Dynamic Behavior of a Fourth-Order Fuzzy Difference Equation

Changyou Wang, Chunyan Ma, Lili Jia

Abstract—**This study investigates the dynamical properties of a fourth-order fuzzy difference equation system, focusing on the stability of equilibria and solution boundedness. By integrating fuzzy set theory with advanced inequality techniques, we establish existence criteria for both the trivial equilibrium (0, 0) and positive constant equilibria. Utilizing linearization theory and eigenvalue analysis, we derive necessary and sufficient conditions for the global asymptotic stability of (0, 0), alongside sufficient conditions for the instability of positive equilibria. Furthermore, refined inequality analyses yield boundedness criteria for positive fuzzy solutions. Numerical simulations in MATLAB validate theoretical findings, demonstrating solution convergence and boundedness in fuzzy environments. The core innovation lies in a novel analytical framework that synergizes fuzzy set theory with classical difference equation methods. This approach pioneers a systematic methodology for studying higher-order fuzzy dynamical systems, advancing the modeling of uncertain processes in biology, ecology, and engineering disciplines.**

Index Terms—**fuzzy difference equation, equilibrium point, boundedness, stability, dynamic analysis**

I. Introduction

Difference equations, the discrete counterparts to differential equations, are fundamental tools for modeling dynamic systems across disciplines such as biology, ecology, engineering, and computer networks (see [1-8]). Their computational efficiency and adaptability to discrete-time systems make them indispensable for simulating complex phenomena, particularly in scenarios requiring real-time data processing or large-scale numerical analysis. However, real-world systems often operate under inherent uncertainties due to incomplete data, measurement errors, or ambiguous parameter definitions, which classical difference equations fail to capture. To address this limitation, fuzzy set theory [9] has been integrated into difference equation, resulting in the development of fuzzy difference equation (FDE). These models leverage fuzzy numbers and membership functions to represent uncertain parameters, enabling a more realistic characterization of dynamic systems with incomplete or imprecise information (see [10-14]).

Previous research has predominantly focused on low-order FDE. For example, Rahman et al. [15] established foundational results on the existence and boundedness of solutions for second-order FDE, while Yalçınkaya et al. [16] extended these insights to third-order system. Applications of FDE have also been demonstrated in medical modeling [9], financial systems [10], and ecological studies [11], underscoring their versatility. Despite these advancements, higher-order FDE, especially those of order 4 and above, remain under explored. Such systems are critical for modeling processes with multi-step memory dependencies, such as population dynamics with delayed regulatory mechanisms or resource allocation systems with cumulative feedback effects (see [3, 17]). Existing studies on stability and boundedness (see [18-19]) primarily address lower-order cases, leaving a significant gap in understanding how these properties scale with increased system complexity and nonlinear interactions.

This paper addresses this gap by rigorously analyzing a fourth-order fuzzy difference equation of the following form:

$$z_{n+1} = \frac{a z_{n-3}}{b + c z_n z_{n-1} z_{n-2} z_{n-3}}, n = 0, 1, 2, \cdots, \quad (1)$$

where the parameters a, b, c and the initial conditions are positive fuzzy numbers. This paper is structured as follows: Section 2 introduces fundamental concepts of fuzzy numbers and difference equations. Section 3 establishes rigorous proofs for stability and boundedness properties. Section 4 provides numerical validations of theoretical results, and Section 5 concludes with key contributions and future research directions.

Remark 1. This study's innovative contributions are threefold: (1) Higher-Order Dynamics: Unlike prior studies confined to second- and third-order systems (see [15–16]), this work tackles the distinct challenges of fourth-order FDE. These include the interplay of multiple delayed terms and nonlinear denominator structures, which complicate stability analysis. (2) Hybrid Methodology: A unified analytical framework is developed, integrating fuzzy set theory, linearization techniques, eigenvalue-based stability criteria, and inequality-driven induction. This approach surmounts limitations of conventional stability analysis, which often neglects fuzzy parameter interactions (see [17–18]). (3) Generalized Theoretical Criteria: The derived conditions for global asymptotic stability and boundedness extend existing

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theories for lower-order FDE (see [11, 15]), providing a scalable framework applicable to diverse fuzzy dynamical systems. In summary, this work bridges the gap between low-order theoretical foundations and high-order practical applications, significantly advancing the analysis of complex, uncertainty-driven difference dynamical systems.

II. Preliminaries and Notations

This section presents fundamental concepts of fuzzy mathematics and difference equations, which serve as the theoretical foundation for the proofs in subsequent sections.

Definition 1. Let X be a non-empty set, assume T is a mapping from X to [0,1], i.e.

$$T: X \to [0,1], x \to T(x), x \in X,$$

then we say T is a fuzzy set on X , $T(x)$ be called a membership function on a fuzzy set T .

Definition 2. Assume T is a fuzzy set and $\alpha \in (0,1]$, the $\alpha-cuts$ of T on R is defined as $[T]_\alpha = \{x \in R \mid T(x) \geq \alpha\}$ and $[T]_0 = \mathrm{supp}(T)$. It is clear that the $[T]_\alpha$ is a bounded closed interval in R for $\alpha \in (0,1]$.

Definition 3. We say that a mapping $T: R \to [0,1]$ is a fuzzy number if it satisfies the following properties (i)-(iv):

(i) T is a normal fuzzy set, i.e., there exists $x \in R$ such that $T(x) = 1$;

(ii) T is a fuzzy convex set, i.e.

$$T(ax+(1-a)y) \geq \min\{T(x),T(y)\}, \forall a \in (0,1), x, y \in R;$$

(iii) T is upper semicontinuous on R ;

(iv) T is compactly supported, i.e.

$$\mathrm{supp}(T) = \overline{\bigcup_{\alpha\in(0,1]}[T]_\alpha} = \overline{\{x \in R \mid T(x) > 0\}}$$

is compact.

