Relationship between Identical Synchronization and Generalized Synchronization in Full Networks of n Linearly Coupled Dynamical Systems of the FitzHugh-Nagumo Type

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***Abstract*—This paper examines the sufficient conditions for identical synchronization and generalized synchronization in full networks of n nodes and their relationship. Each node is linked to others through linear coupling and represented by an ordinary differential system of the FizHugh-Nagumo type. Additionally, this work presents numerical results to confirm the validity of the theoretical findings in the context of this network topology.**

***Index Terms*—full network, generalized synchronization, identical synchronization, FitzHugh-Nagumo model.**

I. Introduction

Synchronization has been extensively studied in various fields and natural phenomena, with numerous studies introducing the concept of synchronization [5], [18], [11], [14], [12], [7], [13]. Mathematically, synchronization typically refers to having the same behavior simultaneously [5]. In recent years, research on complex dynamical networks has gained popularity across multiple domains due to their practicality on a large scale, including applications in information processing, the World Wide Web, biological systems, and neuronal networks [19], [20], [21], [9]. Synchronization is a fundamental issue in cooperative control, requiring all network subsystems to converge to a target state or a common value. There are different types of synchronization in complex networks, including identical synchronization, projective synchronization, phase synchronization, and generalized synchronization [5]. Especially, generalized synchronization is an extension of identical synchronization and is more common in nature and technical applications. However, most theoretical results on synchronization focus on identical synchronization [5], [6], [2], [3], [16], [17]. Due to its complexity, there is a lack of theoretical results for generalized synchronization, but it is gaining special attention. So, research on generalized synchronization in complex dynamic networks is of great practical significance.

In this paper, we are motivated by the discussion above to explore the improvements we have made. We aim to identify the sufficient conditions for the coupling strength required to achieve identical synchronization and generalized synchronization in full (complete) networks. Additionally, our results illustrate the relationship between these types of synchronization, and we provide numerical results to validate the effectiveness of our theoretical findings.

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II. Identical synchronization and generalized synchronization in full networks of n dynamical systems of the FitzHugh-Nagumo type

In 1952, A.L. Hodgkin and A.F. Huxley published a paper introducing a mathematical model of four ordinary differential equations to approximate some properties of neuronal membrane potential [6], [7], [11]. They were awarded a Nobel prize for this remarkable work. Based on their famous study, many scientists searched and found ways to simplify Hodgkin-Huxley's model while retaining the energizing and biological significance properties of the cell. Among them, two scientists named R. FitzHugh and J. Nagumo introduced a new simpler system called the FitzHugh-Nagumo model in 1962 [7], [8], [15]. It is known as a system of two ordinary differential equations, which is simplified from Hodgkin-Huxley's system [6], [12], and could help describe the neuron voltage dynamics. It consists of two equations of two variables, u and v. The first variable, u, represents the transmembrane voltage of the cell. The second one, v, introduces some physical quantities, such as the electrical conductivity of ion currents across the membrane. The system below shows the ordinary differential equations of the FitzHugh-Nagumo type given by [2], [3]:

$$\begin{cases} \varepsilon \dfrac{du}{dt} = \varepsilon u_t = f(u) - v + I, \\ \dfrac{dv}{dt} = v_t = au - bv + c, \end{cases} \tag{1}$$

where $u = u(t), v = v(t)$; a, b, c are constants, especially a and b are strictly positive; $0 < \varepsilon < 1$; $f(u) = -u^3 + 3u$; I presents the external current; t presents the time.

Hereafter, system (1) is considered a model of neuron, and based on this system, we construct a full network of n systems (1) with linear coupling as follows:

$$\begin{cases} \varepsilon u_{it} = f(u_i) - v_i + I - \sum\limits_{j=1, j\neq i}^{n} g_{syn}(u_i - u_j), \\ v_{it} = au_i - bv_i + c, \\ i = 1, 2, ..., n, \end{cases} \tag{2}$$

where $(u_i, v_i), i = 1, 2, ..., n$ is defined as (1); g_{syn} is positive number presenting the coupling strength [7], [6]. Note that the coupling strength could be different among the nodes of networks. However, to make it easy in this work, the paper just investigates the same coupling strength for all nodes, and names it g_{syn}.

Remark 1. A full network means each node connects to all others (see [16], [17]).

A. Identical synchronization

Definition 1 (see [5], [3])**.** Let $\sum_{i=2}^{n} |e_i| + |\overline{e}_i|$ be the identical synchronization error, where $e_i = u_i - u_1, \overline{e}_i = v_i - v_1$, for all $i = 2, ..., n$. We say that the network (2) identically synchronizes if the identical synchronization error reaches zero as t approaches infinity.

Before going to the main results, we need to see a following remark that help to prove our desired results.

Remark 2. The function f satisfies the following condition:

$$|f(u_i) - f(u_j)| \leq \alpha |u_i - u_j|, \; i, j = 1, 2, ..., n, \tag{3}$$

where u_i, u_j present the transmembrane voltages, and α is a positive number.

Proof: For all $u_i, u_j, i, j = 1, 2, ..., n$, we have:

$$\begin{aligned} f(u_i) - f(u_j) &= -u_i^3 + 3u_i + u_j^3 - 3u_j \\ &= (u_i - u_j)\left[3 - (u_i - u_j)^2 - u_i u_j\right]. \end{aligned}$$

Since $u_i, u_j, i, j = 1, 2, ..., n$ are bounded in [4], then we can find a positive constant α such that:

$$|f(u_i) - f(u_j)| \leq \alpha |u_i - u_j|, \; i, j = 1, 2, ..., n.$$

■

Next, we investigate the identical synchronization problem of network (2). The main result is given by the following theorem.

Theorem 1. *Suppose that $g_{syn} > \dfrac{\alpha}{n}$, where α is defined as in Theorem 2. Then the network (2) identically synchronizes in the sense of Definition 1.*

Proof: To prove this theorem, we construct a Lyapunov function as follows:

$$V(t) = \frac{1}{2}\left[\sum_{i=2}^{n}\left(a\varepsilon {e_i}^2 + {\overline{e}_i}^2\right)\right].$$

By calculating the derivative of the function $V(t)$ with respect to the time t, there is the following:

$$\begin{aligned} \frac{dV(t)}{dt} &= \sum_{i=2}^{n}\left[a\varepsilon e_i e_{it} + \overline{e}_i \overline{e}_{it}\right] \\ &= \sum_{i=2}^{n}\Bigg[a e_i \Bigg(f(u_i) - v_i - g_{syn}\sum_{k=1, k\neq i}^{n}(u_i - u_k) \\ &\quad - f(u_1) + v_1 + g_{syn}\sum_{l=2}^{n}(u_1 - u_l)\Bigg) \\ &\quad + \overline{e}_i\left(a e_i - b\overline{e}_i\right)\Bigg] \\ &\leq \sum_{i=2}^{n}\left[a e_i\left(f(u_i) - f(u_1) - n g_{syn} e_i\right) - b{\overline{e}_i}^2\right]. \end{aligned} \tag{4}$$

By using Remark 2, (4) becomes:

$$\frac{dV(t)}{dt} \leq \sum_{i=2}^{n}\left[a\left(\alpha - n g_{syn}\right)e_i^2 \; - b{\overline{e}_i}^2\right]. \tag{5}$$

Since $g_{syn} > \dfrac{\alpha}{n}$, then (5) can be estimated as:

$$\frac{dV(t)}{dt} \leq -\gamma V(t) \Rightarrow V(t) \leq V(0)e^{-\gamma t},$$

where $\gamma = \min\left(2\dfrac{n g_{syn} - \alpha}{\varepsilon}, 2b\right)$. Let the time t approach infinity, the function $V(t)$ approaches zero. Thus, the identical synchronization occurs if the coupling strength satisfies the following condition: $g_{syn} > \dfrac{\alpha}{n}$. ■

B. Generalized synchronization

Definition 2 (see [5])**.** Let $\sum_{i=2}^{n} |e_i| + |\overline{e}_i|$ be the generalized synchronization error, where $e_i = u_i - \phi_i(u_1), \overline{e}_i = v_i - \varphi_i(v_1)$, ϕ_i, φ_i are differentially continuous functions, for all $i = 2, ..., n$. We say that the network (2) generally synchronizes if the generalized synchronization error reaches zero as t approaches infinity.

