TPDC configuration.

The graph V'_n is colored using three distinct colors, c_1, c_2, and c_3, according to the positional classification of its vertices.

- The apex vertex ν_1 and its duplicate vertex υ are assigned color c_1.
- Vertices at even-numbered positions in the first fan graph, i.e., $\{\nu_{2i} \mid 1 \leq i \leq \frac{n}{2}\}$, are assigned color c_2.
- Vertices at even-numbered positions in the second fan graph, i.e., $\{\nu'_{2i} \mid 1 \leq i \leq \frac{n}{2}\}$, are also assigned color c_2.
- Vertices at odd-numbered positions in the first fan graph, i.e., $\{\nu_{2i-1} \mid 1 \leq i \leq \frac{n+1}{2}\}$, are assigned color c_3.
- Vertices at odd-numbered positions in the second fan graph, i.e., $\{\nu'_{2i-1} \mid 1 \leq i \leq \frac{n+1}{2}\}$, are also assigned color c_3.

- All pendant vertices of the star graph $K_{1,n}$, namely $\{\nu_{n+1}, \nu_{n+2}, \ldots, \nu_{2n}\}$, are assigned color c_2.

This procedure guarantees a proper vertex coloring of the graph, ensuring that no two adjacent vertices receive the same color. Hence, it satisfies the fundamental requirement of proper coloring needed for TPDC. Additionally, the assignment of colors ensures that each color class is power dominated by a vertex of a different color class, thereby fulfilling the TPDC condition.

The domination relationships under this coloring scheme are as follows:

- The path vertices from the first and second fan graphs, i.e., $\{\nu_1, \nu_2, \ldots, \nu_n, \nu'_1, \nu'_2, \ldots, \nu'_n\}$, collectively power dominate the color class $c_1 = \{\nu_1, \upsilon\}$, since they are all adjacent to the apex vertex ν_1 and its duplicate υ.
- The pendant vertices of the star graph $K_{1,n}$, namely $\{\nu_{n+1}, \nu_{n+2}, \ldots, \nu_{2n}\}$, are adjacent to both ν_0 and υ, and hence power dominate the color class $c_1 = \{\nu_0, \upsilon\}$.
- The apex vertex ν_0 is adjacent to all path vertices and pendant vertices. Therefore, it power dominates the color classes c_2 and c_3, which include:
  - Even-indexed vertices from both fan paths and all pendant vertices (assigned c_2), and
  - Odd-indexed vertices from both fan paths (assigned c_3).

To show that three colors are necessary, suppose that only two colors are used for the TPDC of V'_n. In such a case, it becomes inevitable that some adjacent vertices must share the same color, thereby violating the condition of proper vertex coloring. Alternatively, even if a proper coloring is somehow preserved with two colors, the power domination condition would be violated, as at least one color class would not be power dominated by any vertex from another color class. Therefore, it is not possible to construct a valid TPDC of V'_n using fewer than three colors.

Hence, the assignment of three colors is both valid and minimal. The color assignment ensures that each vertex in the graph V'_n power dominates all vertices in at least one color class distinct from its own, in accordance with the definition of TPDC. Consequently, every vertex in the graph V'_n either directly or indirectly power dominates a color class other than its own.Therefore, the TPDCN for the graph V'_n, obtained by duplicating an apex or path vertex in the Vanessa graph V_n, is:

$$\chi_{tpd}(V'_n) = 3.$$

Case (ii): Duplicating any arbitrary vertex of any Fan graph in Vanessa graph $\mathcal{V}_n$.

To streamline the discussion while maintaining generality, proceed under the assumption that the arbitrary vertex ν_3 undergoes duplication. This process yields a new vertex υ, which inherits the exact neighborhood of ν_3, thereby maintaining the structural properties of the original graph V_n. As a result, the modified graph V'_n is generated, with the vertex set:

$$\begin{Bmatrix} \upsilon,\ \nu_0,\ \nu_1,\ \nu_2,\ \ldots,\ \nu_n,\ \nu_{n+1},\ \nu_{n+2}, \\ \nu_{n+3},\ \ldots,\ \nu_{2n},\ \nu'_1,\ \nu'_2,\ \ldots,\ \nu'_n \end{Bmatrix}$$

The edge set of the modified graph V'_n is defined as:

$$\begin{aligned} E(V'_n) = &\{\nu_0\nu_i \mid 1 \le i \le 2n\} \\ &\cup \{\nu_i\nu_{i+1} \mid 1 \le i \le n-1\} \\ &\cup \{\nu_0\nu'_i \mid 1 \le i \le n\} \\ &\cup \{\nu'_i\nu'_{i+1} \mid 1 \le i \le n-1\} \\ &\cup \{\upsilon\nu_2,\ \upsilon\nu_4,\ \upsilon\nu_0\}. \end{aligned}$$

The procedure provided below, based on a systematic approach to coloring the graph's vertices, guarantees a resulting coloring that satisfies both the proper coloring condition and the TPDC requirements. Moreover, this procedure ensures that every color class is power dominated by at least one vertex from a different color class, thereby achieving a valid TPDC configuration.