Let us denote by R_f the set of all fuzzy numbers. A fuzzy number v is positive if $\mathrm{supp}(v) \subset (0,+\infty)$, we denote by R_f^+ the space of all positive fuzzy numbers. Similarly, a fuzzy number v is negative if $\mathrm{supp}(v) \subset (-\infty,0)$, we denote by R_f^- the space of all negative fuzzy numbers. If T is a positive real number, then T is also a positive fuzzy number with $[T]_\alpha = [T,T], \alpha \in (0,1]$, and we say that T is a trivial fuzzy number.

Definition 4. For any $u, v \in R_f$, $[u]_\alpha = [u_{l,\alpha}, u_{r,\alpha}]$, $[v]_\alpha = [v_{l,\alpha}, v_{r,\alpha}]$, the sum $u+v$, the scalar product λu , the multiplication uv and division $\dfrac{u}{v}$ in the standard interval arithmetic (SIA) setting are defined by

$$[u+v]_\alpha = [u]_\alpha + [v]_\alpha, [\lambda u]_\alpha = \lambda [u]_\alpha, \forall \alpha \in [0,1], \lambda \in R,$$

$$[uv]_\alpha = [\min\{u_{l,\alpha}v_{l,\alpha}, u_{l,\alpha}v_{r,\alpha}, u_{r,\alpha}v_{l,\alpha}, u_{r,\alpha}v_{r,\alpha}\}, \max\{u_{l,\alpha}v_{l,\alpha}, u_{l,\alpha}v_{r,\alpha}, u_{r,\alpha}v_{l,\alpha}, u_{r,\alpha}v_{r,\alpha}\}],$$

$$\left[\frac{u}{v}\right]_\alpha = \left[\min\left\{\frac{u_{l,\alpha}}{v_{l,\alpha}}, \frac{u_{l,\alpha}}{v_{r,\alpha}}, \frac{u_{r,\alpha}}{v_{l,\alpha}}, \frac{u_{r,\alpha}}{v_{r,\alpha}}\right\}, \max\left\{\frac{u_{l,\alpha}}{v_{l,\alpha}}, \frac{u_{l,\alpha}}{v_{r,\alpha}}, \frac{u_{r,\alpha}}{v_{l,\alpha}}, \frac{u_{r,\alpha}}{v_{r,\alpha}}\right\}\right], 0 \notin [v]_\alpha.$$

Definition 5. Let u, v be fuzzy numbers with $[u]_\alpha = [u_{l,\alpha}, u_{r,\alpha}]$, $[v]_\alpha = [v_{l,\alpha}, v_{r,\alpha}]$, $\alpha \in (0,1]$, then the metric of fuzzy numbers set is defined as follows

$$D(u,v) = \sup_{\alpha\in[0,1]} \max\{|u_{l,\alpha} - v_{l,\alpha}|, |u_{r,\alpha} - v_{r,\alpha}|\},$$

and the norm on fuzzy numbers set is defined as follows

$$\|u\| = \sup_{\alpha\in[0,1]} \max\{|u_{l,\alpha}|, |u_{r,\alpha}|\},$$

then (R_f, D) is a complete metric space.

We define $\hat{0} \in R_f$ as

$$\hat{0} = \begin{cases} 1, x = 0, \\ 0, x \neq 0, \end{cases}$$

thus, $[\hat{0}]_\alpha = [0,0], 0 < \alpha \leq 1$.

Definition 6. The persistence and boundedness of a positive fuzzy number sequence are defined as follows:

(i) A sequence of positive fuzzy numbers $\{x_n\}$ is persistent (resp. is bounded) if there exists a positive real numbers $M\,(resp. N)$ such that $\mathrm{supp}\, x_n \in [M,\infty)\,(resp. \mathrm{supp}\, x_n \in [0,N)),\ n = 1,2,\cdots;$

(ii) A sequence of positive fuzzy numbers $\{x_n\}$ is bounded and persistent if there exist some positive real numbers $M, N > 0$ such that $\mathrm{supp}\, x_n \in [M,N), n = 1,2,\cdots.$

For the following system of difference equations

$$\begin{cases} x_{n+1} = f(x_n, x_{n-1}, \cdots, x_{n-k}, y_n, y_{n-1}, \cdots, y_{n-l}), \\ y_{n+1} = g(x_n, x_{n-1}, \cdots, x_{n-k}, y_n, y_{n-1}, \cdots, y_{n-l}), \end{cases} \tag{2}$$

where I_x, I_y are the interval of real numbers, $f: I_x^{k+1} \times I_y^{l+1} \to I_x$, $g: I_x^{k+1} \times I_y^{l+1} \to I_y$ are continuous function.

Definition 7. If there exists a point $(\bar{x}, \bar{y})$ that satisfies the following equation

$$\bar{x} = f(\bar{x}, \bar{x}, \cdots, \bar{x}, \bar{y}, \bar{y}, \cdots, \bar{y}),$$

$$\bar{y} = g(\bar{x}, \bar{x}, \cdots, \bar{x}, \bar{y}, \bar{y}, \cdots, \bar{y}),$$

then this point is called an equilibrium point of the equations (2). This means that the equilibrium solution of equations (2) is the solution that satisfies $x_n = \bar{x}, y_n = \bar{y}$.

