Modeling and Analysis of Hybrid Systems Based on Time Interval Hybrid Petri Nets for State Estimation

Khouloud Soltani, Hajer Mlayeh, Atef Khedher

Abstract—**The aim of this study is to analyze hybrid systems modeled using hybrid Petri nets at different time intervals. A new form of the state model is thus proposed, which explains the time-changing of the system. It provides the limit bounds for each element of the state vector. A new form of the state calculator is then set up to evaluate the overall state of the system as well as the inputs. The limit bounds are also provided by this estimator. Counting and dating methods are used to develop the state model and estimator. In the counting method, the system state is the number of triggered transitions, while in the dating method, it is the dates of these triggers. The obtained model and the estimator have the form $b_{min} \leq Ax \leq b_{max}$. They are applied to analyze a production process modeled by a time interval hybrid Petri net. This application highlights the robustness of the estimates of the state and the inputs. Finally, these models are also used to detect, locate and estimate defects, and the obtained results can be considered satisfactory.**

Index Terms—**hybrid Petri Nets, time intervals, state model, state observer, defect detection, defect estimation**

I. Introduction

Regarding Discrete Event Systems (DESs), it is essential to evaluate and detect faults in order to ensure the smooth operation and reliability of industrial processes [3], [4]. As modeling tools, Petri Nets (PNs) represents these systems graphically and mathematically, which allows a better understanding of their dynamics [23], [24]. This approach is extended by the use of Hybrid PNs (HPNs) to model hybrid systems that combine continuous and discrete behaviors. Time intervals are considered when it is crucial to model systems where the timing of events significantly impacts the overall behavior [1], [2], [15].

HPNs with time intervals take into account specific temporal constraints, such as the start delays of transitions and the residence times of tokens in places. Thanks to this temporal modeling, it is possible to simulate the system evolution while identifying undesirable situations. It is therefore significant to estimate the system state and detect any potential failures in order to avoid costly breakdowns and ensure the continuity of operations [5], [6]. In this situation, the analysis of the system state relies on advanced estimation methods, which provide valuable data on the performance and reliability of the processes. Furthermore, to detect defects, it is necessary to use robust methods to identify, locate and assess anomalies that could disrupt the proper functioning of the system. In collaboration, these methods promote the improvement of hybrid system management by providing a proactive response to potential failures [7]–[9].

This work aims to evaluate and estimate the state of hybrid systems modeled using HPNs at several time intervals. By integrating the temporal aspect into modeling, our goal is to propose a new state model and a new observer to analyze the system behavior. These model and observer can be used to enhance the monitoring, control and diagnosis of hybrid systems.

Our contribution is to use dating and counting approaches for hybrid systems modeled by Time Interval HPNs (TIHPNs) to propose a new state model and an observer. The obtained state model and the observer take into account the discrete and continuous behaviors and have the form $b_{min} \leq Ax \leq b_{max}$ where b_{min} and b_{max} can be computed using system matrices. The elaborated model allows analyzing the time evolution of the system state, and the obtained observer estimates the system state over time. This kind of applications has not been done in the literature in the context of TIHPNs. For the counting method, the elements of the state vector are the numbers of triggers for each transition, whereas for the dating approach, they refer to the precise moments when these triggers occur. The state model and the observer are applied to a manufacturer system modeled by a TIHPN.

The paper is structured as follows. Section 2 sums up the related studies and provides the main differences between theses studies and our work. The fundamental concept of HPNs and TIHPNs is presented and detailed in section 3, and the used notations are also presented. The elaborated state model is detailed in section 4. This model is conceived following the counting and dating methods. Section 5 describes the design approach of the state observer, following also the counting and dating methods. The application of the obtained model and observer to a production system, already modeled by TIHPN, is presented in section 6, where both results of simulation and estimation are given in appropriate tables. The simulation results show the accuracy of the proposed model and observer for state and fault estimation.

II. Discussion of related studies

Numerous studies have addressed the issue of diagnosing and estimating the state of DESs modeled by various PNs classes in relation to the current marking of PNs [10]–[13].

Manuscript received January 06, 2025; revised June 12, 2025.

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In [10], the authors used a P-timed PN model and applied it to the validation of critical systems. This application required some temporal validations and did not include system simulation and state estimation. In [11], the authors used labeled timed PNs to an online diagnosis of DESs and gave several reduction rules to simplify the considered system. The work presented in [12] used timed PNs for the design of pipelined circuits based on synchronization and temporal verification. In [13], the authors used Interval PNs for the regulation of industrial quality and the management of temporal intervals.

In [20], the authors modeled a biological regulatory network by stochastic PNs in order to simulate and survey it. Distributed systems were considered in [21] for the survey of performance and the evaluation of models . In [22], the authors proposed an adaptive PN to model a complex system.

Previous work has focused on modeling issues, temporal validation, or specific industrial applications. They modeled DESs by severel kinds of PNs such as stochastic, time or timed PNs. In our work, we consider hybrid systems modeled by TIHPNs. Hybrid systems combine a continuous behavior and a discrete event behavior which makes our work more general. Time intervals can also help consider system uncertainties. In summary, our research introduces a new state model and an observer able to simulate and estimate the state and detect faults in hybrid systems modeled by TIHPNs.

III. Fundamental concepts of TIHPNs

A. Notations

A TIHPN represents a variant of traditional PNs, which helps represent both continuous and discrete dynamics. It is characterized by the following tuple [14] :

$$N = (P, T, \text{Pre}, \text{Post}, M_0, \lambda, \xi)$$

where:

- $P = P_d \cup P_c$ represents the places set, where P_d denotes the discrete places and P_c denotes the continuous places,
- $T = T_d \cup T_c$ represents the transitions set, where T_d denotes the discrete transitions and T_c denotes the continuous ones,
- $\text{Pre} : P \times T \to \mathbb{N}$ is the pre-incidence function,
- $\text{Post} : P \times T \to \mathbb{N}$ is the post-incidence function,
- $M_0 : P \to \mathbb{R}^+$ is the initial marking,
- $\lambda : T_c \to \mathbb{R}^+$ is the rate function for continuous transitions,
- $\xi : T_d \to \mathbb{N}$ is the time function for discrete transitions,
- The M marking is defined as follows: $M : P \to \mathbb{R}^+$.
  - For a discrete place $p \in P_d$, $M(p) \in \mathbb{N}$.
  - For a continuous place $p \in P_c$, $M(p) \in \mathbb{R}^+$.

The incidence vector C is defined as follows:

$$C(p, t) = \text{Post}(p, t) - \text{Pre}(p, t)$$

The result of a transition $t \in T$ is possible if:

$$M(p) \geq \text{Pre}(p, t) \quad \forall p \in P$$

Once t is drawn, the M mark is updated as follows:

$$M^*(p) = M(p) + C(p, t) \quad \forall p \in P$$

For a continuous transition $t \in T_c$, the flow can be represented by:

$$f(t) = \lambda(t) \cdot \min_{p \in P_c} \left(\frac{M(p)}{\text{Pre}(p, t)} \right)$$

It is necessary to initiate a transition t within a given time interval $[d_{\text{min}}, d_{\text{max}}]$ after its activation.

$$d_{\text{min}} \leq t_{\text{current}} - t_{\text{activation}} \leq d_{\text{max}}.$$

If t is not triggered before d_{max}, a specific action may be enforced (e.g. forcing, cancellation, etc.).

Example: A sensor must send a signal between 2 and 10 seconds after detection.

In this paper, we denote by x the internal transitions, u the input transition, and y the output transition. The set of places can be decomposed into four subsets:

- P_{ux}: the set of places between the input and internal transitions,
- P_{xx}: the set of places between the internal transitions,
- P_{xy}: the set of places between the internal and output transitions,
- P_{uy}: the set of places directly linking the input and the output.

Similarly, we define:

- T_{ux}: the timing constraints of set places P_{ux},
- T_{xx}: the timing constraints of set places P_{xx},
- T_{xy}: the timing constraints of set places P_{xy},
- T_{uy}: the timing constraints of set places P_{uy}.

We also define:

- m_{ux}: the marking of set places P_{ux},
- m_{xx}: the marking of set places P_{xx},
- m_{xy}: the marking of set places P_{xy},
- m_{uy}: the marking of set places P_{uy}.

Additionally, we introduce:

- T^-: a column vector of the minimal residence times of a token in the places,
- T^+: a column vector of the maximal residence times of a token in the places,
- $x_i(t)$: a timing variable associated with transition x_i in a timed event graph,
- $\mathcal{X}_j(t)$: the set of transitions preceding transition $x_i(t)$,
- $\mathcal{X}_q(t)$: the set of transitions following transition $x_i(t)$.

For TIHPNs, transitions can be continuous or discrete. Therefore, the following notations are used:

$$u(t) = \begin{bmatrix} u_c(t) \\ u_d(t) \end{bmatrix}, \quad x(t) = \begin{bmatrix} x_c(t) \\ x_d(t) \end{bmatrix}, \quad y(t) = \begin{bmatrix} y_c(t) \\ y_d(t) \end{bmatrix}$$

where index c refers to continuous transitions and index d refers to discrete transitions. Moreover, index cc models the link between continuous places, index dd models the link between discrete places, index dc describes the link from discrete to continuous places, and index cd models the link from continuous to discrete places.

IV. Design of new state model

A. Objective

This part aims to design a new form of a state model which can be used to analyze and simulate the evolution of the state and the outputs of the system over time. The obtained state model is composed of two inequalities. The calculation of the limit bounds of the state is possible thanks to the first inequality, whereas the second one determines the limit bounds of the system outputs. This suggested state model is an extension of the model developed for classic HPNs in [15]–[19]. This model is constructed following the counting and dating methods.