To have the generalized synchronization, we need to define the controllers for the network (2). Specifically, to synchronize the first neuron and neuron ith of network (2), we need to construct and add the controllers into neuron ith as follows:

$$\begin{cases} \varepsilon u_{1t} = f(u_1) - v_1 + I - g_{syn}\displaystyle\sum_{l=2}^{n}(u_1 - u_l), \\ v_{1t} = a u_1 - b v_1 + c, \\ \varepsilon u_{it} = f(u_i) - v_i + I - g_{syn}\displaystyle\sum_{h=1, h\neq i}^{n}(u_i - u_h) + w_i, \\ v_{it} = a u_i - b v_i + c + \overline{w}_i, \\ i = 2, ..., n, \end{cases} \tag{6}$$

where the controllers $w_i = w_i(t)$ are $\overline{w}_i = \overline{w}_i(t)$ are designed as follows:

$$\begin{cases} w_i = \varepsilon\dfrac{\partial\phi_i(u_1)}{\partial u_1}u_{1t} - f(\phi_i(u_1)) + \varphi_i(v_1) - I \\ \qquad + g_{syn}\displaystyle\sum_{m=1, m\neq i}^{n}(\phi_i(u_1) - u_m) - k_i e_i, \\ \overline{w}_i = \dfrac{\partial\varphi_i(v_1)}{\partial v_1}v_{1t} - a\phi_i(u_1) - b\varphi_i(v_1) - c, \end{cases} \tag{7}$$

with the updated rules defined as follows:

$$k_{it} = r_i e_i^2, \tag{8}$$

where $k_i = k_i(t)$, r_i is a arbitrary positive constant, for $i = 2, ..., n$.

Under the action of the controllers designed as above, the error dynamic equations of the system (6) are described as:

$$\begin{aligned}
\varepsilon e_{it} &= \varepsilon\left(u_{it} - \frac{\partial\phi_i(u_1)}{\partial u_1}u_{1t}\right)\\
&= f(u_i) - v_i + I - g_{syn}\sum_{h=1,h\neq i}^{n}(u_i-u_h) - f(\phi_i(u_1))\\
&+\varphi_i(v_1) - I + g_{syn}\sum_{m=1,m\neq i}^{n}(\phi_i(u_1)-u_m) - k_ie_i\\
&= f(u_i) - f(\phi_i(u_1)) - (v_i - \varphi_i(v_1))\\
&\qquad -ng_{syn}(u_i-\phi_i(u_1)) - k_ie_i\\
&= f(u_i) - f(\phi_i(u_1)) - \overline{e}_i - ng_{syn}e_i - k_ie_i,
\end{aligned} \tag{9}$$

and

$$\begin{aligned}
\overline{e}_{it} &= v_{it} - \frac{\partial\varphi_i(v_1)}{\partial v_1}v_{1t}\\
&= au_i - bv_i + c - a\phi_i(u_1) - b\varphi_i(v_1) - c\\
&= ae_i - b\overline{e}_i,
\end{aligned} \tag{10}$$

for $i = 2, ..., n$.

Remark 3. The function f satisfies the following condition:

$$|f(u_i) - f(\phi_i(u_j))| \leq \beta\,|u_i - \phi_i(u_j)|, \tag{11}$$

where u_i, u_j present the transmembrane voltages, ϕ_i is defined as in Definition 2, $i, j = 1, 2, ..., n$ and β is a positive number.

Proof: For all $u_i, u_j, i, j = 1, 2, ..., n$, we have:

$$\begin{aligned}
f(u_i) - f(\phi_i(u_j)) &= -u_i^3 + 3u_i + \phi_i(u_j)^3 - 3\phi_i(u_j)\\
&= (u_i - \phi_i(u_j))\Big[3 - (u_i - \phi_i(u_j))^2\\
&\quad -u_i\phi_i(u_j)\Big].
\end{aligned}$$

Since $u_i, i = 1, 2, ..., n$ is bounded and belongs to a compact set that is a global attractor (see [4]), then $\phi_i(u_i)$ is also bounded since ϕ_i is continuous. Therefore, we can find a positive constant β such that:

$$|f(u_i) - f(\phi_i(u_i))| \leq \beta\,|u_i - \phi_i(u_j)|,\ i, j = 1, 2, ..., n.$$

■

Next, we investigate the generalized synchronization problem of network (2). The main result is given by the following theorem.

Theorem 2. *Suppose that $g_{syn} > \dfrac{\beta}{n}$, then we can achieve generalized synchronization for network* (2) *under the adaptive controllers* (7) *and updated rules* (8).

Proof: To prove this theorem , we construct the Lyapunov function as follows:

$$V(t) = \frac{1}{2}\sum_{i=2}^{n}\left(a\varepsilon {e_i}^2 + \overline{e}_i^2 + \frac{a}{r_i}{k_i}^2\right). \tag{12}$$

By calculating the time derivative of $V(t)$ along the error systems (9) and (10), we get:

$$\begin{aligned}
\frac{dV(t)}{dt} &= \sum_{i=2}^{n}\left[a\varepsilon e_ie_{it} + \overline{e}_i\overline{e}_{it} + \frac{a}{r_i}k_ik_{it}\right]\\
&= \sum_{i=2}^{n}\big[ae_i\left(f(u_i) - f(\phi_i(u_1)) - \overline{e}_i - ng_{syn}e_i - k_ie_i\right)\\
&\qquad +\overline{e}_i(ae_i - b\overline{e}_i) + ak_ie_i^2\big]\\
&= \sum_{i=2}^{n}\big[ae_i\left(f(u_i) - f(\phi_i(u_1))\right) - ae_i\overline{e}_i - nag_{syn}e_i^2\\
&\qquad -ak_ie_i^2 + \overline{e}_iae_i - b\overline{e}_i^2 + ak_ie_i^2\big]\\
&= \sum_{i=2}^{n}\left[ae_i\left(f(u_i) - f(\phi_i(u_1))\right) - nag_{syn}e_i^2 - b\overline{e}_i^2\right].
\end{aligned} \tag{13}$$

By using Remark 3, it is easy to obtain:

$$\begin{aligned}
\frac{dV(t)}{dt} &\leq \sum_{i=2}^{n}\left[a\beta e_i^2 - nag_{syn}e_i^2 - b\overline{e}_i^2\right]\\
&\leq \sum_{i=2}^{n}\left[a(\beta - ng_{syn})e_i^2 - b\overline{e}_i^2\right].
\end{aligned} \tag{14}$$

Since $g_{syn} > \dfrac{\beta}{n}$, then (14) can be estimated as:

$$\frac{dV(t)}{dt} \leq -\gamma\sum_{i=1}^{n}\int_{\Omega}\left[\frac{1}{2}(a\varepsilon e_i^2 + \overline{e}_i^2)\right]dx, \tag{15}$$

where

$$\gamma = \min\left\{\frac{2(ng_{syn} - \beta)}{\varepsilon}; 2b\right\}.$$

From (15), It can be seen that $0 \leq V(t) \leq V(0)$, this together with (12) implies that $V(t)$ is bounded. It is based on Lyapunov stability theory and LaSalle's invariance principle [1], we have:

$$\lim_{t\to+\infty}\sum_{i=2}^{n}|e_i| + |\overline{e}_i| = 0.$$

Therefore, it implies that the network (2) achieves generalized synchronization in the sense of Definition 2. The theorem is proved. ■

Remark 4. If $\phi_i(u_j) = u_j$ and $\varphi_i(v_j) = v_j, i, j = 1, 2, ..., n$, then generalized synchronization becomes identical synchronization. Moreover, the positive number β in Remark 3 becomes α in Remark 2, and Theorem 2 becomes Theorem 1.

III. Numerical results and discussion

In this section, we concretely shows some examples to check if the proposed method in the theoretical section is effective. The integration is realized by using the programming R. The simulation results are obtained with the following parameter values:

$$a = 1, b = 0.001, c = 0, I = 0, \varepsilon = 0.1.$$

A. Example 1.

In this example, we take a full network of 2 nodes and search for a necessary value of coupling strength to get the identical synchronization.

Specifically, a full network of two neurons with linear coupling is given by the following system:

$$\begin{cases} \varepsilon u_{1t} = f(u_1) - v_1 + I - g_{syn}(u_1 - u_2), \\ v_{1t} = au_1 - bv_1 + c, \\ \varepsilon u_{2t} = f(u_2) - v_2 + I - g_{syn}(u_2 - u_1), \\ v_{2t} = au_2 - bv_2 + c. \end{cases} \tag{16}$$

Let $|e_2| + |\overline{e}_2| = |u_2 - u_1| + |v_2 - v_1|$ be the identical synchronization error of the network (16). We say that this network identically synchronizes if the identical synchronization error reaches zero as t approaches infinity.