Definition 8. Assume that $(\bar{x}, \bar{y})$ be an equilibrium point of the equations (2). Then, we have

(i) An equilibrium point $(\bar{x}, \bar{y})$ is called locally stable if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any initial conditions $(x_i, y_j) \in I_x \times I_y (i = -k, \cdots, 0, \quad j = -l, \cdots,$

$0)$, with $\sum_{i=-k}^{0}|x_i-\bar{x}|<\delta$, $\sum_{j=-l}^{0}|y_j-\bar{y}|<\delta$, we have $|x_n-\bar{x}|<\varepsilon,\ |y_n-\bar{y}|<\varepsilon$ for any $n>0$.

(ii) An equilibrium point $(\bar{x},\bar{y})$ is called globally attractor if $\lim_{n\to\infty}x_n=\bar{x}$, $\lim_{n\to\infty}y_n=\bar{y}$ for any initial conditions $(x_i,y_j)\in I_x\times I_y(i=-k,\cdots,0,j=-l,\cdots,0)$.

(iii) An equilibrium point $(\bar{x},\bar{y})$ is called globally asymptotically stable if it is locally stable and globally attractor.

(iv) An equilibrium point $(\bar{x},\bar{y})$ is called unstable if it is not locally stable.

Definition 9. Assume that $(\bar{x},\bar{y})$ be an equilibrium point of the equations (2), and $f,g\in C^1(R^{k+1}\times R^{k+1},R)$. Then, the linearized equation of equations (2) around the equilibrium point $(\bar{x},\bar{y})$ is

$$X_{n+1}=F(X_n)=F_j\times X_n, \tag{3}$$

where F_j is the Jacobian matrix of the equations (2) about $(\bar{x},\bar{y})$ and the characteristic equation of equation (3) is $\lambda^{k+1}=a_0\lambda^k+a_1\lambda^{k-1}+\cdots+a_{k-1}\lambda+a_k$.

Lemma 1. Assume that $X(n+1)=F(X(n)),n=0,1,\cdots,$ is a difference equation and $\bar{X}$ is the equilibrium point of this equation. Then, we have

(i) If all eigenvalues of the Jacobian matrix J_F about $\bar{X}$ lie inside the open unit disk, i.e., $|\lambda|<1$, then $\bar{X}$ is locally asymptotically stable.

(ii) If one of eigenvalues of the Jacobian matrix J_F about $\bar{X}$ has norm greater than one, then $\bar{X}$ is unstable.

Lemma 2. Assume that $X(n+1)=F(X(n)),n=0,1,\cdots$, is a difference equation and $\bar{X}$ is the equilibrium point of this equation. The characteristic equation of the equation with respect to $\bar{X}$ is $P(\lambda)=a_0\lambda^n+a_1\lambda^{n-1}+\cdots+a_{n-1}\lambda$ $+a_n=0,a_0>0$, and the necessary and sufficient condition for all characteristic roots $|\lambda|<1$ is that all $\Delta_k>0,k=1,$ $2,\cdots,n$, where Δ_k is the $k-\text{th}$ principal minor of an $n-\text{th}$ order matrix,

$$\Delta_n=\begin{bmatrix} a_1 & a_3 & a_5 & \cdots & 0\\ a_0 & a_2 & a_4 & \cdots & 0\\ 0 & a_1 & a_3 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & a_n \end{bmatrix}.$$

Lemma 3. Let $f:R^+\times R^+\times R^+\to R^+$ be a continuous function and A,B,C are positive fuzzy numbers, then

$$[f(A,B,C)]_\alpha=f([A]_\alpha,[B]_\alpha,[C]_\alpha),\alpha\in(0,1].$$

Lemma 4. Let $u\in R_f$, write $[u]_\alpha=[u_{l,\alpha},u_{r,\alpha}],\alpha\in(0,$ $1]$, then $u_{l,\alpha},u_{r,\alpha}$ are function on $[0,1]$, which satisfy the following conditions:

(i) $u_{l,\alpha}$ is nondecreasing and left continuous;

(ii) $u_{r,\alpha}$ is nonincreasing and left continuous;

(iii) $u_{l,\alpha}\le u_{r,\alpha}$.

Lemma 5. For the system of equations (2), if the initial value $(x_i,y_j)\in I_x\times I_y(i=-k,\cdots,0,j=-l,\cdots,0)$ is given, then there exists a sequence of real numbers $\{(x_i,y_j)\}_{i=-k,j=-l}^{+\infty,+\infty}$ such that equation (2) holds.

III. MAIN RESULTS AND PROOFS

This section investigates the dynamical properties of the fourth-order fuzzy difference equation (1) by employing fuzzy set theory, linearization techniques, mathematical induction, inequality analysis, and the aforementioned lemmas.

Assume $\{z_n\}$ is the solution of equation (1) with initial conditions z_{-3},z_{-2},z_{-1},z_0 and satisfies

$$[z_n]_\alpha=[L_{n,\alpha},R_{n,\alpha}],\alpha\in(0,1],n=0,1,\cdots,$$

then $(L_{n,\alpha},R_{n,\alpha})$ satisfies the following equations

$$\begin{aligned} L_{n+1,\alpha}&=\frac{a_{l,\alpha}L_{n-3,\alpha}}{b_{r,\alpha}+c_{r,\alpha}R_{n,\alpha}R_{n-1,\alpha}R_{n-2,\alpha}R_{n-3,\alpha}},\\ R_{n+1,\alpha}&=\frac{a_{r,\alpha}R_{n-3,\alpha}}{b_{l,\alpha}+c_{l,\alpha}L_{n,\alpha}L_{n-1,\alpha}L_{n-2,\alpha}L_{n-3,\alpha}}. \end{aligned} \tag{4}$$

To study the asymptotic behavior of the solution for fuzzy difference equation (1), the above expression is simplified into the following rational difference equations system:

$$\begin{aligned} x_{n+1}&=\frac{Ax_{n-3}}{B+Cy_ny_{n-1}y_{n-2}y_{n-3}},\\ y_{n+1}&=\frac{Dy_{n-3}}{E+Fx_nx_{n-1}x_{n-2}x_{n-3}},n=0,1,\cdots, \end{aligned} \tag{5}$$

where the parameters A,B,C,D,E,F and the initial conditions $x_{-3},x_{-2},x_{-1},x_0,y_{-3},y_{-2},y_{-1},y_0$ are positive real numbers. In the meantime, it holds that $x_n\le y_n$, $A\le D,E\le B,F\le C$.