B. Counting approach

The counting approach makes it possible to estimate the number of firings of each transition. Indeed, each element of the state vector corresponds to the number of firings of the relevant transition at each moment t. The system behavior can be described, for the four P_{ux}, P_{xx}, P_{xy} and P_{uy} sets, by inequalities (1) to (8), where the matrices w^+_{ux}, w^-_{ux}, w^+_{xx}, w^-_{xx}, w^+_{xy}, w^-_{xy}, w^+_{uy} and w^-_{uy} are the elements of the incidence matrices W^+ and W^- and with $\theta = t - T$.

$$(m_{ux}) + (w^{+cc}_{ux} u_c(\theta^+_{l,ux}) + w^{+cd}_{ux} u_d(\theta^+_{l,ux})) \leq w^{-cc}_{ux} x_c(t) + w^{-cd}_{ux} x_d(t) \leq (m_{ux}) + (w^{+cc}_{ux} u_c(\theta^-_{l,ux}) + w^{+cd}_{ux} u_d(\theta^-_{l,ux})) \quad (1)$$

$$(m_{ux}) + (w^{+dc}_{ux} u_c(\theta^+_{l,ux}) + w^{+dd}_{ux} u_d(\theta^+_{l,ux})) \leq w^{-dc}_{ux} x_c(t) + w^{-dd}_{ux} x_d(t) \leq (m_{ux}) + (w^{+dc}_{ux} u_c(\theta^-_{l,ux}) + w^{+dd}_{ux} u_d(\theta^-_{l,ux})) \quad (2)$$

$$(m_{xx}) + (w^{+cc}_{xx} x_c(\theta^+_{l,xx}) + w^{+cd}_{xx} x_d(\theta^+_{l,xx})) \leq w^{-cc}_{xx} x_c(t) + w^{-cd}_{xx} x_d(t) \leq (m_{xx}) + (w^{+cc}_{xx} x_c(\theta^-_{l,ux}) + w^{+cd}_{xx} x_d(\theta^-_{l,xx})) \quad (3)$$

$$(m_{xx}) + (w^{+dc}_{xx} x_c(\theta^+_{l,xx}) + w^{+dd}_{xx} x_d(\theta^+_{l,xx})) \leq w^{-dc}_{xx} x_c(t) + w^{-dd}_{xx} x_d(t) \leq (m_{xx}) + (w^{+dc}_{xx} x_c(\theta^-_{l,xx}) + w^{+dd}_{xx} x_d(\theta^-_{l,xx})) \quad (4)$$

$$(m_{xy}) + (w^{+cc}_{xy} x_c(\theta^+_{l,xy}) + w^{+cd}_{xy} x_d(\theta^+_{l,xy})) \leq w^{-cc}_{xy} y_c(t) + w^{-cd}_{xy} y_d(t) \leq (m_{xy}) + (w^{+cc}_{xy} x_c(\theta^-_{l,xy}) + w^{+cd}_{xy} x_d(\theta^-_{l,xy})) \quad (5)$$

$$(m_{xy}) + (w^{+dc}_{xy} x_c(\theta^+_{l,xy}) + w^{+dd}_{xy} x_d(\theta^+_{l,xy})) \leq w^{-dc}_{xy} y_c(t) + w^{-dd}_{xy} y_d(t) \leq (m_{xy}) + (w^{+dc}_{xy} x_c(\theta^-_{l,xy}) + w^{+dd}_{xy} x_d(\theta^-_{l,xy})) \quad (6)$$

$$(m_{uy}) + (w^{+cc}_{uy} u_c(\theta^+_{l,uy}) + w^{+cd}_{uy} u_d(\theta^+_{l,uy})) \leq w^{-cc}_{uy} y_c(t) + w^{-cd}_{uy} y_d(t) \leq (m_{uy}) + (w^{+cc}_{uy} u_c(\theta^-_{l,uy}) + w^{+cd}_{uy} u_d(\theta^-_{l,uy})) \quad (7)$$

$$(m_{uy}) + (w^{+dc}_{uy} u_c(\theta^+_{l,uy}) + w^{+dd}_{uy} u_d(\theta^+_{l,uy})) \leq w^{-dc}_{uy} y_c(t) + w^{-dd}_{uy} y_d(t) \leq (m_{uy}) + (w^{+dc}_{uy} u_c(\theta^-_{l,uy}) + w^{+dd}_{uy} u_d(\theta^-_{l,uy})) \quad (8)$$

The system must evolve in a non-decreasing way, so the following inequalities (9) to (12) are added:

$$x_c(\theta^+_{l,xx}) \leq x_c(t) \leq x_c(\theta^-_{l,xx}), \quad (9)$$
$$x_d(\theta^+_{l,xx}) \leq x_d(t) \leq x_d(\theta^-_{l,xx}), \quad (10)$$
$$y_c(\theta^+_{l,uy}) \leq y_c(t) \leq y_c(\theta^-_{l,uy}), \quad (11)$$
$$y_d(\theta^+_{l,uy}) \leq y_d(t) \leq y_d(\theta^-_{l,uy}). \quad (12)$$

Inequalities (31) to (12) can be aggregated to design state model (13). The new state model is provided by Theorem 1.

Theorem 1: A TIHPN can be described by state model (13) using the counting method:

$$\begin{cases} \Phi^c_m \leq \mathcal{A}^-_{cd} \begin{bmatrix} x_c(t) \\ x_d(t) \end{bmatrix} \leq \Phi^c_M \\ \Psi^c_m \leq \mathcal{C}^-_{cd} \begin{bmatrix} y_c(t) \\ y_d(t) \end{bmatrix} \leq \Psi^c_M \end{cases} \quad (13)$$

with

$$\begin{cases} \Phi^c_m = \mathcal{A}^+_{cd} \begin{bmatrix} x_c(\theta^+_{l,xx}) \\ x_d(\theta^+_{l,xx}) \end{bmatrix} + \mathcal{B}^+_{cd} \begin{bmatrix} u_c(\theta^+_{l,ux}) \\ u_d(\theta^+_{l,ux}) \end{bmatrix} + [M_x] \\ \Phi^c_M = \mathcal{A}^+_{cd} \begin{bmatrix} x_c(\theta^-_{l,xx}) \\ x_d(\theta^-_{l,xx}) \end{bmatrix} + \mathcal{B}^+_{cd} \begin{bmatrix} u_c(\theta^-_{l,ux}) \\ u_d(\theta^-_{l,ux}) \end{bmatrix} + [M_x] \end{cases} \quad (14)$$

and

$$\begin{cases} \Psi^c_m = \mathcal{C}^+_{cd} \begin{bmatrix} x_c(\theta^+_{l,xy}) \\ x_d(\theta^+_{l,xy}) \end{bmatrix} + \mathcal{D}^+_{cd} \begin{bmatrix} u_c(\theta^+_{l,uy}) \\ u_d(\theta^+_{l,uy}) \end{bmatrix} + \mathcal{I} \begin{bmatrix} y_c(\theta^+_{l,uy}) \\ y_d(\theta^+_{l,uy}) \end{bmatrix} + [M_y] \\ \Psi^c_M = \mathcal{C}^+_{cd} \begin{bmatrix} x_c(\theta^-_{l,xy}) \\ x_d(\theta^-_{l,xy}) \end{bmatrix} + \mathcal{D}^+_{cd} \begin{bmatrix} u_c(\theta^-_{l,uy}) \\ u_d(\theta^-_{l,uy}) \end{bmatrix} + \mathcal{I} \begin{bmatrix} y_c(\theta^-_{l,uy}) \\ y_d(\theta^-_{l,uy}) \end{bmatrix} + [M_y] \end{cases} \quad (15)$$

where : I is the identity matrix and :

$$\mathcal{A}^+_{cd} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ w^{+cc}_{xx} & w^{+cd}_{xx} \\ w^{+dc}_{xx} & w^{+dd}_{xx} \\ I & I \end{bmatrix}, \mathcal{B}^+_{cd} = \begin{bmatrix} w^{+cc}_{ux} & w^{+cd}_{ux} \\ w^{+dc}_{ux} & w^{+dd}_{ux} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, M_x = \begin{bmatrix} m_{ux} \\ m_{ux} \\ m_{xx} \\ m_{xx} \\ 0 \end{bmatrix}$$

$$\mathcal{C}^+_{cd} = \begin{bmatrix} w^{+cc}_{xy} & w^{+cd}_{xy} \\ w^{+dc}_{xy} & w^{+dd}_{xy} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \mathcal{D}^+_{cd} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ w^{+cc}_{uy} & w^{+cd}_{uy} \\ w^{+dc}_{uy} & w^{+dd}_{uy} \\ 0 & 0 \end{bmatrix}, M_y = \begin{bmatrix} m_{xy} \\ m_{xy} \\ m_{uy} \\ m_{uy} \\ 0 \end{bmatrix}$$

$$\mathcal{A}^-_{cd} = \begin{bmatrix} w^{-cc}_{ux} & w^{-cd}_{ux} \\ w^{-cc}_{xx} & w^{-cd}_{xx} \\ w^{-dc}_{ux} & w^{-dd}_{ux} \\ w^{-dc}_{xx} & w^{-dd}_{xx} \\ I & I \end{bmatrix}, \mathcal{C}^-_{cd} = \begin{bmatrix} w^{-cc}_{xy} & w^{-cd}_{xy} \\ w^{-cc}_{uy} & w^{-cd}_{uy} \\ w^{-dc}_{xy} & w^{-dd}_{xy} \\ w^{-dc}_{uy} & w^{-dd}_{uy} \\ I & I \end{bmatrix}, \text{and} \mathcal{I} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I \end{bmatrix} \blacksquare$$

The proof of Theorem 1 is given in Appendix A.