The procedure outlined below is based on a systematic approach to coloring the vertices of the graph and guarantees a coloring that satisfies both the proper coloring and the TPDC requirements. Specifically, this method ensures that every color class is power dominated by at least one vertex from a distinct color class, thereby producing a valid TPDC configuration.

The graph V'_n is colored using three colors, c_1, c_2, and c_3, according to the positions of the vertices:

- The apex vertex ν_0 is assigned color c_1.
- Vertices at even-numbered positions in the two fan graphs, i.e., $\{\nu_{2i} \mid 1 \le i \le \frac{n}{2}\} \cup \{\nu'_{2i} \mid 1 \le i \le \frac{n}{2}\}$, are assigned color c_2.
- Vertices at odd-numbered positions in the fan graphs, i.e., $\{\nu_{2i-1} \mid 1 \le i \le \frac{n+1}{2}\} \cup \{\nu'_{2i-1} \mid 1 \le i \le \frac{n+1}{2}\}$, along with the newly introduced vertex υ, are assigned color c_3.
- All pendant vertices of the star graph $K_{1,n}$, i.e., $\{\nu_{n+2}, \nu_{n+3}, \ldots, \nu_{2n+1}\}$, are also assigned color c_2.

This procedure guarantees a proper vertex coloring of the graph, ensuring that no two adjacent vertices receive the same color. Hence, it satisfies the fundamental requirement of proper coloring needed for TPDC. Additionally, the assignment of colors ensures that each color class is power dominated by a vertex from a different color class, thereby fulfilling the TPDC condition.

The domination relationships under this coloring scheme are as follows:

- The path vertices from both fan graphs, i.e., $\{\nu_1, \nu_2, \ldots, \nu_n,\ \nu'_1, \nu'_2, \ldots, \nu'_n\}$, along with the duplicate vertex υ, power dominate the color class $c_1 = \{\nu_0\}$, as they are all adjacent to the apex vertex.

- All pendant vertices of the star graph, $\{\nu_{n+2}, \nu_{n+3}, \ldots, \nu_{2n+1}\}$, are also adjacent to ν_0, and hence contribute to dominating the color class c_1.
- The apex vertex ν_0, being adjacent to every fan path vertex and every pendant vertex, power dominates the color classes c_2 and c_3.

To show that three colors are necessary, suppose that only two colors are used for the TPDCof V_n'. In such a case, some adjacent vertices must necessarily share the same color, thereby violating the condition of proper vertex coloring. Alternatively, even if proper coloring is somehow maintained, the power domination condition will be violated, as at least one color class will not be power dominated by any vertex from another color class. Therefore, it is not possible to construct a valid TPDCof V_n' using fewer than three colors.

Therefore, the color assignment is both valid and minimal, as it ensures that each vertex in the graph power dominates all vertices in at least one color class different from its own, in accordance with the definition of TPDC. Hence, every vertex in the graph V_n' power-dominates all vertices in at least one distinct color class. The TPDCN for the graph V_n', created through the process of duplicating an arbitrary vertex in the fan graph of the Vanessa graph V_n, is 3. That is,

$$\chi_{tpd}(V_n') = 3.$$

Case (iii): Duplicating any arbitrary pendant vertex of the Star graph in the Vanessa graph ν_n.

To streamline the discussion while maintaining generality, proceed under the assumption that a pendant vertex of the star graph $K_{1,n}$, say ν_{n+2}, undergoes duplication. This process yields a new vertex υ, which inherits the exact neighborhood of ν_{n+2}, that is, it is adjacent to the apex vertex ν_0. The structural properties of the original graph ν_n are thus preserved, and the resulting modified graph is denoted by ν_n'. The vertex set of ν_n' becomes:

$$\begin{Bmatrix} \upsilon,\ \nu_0,\ \nu_1,\ \ldots,\ \nu_n,\ \nu_1',\ \nu_2',\ \ldots,\ \nu_n', \\ \nu_{n+1},\ \nu_{n+2},\ \ldots,\ \nu_{2n+1} \end{Bmatrix}$$

The edge set of the graph ν_n' is defined as:

$$\begin{aligned} E(\nu_n') = &\{\nu_0\nu_i \mid 1 \le i \le 2n+1\} \\ &\cup \{\nu_i\nu_{i+1} \mid 1 \le i \le n-1\} \\ &\cup \{\nu_0\nu_i' \mid 1 \le i \le n\} \\ &\cup \{\nu_i'\nu_{i+1}' \mid 1 \le i \le n-1\} \cup \{\upsilon\nu_0\}. \end{aligned}$$

The procedure outlined below is based on a systematic approach to coloring the vertices of the graph and guarantees a coloring that satisfies both proper coloring and TPDC requirements. Specifically, this method ensures that every color class is power dominated by at least one vertex from a distinct color class, thereby producing a valid TPDC configuration.