According to Lemma 5, it can be known that for any given initial values, the equations (5) have a unique solution (x_n,y_n), and it is easy to know that the equations (5) has an equilibrium point $\bar{Z}_1=(\bar{x}_1,\bar{y}_1)=(0,0)$. When $A>B$ and $D>E$, the equations (5) has second equilibrium point

$$\bar{Z}_2=(\bar{x}_2,\bar{y}_2)=\left(\sqrt[4]{\frac{D-E}{F}},\sqrt[4]{\frac{A-B}{C}}\right).$$

Theorem 1. For the equilibrium point $\bar{Z}_1=(0,0)$ of difference equations (5), we have the following conclusions:

(i) When $B>A$ and $E>D$, the equilibrium $(0,0)$ is locally asymptotically stable.

(ii) When $A>B$ or $D>E$, the equilibrium $(0,0)$ is unstable.

Proof. Define functions $F:(R^+)^5 \to R^+, H:(R^+)^5 \to R^+$ as

$$F(x_{n-3}, y_n, y_{n-1}, y_{n-2}, y_{n-3}) = \frac{Ax_{n-3}}{B + Cy_n y_{n-1} y_{n-2} y_{n-3}},$$

$$H(y_{n-3}, x_n, x_{n-1}, x_{n-2}, x_{n-3}) = \frac{Dy_{n-3}}{E + Fx_n x_{n-1} x_{n-2} x_{n-3}}.$$

To find the partial derivatives of the above functions, we have

$$\begin{aligned}
F_{x_{n-3}} &= \frac{A}{B + Cy_n y_{n-1} y_{n-2} y_{n-3}},\\
F_{y_n} &= \frac{-ACx_{n-3} y_{n-1} y_{n-2} y_{n-3}}{(B + Cy_n y_{n-1} y_{n-2} y_{n-3})^2},\\
F_{y_{n-1}} &= \frac{-ACx_{n-3} y_n y_{n-2} y_{n-3}}{(B + Cy_n y_{n-1} y_{n-2} y_{n-3})^2},\\
F_{y_{n-2}} &= \frac{-ACx_{n-3} y_n y_{n-1} y_{n-3}}{(B + Cy_n y_{n-1} y_{n-2} y_{n-3})^2},\\
F_{y_{n-3}} &= \frac{-ACx_{n-3} y_n y_{n-1} y_{n-2}}{(B + Cy_n y_{n-1} y_{n-2} y_{n-3})^2},
\end{aligned} \tag{6}$$

and

$$\begin{aligned}
H_{y_{n-3}} &= \frac{D}{E + Fx_n x_{n-1} x_{n-2} x_{n-3}},\\
H_{x_n} &= \frac{-DFy_{n-3} x_{n-1} x_{n-2} x_{n-3}}{(E + Fx_n x_{n-1} x_{n-2} x_{n-3})^2},\\
H_{x_{n-1}} &= \frac{-DFy_{n-3} x_n x_{n-2} x_{n-3}}{(E + Fx_n x_{n-1} x_{n-2} x_{n-3})^2},\\
H_{x_{n-2}} &= \frac{-DFy_{n-3} x_n x_{n-1} x_{n-3}}{(E + Fx_n x_{n-1} x_{n-2} x_{n-3})^2},\\
H_{x_{n-3}} &= \frac{-DFy_{n-3} x_n x_{n-1} x_{n-2}}{(E + Fx_n x_{n-1} x_{n-2} x_{n-3})^2}.
\end{aligned} \tag{7}$$

From (6) and (7), the linear equation of equations (5) about the equilibrium point $(0,0)$ is

$$\varphi_{n+1} = A_1 \varphi_n, \tag{8}$$

where $\varphi_n = (x_n, x_{n-1}, x_{n-2}, x_{n-3}, y_n, y_{n-1}, y_{n-2}, y_{n-3})$,

$$A_1 = \begin{bmatrix}
0 & 0 & 0 & \frac{A}{B} & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{D}{E}\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}.$$

Then the characteristic equation of equation (8) is

$$f(\lambda) = (\lambda^4 - \frac{A}{B})(\lambda^4 - \frac{D}{E}) = 0 \tag{9}$$

Therefore, if $B > A$ and $E > D$, then all $|\lambda| < 1$ can be obtained. Thus, from Lemma 1, the equilibrium point $(0,0)$ is locally asymptotically stable. If $A > B$ or $D > E$, then the equilibrium point $(0,0)$ is unstable.

The proof of Theorem 1 is completed. □

Theorem 2. When $A > B$ and $D > E$, the equilibrium point $\overline{Z}_2 = (\sqrt[4]{\frac{D-E}{F}}, \sqrt[4]{\frac{A-B}{C}})$ of difference equations (5) is unstable.