C. Dating approach

The dating method assigns a specific date to each transition when it can be initiated. Each element of the state vector indicates the possible start dates for the relevant transition. The system behavior can be described, for the four P_{ux}, P_{xx}, P_{xy} and P_{uy} sets, by the following inequalities, with $\mu = k - m$.

$$(T^-_{ux}) + (w^{+cc}_{ux} u_c(\mu^-_{ux}) + w^{+cd}_{ux} u_d(\mu^-_{ux})) \leq w^{-cc}_{ux} x_c(k) + w^{-cd}_{ux} x_d(k) \leq (T^+_{ux}) + (w^{+cc}_{ux} u_c(\mu^+_{ux}) + w^{+cd}_{ux} u_d(\mu^+_{ux})) \quad (16)$$

$$(T^-_{ux}) + (w^{+dc}_{ux} u_c(\mu^-_{ux}) + w^{+dd}_{ux} u_d(\mu^-_{ux})) \leq w^{-dc}_{ux} x_c(k) + w^{-dd}_{ux} x_d(k) \leq (T^+_{ux}) + (w^{+dc}_{ux} u_c(\mu^+_{ux}) + w^{+dd}_{ux} u_d(\mu^+_{ux})) \quad (17)$$

$$(T^-_{xx}) + (w^{+cc}_{xx} u_c(\mu^-_{xx}) + w^{+cd}_{xx} u_d(\mu^-_{xx})) \leq w^{-cc}_{xx} x_c(k) + w^{-cd}_{xx} x_d(k) \leq (T^+_{xx}) + (w^{+cc}_{xx} u_c(\mu^+_{ux}) + w^{+cd}_{xx} u_d(\mu^+_{xx})) \quad (18)$$

$$(T^-_{xx}) + (w^{+dc}_{xx} u_c(\mu^-_{xx}) + w^{+dd}_{xx} u_d(\mu^-_{xx})) \leq w^{-dc}_{xx} x_c(k) + w^{-dd}_{xx} x_d(k) \leq (T^+_{xx}) + (w^{+dc}_{xx} u_c(\mu^-_{xx}) + w^{+dd}_{xx} u_d(\mu^-_{xx})) \quad (19)$$

$$(T^-_{xy}) + (w^{+cc}_{xy} u_c(\mu^-_{xy}) + w^{+cd}_{xy} u_d(\mu^-_{xy})) \leq w^{-cc}_{xy} y_c(k) + w^{-cd}_{xy} y_d(k) \leq (T^+_{xy}) + (w^{+cc}_{xy} u_c(\mu^+_{xy}) + w^{+cd}_{xy} u_d(\mu^+_{xy})) \quad (20)$$

$$(T^-_{xy}) + (w^{+dc}_{xy} u_c(\mu^-_{xy}) + w^{+dd}_{xy} u_d(\mu^-_{xy})) \leq w^{-dc}_{xy} y_c(k) + w^{-dd}_{xy} y_d(k) \leq (T^+_{xy}) + (w^{+dc}_{xy} u_c(\mu^+_{xy}) + w^{+dd}_{xy} u_d(\mu^+_{xy})) \quad (21)$$

$$(T^-_{uy}) + (w^{+cc}_{uy} u_c(\mu^-_{uy}) + w^{+cd}_{uy} u_d(\mu^-_{uy})) \leq w^{-cc}_{uy} y_c(k) + w^{-cd}_{uy} y_d(k) \leq (T^+_{uy}) + (w^{+cc}_{uy} u_c(\mu^+_{uy}) + w^{+cd}_{uy} u_d(\mu^+_{uy})) \quad (22)$$

$$(T^-_{uy}) + (w^{+dc}_{uy} u_c(\mu^-_{uy}) + w^{+dd}_{uy} u_d(\mu^-_{uy})) \leq w^{-dc}_{uy} y_c(k) + w^{-dd}_{uy} y_d(k) \leq (T^+_{uy}) + (w^{+dc}_{uy} u_c(\mu^+_{uy}) + w^{+dd}_{uy} u_d(\mu^+_{uy})) \quad (23)$$

To guarantee a non decreasing system behavior, the following inequalities are added:

$$T^-_{xx} + x_c(\mu^-_{xx}) \leq x_c(k) \leq T^+_{xx} + x_c(\mu^+_{xx}), \quad (24)$$
$$T^-_{xx} + x_d(\mu^-_{xx}) \leq x_d(k) \leq T^+_{xx} + x_d(\mu^+_{xx}), \quad (25)$$
$$T^-_{uy} + y_c(\mu^-_{uy}) \leq y_c(k) \leq T^+_{uy} + y_c(\mu^+_{uy}), \quad (26)$$
$$T^-_{uy} + y_d(\mu^-_{uy}) \leq y_d(k) \leq T^+_{uy} + y_d(\mu^+_{uy}). \quad (27)$$

Theorem 2: A TIHPN can be described by state model (28) using the dating method:

$$\begin{cases} \Phi_m^d \leq \mathcal{A}_{cd}^- \begin{bmatrix} x_c(k) \\ x_d(k) \end{bmatrix} \leq \Phi_M^d \\ \Psi_m^d \leq \mathcal{C}_{cd}^- \begin{bmatrix} y_c(k) \\ y_d(k) \end{bmatrix} \leq \Psi_M^d \end{cases} \tag{28}$$

with

$$\begin{cases} \Phi_m^d = \mathcal{A}_{cd}^+ \begin{bmatrix} x_c(\mu_{xx}^-) \\ x_d(\mu_{xx}^-) \end{bmatrix} + \mathcal{B}_{cd}^+ \begin{bmatrix} u_c(\mu_{ux}^-) \\ u_d(\mu_{ux}^-) \end{bmatrix} + \left[T_x^-\right] \\ \Phi_M^d = \mathcal{A}_{cd}^+ \begin{bmatrix} x_c(\mu_{xx}^+) \\ x_d(\mu_{xx}^+) \end{bmatrix} + \mathcal{B}_{cd}^+ \begin{bmatrix} u_c(\mu_{ux}^+) \\ u_d(\mu_{ux}^+) \end{bmatrix} + \left[T_x^+\right] \end{cases} \tag{29}$$

and

$$\begin{cases} \Psi_m^d = \mathcal{C}_{cd}^+ \begin{bmatrix} x_c(\mu_{xy}^-) \\ x_d(\mu_{xy}^-) \end{bmatrix} + \mathcal{D}_{cd}^+ \begin{bmatrix} u_c(\mu_{uy}^-) \\ u_d(\mu_{uy}^-) \end{bmatrix} + \mathcal{I} \begin{bmatrix} u_c(\mu_{uy}^-) \\ u_d(\mu_{uy}^-) \end{bmatrix} + \left[T_y^-\right] \\ \Psi_M^d = \mathcal{C}_{cd}^+ \begin{bmatrix} x_c(\mu_{xy}^+) \\ x_d(\mu_{xy}^+) \end{bmatrix} + \mathcal{D}_{cd}^+ \begin{bmatrix} u_c(\mu_{uy}^+) \\ u_d(\mu_{uy}^+) \end{bmatrix} + \mathcal{I} \begin{bmatrix} uc(\mu_{uy}^+) \\ u_d(\mu_{uy}^+) \end{bmatrix} + \left[T_y^+\right] \end{cases} \tag{30}$$

with

$$T_x^+ = \begin{bmatrix} T_{ux}^+ \\ T_{ux}^+ \\ T_{xx}^+ \\ T_{xx}^+ \\ 0 \end{bmatrix}, T_x^- = \begin{bmatrix} T_{ux}^- \\ T_{ux}^- \\ T_{xx}^- \\ T_{xx}^- \\ 0 \end{bmatrix}, T_y^+ = \begin{bmatrix} T_{uy}^+ \\ T_{uy}^+ \\ T_{xy}^+ \\ T_{xy}^+ \\ 0 \end{bmatrix}, T_y^- = \begin{bmatrix} T_{uy}^- \\ T_{uy}^- \\ T_{xy}^- \\ T_{xy}^- \\ 0 \end{bmatrix},$$

$$\mathcal{A}_{cd}^+ = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ w_{xx}^{+cc} & w_{xx}^{+cd} \\ w_{xx}^{+dc} & w_{xx}^{+dd} \\ I & I \end{bmatrix} \mathcal{B}_{cd}^+ = \begin{bmatrix} w_{ux}^{+cc} & w_{ux}^{+cd} \\ w_{ux}^{+dc} & w_{ux}^{+dd} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \mathcal{I} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I \end{bmatrix}$$

$$\mathcal{C}_{cd}^+ = \begin{bmatrix} w_{xy}^{+cc} & w_{xy}^{+cd} \\ w_{xy}^{+dc} & w_{xy}^{+dd} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \mathcal{D}_{cd}^+ = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ w_{uy}^{+cc} & w_{uy}^{+cd} \\ w_{uy}^{+dc} & w_{uy}^{+dd} \\ 0 & 0 \end{bmatrix}$$

$$\mathcal{A}_{cd}^- = \begin{bmatrix} w_{ux}^{-cc} & w_{ux}^{-cd} \\ w_{xx}^{-cc} & w_{xx}^{-cd} \\ w_{ux}^{-dc} & w_{ux}^{-dd} \\ w_{xx}^{-dc} & w_{xx}^{-dd} \\ I & I \end{bmatrix} \text{and} \mathcal{C}_{cd}^- = \begin{bmatrix} w_{xy}^{-cc} & w_{xy}^{-cd} \\ w_{uy}^{-cc} & w_{uy}^{-cd} \\ w_{xy}^{-dc} & w_{xy}^{-dd} \\ w_{uy}^{-dc} & w_{uy}^{-dd} \\ I & I \end{bmatrix} \blacksquare$$

The proof of Theorem 2 is given in Appendix B.