Fig. 1(a), 1(b) represent the identical synchronization errors of the network (16) with respect to different values of coupling strength, $t \in [0; 4000]$ and the initial conditions as follows:

$$(u_1(0), v_1(0), u_2(0), v_2(0)) = (-0.1, 0, 0, 0.1).$$

In Fig. 1(a) with $g_{syn} = 0.0001$, the simulation shows that the identical synchronization error does not reach zero, which means the identical synchronization phenomenon does not occur.

In Fig. 1(b) with $g_{syn} = 0.1$, the simulation shows that the identical synchronization error reaches zero, which means:

$$u_1(t) \approx u_2(t),\ v_1(t) \approx v_2(t).$$

In other words, the identical synchronization phenomenon occurs when the coupling stength is more than its threshold value. It actually meets with the result in Theorem 1.

B. Example 2.

In this example, we take a full network of 3 nodes and search for a necessary value of coupling strength to get the identical synchronization.

Specifically, a full network of three neurons with linear coupling is given by the following system:

$$\begin{cases} \varepsilon u_{1t} = f(u_1) - v_1 + I - g_{syn}(u_1 - u_2) - g_{syn}(u_1 - u_3), \\ v_{1t} = au_1 - bv_1 + c, \\ \varepsilon u_{2t} = f(u_2) - v_2 + I - g_{syn}(u_2 - u_1) - g_{syn}(u_2 - u_3), \\ v_{2t} = au_2 - bv_2 + c, \\ \varepsilon u_{3t} = f(u_3) - v_3 + I - g_{syn}(u_3 - u_1) - g_{syn}(u_3 - u_2), \\ v_{3t} = au_3 - bv_3 + c. \end{cases} \tag{17}$$

Let $|e_2| + |\overline{e}_2| + |e_3| + |\overline{e}_3| = |u_2 - u_1| + |v_2 - v_1| + |u_3 - u_1| + |v_3 - v_1|$ be the identical synchronization error of the network (17). We say that this network identically synchronizes if the identical synchronization error reaches zero as t approaches infinity.

Fig. 2(a), 2(b) represent the identical synchronization errors of the network (17) with respect to different values of coupling strength, $t \in [0; 10000]$ and the initial conditions as follows:

$$(u_1(0), v_1(0), u_2(0), v_2(0), u_3(0), v_3(0)) =$$

$$(-0.1, 0, 0, 0.1, 0, -0.1).$$

In Fig. 2(a) with $g_{syn} = 0.001$, the simulation shows that the identical synchronization error does not reach zero, which means the identical synchronization phenomenon does not occur.

In Fig. 2(b) with $g_{syn} = 0.05$, the simulation shows that the identical synchronization error reaches zero, which means:

$$u_1(t) \approx u_2(t),\ v_1(t) \approx v_2(t),$$

$$u_1(t) \approx u_3(t),\ v_1(t) \approx v_3(t).$$

In other words, the identical synchronization phenomenon occurs when the coupling stength is large enough. It actually meets with the result in Theorem 1.

C. Example 3.

In this example, we take a full network of 2 nodes and search for a necessary value of coupling strength to get the generalized synchronization. Moreover, it needs to construct some controllers for this network as the theoretical results above (7)-(8). Hence, we have to investigate if those controllers numerically work. Specifically, a full network of two nodes with controllers to get the generalized synchronization is given by the following system:

$$\begin{cases} \varepsilon u_{1t} = f(u_1) - v_1 + I - g_{syn}(u_1 - u_2), \\ v_{1t} = au_1 - bv_1 + c, \\ \varepsilon u_{2t} = f(u_2) - v_2 + I - g_{syn}(u_2 - u_1) + w_2, \\ v_{2t} = au_2 - bv_2 + c + \overline{w}_2, \end{cases} \tag{18}$$

where

$$\begin{cases} w_2 = \varepsilon \dfrac{\partial \phi_2(u_1)}{\partial u_1} u_{1t} - f(\phi_2(u_1)) + \varphi_2(v_1) - I \\ \qquad\qquad + g_{syn}(\phi_2(u_1) - u_1) - k_2 e_2, \\ \overline{w}_2 = \dfrac{\partial \varphi_2(v_1)}{\partial v_1} v_{1t} - a\phi_2(u_1) + b\varphi_2(v_1) - c, \end{cases} \tag{19}$$

with the updated rule $k_{2t} = r_2 e_2^2$, and $e_2 = u_2 - \phi_2(u_1), \overline{e}_2 = v_2 - \varphi_2(v_1)$, where ϕ_2, φ_2 are the continuously differential functions. Let $|e_2| + |\overline{e}_2| = |u_2 - \phi_2(u_1)| + |v_2 - \varphi_2(v_1)|$ be the generalized synchronization error. We say that the network (18) generally synchronizes if the generalized synchronization error reaches zero as t approaches infinity.

Fig. 3 represents the generalized synchronization error between nodes of network (18), where we take the initial conditions as follows:

$$(u_1(0), v_1(0), u_2(0), v_2(0)) = (-0.1, 0, 0, 0.1);$$

$$\phi_2(x) = -\cos x - 1; \varphi_2(x) = x^2 + 1;$$

and

$$r_2 = 0.1; t \in [0; 10000].$$

Fig. 3(a) presents the generalized synchronization error with respect to t of the network (18) without controller (19). We can see that it does not reach zero, which means the generalized synchronization does not occur. However, Fig. 3(b) presents the generalized synchronization error of the network (18) with controller (19), and it actually reaches zero. In other words, the network (18) achieves generalized synchronization. It means the controller added in such a network effectively works.

Specifically, we first simulate the system (18) without controller (19). In Fig. 3(a), we take $g_{syn} = 0.01$, the

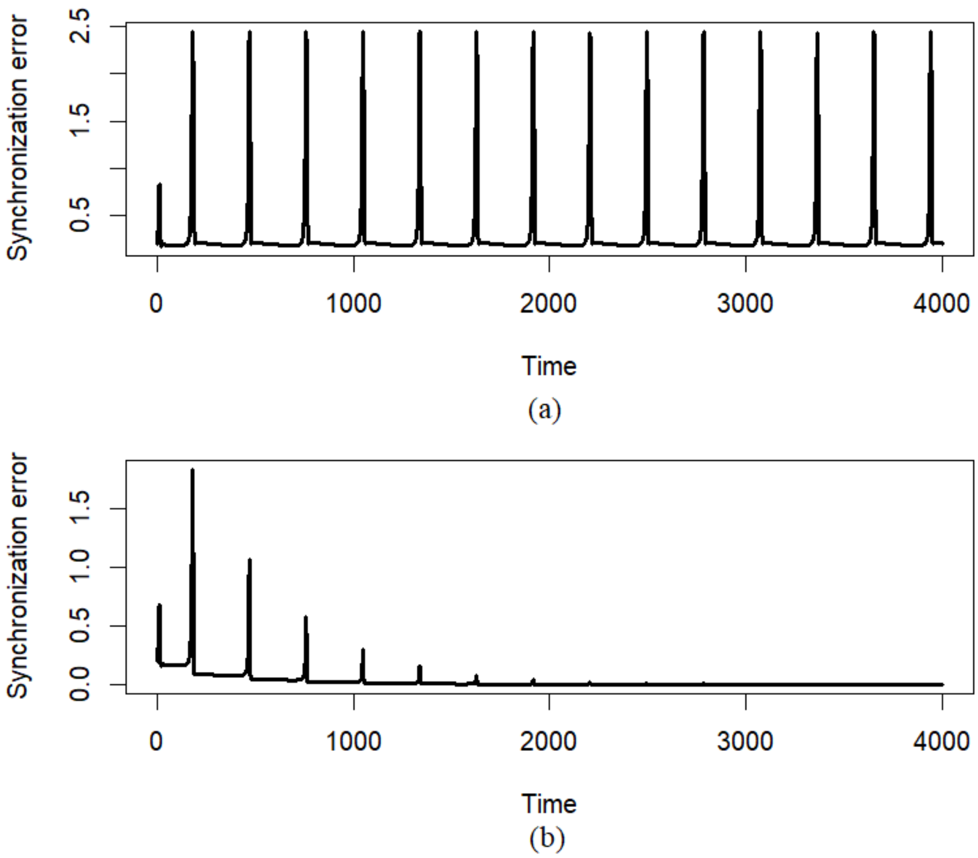

Fig. 1. Identical synchronization errors of the network (16) with respect to different values of coupling strength: (a) $g_{syn} = 0.0001$; (b) $g_{syn} = 0.1$.

generalized synchronization error does not reach zero. Even if we take a very large value of $g_{syn} = 5.5$, see Fig. 3(b), the generalized synchronization error also does not reach zero which means the generalized synchronization does not occur without controller. Clearly, Fig. 4 represents the time series of all variables of the system (18) without controller (19). In Fig. 4(a), the variable $\phi_2(u_1)$ is presented by the solid line, and the dotted line for u_2 (respectively, $\varphi_2(v_1)$ and v_2 in Fig. 4(b)). We can see that the solid line does not copy the behaviour of the dotted one. In other words, the generalized synchronization phenomenon does not occur in this case.