Proof. According to (6) and (7), the linear equation of equations (5) about the equilibrium point $\overline{Z}_2 = (\sqrt[4]{\frac{D-E}{F}}, \sqrt[4]{\frac{A-B}{C}})$ is

$$\varphi_{n+1} = A_2 \varphi_n, \tag{10}$$

where

$$A_2 = \begin{bmatrix}
0 & 0 & 0 & 1 & -\frac{CM}{A} & -\frac{CM}{A} & -\frac{CM}{A} & -\frac{CM}{A}\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
-\frac{FN}{D} & -\frac{FN}{D} & -\frac{FN}{D} & -\frac{FN}{D} & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix},$$

where

$$M = \sqrt[4]{\frac{D-E}{F}}(\frac{A-B}{C})^{\frac{3}{4}}, \quad N = \sqrt[4]{\frac{A-B}{C}}(\frac{D-E}{F})^{\frac{3}{4}}.$$

Let $\frac{CFMN}{AD} = T$, then the characteristic equation of equation (10) is

$$\begin{aligned} f(\lambda) = \lambda^8 - T\lambda^6 - 2T\lambda^5 - (3T+2)\lambda^4 - 4T\lambda^3 \\ -3T\lambda^2 - 2T\lambda - T + 1 = 0. \end{aligned} \tag{11}$$

From (11), we have

$$f(1) = -17T < 0,$$

and

$$\lim_{\lambda \to +\infty} f(\lambda) = +\infty.$$

Thus, the equation (11) has at least one root in the interval $(1, +\infty)$. According to Lemma 1, the equilibrium point $\overline{Z}_2$ of equations (5) is unstable.

The proof of Theorem 2 is completed. □

Theorem 3. For the fuzzy difference equation (1), where the parameters a, b, c and the initial conditions $z_{-3}, z_{-2}, z_{-1}, z_0$, are positive fuzzy numbers. If $a_{r,\alpha} < b_{l,\alpha}$, then the equilibrium point $\overline{Z}_1 = (0,0)$ of equation (1) is globally asymptotically stable.

Proof. According to Theorem 1, if $a_{r,\alpha} < b_{l,\alpha}$, it is easy to know that the equilibrium point $\overline{Z}_1 = (0,0)$ of equation (1) is locally asymptotically stable. According to the equation (4), we have

$$\begin{aligned} L_{n+1,\alpha} &= \frac{a_{l,\alpha} L_{n-3,\alpha}}{b_{r,\alpha} + c_{r,\alpha} R_{n,\alpha} R_{n-1,\alpha} R_{n-2,\alpha} R_{n-3,\alpha}} \\ &\le \frac{a_{l,\alpha} L_{n-3,\alpha}}{b_{r,\alpha}}, \\ R_{n+1,\alpha} &= \frac{a_{r,\alpha} R_{n-3,\alpha}}{b_{l,\alpha} + c_{l,\alpha} L_{n,\alpha} L_{n-1,\alpha} L_{n-2,\alpha} L_{n-3,\alpha}} \\ &\le \frac{a_{r,\alpha} R_{n-3,\alpha}}{b_{l,\alpha}}. \end{aligned} \tag{12}$$

Let $\frac{a_{l,\alpha}}{b_{r,\alpha}} = k$, from (12), we get

$$L_{n+1,\alpha} \le k L_{n-3,\alpha}. \tag{13}$$

From (13), we have

$$L_{4m,\alpha} \le k L_{4(m-1),\alpha} \le k^2 L_{4(m-2),\alpha} \le \cdots \le k^m L_{0,\alpha}, n = 4m,$$
$$L_{4m+1,\alpha} \le k L_{4(m-1)+1,\alpha} \le k^2 L_{4(m-2)+1,\alpha} \le \cdots \le k^m L_{1,\alpha}, n = 4m+1,$$
$$L_{4m+2,\alpha} \le k L_{4(m-1)+2,\alpha} \le k^2 L_{4(m-2)+2,\alpha} \le \cdots \le k^m L_{2,\alpha}, n = 4m+2,$$
$$L_{4m+3,\alpha} \le k L_{4(m-1)+3,\alpha} \le k^2 L_{4(m-2)+3,\alpha} \le \cdots \le k^m L_{3,\alpha}, n = 4m+3,$$

where $m = 0,1,2,\cdots$, $L_{0,\alpha}, L_{1,\alpha}, L_{2,\alpha}, L_{3,\alpha}$ are positive real numbers and $0 < k < 1$, so

$$\lim_{m\to+\infty} L_{4m,\alpha} = 0, \lim_{m\to+\infty} L_{4m+1,\alpha} = 0,$$
$$\lim_{m\to+\infty} L_{4m+2,\alpha} = 0, \lim_{m\to+\infty} L_{4m+3,\alpha} = 0.$$

Thus, we get $\lim_{n\to+\infty} L_{n,\alpha} = 0$. In the same way, we can obtain $\lim_{n\to+\infty} R_{n,\alpha} = 0$.

Consequently, we have proven that the equilibrium point $(0,0)$ is both locally asymptotically stable and a global attractor. Thus, by Definition 8, it is globally asymptotically stable.

The proof of Theorem 3 is completed. □

Theorem 4. Assume (x_n, y_n) be a positive solution of the difference equations (5), if $A \le D \le E \le B$, the positive solution of the equations (5) is bounded.