Remark 1: Systems (13) and (28) describe the obtained model for the counting and dating methods. These models help search the limit bounds of system state x and system output y. They guarantee that these limit bounds are finite and that the system trajectory is non-decreasing. At each moment, the state vector belong to the interval between the computed upper and the lower bounds. ■

V. Design of a State and Input Estimator

A. Objective

This part aims to conceive, following the counting and dating methods, a new estimator able to estimate the limit bounds of the system state and the input. This estimator is composed of two inequalities. The first inequality estimates the limit bounds of the state, whereas the second one estimates the limit bounds of the input. This observer extends the one developed in [15]–[19] to TIHPNs. This observer is elaborated following the counting and dating methods. For estimation, it is supposed that the system output is known.

B. Counting approach

Inequalities (1) to (8) can be rewritten in the following forms, with $\tau = t + T$.

$$\left(w_{ux}^{-cc}\hat{x}_c(\tau_{l,ux}^-) + w_{ux}^{-cd}\hat{x}_d(\tau_{l,ux}^-)\right) - (m_{ux}) \leq w_{ux}^{+cc}\hat{u}_c(t) + w_{ux}^{+cd}\hat{u}_d(t) \leq \left(w_{ux}^{-cc}\hat{x}_c(\tau_{l,ux}^+) + w_{ux}^{-cd}\hat{x}_d(\tau_{l,ux}^+)\right) - (m_{ux}) \tag{31}$$

$$\left(w_{ux}^{-dc}\hat{x}_c(\tau_{l,ux}^-) + w_{ux}^{-dd}\hat{x}_d(\tau_{l,ux}^-)\right) - (m_{ux}) \leq w_{ux}^{+dc}\hat{u}_c(t) + w_{ux}^{+dd}\hat{u}_d(t) \leq \left(w_{ux}^{-dc}\hat{x}_c(\tau_{l,ux}^+) + w_{ux}^{-dd}\hat{x}_d(\tau_{l,ux}^+)\right) - (m_{ux}) \tag{32}$$

$$\left(w_{xx}^{-cc}\hat{x}_c(\tau_{l,xx}^-) + w_{xx}^{-cd}\hat{x}_d(\tau_{l,xx}^-)\right) - (m_{xx}) \leq w_{xx}^{+cc}\hat{x}_c(t) + w_{xx}^{+cd}\hat{x}_d(t) \leq \left(w_{xx}^{-cc}\hat{x}_c(\tau_{l,ux}^+) + w_{xx}^{-cd}\hat{x}_d(\tau_{l,xx}^+)\right) - (m_{xx}) \tag{33}$$

$$\left(w_{xx}^{-dc}\hat{x}_c(\tau_{l,xx}^-) + w_{xx}^{-dd}\hat{x}_d(\tau_{l,xx}^-)\right) - (m_{xx}) \leq w_{xx}^{+dc}\hat{x}_c(t) + w_{xx}^{+dd}\hat{x}_d(t) \leq \left(w_{xx}^{-dc}\hat{x}_c(\tau_{l,xx}^+) + w_{xx}^{-dd}\hat{x}_d(\tau_{l,xx}^+)\right) - (m_{xx}) \tag{34}$$

$$\left(w_{xy}^{-cc}\hat{y}_c(\tau_{l,xy}^-) + w_{xy}^{-cd}\hat{y}_d(\tau_{l,xy}^-)\right) - (m_{xy}) \leq w_{xy}^{+cc}\hat{x}_c(t) + w_{xy}^{+cd}\hat{x}_d(t) \leq \left(w_{xy}^{-cc}\hat{y}_c(\tau_{l,xy}^+) + w_{xy}^{-cd}\hat{y}_d(\tau_{l,xy}^+)\right) - (m_{xy}) \tag{35}$$

$$\left(w_{xy}^{-dc}\hat{y}_c(\tau_{l,xy}^-) + w_{xy}^{-dd}\hat{y}_d(\tau_{l,xy}^-)\right) - (m_{xy}) \leq w_{xy}^{+dc}\hat{x}_c(t) + w_{xy}^{+dd}\hat{x}_d(t) \leq \left(w_{xy}^{-dc}\hat{y}_c(\tau_{l,xy}^+) + w_{xy}^{-dd}\hat{y}_d(\tau_{l,xy}^+)\right) - (m_{xy}) \tag{36}$$

$$\left(w_{uy}^{-cc}\hat{y}_c(\tau_{l,uy}^-) + w_{uy}^{-cd}\hat{y}_d(\tau_{l,uy}^-)\right) - (m_{uy}) \leq w_{uy}^{+cc}\hat{u}_c(t) + w_{uy}^{+cd}\hat{u}_d(t) \leq \left(w_{uy}^{-cc}\hat{y}_c(\tau_{l,uy}^+) + w_{uy}^{-cd}\hat{y}_d(\tau_{l,uy}^+)\right) - (m_{uy}) \tag{37}$$

$$\left(w_{uy}^{-dc}\hat{y}_c(\tau_{l,uy}^-) + w_{uy}^{-dd}\hat{y}_d(\tau_{l,uy}^-)\right) - (m_{uy}) \leq w_{uy}^{+dc}\hat{u}_c(t) + w_{uy}^{+dd}\hat{u}_d(t) \leq \left(w_{uy}^{-dc}\hat{y}_c(\tau_{l,uy}^+) + w_{uy}^{-dd}\hat{y}_d(\tau_{l,uy}^+)\right) - (m_{uy}) \tag{38}$$

The system must evolve in a non-decreasing way, so, the following conditions are added:

$$(m_{xx}) + (\hat{x}_c(\tau_{l,xx}^-)) \leq \hat{x}_c(t) \leq (m_{xx}) + (\hat{x}_c(\tau_{l,xx}^+)), \tag{39}$$

$$(m_{xx}) + (\hat{x}_d(\tau_{l,xx}^-)) \leq \hat{x}_d(t) \leq (m_{xx}) + (\hat{x}_d(\tau_{l,xx}^+)), \tag{40}$$

$$(m_{uy}) + (\hat{u}_c(\tau_{l,uy}^-)) \leq \hat{u}_c(t) \leq (m_{uy}) + (\hat{u}_c(\tau_{l,uy}^+)), \tag{41}$$

$$(m_{uy}) + (\hat{u}_d(\tau_{l,uy}^-)) \leq \hat{u}_d(t) \leq (m_{uy}) + (\hat{u}_d(\tau_{l,uy}^+)). \tag{42}$$

Inequalities (31) to (42) can be gathered to design state observer (43).

Theorem 3: A state estimator able to estimate the state of a TIHPN is given by equation (43) using the counting method:

$$\begin{cases} \hat{\Phi}_m^c \leq \mathcal{A}_{cd}^- \begin{bmatrix} \hat{x}_c(t) \\ \hat{x}_d(t) \end{bmatrix} \leq \hat{\Phi}_M^c \\ \hat{\Psi}_m^c \leq \mathcal{B}_{cd}^- \begin{bmatrix} \hat{u}_c(t) \\ \hat{u}_d(t) \end{bmatrix} \leq \hat{\Psi}_M^c \end{cases} \tag{43}$$

with

$$\begin{cases} \hat{\Phi}_m^c = \mathcal{A}_{cd}^+ \begin{bmatrix} \hat{x}_c(\tau_{l,xx}^-) \\ \hat{x}_d(\tau_{l,xx}^-) \end{bmatrix} - \mathcal{C}_{cd}^+ \begin{bmatrix} \hat{y}_c(\tau_{l,xy}^-) \\ \hat{y}_d(\tau_{l,xy}^-) \end{bmatrix} - \left[M_x\right] \\ \hat{\Phi}_M^c = \mathcal{A}_{cd}^+ \begin{bmatrix} \hat{x}_c(\tau_{l,xx}^+) \\ \hat{x}_d(\tau_{l,xx}^+) \end{bmatrix} + \mathcal{C}_{cd}^+ \begin{bmatrix} \hat{y}_c(\tau_{l,xy}^+) \\ \hat{y}_d(\tau_{l,xy}^+) \end{bmatrix} - \left[M_x\right] \end{cases} \tag{44}$$

and

$$\begin{cases} \hat{\Psi}_m^c = \mathcal{I} \begin{bmatrix} \hat{u}_c(\tau_{l,uy}^-) \\ \hat{u}_d(\tau_{l,uy}^-) \end{bmatrix} + \mathcal{B}_{cd}^+ \begin{bmatrix} \hat{x}_c(\tau_{l,ux}^-) \\ \hat{x}_d(\tau_{l,ux}^-) \end{bmatrix} + \mathcal{D}_{cd}^+ \begin{bmatrix} \hat{y}_c(\tau_{l,uy}^-) \\ \hat{y}_d(\tau_{l,uy}^-) \end{bmatrix} - \left[M_y\right] \\ \hat{\Psi}_M^c = \mathcal{I} \begin{bmatrix} \hat{u}_c(\tau_{l,uy}^+) \\ \hat{u}_d(\tau_{l,uy}^+) \end{bmatrix} + \mathcal{B}_{cd}^+ \begin{bmatrix} \hat{x}_c(\tau_{l,ux}^+) \\ \hat{x}_d(\tau_{l,ux}^+) \end{bmatrix} + \mathcal{D}_{cd}^+ \begin{bmatrix} \hat{y}_c(\tau_{l,uy}^+) \\ \hat{y}_d(\tau_{l,uy}^+) \end{bmatrix} - \left[M_y\right] \end{cases} \tag{45}$$