Next, we simulate the system (18) with controller (19). In Fig. 3(c), we take $g_{syn} = 0.01$, the generalized synchronization error reaches zero which means the generalized synchronization occurs with controller. In other words, the controller is effective. Clearly, Fig. 5 represents the time series of all variables of the system (18) with controller (19). In Fig. 5(a), the variable $\phi_2(u_1)$ is presented by the solid line, and the dotted line for u_2 (respectively, $\varphi_2(v_1)$ and v_2 in Fig. 5(b)). We can see that the solid line copies the behaviour of the dotted one. In other words, the generalized synchronization phenomenon occurs in this case.

D. Example 4.

Similarly, we take a full network of 3 nodes and search for a necessary value of coupling strength to get the generalized synchronization and investigate if the controllers (7) and (8) constructed as the theoretical part numerically work. Specifically, a full network of three nodes with controllers to get the generalized synchronization is given by the following system:

$$\begin{cases} \varepsilon u_{1t} = f(u_1) - v_1 + I \\ \qquad -g_{syn}(u_1 - u_2) - g_{syn}(u_1 - u_3), \\ u_{1t} = au_1 - bv_1 + c, \\ \varepsilon u_{2t} = f(u_2) - v_2 + I \\ \qquad -g_{syn}(u_2 - u_1) - g_{syn}(u_2 - u_3) + w_2, \\ v_{2t} = au_2 - bv_2 + c + \overline{w}_2, \\ \varepsilon u_{3t} = f(u_3) - v_3 + I \\ \qquad -g_{syn}(u_3 - u_1) - g_{syn}(u_3 - u_2) + w_3, \\ v_{3t} = au_3 - bv_3 + c + \overline{w}_3, \end{cases} \tag{20}$$

where

$$\begin{cases} w_2 = \varepsilon \dfrac{\partial \phi_2(u_1)}{\partial u_1} u_{1t} - f(\phi_2(u_1)) + \varphi_2(v_1) - I \\ +g_{syn}(\phi_2(u_1) - u_1) + g_{syn}(\phi_2(u_1) - u_3) - k_2 e_2, \\ \overline{w}_2 = \dfrac{\partial \varphi_2(v_1)}{\partial v_1} v_{1t} - a\phi_2(u_1) + b\varphi_2(v_1) - c, \\ w_3 = \varepsilon \dfrac{\partial \phi_3(u_1)}{\partial u_1} u_{1t} - f(\phi_3(u_1)) + \varphi_3(v_1) - I \\ +g_{syn}(\phi_3(u_1) - u_1) + g_{syn}(\phi_3(u_1) - u_2) - k_3 e_3, \\ \overline{w}_3 = \dfrac{\partial \varphi_3(v_1)}{\partial v_1} v_{1t} - a\phi_3(u_1) + b\varphi_3(v_1) - c, \end{cases} \tag{21}$$

with the updated rule $k_{it} = r_i e_i^2, i = 2, 3$, and $e_i = u_i - \phi_i(u_1), \overline{e}_i = v_i - \varphi_i(v_1)$, where $\phi_i, \varphi_i, i = 2, 3$ are the continuously differential functions. Let $\sum_{i=2}^{3} |e_i| + |\overline{e}_i| =$

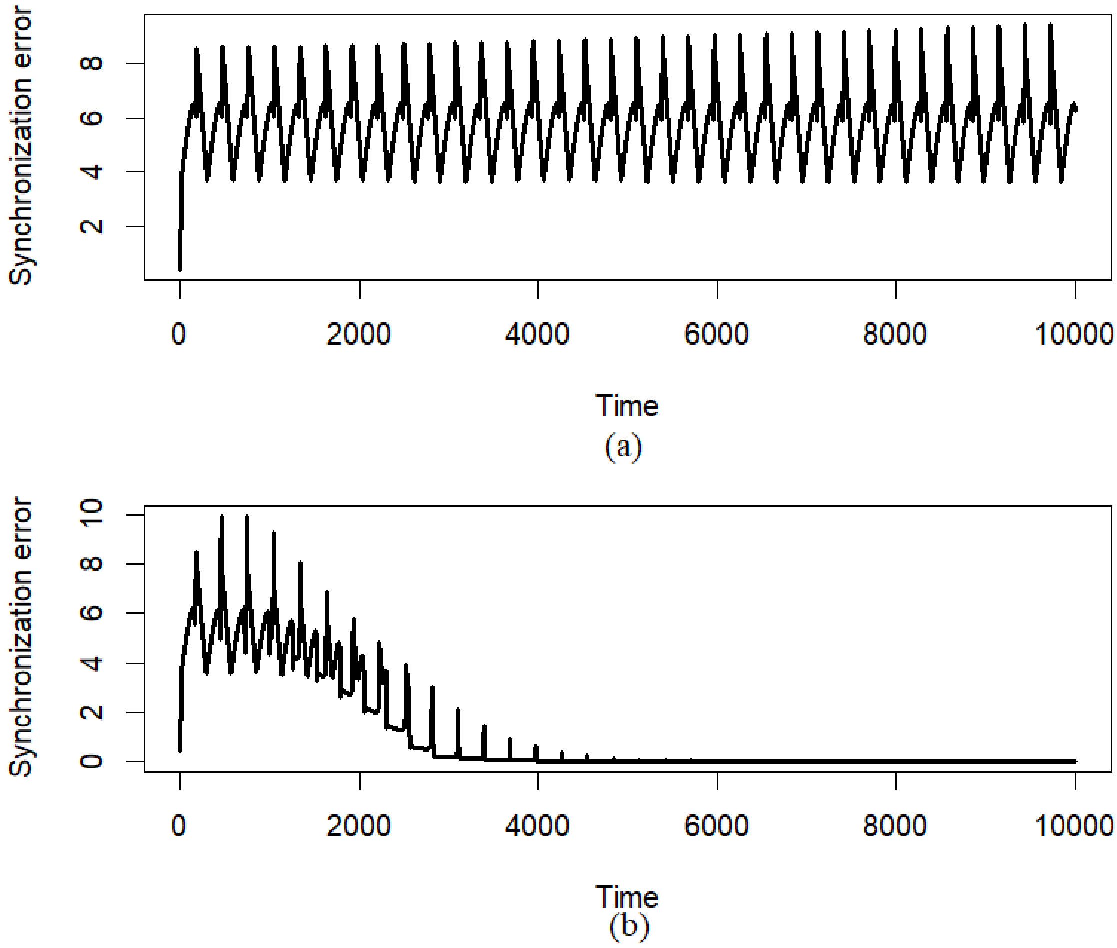

Fig. 2. Identical synchronization errors of the network (17) with respect to different values of coupling strength: (a) $g_{syn} = 0.001$; (b) $g_{syn} = 0.05$.

$\sum_{i=2}^{3} |u_i - \phi_i(u_1)| + |v_i - \varphi_i(v_1)|$ be generalized the synchronization error. We say that the network (20) generally synchronizes if the generalized synchronization error reaches zero as t approaches infinity.

Fig. 6 represents the generalized synchronization error between nodes of network (20), where we take the initial conditions as follows:

$$(u_1(0), v_1(0), u_2(0), v_2(0), u_3(0), v_3(0)) =$$
$$(-0.1, 0, 0, 0.1, 0, -0.1);$$
$$\phi_2(x) = -\cos^2 x + x; \varphi_2(x) = x^2 + 1;$$
$$\phi_3(x) = \sin x + x; \varphi_3(x) = 10x^2 + 1;$$

and

$$r_2 = 0.1, r_3 = 0.2, t \in [0; 50000].$$

Fig. 6(a) presents the generalized synchronization error with respect to t of the network (20) without controller (21). We can see that it does not reach zero, which means the generalized synchronization does not occur. However, Fig. 6(b) presents the generalized synchronization error of the network (20) with controller (21), and it actually reaches zero. In other words, the network (20) achieves generalized synchronization. It means the controller added in such a network effectively works.