Proof. If the following inequality holds, then the positive solution of the system of equations (5) is bounded.

$$0 \le x_n \le \left(\frac{A}{B}\right)^{m+1} x_{-1} \le x_{-1}, n = 4m+3, m = 0,1,2,\cdots,$$
$$0 \le x_n \le \left(\frac{A}{B}\right)^{m+1} x_{-2} \le x_{-2}, n = 4m+2, m = 0,1,2,\cdots,$$
$$0 \le x_n \le \left(\frac{A}{B}\right)^{m+1} x_{-3} \le x_{-3}, n = 4m+1, m = 0,1,2,\cdots,$$
$$0 \le x_n \le \left(\frac{A}{B}\right)^{m} x_{0} \le x_{0}, n = 4m, m = 0,1,2,\cdots,$$

and

$$0 \le y_n \le \left(\frac{D}{E}\right)^{m+1} y_{-1} \le y_{-1}, n = 4m+3, m = 0,1,2,\cdots,$$
$$0 \le y_n \le \left(\frac{D}{E}\right)^{m+1} y_{-2} \le y_{-2}, n = 4m+2, m = 0,1,2,\cdots,$$
$$0 \le y_n \le \left(\frac{D}{E}\right)^{m+1} y_{-3} \le y_{-3}, n = 4m+1, m = 0,1,2,\cdots,$$
$$0 \le y_n \le \left(\frac{D}{E}\right)^{m} y_{0} \le y_{0}, n = 4m, m = 0,1,2,\cdots.$$

When $m = 0$, the above inequalities obviously hold, that is

$$0 \le x_3 \le \frac{A}{B} x_{-1} \le x_{-1},\ 0 \le x_2 \le \frac{A}{B} x_{-2} \le x_{-2},$$
$$0 \le x_1 \le \frac{A}{B} x_{-3} \le x_{-3},\ 0 \le x_0 \le x_0,$$

and

$$0 \le y_3 \le \frac{D}{E} y_{-1} \le y_{-1},\ 0 \le y_2 \le \frac{D}{E} y_{-2} \le y_{-2},$$
$$0 \le y_1 \le \frac{D}{E} y_{-3} \le y_{-3},\ 0 \le y_0 \le y_0.$$

We assume that the above inequality also holds when $m = k$, that is

$$0 \le x_n \le \left(\frac{A}{B}\right)^{k+1} x_{-1} \le x_{-1}, n = 4k+3, k = 0,1,2,\cdots,$$
$$0 \le x_n \le \left(\frac{A}{B}\right)^{k+1} x_{-2} \le x_{-2}, n = 4k+2, k = 0,1,2,\cdots,$$
$$0 \le x_n \le \left(\frac{A}{B}\right)^{k+1} x_{-3} \le x_{-3}, n = 4k+1, k = 0,1,2,\cdots,$$
$$0 \le x_n \le \left(\frac{A}{B}\right)^{k} x_{0} \le x_{0}, n = 4k, k = 0,1,2,\cdots,$$

and

$$0 \le y_n \le \left(\frac{D}{E}\right)^{k+1} y_{-1} \le y_{-1}, n = 4k+3, k = 0,1,2,\cdots,$$
$$0 \le y_n \le \left(\frac{D}{E}\right)^{k+1} y_{-2} \le y_{-2}, n = 4k+2, k = 0,1,2,\cdots,$$
$$0 \le y_n \le \left(\frac{D}{E}\right)^{k+1} y_{-3} \le y_{-3}, n = 4k+1, k = 0,1,2,\cdots,$$
$$0 \le y_n \le \left(\frac{D}{E}\right)^{k} y_{0} \le y_{0}, n = 4k, k = 0,1,2,\cdots.$$

When $m=k+1$, we have

$$0\le x_{4(k+1)+3}\le \frac{A}{B}x_{4(k+1)-1}\le \left(\frac{A}{B}\right)^{k+2}x_{-1},$$

$$0\le x_{4(k+1)+2}\le \frac{A}{B}x_{4(k+1)-2}\le \left(\frac{A}{B}\right)^{k+2}x_{-2},$$

$$0\le x_{4(k+1)+1}\le \frac{A}{B}x_{4(k+1)-3}\le \left(\frac{A}{B}\right)^{k+2}x_{-3},$$

$$0\le x_{4(k+1)}\le \frac{A}{B}x_{4k}\le \left(\frac{A}{B}\right)^{k+1}x_{0},$$

and

$$0\le y_{4(k+1)+3}\le \frac{D}{E}y_{4(k+1)-1}\le \left(\frac{D}{E}\right)^{k+2}y_{-1},$$

$$0\le y_{4(k+1)+2}\le \frac{D}{E}y_{4(k+1)-2}\le \left(\frac{D}{E}\right)^{k+2}y_{-2},$$

$$0\le y_{4(k+1)+1}\le \frac{D}{E}y_{4(k+1)-3}\le \left(\frac{D}{E}\right)^{k+2}y_{-3},$$

$$0\le y_{4(k+1)}\le \frac{D}{E}y_{4k}\le \left(\frac{D}{E}\right)^{k+1}y_{0}.$$

Therefore, according to mathematical induction, if $A\le D\le E\le B$, the positive solution of the equations (5) is bounded, the proof of Theorem 4 is completed. □

Theorem 5. For the fuzzy difference equation (1), where the parameters a,b,c and the initial conditions z_{-3},z_{-2},z_{-1}, z_0 are positive fuzzy numbers. If the parameters satisfy $a_{l,\alpha}\le a_{r,\alpha}\le b_{l,\alpha}\le b_{r,\alpha}$, then the positive solutions of the difference equation (1) are all bounded.