with

$$\mathcal{A}_{cd}^{+}=\begin{bmatrix} w_{xx}^{-cc} & w_{xx}^{-cd} \\ w_{xx}^{-dc} & w_{xx}^{-dd} \\ 0 & 0 \\ 0 & 0 \\ I & I \end{bmatrix}, \mathcal{B}_{cd}^{+}=\begin{bmatrix} w_{ux}^{-cc} & w_{ux}^{-cd} \\ w_{ux}^{-dc} & w_{ux}^{-dd} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, M_x=\begin{bmatrix} m_{ux} \\ m_{ux} \\ m_{xx} \\ m_{xx} \\ 0 \end{bmatrix},$$

$$\mathcal{C}_{cd}^{+}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ w_{xy}^{-cc} & w_{xy}^{-cd} \\ w_{xy}^{-dc} & w_{xy}^{-dd} \\ 0 & 0 \end{bmatrix}, \mathcal{D}_{cd}^{+}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ w_{uy}^{-cc} & w_{uy}^{-cd} \\ w_{uy}^{-dc} & w_{uy}^{-dd} \\ 0 & 0 \end{bmatrix}, M_y=\begin{bmatrix} m_{xy} \\ m_{xy} \\ m_{uy} \\ m_{uy} \\ 0 \end{bmatrix},$$

$$\mathcal{A}_{cd}^{-}=\begin{bmatrix} w_{xx}^{+cc} & w_{xx}^{+cd} \\ w_{xx}^{+cc} & w_{xx}^{+cd} \\ w_{xy}^{+dc} & w_{xy}^{+dd} \\ w_{xy}^{+dc} & w_{xy}^{+dd} \\ I & I \end{bmatrix}, \mathcal{B}_{cd}^{-}=\begin{bmatrix} w_{ux}^{+cc} & w_{ux}^{+cd} \\ w_{ux}^{+dc} & w_{ux}^{+dd} \\ w_{uy}^{+cc} & w_{uy}^{+cd} \\ w_{uy}^{+dc} & w_{uy}^{+dd} \\ I & I \end{bmatrix} \text{ and } \mathcal{I}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I \end{bmatrix} \blacksquare$$

The proof of the Theorem is given in Appendix C.

C. Dating approach

Inequalities (16) to (23) can be rewritten in the following form, with $\vartheta = k + m$:

$$\big(w_{ux}^{-cc}\hat{x}_c(\vartheta_{ux}^{+}) + w_{ux}^{-cd}\hat{x}_d(\vartheta_{ux}^{+})\big) - (T_{ux}^{+}) \le w_{ux}^{+cc}\hat{u}_c(k) + w_{ux}^{+cd}\hat{u}_d(k) \le \big(w_{ux}^{-cc}\hat{x}_c(\vartheta_{ux}^{-}) + w_{ux}^{-cd}\hat{x}_d(\vartheta_{ux}^{-})\big) - (T_{ux}^{-}) \quad (46)$$

$$\big(w_{ux}^{-dc}\hat{x}_c(\vartheta_{ux}^{+}) + w_{ux}^{-dd}\hat{x}_d(\vartheta_{ux}^{+})\big) - (T_{ux}^{+}) \le w_{ux}^{+dc}\hat{u}_c(k) + w_{ux}^{+dd}\hat{u}_d(k) \le \big(w_{ux}^{-dc}\hat{x}_c(\vartheta_{ux}^{-}) + w_{ux}^{-dd}\hat{x}_d(\vartheta_{ux}^{-})\big) - (T_{ux}^{-}) \quad (47)$$

$$\big(w_{xx}^{-cc}\hat{x}_c(\vartheta_{xx}^{+}) + w_{xx}^{-cd}\hat{x}_d(\vartheta_{xx}^{+})\big) - (T_{xx}^{+}) \le w_{xx}^{+cc}\hat{x}_c(k) + w_{xx}^{+cd}\hat{x}_d(k) \le \big(w_{xx}^{-cc}\hat{x}_c(\vartheta_{xx}^{-}) + w_{xx}^{-cd}\hat{x}_d(\vartheta_{xx}^{-})\big) - (T_{xx}^{-}) \quad (48)$$

$$\big(w_{xx}^{-dc}\hat{x}_c(\vartheta_{xx}^{+}) + w_{xx}^{-dd}\hat{x}_d(\vartheta_{xx}^{+})\big) - (T_{xx}^{+}) \le w_{xx}^{+dc}\hat{x}_c(k) + w_{xx}^{+dd}\hat{x}_d(k) \le \big(w_{xx}^{-dc}\hat{x}_c(\vartheta_{xx}^{-}) + w_{xx}^{-dd}\hat{x}_d(\vartheta_{xx}^{-})\big) - (T_{xx}^{-}) \quad (49)$$

$$\big(w_{xy}^{-cc}\hat{y}_c(\vartheta_{xy}^{+}) + w_{xy}^{-cd}\hat{y}_d(\vartheta_{xy}^{+})\big) - (T_{xy}^{+}) \le w_{xy}^{+cc}\hat{x}_c(k) + w_{xy}^{+cd}\hat{x}_d(k) \le \big(w_{xy}^{-cc}\hat{y}_c(\vartheta_{xy}^{-}) + w_{xy}^{-cd}\hat{y}_d(\vartheta_{xy}^{-})\big) - (T_{xy}^{-}) \quad (50)$$

$$\big(w_{xy}^{-dc}\hat{y}_c(\vartheta_{xy}^{+}) + w_{xy}^{-dd}\hat{y}_d(\vartheta_{xy}^{+})\big) - (T_{xy}^{+}) \le w_{xy}^{+dc}\hat{x}_c(k) + w_{xy}^{+dd}\hat{x}_d(k) \le \big(w_{xy}^{-dc}\hat{y}_c(\vartheta_{xy}^{-}) + w_{xy}^{-dd}\hat{y}_d(\vartheta_{xy}^{-})\big) - (T_{xy}^{-}) \quad (51)$$

$$\big(w_{uy}^{-cc}\hat{y}_c(\vartheta_{uy}^{+}) + w_{uy}^{-cd}\hat{y}_d(\vartheta_{uy}^{+})\big) - (T_{uy}^{+}) \le w_{uy}^{+cc}\hat{u}_c(k) + w_{uy}^{+cd}\hat{u}_d(k) \le \big(w_{uy}^{-cc}\hat{y}_c(\vartheta_{uy}^{-}) + w_{uy}^{-cd}\hat{y}_d(\vartheta_{uy}^{-})\big) - (T_{uy}^{-}) \quad (52)$$

$$\big(w_{uy}^{-dc}\hat{y}_c(\vartheta_{uy}^{+}) + w_{uy}^{-dd}\hat{y}_d(\vartheta_{uy}^{+})\big) - (T_{uy}^{+}) \le w_{uy}^{+dc}\hat{u}_c(k) + w_{uy}^{+dd}\hat{u}_d(k) \le \big(w_{uy}^{-dc}\hat{y}_c(\vartheta_{uy}^{-}) + w_{uy}^{-dd}\hat{y}_d(\vartheta_{uy}^{-})\big) - (T_{uy}^{-}) \quad (53)$$

The system must evolve in a non-decreasing way, so the following conditions are added:

$$T_{xx}^{-} + \hat{x}_c(\vartheta_{xx}^{-}) \le \hat{x}_c(k) \le T_{xx}^{+} + \hat{x}_c(\vartheta_{xx}^{+}), \quad (54)$$

$$T_{xx}^{-} + \hat{x}_d(\vartheta_{xx}^{-}) \le \hat{x}_d(k) \le T_{xx}^{+} + \hat{x}_d(\vartheta_{xx}^{+}), \quad (55)$$

$$T_{uy}^{-} + \hat{u}_c(\vartheta_{uy}^{-}) \le \hat{u}_c(k) \le T_{uy}^{+} + \hat{u}_c(\vartheta_{uy}^{+}), \quad (56)$$

$$T_{uy}^{-} + \hat{u}_d(\vartheta_{uy}^{-}) \le \hat{u}_d(k) \le T_{uy}^{+} + \hat{u}_d(\vartheta_{uy}^{+}). \quad (57)$$

Inequalities (46) to (57) can be aggregated to design observer (58):

Theorem 4: A state estimator able to estimate the state of a TIHPN is given by equation (58) using the dating method:

$$\begin{cases} \hat{\Phi}_m^d \le \mathcal{A}_{cd}^{-}\begin{bmatrix} \hat{x}_c(k) \\ \hat{x}_d(k) \end{bmatrix} \le \hat{\Phi}_M^d \\ \hat{\Psi}_m^d \le \mathcal{B}_{cd}^{-}\begin{bmatrix} \hat{u}_c(k) \\ \hat{u}_d(k) \end{bmatrix} \le \hat{\Psi}_M^d \end{cases} \quad (58)$$

with

$$\begin{cases} \hat{\Phi}_m^d = \mathcal{A}_{cd}^{+}\begin{bmatrix} \hat{x}_c(\vartheta_{xx}^{+}) \\ \hat{x}_d(\vartheta_{xx}^{+}) \end{bmatrix} - \mathcal{C}_{cd}^{+}\begin{bmatrix} \hat{y}_c(\vartheta_{xy}^{-}) \\ \hat{y}_d(\vartheta_{xy}^{-}) \end{bmatrix} - [T_{xx}^{+}] \\ \hat{\Phi}_M^d = \mathcal{A}_{cd}^{+}\begin{bmatrix} \hat{x}_c(\vartheta_{xx}^{+}) \\ \hat{x}_d(\vartheta_{xx}^{+}) \end{bmatrix} + \mathcal{C}_{cd}^{+}\begin{bmatrix} \hat{y}_c(\vartheta_{xy}^{-}) \\ \hat{y}_d(\vartheta_{xy}^{-}) \end{bmatrix} - [T_{xx}^{-}] \end{cases} \quad (59)$$