Specifically, we first simulate the system (20) without controller (21). In Fig. 6(a), we take $g_{syn} = 0.001$, the generalized synchronization error does not reach zero. Even if we take a very large value of $g_{syn} = 9.5$, see Fig. 6(b), the generalized synchronization error also does not reach zero which means the generalized synchronization does not occur without controller. Clearly, Fig. 7 and Fig. 8 represent the time series of all variables of the system (20) without controller (21). In Fig. 7(a), the variable $\phi_2(u_1)$ is presented by the solid line, and the dotted line for u_2 (respectively, $\varphi_2(v_1)$ and v_2 in Fig. 7(b)). In Fig. 8(a), the variable $\phi_3(u_1)$ is presented by the solid line, and the dotted line for u_3 (respectively, $\varphi_3(v_1)$ and v_3 in Fig. 8(b)). We can see that the solid lines do not copy the behaviour of the dotted ones. In other words, the generalized synchronization phenomenon does not occur in this case.

Next, we simulate the system (20) with controller (21). In Fig. 6(c), we take $g_{syn} = 0.001$, the generalized synchronization error reaches zero which means the generalized synchronization occurs with controller. In other words, the controller is effective. Clearly, Fig. 9 and Fig. 10 represent the time series of all variables of the system (20) without controller (21). In Fig. 9(a), the variable $\phi_2(u_1)$ is presented by the solid line, and the dotted line for u_2 (respectively, $\varphi_2(v_1)$ and v_2 in Fig. 9(b)). In Fig. 10(a), the variable $\phi_3(u_1)$ is presented by the solid line, and the dotted line for u_3 (respectively, $\varphi_3(v_1)$ and v_3 in Fig. 10(b)). We can see that

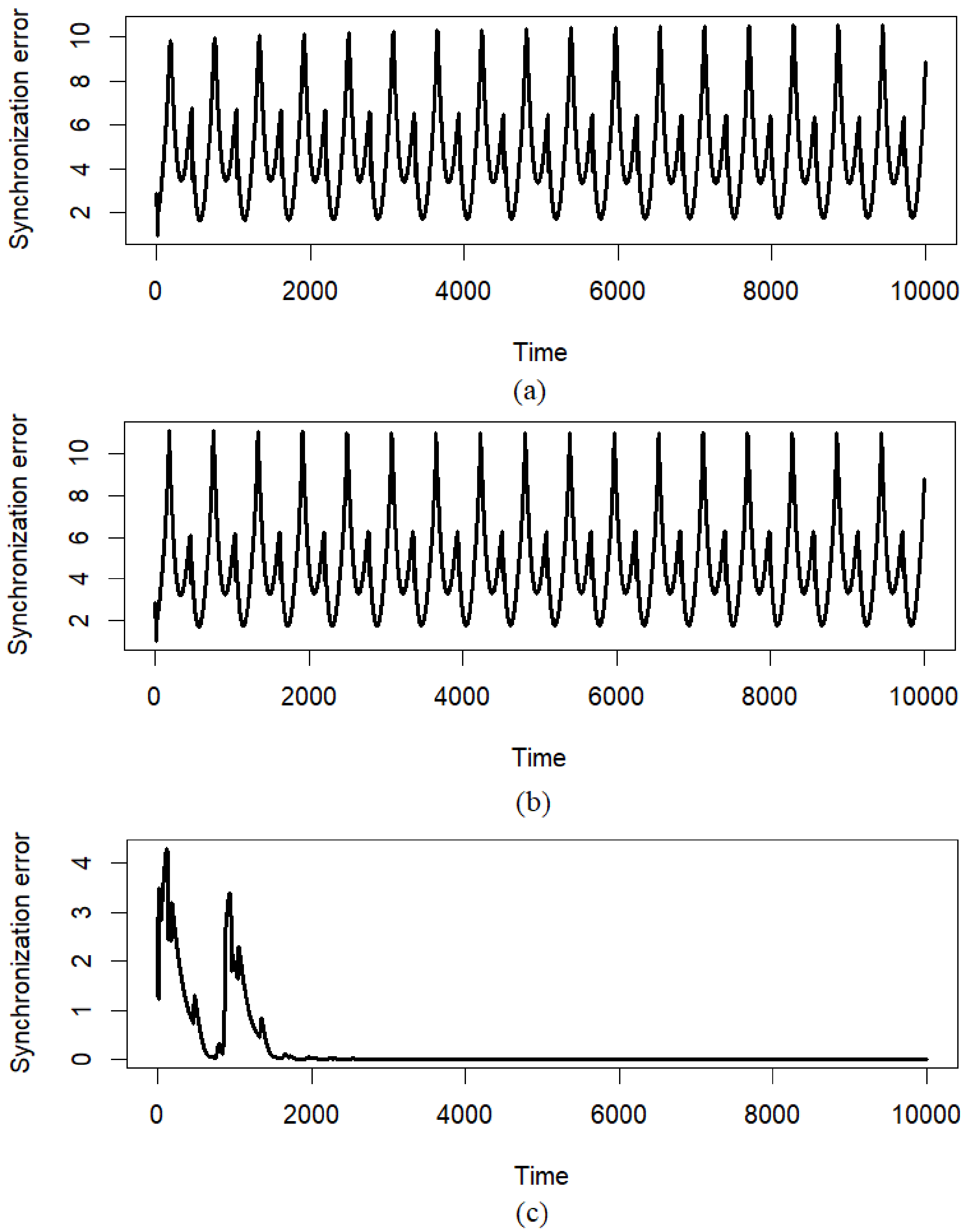

Fig. 3. Generalized synchronization errors between nodes of the network (18).

the solid lines copies the behaviour of the dotted ones. In other words, the generalized synchronization phenomenon occurs in this case.

IV. Conclusion

This paper presents sufficient conditions on the coupling strength required to achieve identical synchronization and generalized synchronization in full networks of n linearly coupled dynamical systems of the FitzHugh-Nagumo type. The study also explores the relationship between these types of synchronization. The results indicate that both types of synchronization occur when the coupling strength exceeds certain threshold values. Additionally, identical synchronization is found to be a specific case of generalized synchronization. To achieve generalized synchronization, controllers were designed for the network, and numerical results demonstrate their effective implementation. Moreover, further research is needed to investigate different synchronization regimes for various network topologies.

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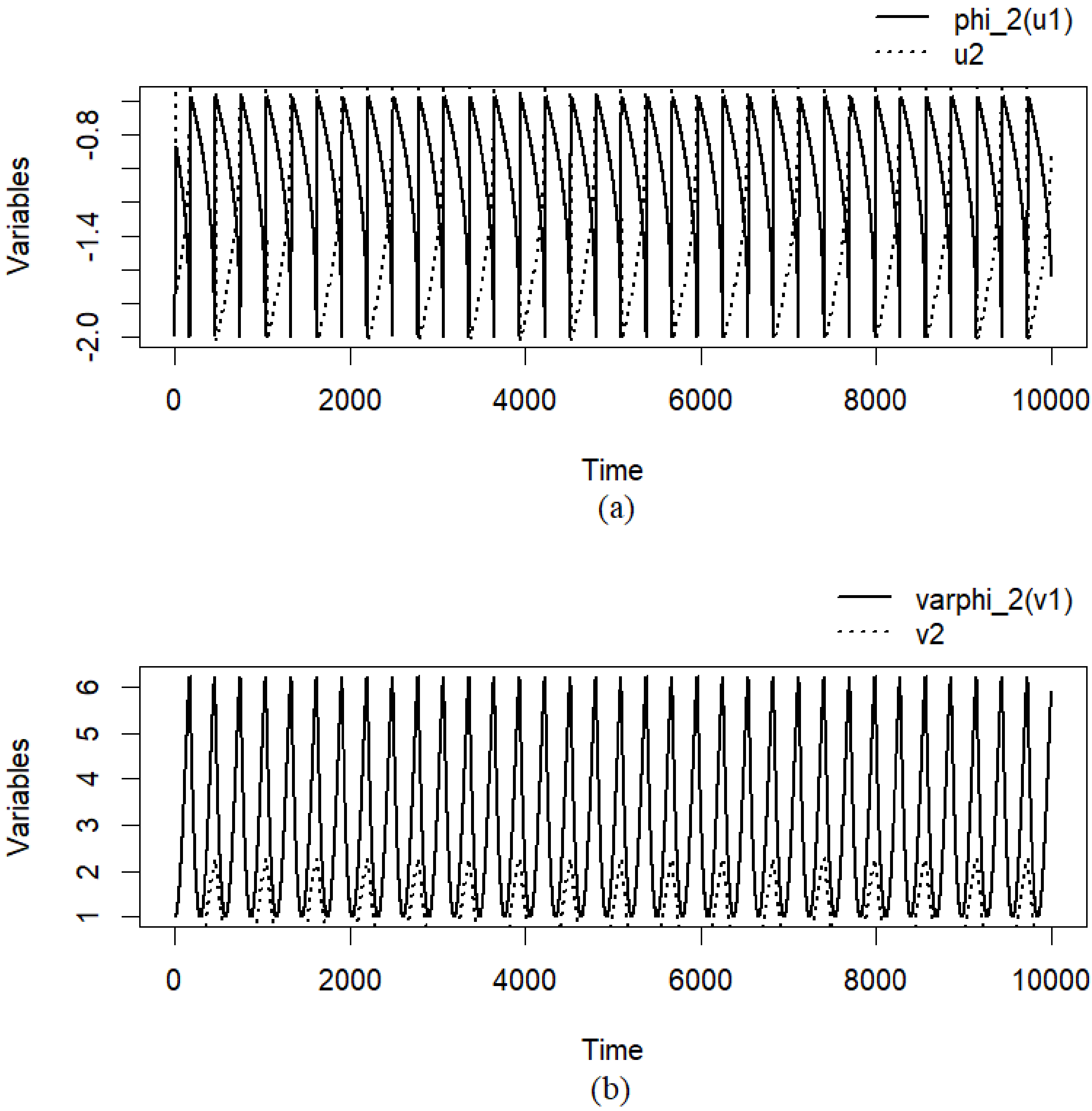