Proof. Assume z_n is a positive solution of fuzzy difference equation (1) based on the initial conditions z_{-3},z_{-2},z_{-1},z_0, satisfying $[z_n]_\alpha=[L_{n,\alpha},R_{n,\alpha}]$, $\alpha\in(0,1]$, $n=0,1,\cdots$, and the parameters a,b,c are positive fuzzy numbers. Therefore, we have

$[a]_\alpha=[a_{l,\alpha},a_{r,\alpha}],[b]_\alpha=[b_{l,\alpha},b_{r,\alpha}],$

$[c]_\alpha=[c_{l,\alpha},c_{r,\alpha}],[z_n]_\alpha=[L_{n,\alpha},R_{n,\alpha}],n=-3,-2,\cdots.$

According to Lemma 4, we can get

$L_{4m+3,\alpha}\le L_{-1,\alpha},L_{4m+2,\alpha}\le L_{-2,\alpha},$

$L_{4m+1,\alpha}\le L_{-3,\alpha},L_{4m,\alpha}\le L_{0,\alpha},$

$R_{4m+3,\alpha}\le R_{-1,\alpha},R_{4m+2,\alpha}\le R_{-2,\alpha},$

$R_{4m+1,\alpha}\le R_{-3,\alpha},R_{4m,\alpha}\le R_{0,\alpha},m=0,1,\cdots,$

namely there are two positive real numbers satisfying

$$0\le L_{n,\alpha}\le u,0\le R_{n,\alpha}\le v,$$

where

$$u=\max\{L_{-3,\alpha},L_{-2,\alpha},L_{-1,\alpha},L_{0,\alpha}\},$$

$$v=\max\{R_{-3,\alpha},R_{-2,\alpha},R_{-1,\alpha},R_{0,\alpha}\}.$$

Therefore, set $N=\max\{u,v\}$, we have

$$\operatorname{supp}(z_n)=\overline{\bigcup_{\alpha\in(0,1]}[z_n]_\alpha}=\bigcup_{\alpha\in(0,1]}[L_{n,\alpha},R_{n,\alpha}]\subset(0,N].$$

Thus, when $a_{l,\alpha}\le a_{r,\alpha}\le b_{l,\alpha}\le b_{r,\alpha}$, the positive solutions of the difference equation (1) are all bounded, the proof of Theorem 5 is completed. □

IV. Numerical Simulation

Example 1. Consider the following fuzzy difference equation:

$$z_{n+1}=\frac{az_{n-3}}{b+cz_nz_{n-1}z_{n-2}z_{n-3}},n=0,1,2,\cdots, \tag{14}$$

where the parameters a,b,c are positive fuzzy numbers, we define the parameters a,b,c that satisfy $a_{r,\alpha}<b_{l,\alpha}$ as follows:

$$[a]_\alpha=[2+\alpha,5-2\alpha],[b]_\alpha=[11+\alpha,13-\alpha],$$
$$[c]_\alpha=[3+\alpha,5-\alpha],$$

and define the initial conditions as follow:

$$z_0(z)=\begin{cases}10z-2,0.2\le z\le 0.3,\\ -10z+4,0.3\le z\le 0.4,\end{cases}\quad z_{-1}(z)=\begin{cases}\frac{1}{2}z-\frac{5}{2},5\le z\le 7,\\ -\frac{1}{2}z+\frac{9}{2},7\le z\le 9,\end{cases}$$
$$z_{-2}(z)=\begin{cases}5z-1.5,0.3\le z\le 0.5,\\ -5z+3.5,0.5\le z\le 0.7,\end{cases}\quad z_{-3}(z)=\begin{cases}\frac{1}{2}z-\frac{1}{2},1\le z\le 3,\\ -\frac{1}{2}z+\frac{5}{2},3\le z\le 5.\end{cases} \tag{15}$$

From (15), we have

$$[z_0]_\alpha=[0.2+0.1\alpha,0.4-0.1\alpha],\ [z_{-1}]_\alpha=[5+2\alpha,9-2\alpha],$$
$$[z_{-2}]_\alpha=[0.3+0.2\alpha,0.7-0.2\alpha],\ [z_{-3}]_\alpha=[1+2\alpha,5-2\alpha].$$

According to the equation (14), we can establish the following difference symmetric system with parameter α:

$$\begin{aligned}L_{n+1,\alpha}&=\frac{a_{l,\alpha}L_{n-3,\alpha}}{b_{r,\alpha}+c_{r,\alpha}R_{n,\alpha}R_{n-1,\alpha}R_{n-2,\alpha}R_{n-3,\alpha}},\\ R_{n+1,\alpha}&=\frac{a_{r,\alpha}R_{n-3,\alpha}}{b_{l,\alpha}+c_{l,\alpha}L_{n,\alpha}L_{n-1,\alpha}L_{n-2,\alpha}L_{n-3,\alpha}}.\end{aligned} \tag{16}$$

According to the Theorem 5, every positive solution of the fuzzy difference equation (14) is bounded. From Theorem 3, we can obtain that the trivial solution $z=\hat{0}$ of equation (14) is globally asymptotically stable with respect to the initial conditions (15). (see Figs. 1-4).

V. Conclusion

This paper conducts a systematic investigation into the dynamical properties of fourth-order fuzzy difference equations, focusing on equilibrium point stability and solution boundedness. The key findings are summarized as follows:

1. Equilibrium Classification: This study characterizes two fundamental classes of equilibrium points in fourth-order fuzzy difference equations: trivial equilibria (e.g., (0,0)) and nontrivial equilibria

$$\bar{Z}_2 = (\bar{x}_2, \bar{y}_2) = (\sqrt[4]{\frac{D-E}{F}}, \sqrt[4]{\frac{A-B}{C}}).$$

2. Stability Criteria: (1) Under specific fuzzy parameter constraints, the trivial equilibrium (0,0) is globally asymptotically stable, guaranteeing convergence of all solutions to zero. (2) Violation of these constraints induces instability in the trivial equilibrium. (3) For both equilibrium classes, rigorous stability conditions are established via linearization techniques and inequality-based analyses.

3. Boundedness Analysis: This work obtains sufficient conditions ensuring boundedness of positive fuzzy solutions under prescribed parameter constraints, which enforce solution trajectories to remain within finite bounds. These conditions are rigorously derived through mathematical induction and advanced inequality analyses.