and

$$\begin{cases} \hat{\Psi}_m^d = \mathcal{B}_{cd}^{+}\begin{bmatrix} \hat{x}_c(\vartheta_{ux}^{+}) \\ \hat{x}_d(\vartheta_{ux}^{+}) \end{bmatrix} + \mathcal{D}_{cd}^{+}\begin{bmatrix} \hat{y}_c(\vartheta_{uy}^{+}) \\ \hat{y}_d(\vartheta_{uy}^{+}) \end{bmatrix} + \mathcal{I}\begin{bmatrix} \hat{u}_c(\vartheta_{uy}^{+}) \\ \hat{u}_d(\vartheta_{uy}^{+}) \end{bmatrix} - [T_u^{+}] \\ \hat{\Psi}_M^d = \mathcal{B}_{cd}^{+}\begin{bmatrix} \hat{x}_c(\vartheta_{ux}^{-}) \\ \hat{x}_d(\vartheta_{ux}^{-}) \end{bmatrix} + \mathcal{D}_{cd}^{+}\begin{bmatrix} \hat{y}_c(\vartheta_{uy}^{-}) \\ \hat{y}_d(\vartheta_{uy}^{-}) \end{bmatrix} + \mathcal{I}\begin{bmatrix} \hat{u}_c(\vartheta_{uy}^{-}) \\ \hat{u}_d(\vartheta_{uy}^{-}) \end{bmatrix} - [T_u^{-}] \end{cases} \quad (60)$$

with

$$T_x^{+}=\begin{bmatrix} T_{ux}^{+} \\ T_{ux}^{+} \\ T_{xx}^{+} \\ T_{xx}^{+} \\ 0 \end{bmatrix}, T_x^{-}=\begin{bmatrix} T_{ux}^{-} \\ T_{ux}^{-} \\ T_{xx}^{-} \\ T_{xx}^{-} \\ 0 \end{bmatrix}, T_u^{+}=\begin{bmatrix} T_{ux}^{+} \\ T_{ux}^{+} \\ T_{uy}^{+} \\ T_{uy}^{+} \\ 0 \end{bmatrix}, T_u^{-}=\begin{bmatrix} T_{ux}^{-} \\ T_{ux}^{-} \\ T_{uy}^{-} \\ T_{uy}^{-} \\ 0 \end{bmatrix},$$

$$\mathcal{A}_{cd}^{+}=\begin{bmatrix} w_{xx}^{-cc} & w_{xx}^{-cd} \\ w_{xx}^{-dc} & w_{xx}^{-dd} \\ 0 & 0 \\ 0 & 0 \\ I & I \end{bmatrix}, \mathcal{B}_{cd}^{+}=\begin{bmatrix} w_{ux}^{-cc} & w_{ux}^{-cd} \\ w_{ux}^{-dc} & w_{ux}^{-dd} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, I=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I \end{bmatrix},$$

$$\mathcal{C}_{cd}^{+}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ w_{xy}^{-cc} & w_{xy}^{-cd} \\ w_{xy}^{-dc} & w_{xy}^{-dd} \\ 0 & 0 \end{bmatrix}, \mathcal{D}_{cd}^{+}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ w_{uy}^{-cc} & w_{uy}^{-cd} \\ w_{uy}^{-dc} & w_{uy}^{-dd} \\ 0 & 0 \end{bmatrix}$$

$$\mathcal{A}_{cd}^{-}=\begin{bmatrix} w_{xx}^{+cc} & w_{xx}^{+cd} \\ w_{xx}^{+cc} & w_{xx}^{+cd} \\ w_{xy}^{+dc} & w_{xy}^{+dd} \\ w_{xy}^{+dc} & w_{xy}^{+dd} \\ I & I \end{bmatrix} \text{ and } \mathcal{B}_{cd}^{-}=\begin{bmatrix} w_{ux}^{+cc} & w_{ux}^{+cd} \\ w_{ux}^{+dc} & w_{ux}^{+dd} \\ w_{uy}^{+cc} & w_{uy}^{+cd} \\ w_{uy}^{+dc} & w_{uy}^{+dd} \\ I & I \end{bmatrix} \blacksquare$$

The proof of Theorem 4 is given in Appendix D.

Remark 2: Observers (43) and (58) estimate of the limit bounds of the system state and input, which can be useful for fault estimation, detection and localization. Moreover, they guarantee that the limit bounds of the estimation of x and u are finite and that the system trajectory estimation is non-decreasing. At each moment, state vector estimation belongs to the interval between the upper and the lower bounds. ■

VI. Application to production system

A. System description

Let consider the production system modeled by a TIHPN [19], where two pallets, A and B, transport parts. The cycle of palette A consists of operation A_1, which lasts between 3 and 4 time units, followed by operation A_2, which takes place on a shared machine with palette B and lasts between 2 and 9 units. In the same way, palette B performs operation B_1, taking 3 to 5 time units, and then carries out operation B_2 on the shared machine. Priority is given to operation B_2 over A_2, which means that A_2 is interrupted as soon as B_1 is completed, without losing any processing time for A_2 [19]. The time interval corresponding to A_2 is [a = 2, b = 9], with values a = 2 and b - a = 7 recorded in positions P_8 and

P_9. Once machine M is not used for B_2, T_9 is pulled at a speed of 1, and once the 2-time units pass (with possible interruptions), T_3 is pulled, which allows operation A_2 to be completed. At the end of operation A_2, T_4 or T_5 is drawn. It is possible to draw T_4 at any time after the minimum duration of A_2. If the maximum remaining time for A_2 reaches zero, then T_5 is taken immediately.

Figure 1 [19] shows the TIHPN that models the system, where timed arcs and transitions represent the evolution of operations on the two pallets.

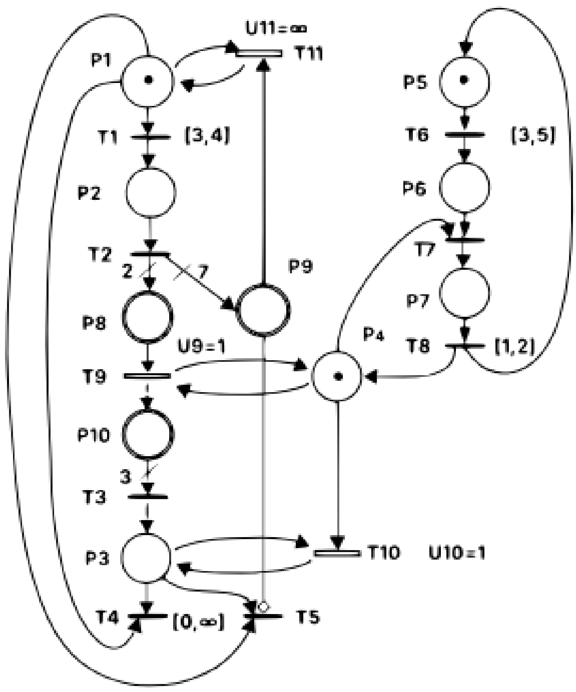

Fig. 1. The production system modeled by TIHPN

The following inequalities describe the system behavior:

$$T_1(t) + 3 \le T_2(t) \le T_1(t) + 4 \tag{61a}$$
$$T_9(t) + 1 \le T_3(t) \le T_9(t) + 2 \tag{61b}$$
$$T_3(t) + 2 \le T_4(t) + T_5(t) \le T_3(t) + 7 \tag{61c}$$
$$T_{11}(t) + 0 \le T_5(t) \le T_{11}(t) + 4 \tag{61d}$$
$$T_7(t) + 1 \le T_6(t) \le T_7(t) + 2 \tag{61e}$$
$$T_8(t) + 1.2 \le T_7(t) + T_9(t) \le T_8(t) + 2.4 \tag{61f}$$
$$T_6(t) + 3 \le T_7(t) \le T_6(t) + 5 \tag{61g}$$
$$T_9(t) + 1 \le T_7(t) \le T_9(t) + 2 \tag{61h}$$
$$T_7(t) + 1 \le T_8(t) \le T_7(t) + 2 \tag{61i}$$
$$T_3(t) + 2 \le T_{10}(t) \le T_3(t) + 7 \tag{61j}$$
$$T_2(t) + 2.5 \le T_{10}(t) \le T_2(t) + 9.2 \tag{61k}$$
$$T_9(t) + 1 \le T_{10}(t) \le T_9(t) + 2 \tag{61l}$$
$$T_1(t) + 3.1 \le T_{11}(t) \le T_1(t) + 4.2 \tag{61m}$$

The system matrices are:

$$w_{ux}^- = w_{xy}^+ = w_{uy}^+ = w_{uy}^- = 0 \tag{62}$$

$$w_{ux}^+ = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad w_{xx}^+ = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \tag{63}$$

$$w_{xy}^- = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad w_{xx}^- = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{64}$$

The simulation results are presented in Table I. It is supposed that input $T_1(k)$ is known. Table I gives the limit bounds for each state and output, which are obtained based on equation (28), where matrices w_{ux}^+, w_{ux}^-, w_{xx}^+, w_{xx}^-, w_{xy}^+, w_{xy}^-, w_{uy}^+ and w_{uy}^- are given in (62), (63) and (64).