The graph ν_n' is colored using three colors, c_1, c_2, and c_3, based on the vertex positions:

- The apex vertex ν_0 is assigned color c_1.
- Vertices at even-numbered positions in the fan graphs, i.e., $\{\nu_{2i} \mid 1 \le i \le \frac{n}{2}\} \cup \{\nu_{2i}' \mid 1 \le i \le \frac{n}{2}\}$, are assigned color c_2.
- Vertices at odd-numbered positions in the fan graphs, i.e., $\{\nu_{2i-1} \mid 1 \le i \le \frac{n+1}{2}\} \cup \{\nu_{2i-1}' \mid 1 \le i \le \frac{n+1}{2}\}$, are assigned color c_3.
- All pendant vertices of the star graph, i.e., $\{\nu_{n+2}, \nu_{n+3}, \ldots, \nu_{2n+1}, \quad \upsilon\}$, are assigned color c_2.

This procedure guarantees a proper vertex coloring of the graph, ensuring that no two adjacent vertices receive the same color. Hence, it satisfies the fundamental requirement of proper coloring needed for TPDC. Additionally, the assignment of colors ensures that each color class is power dominated by a vertex of a different color class, thereby fulfilling the TPDC condition.

The domination relationships under this coloring scheme are as follows:

- The path vertices from both fan graphs, $\{\nu_1, \nu_2, \ldots, \nu_n, \quad \nu_1', \nu_2', \ldots, \nu_n'\}$, are adjacent to the apex vertex ν_0, and hence collectively power dominate the color class $c_1 = \{\nu_0\}$.
- The pendant vertices of the star graph, including the duplicate vertex υ, i.e., $\{\nu_{n+2}, \ldots, \nu_{2n+1}, \upsilon\}$, are also adjacent to ν_0, and thus additionally power dominate the color class $c_1 = \{\nu_0\}$.
- The apex vertex ν_0, being adjacent to all fan vertices and all pendant vertices, power dominates the color classes c_2 and c_3.

To show that three colors are necessary, suppose that only two colors are used for the TPDCof ν_n'. In such a case, some adjacent vertices must necessarily share the same color, thereby violating the condition of proper vertex coloring. Alternatively, even if a proper coloring is somehow maintained, the power domination condition will be violated, as at least one color class will not be power dominated by any vertex from another color class. Therefore, it is not possible to construct a valid TPDC of ν_n' using fewer than three colors.

Therefore, the color assignment is both valid and minimal, as it ensures that each vertex in the graph power dominates all vertices in at least one color class different from its own, in accordance with the definition of TPDC. Hence, every vertex in the graph ν_n' power dominates all vertices in at least one distinct color class. The TPDCN for the graph ν_n', created through the process of duplicating a pendant vertex of the star in the Vanessa graph ν_n, is:

$$\chi_{tpd}(\nu_n') = 3.$$

In all three scenarios considered in the modified Vanessa graph ν_n'—namely, duplicating an apex vertex, duplicating any arbitrary path vertex from the fan components, and duplicating a pendant vertex of the star graph $K_{1,n}$—a

TPDC using exactly three colors can be successfully constructed.

In each case, the coloring satisfies both the proper vertex coloring condition, ensuring that adjacent vertices receive distinct colors, and the power domination requirement, ensuring that each color class is power dominated by at least one vertex from a different color class. Therefore, the coloring is both valid and minimal in all three structural variations. Consequently, the TPDCN of the modified Vanessa graph ν_n' is

$$\chi_{tpd}(\nu_n') = 3.$$

■

Example 12. *In figure 12, the TPDC for the graph* $\nu_3^{'}$ *through the process of duplicating a vertex* $\nu_2^{'}$ *by* υ *in Vanessa graph* ν_3 *is shown.*