4. Numerical Validation: MATLAB-based simulations substantiate theoretical findings, graphically demonstrating solution convergence and boundedness under fuzziness.

This research delivers dual-impact contributions bridging theoretical and practical domains. Theoretically, it addresses a critical literature gap by pioneering analytical frameworks for higher-order fuzzy difference equations. The novel methodology synergizes fuzzy set theory, linearization techniques, and inequality-based analyses, enhancing comprehension of uncertain dynamical systems while establishing scalable protocols for diverse fuzzy systems. Practically, the findings enable biological applications (multi-step population dynamics modeling), ecological implementations (resource allocation feedback analysis), and engineering solutions under incomplete information. Fuzzy parameters facilitate robust real-world uncertainty quantification. Methodologically, the hybrid approach transcends classical stability analysis by elucidating fuzzy parameter interactions. The derived stability/boundedness criteria extend existing lower-order theories, laying foundations for advanced fuzzy dynamical models. Given higher-order fuzzy systems' prevalence in real-world scenarios, subsequent work will investigate complex fuzzy difference systems to generate theoretical guidance for practical problem-solving.

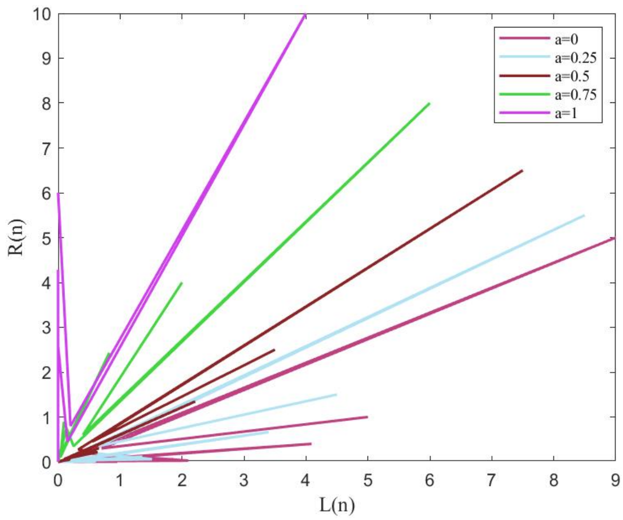

Fig.1. The dynamics of difference equations (16) .

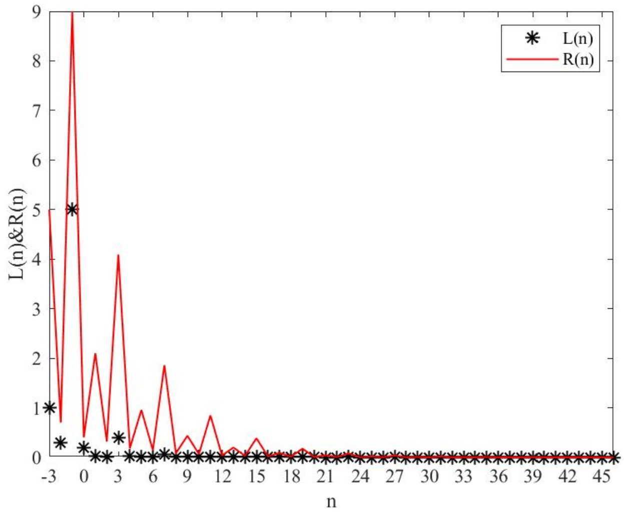

Fig.2. The solutions of difference equations (16) with $\alpha = 0$.

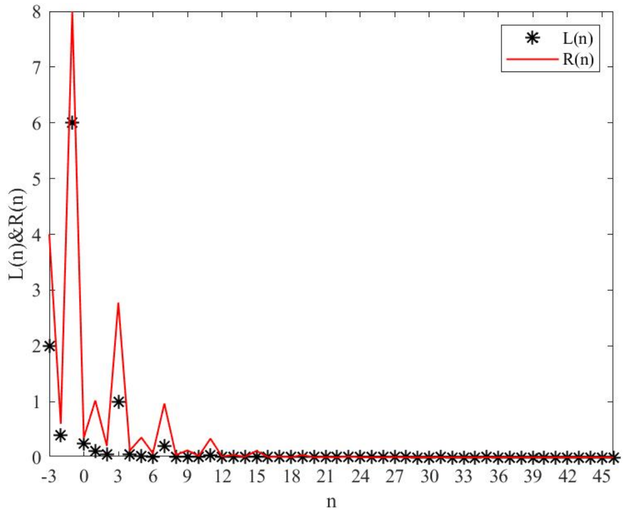

Fig.3. The solutions of difference equations (16) with $\alpha = 0.5$.

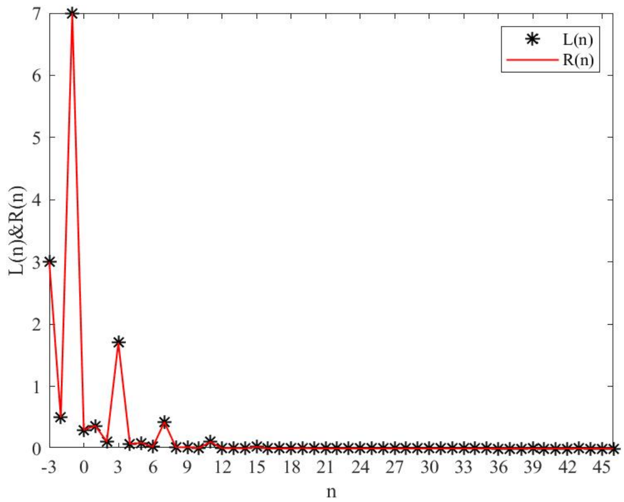

Fig.4. The solutions of difference equations (16) with $\alpha = 1$.

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