B. State estimation

The following inequalities describe the behavior of the state estimator:

$$T_2(k) - 4 \le T_1(k) \le T_2(k) - 3 \tag{65a}$$
$$T_{10}(k) - 9 \le T_2(k) \le T_{10}(k) - 2 \tag{65b}$$
$$T_{10}(k) - 7 \le T_3(k) \le T_{10}(k) - 2 \tag{65c}$$
$$T_3(k) - 7 \le T_4(k) + T_5(k) \le T_3(k) - 2 \tag{65d}$$
$$T_7(k) - 5 \le T_6(k) \le T_7(k) - 3 \tag{65e}$$
$$T_8(k) - 2 \le T_7(k) \le T_8(k) - 1 \tag{65f}$$
$$T_7(k) + T_9(k) - 2 \le T_8(k) \le T_7(k) + T_9(k) - 1 \tag{65g}$$
$$T_{10}(k) + T_3(k) - 2 \le T_9(k) \le T_{10}(k) + T_3(k) - 1 \tag{65h}$$
$$T_2(k) - 9.2 \le T_{10}(k) \le T_2(k) - 2 \tag{65i}$$
$$T_5(k) - 4 \le T_{11}(k) \le T_5(k) - 0 \tag{65j}$$

Table II gives the estimation result of the system state and input. Symbol (est) means that it is an estimated variable. For estimation, it is supposed that only system output $T_{10}(k)$ is known. The results provided in Table II are obtained using the estimator proposed in equation (58).

Table II gives a possible estimation of the state vector at each date k. The estimation is based on the interval of $T_{10}(k)$.

Figure 2 highlights the contrast between the simulation and the estimation results for each transition (T_1 to T_{11}). Simulated intervals are represented by dark-color, whereas estimated intervals are represented by light-color. This differentiation allows emphasizing the accuracy of the estimates in comparison to the simulated values. The points

TABLE I
SYSTEM SIMULATION

Dates	1	2	3	4	5	6	7	8	9	10
$T_1(k)$	[1,3]	[2,4]	[3,5]	[4,6]	[6,8]	[8,10]	[9,12]	[11,13]	[11,15]	[13,17]
$T_2(k)$	[4,7]	[5,8]	[6,9]	[7,10]	[9,12]	[11,14]	[12,16]	[14,17]	[14,19]	[16,21]
$T_3(k)$	[0,14]	[0,15]	[1,16]	[2,17]	[4,19]	[6,21]	[7,23]	[9,24]	[9,26]	[11,28]
$T_4(k)$	[2,21]	[2,22]	[3,23]	[4,24]	[6,26]	[8,28]	[9,30]	[11,31]	[11,33]	[13,35]
$T_5(k)$	[4,11]	[5,12]	[6,13]	[7,14]	[9,16]	[11,18]	[12,20]	[14,21]	[14,23]	[16,25]
$T_6(k)$	[6,19]	[7,20]	[8,21]	[9,22]	[11,24]	[13,26]	[14,28]	[16,29]	[18,31]	[18,33]
$T_7(k)$	[5,17]	[6,18]	[7,19]	[8,20]	[10,22]	[12,24]	[13,26]	[15,27]	[17,29]	[17,31]
$T_8(k)$	[6,19]	[7,20]	[8,21]	[9,22]	[11,24]	[13,26]	[14,28]	[16,29]	[18,31]	[18,33]
$T_9(k)$	[4.5,15.2]	[5.5,16.2]	[6.5,17.2]	[7.5,18.2]	[9.5,20.2]	[11.5,22.2]	[12.5,24.2]	[14.5,25.2]	[16.5,27.2]	[16.5,29.2]
$T_{11}(k)$	[4.1,7.2]	[5.1,8.2]	[6.1,9.2]	[7.1,10.2]	[9.1,12.2]	[11.1,14.2]	[12.1,16.2]	[14.1,17.2]	[14.1,19.2]	[16.1,21.2]
$T_{10}(k)$	[6.5,16.2]	[7.5,17.2]	[8.5,18.2]	[9.5,19.2]	[11.5,21.2]	[13.5,23.2]	[14.5,25.2]	[16.5,26.2]	[16.5,28.2]	[18.5,30.2]

TABLE II
STATE ESTIMATION

Dates	1	2	3	4	5	6	7	8	9	10
$T_{1_{est}}(k)$	[0,11]	[0,12]	[0,13]	[0,14]	[0,16]	[0,18]	[1,20]	[3,21]	[3,23]	[5,25]
$T_2(k)$	[0,14]	[0,15]	[0,16]	[0,17]	[2,19]	[4,21]	[5,23]	[7,24]	[7,26]	[9,28]
$T_3(k)$	[0,14]	[0,15]	[1,16]	[2,17]	[4,19]	[6,21]	[7,23]	[7,24]	[7,26]	[9,28]
$T_4(k)$	[0,12]	[0,13]	[0,14]	[0,15]	[0,17]	[0,19]	[0,21]	[2,22]	[2,24]	[4,26]
$T_5(k)$	[0,12]	[0,13]	[0,14]	[0,15]	[0,17]	[0,19]	[0,21]	[2,22]	[2,24]	[4,26]
$T_6(k)$	[0,11]	[0,12]	[0,13]	[0,14]	[2,16]	[4,18]	[5,20]	[7,21]	[7,23]	[9,25]
$T_7(k)$	[2,14]	[3,15]	[4,16]	[5,17]	[7,19]	[9,21]	[10,23]	[12,24]	[12,26]	[14,28]
$T_8(k)$	[4,25]	[6,30]	[8,32]	[10,34]	[14,38]	[18,42]	[20,46]	[24,48]	[24,52]	[28,56]
$T_9(k)$	[4.5,15.2]	[5.5,16.2]	[6.5,17.2]	[7.5,18.2]	[9.5,20.2]	[11.5,22.2]	[12.5,24.2]	[14.5,25.2]	[14.5,27.2]	[16.5,29.2]
$T_{11}(k)$	[0,6.2]	[0,7.2]	[0,8.2]	[0,9.2]	[0,10.2]	[0,12.2]	[0,14.2]	[0,16.2]	[0,17.2]	[2,19.2]
$T_{10}(k)$	[6.5,16.2]	[7.5,17.2]	[8.5,18.2]	[9.5,19.2]	[11.5,21.2]	[13.5,23.2]	[14.5,25.2]	[16.5,26.2]	[16.5,28.2]	[18.5,30.2]

of overlap or the differences between intervals indicate an alignment or a divergence, hence understanding the time margins for each transition.

C. Discussion of results

The results presented in Tables I and II, as well as in Figure 2, allow us to evaluate the performance of the production system modeled by TIHPN in terms of simulation and state estimation. The comparative study between the time bounds obtained by simulation and those obtained by estimation shows strong consistency between the two approaches.

On the one hand, simulation, carried out on the assumption that the input $T_1(k)$ is known, provides precise time intervals for each transition. On the other hand, estimation, based solely on the output $T_{10}(k)$, produces time bounds very close to those obtained by simulation. This closeness of the intervals confirms the reliability of the estimate, despite the limited access to information.

For the T_3 transition, for example, the estimated intervals are globally included in those provided by simulation, with reduced margins of error. Similarly, critical transitions such as T_5, T_7 or T_{11}, which are strongly influenced by the dynamics of machine sharing between pallets A and B, are correctly estimated. This demonstrates the accuracy of the estimation algorithm even in complex configurations with priorities and interruptions.

The small deviation between the upper and lower bounds of the estimated intervals prooves also good convergence of the estimator, limiting temporal uncertainty. This is essential for the supervision of real-time production systems, where decisions have to be taken quickly on the basis of partial information.

The obtained results validate the ability of the TIHPN model to provide efficient, reliable and accurate temporal estimation of system behavior. This approach therefore represents a robust solution for monitoring and controlling hybrid discrete and continuous event systems.

For all transitions, the intersections between simulated and estimated intervals are not empty, so we can conclude that the state model and the observer give acceptable results with high accuracy.

D. Fault detection and localization

An actual defect $\alpha(k)$ is introduced into the production system. Estimating the transition under fault circumstances, simulating the influence on the transition $T_9(k)$ and computing the related estimated fault $\hat{\alpha}(k)$ are the objectives in this section. This defect is considered as an unknown input $\alpha(k)$ that affects the transition $T_9(k)$ such that:

$$T_9^{\text{inf}}(k) = T_3(k) + \alpha(k), \quad T_9^{\text{sup}}(k) = T_3(k) + \alpha(k) + 1$$

with:

$$\alpha(k) = \begin{cases} 5000 & \text{if } k = 3, 6, 8 \\ 0 & \text{otherwise} \end{cases}$$

1) Simulation with fault: Given the simulated values of $T_3(k)$ and $\alpha(k)$:

$$T_3(k) = [0,\ 14],\ [0,\ 15],\ [1,\ 16],\ [2,\ 17],\ [4,\ 19],\\ [6,\ 21],\ [7,\ 23],\ [9,\ 24],\ [9,\ 26],\ [11,\ 28]$$

$$\alpha(k) = [0,\ 0,\ 5000,\ 0,\ 0,\ 5000,\ 0,\ 5000,\ 0,\ 0]$$

the bounds of $T_9(k)$ under fault are computed as follows:

$$T_9^{\text{inf}}(k) = T_3(k) + \alpha(k), \quad T_9^{\text{sup}}(k) = T_3(k) + \alpha(k) + 1$$