Fig. 4. Time series of all variables of the system (18) without controller (19) accroding to the coupling strength $g_{syn} = 0.01$, and $t \in [0; 10000]$.

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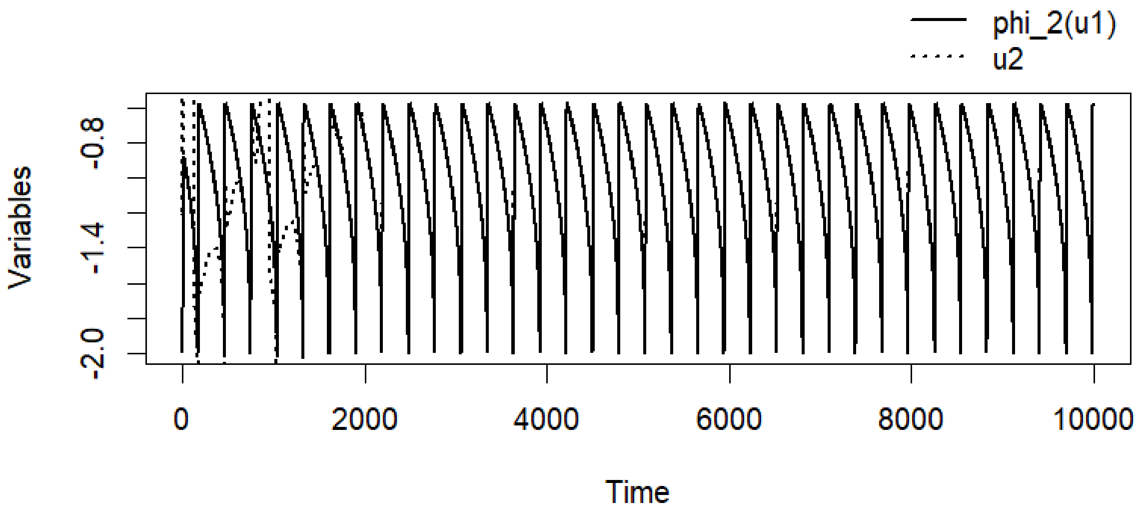
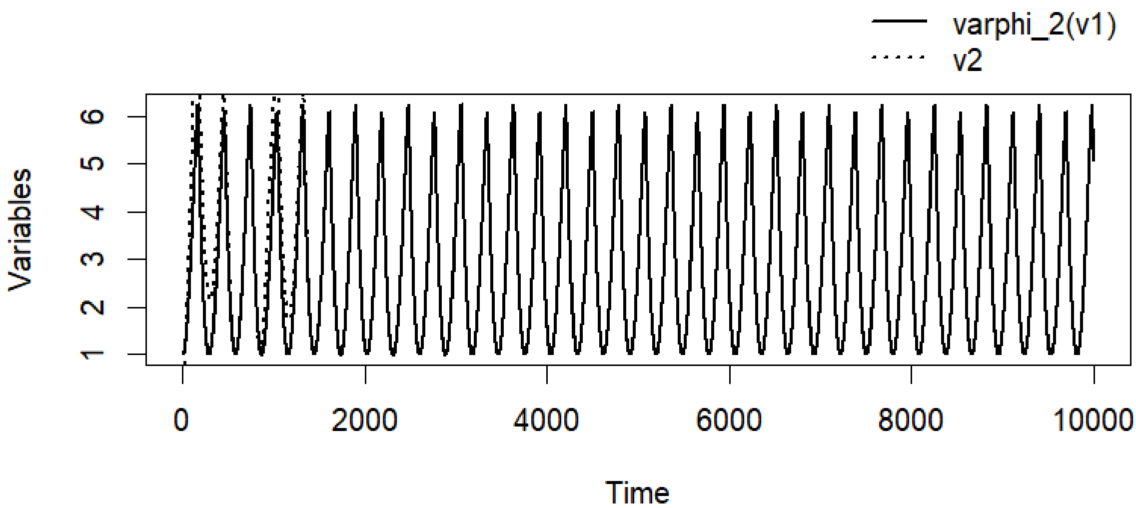


(a)



(b)

Fig. 5. Time series of all variables of the system (18) with controller (19) accroding to the coupling strength $g_{syn} = 0.01$, and $t \in [0; 10000]$.

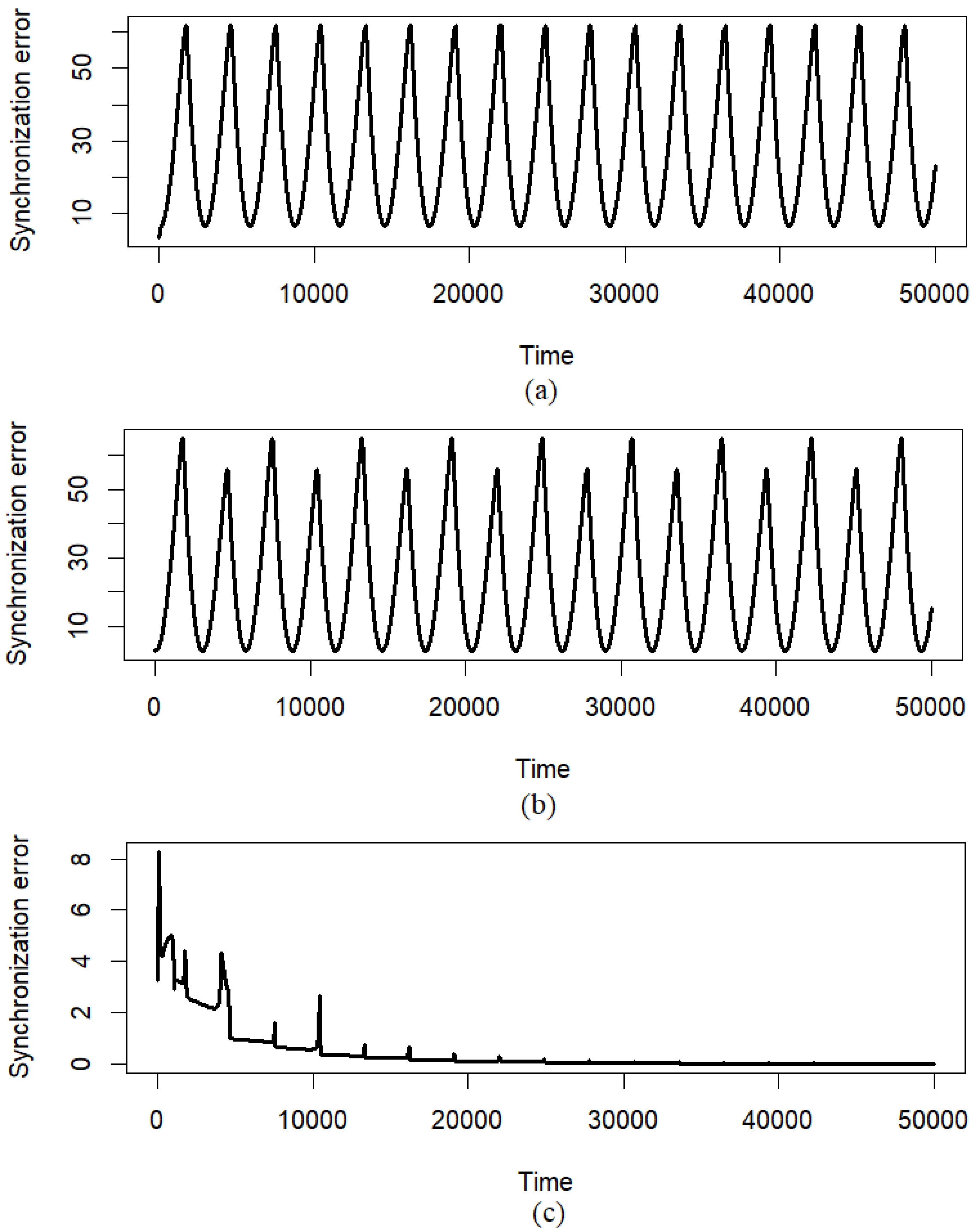


Fig. 6. Generalized synchronization errors between nodes of the network (20).