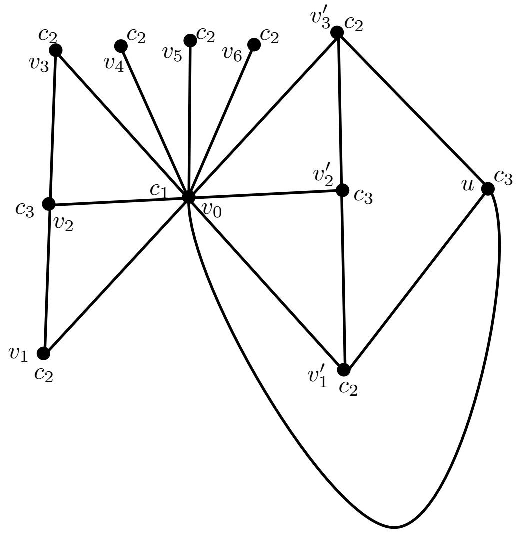

Fig. 12. The modified Vanessa graph $\nu_3^{'}$ created through the process of duplicating a pendent vertex $\nu_2^{'}$, the color classes of the $\nu_3^{'}$ are $c_1 = \{\nu_0\}$, $c_2 = \{\nu_1, \nu_3, \nu_4, \nu_5, \nu_6, \nu_1', \nu_3'\}$ and $c_3 = \{\nu_2, \nu_2', \upsilon\}$ Then $\chi_{tpd}(\nu_3^{'})$=3.

Remark 4. *In the case of graphs which were discussed in Theorems 7, 8, 9 and 10, for* $n = 3$ *is a complete graph, for which we gave proofs systematically in Theorem 4.*

When $n = 1$ *and* $n = 2$ *for the theorems stated in 7, 8, 9 and 10, we will have path graphs, for which we have proved the in Theorem 1 and 2.*

IV. Table of Results

The folowing table compiles the results of this article and compares them χ_{tpd} with their chromatic number χ.
Table 1 summarizes the chromatic number χ, the power domination chromatic number χ_{pd}, and the TPDCN χ_{tpd} for eight different graph families obtained through the process of vertex duplication. These values reflect the coloring characteristics under standard proper coloring, power domination constraints, and power domination constraints, respectively.

For most of the graphs examined—namely K_n', $K_{m,n}'$, F_n', DF_n', O_n', FP_n', and V_n'—it is observed that the chromatic number χ and the TPDCN χ_{tpd} are equal. This implies that in these cases, the minimum number of colors required to achieve a proper coloring is also sufficient to satisfy the more stringent conditions imposed by TPDC .

TABLE I
Chromatic Numbers, PDN's, and TPDCN's for Duplicated Graphs

S.No	Graph	χ	χ_{pd}	χ_{tpd}
1	$C_n^{'}$	$\chi = 2$ for odd n $\chi = 3$ for even n	$\chi_{pd} = 2$ for odd n $\chi_{pd} = 3$ for even n	$\chi_{tpd} = 3$
2	$K_n^{'}$	$\chi = n$	$\chi_{pd} = n$	$\chi_{tpd} = n$
3	$K_{m,n}^{'}$	$\chi = 2$	$\chi_{pd} = 2$	$\chi_{tpd} = 2$
4	$F_n^{'}$	$\chi = 3$	$\chi_{pd} = 3$	$\chi_{tpd} = 3$
5	$DF_n^{'}$	$\chi = 3$	$\chi_{pd} = 3$	$\chi_{tpd} = 3$
6	$O_n^{'}$	$\chi = 3$	$\chi_{pd} = 3$	$\chi_{tpd} = 3$
7	$FP_n^{'}$	$\chi = 3$	$\chi_{pd} = 3$	$\chi_{tpd} = 3$
8	$V_n^{'}$	$\chi = 3$	$\chi_{pd} = 3$	$\chi_{tpd} = 3$

However, an exception occurs in the case of the duplicated cycle graph C_n'. Here, the chromatic number χ depends on the parity of n; that is, $\chi = 2$ when n is odd and $\chi = 3$ when n is even. Regardless of this variation, the TPDCN for C_n' remains fixed at $\chi_{tpd} = 3$. This indicates that while the standard coloring number can vary with structural properties like parity, the TPDC condition introduces a stricter constraint that necessitates the use of at least three colors for all $n \geq 3$.

V. Conclusion

In this study, the TPDC concept, which is a blend of power domination and graph coloring, was explored for various classes of graphs. We determined the total power dominator chromatic number for various types of graphs, such as paths, cycles, complete graphs, bipartite graphs, double fan graphs, octopus graphs, and Venessa graphs. The effect of vertex duplication was particularly highlighted, which had different effects on the chromatic number depending on the particular structure of the graph.

The results of this research confirm the complexity of this parameter and its structural sensitivity. This research sheds light on domination-coloring relations and lays the ground for research on more complex graph families. Snake graph-related graphs could be the subject of future research, as they present intriguing structural configurations worthy of research in this field.

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