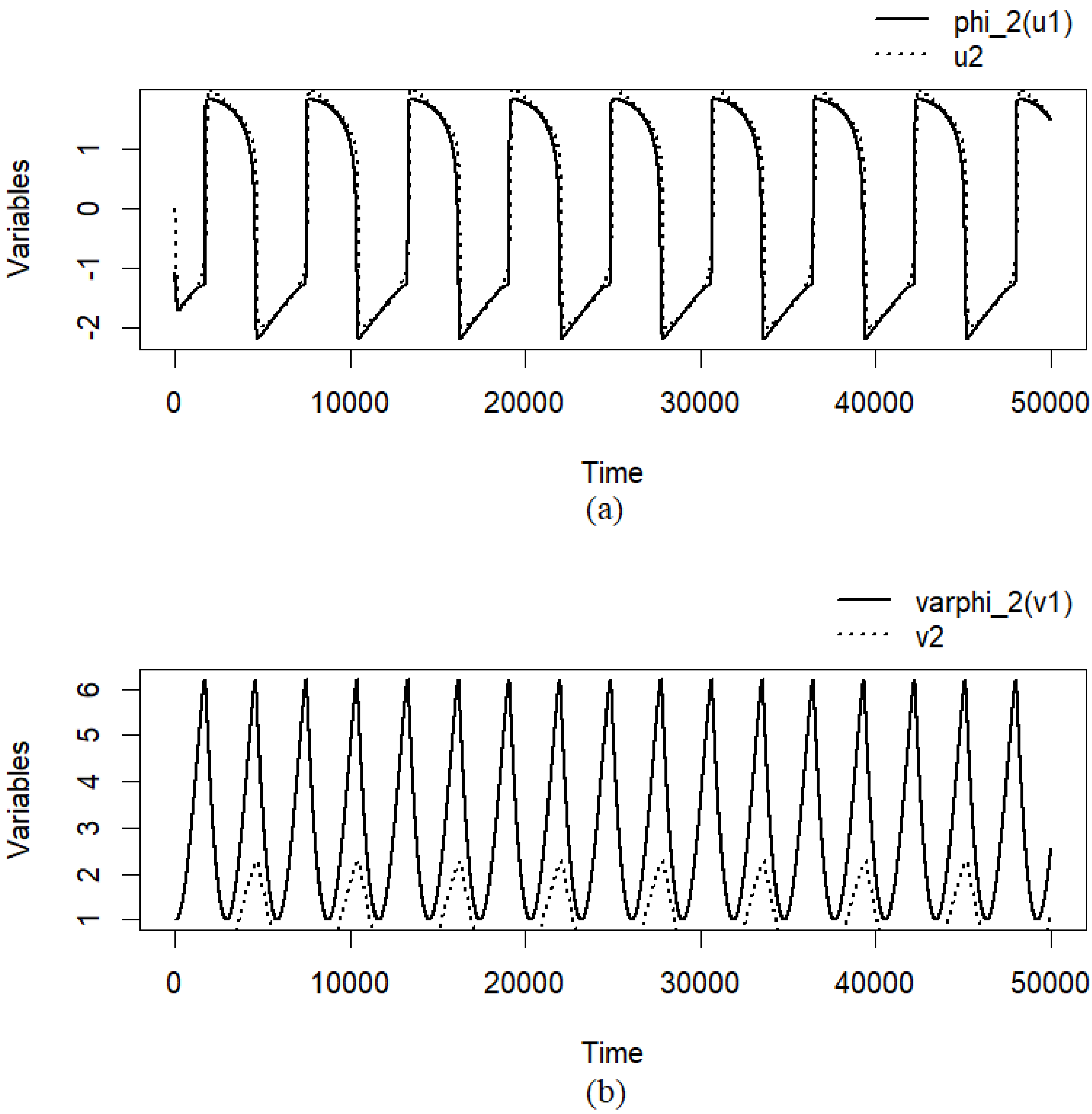
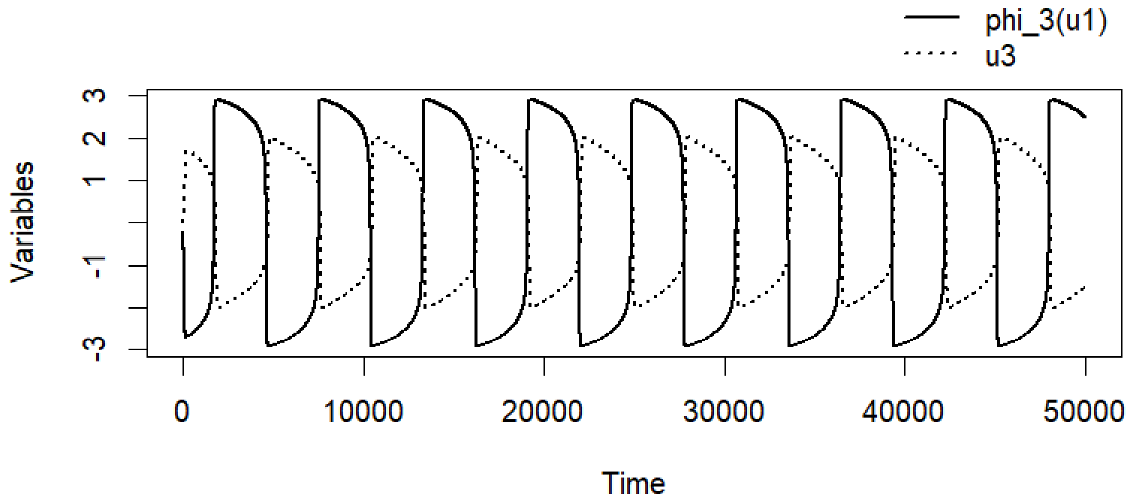
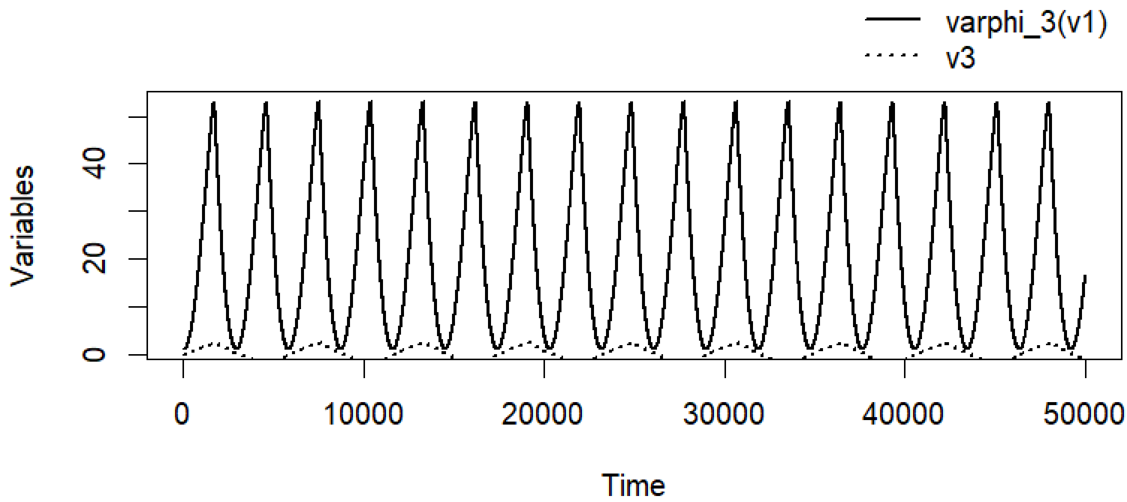


Fig. 7. Time series of all variables of the system (20) without controller (21) accroding to the coupling strength $g_{syn} = 0.001$, and $t \in [0; 50000]$.



(a)



(b)

Fig. 8. Time series of all variables of the system (20) without controller (21) accroding to the coupling strength $g_{syn} = 0.001$, and $t \in [0; 50000]$.

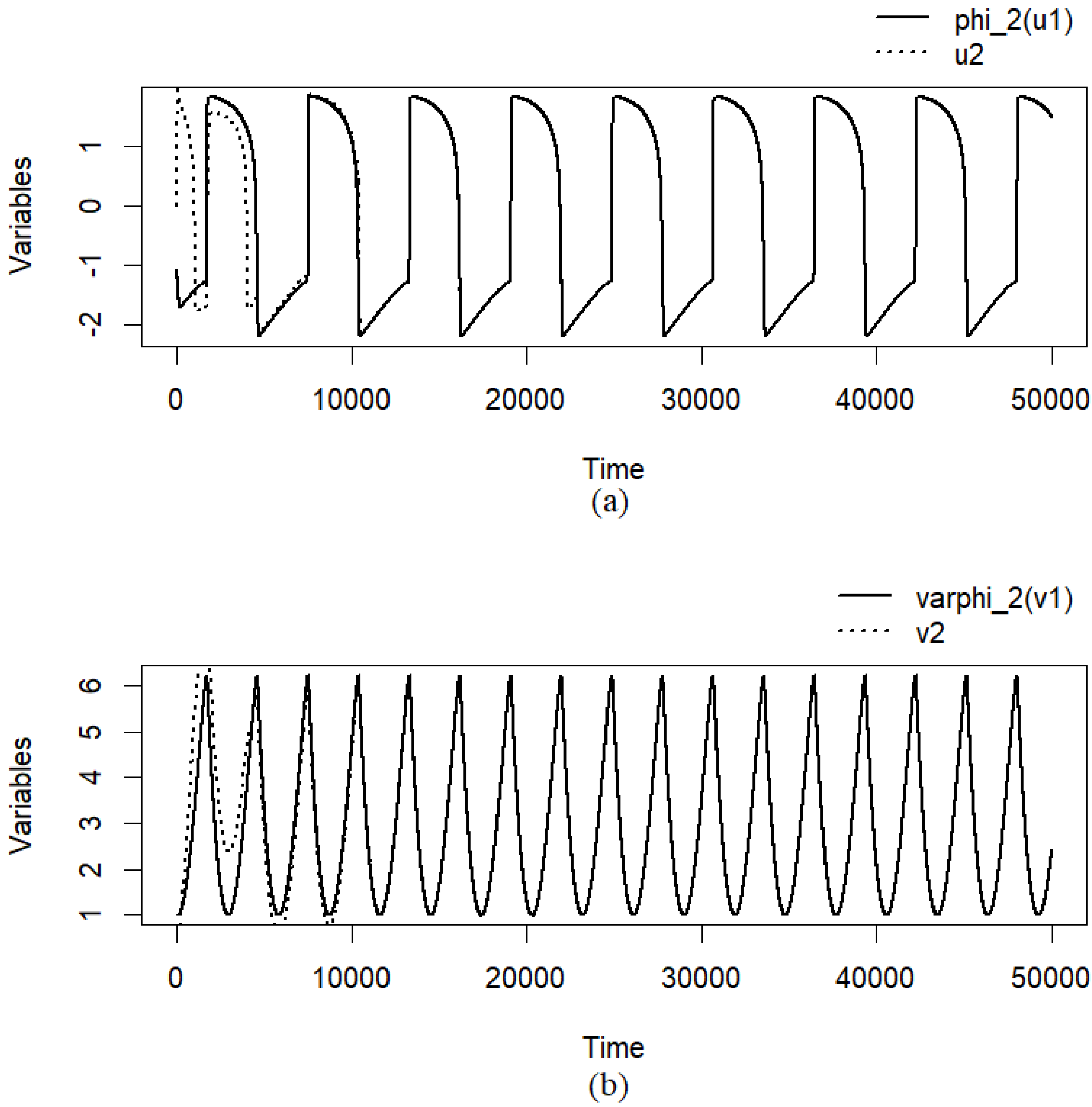


Fig. 9. Time series of all variables of the system (20) with controller (21) accroding to the coupling strength $g_{syn} = 0.001$, and $t \in [0; 50000]$.

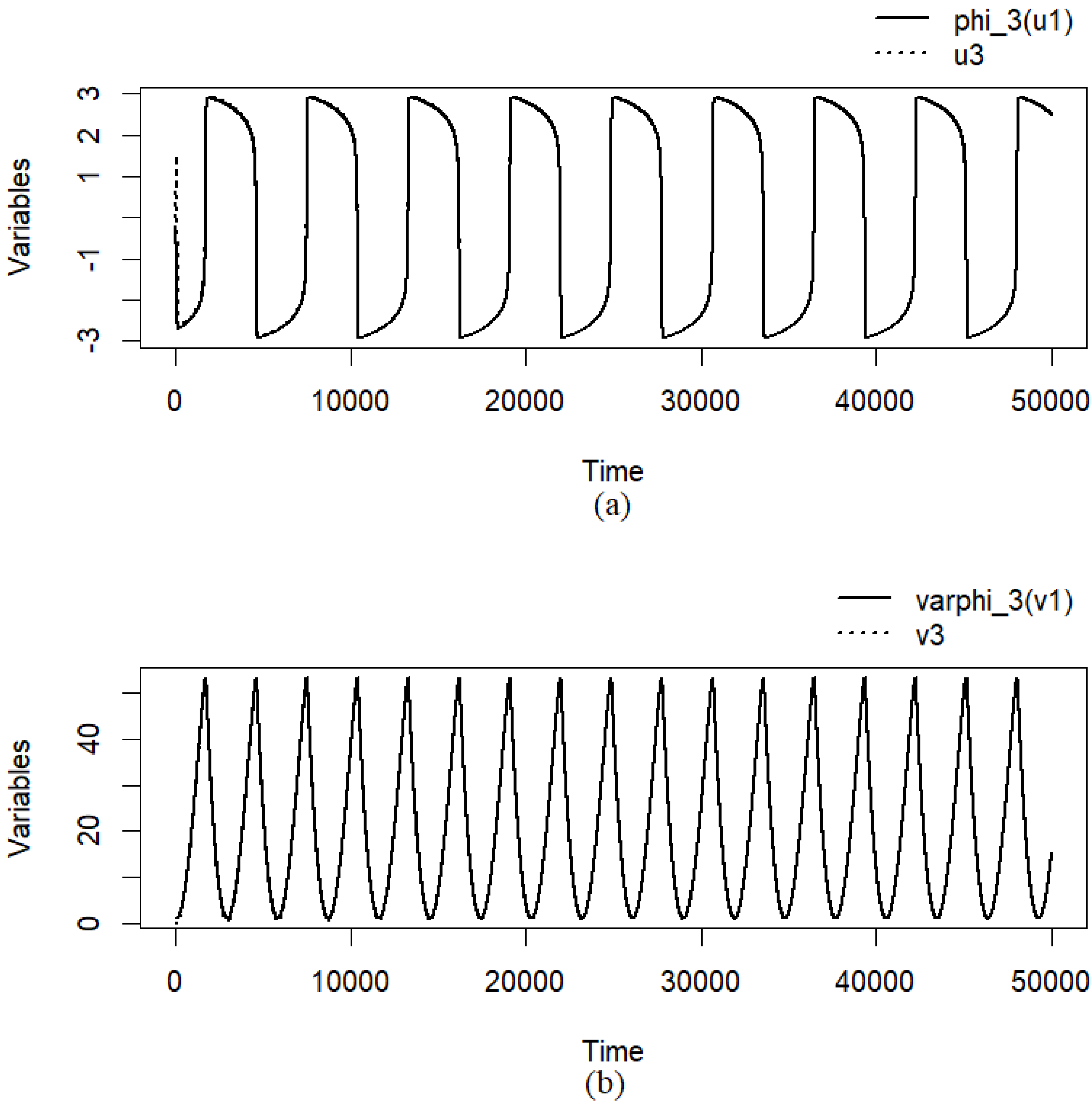


Fig. 10. Time series of all variables of the system (20) with controller (21) accroding to the coupling strength $g_{syn} = 0.001$, and $t \in [0; 50000]$.