The results of simulation of the bounds of $T_9(k)$ are given in Table III

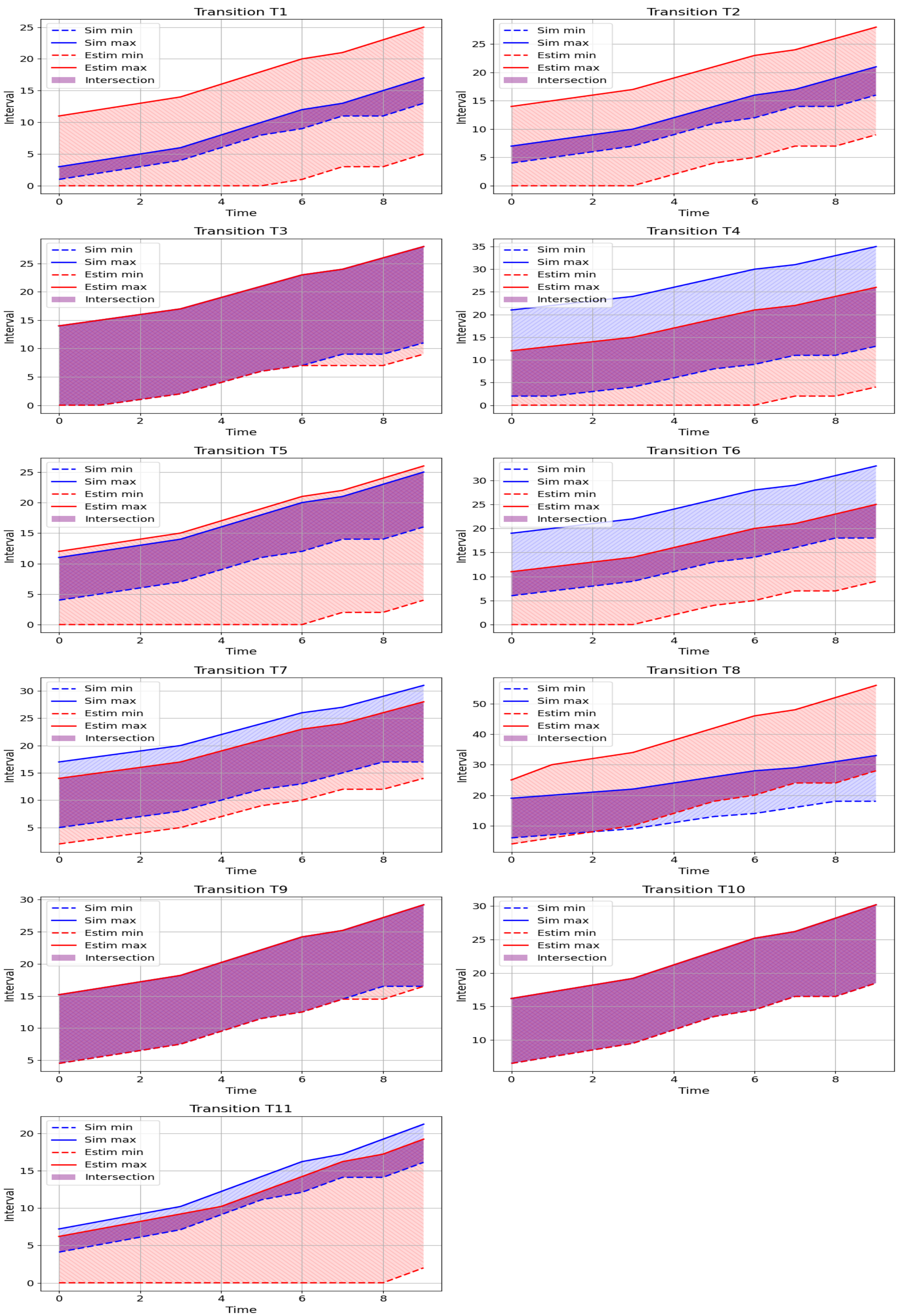

Fig. 2. Comparison between simulated and estimated values

TABLE III
SIMULATION OF $T_9(k)$ WITH INJECTED FAULT

k	$T_3(k)$	$\alpha(k)$	$T_9(k)$ inf	$T_9(k)$ sup
1	[0, 14]	0	[0, 14]	[1, 15]
2	[0, 15]	0	[0, 15]	[1, 16]
3	[1, 16]	5000	[5001, 5016]	[5002, 5017]
4	[2, 17]	0	[2, 17]	[3, 18]
5	[4, 19]	0	[4, 19]	[5, 20]
6	[6, 21]	5000	[5006, 5021]	[5007, 5022]
7	[7, 23]	0	[7, 23]	[8, 24]
8	[9, 24]	5000	[5009, 5024]	[5010, 5025]
9	[9, 26]	0	[9, 26]	[10, 27]
10	[11, 28]	0	[11, 28]	[12, 29]

2) Estimation with fault: Using the estimated intervals of $T_3(k)$ and $T_9(k)$, the bounds of the estimated fault $\hat{\alpha}(k)$ are computed as follows:

$$\hat{\alpha}_{\min}(k) = \max(0, \min(T_9^{\text{inf}}(k)) - \max(T_3^{\text{sup}}(k)) - 1)$$

$$\hat{\alpha}_{\max}(k) = \max(0, \max(T_9^{\text{sup}}(k)) - \min(T_3^{\text{inf}}(k)))$$

$\hat{\alpha}_{\min}(k)$ and $\hat{\alpha}_{\max}(k)$ are given in Table IV

TABLE IV
ESTIMATED VALUES OF $\hat{\alpha}(k)$

k	$T_3^{\text{est}}(k)$	$T_9^{\text{est}}(k)$	$\hat{\alpha}_{\min}(k)$	$\hat{\alpha}_{\max}(k)$
1	[0, 14]	[4.5, 15.2]	0.0	15.2
2	[0, 15]	[5.5, 16.2]	0.0	16.2
3	[1, 16]	[5006.5, 5017.2]	4990.5	5016.2
4	[2, 17]	[7.5, 18.2]	0.0	16.2
5	[4, 19]	[9.5, 20.2]	0.0	16.2
6	[6, 21]	[5008.5, 5022.2]	4986.5	5016.2
7	[7, 23]	[12.5, 24.2]	0.0	24.2
8	[9, 24]	[5014.5, 5025.2]	4990.5	5016.2
9	[9, 26]	[16.5, 27.2]	0.0	18.2
10	[11, 28]	[16.5, 29.2]	0.0	29.2

We note that moments $k = 3, 6, 8$ display a notably high value of $\hat{\alpha}(k)$, demonstrating the detection of a defect introduced at these instants. For the other moments, the limits remain close to zero, indicating the absence of a defect.

VII. CONCLUSION AND FUTURE WORK

In this study, we have presented a comprehensive method for assessing the state, the inputs and the outputs of hybrid systems modeled using TIHPNs. This assessing, is developed following the counting and dating methods. Furthermore, we have used a status indicator that not only assesses these limits but also can play a crucial role in detecting and locating faults within a system.

Unlike previous work that has focused on system validation or diagnosis using classical PNs, our approach has considered a hybrid system framework that combine continuous and discrete-event behaviors, and taken into account time intervals.

The concepts and approaches developed in this study have created robust fault detection mechanisms, particularly in systems where time constraints and hybrid behaviors are essential. Future studies may focus on improving the computational efficiency of estimators and expanding their applications to more complex and larger-scale systems. Our work can be continued by analyzing the effectiveness of the estimation using linear programming techniques, which allows for the integration of additional criteria into the simulation and estimation processes. Ultimately, this study contributes to the field by enhancing our ability to ensure the safety and reliability of systems through better failure management and state monitoring. It has a follow-up on the states.

APPENDIX A
PROOF OF THEOREM 1.

Proof: Inequalities (1), (2), (3), (4), (9), and (10) define the boundary constraints for system state $x(t)$.

$$\begin{pmatrix} m_{ux} \\ m_{xx} \\ m_{ux} \\ m_{xx} \\ 0 \end{pmatrix} + \begin{pmatrix} w_{ux}^{+cc} & w_{ux}^{+cd} & 0 & 0 \\ w_{ux}^{+dc} & w_{ux}^{+dd} & 0 & 0 \\ 0 & 0 & w_{xx}^{+cc} & w_{xx}^{+cd} \\ 0 & 0 & w_{xx}^{+dc} & w_{xx}^{+dd} \\ 0 & 0 & I & I \end{pmatrix} \begin{pmatrix} u_c(\theta_{l,ux}^+) \\ u_d(\theta_{l,ux}^+) \\ x_c(\theta_{l,xx}^+) \\ x_d(\theta_{l,xx}^+) \end{pmatrix} \leq \begin{pmatrix} w_{ux}^{-cc} & w_{ux}^{-cd} \\ w_{ux}^{-dc} & w_{ux}^{-dd} \\ w_{xx}^{-cc} & w_{xx}^{-cd} \\ w_{xx}^{-dc} & w_{xx}^{-dd} \\ I & I \end{pmatrix} \begin{pmatrix} x_c(t) \\ x_d(t) \end{pmatrix}$$
$$\leq \begin{pmatrix} m_{ux} \\ m_{xx} \\ m_{ux} \\ m_{xx} \\ 0 \end{pmatrix} + \begin{pmatrix} w_{ux}^{+cc} & w_{ux}^{+cd} & 0 & 0 \\ w_{ux}^{+dc} & w_{ux}^{+dd} & 0 & 0 \\ 0 & 0 & w_{xx}^{+cc} & w_{xx}^{+cd} \\ 0 & 0 & w_{xx}^{+dc} & w_{xx}^{+dd} \\ 0 & 0 & I & I \end{pmatrix} \begin{pmatrix} u_c(\theta_{l,ux}^-) \\ u_d(\theta_{l,ux}^-) \\ x_c(\theta_{l,xx}^-) \\ x_d(\theta_{l,xx}^- \end{pmatrix} \quad (66)$$

Inequality (66) allows for the computation of the limit bounds of system state $x(t)$.
Similarly, inequality (67) may be used to describe inequalities (5), (6), (7), (8), (11) and (12) that identify the upper and

lower bounds of system output $y(t)$.

$$\begin{pmatrix} m_{xy} \\ m_{uy} \\ m_{xy} \\ m_{uy} \\ 0 \end{pmatrix} + \begin{pmatrix} w_{xy}^{+cc} & w_{xy}^{+cd} & 0 & 0 & 0 & 0 \\ w_{xy}^{+dc} & w_{xy}^{+dd} & 0 & 0 & 0 & 0 \\ 0 & 0 & w_{uy}^{+cc} & w_{uy}^{+cd} & 0 & 0 \\ 0 & 0 & w_{uy}^{+dc} & w_{uy}^{+dd} & 0 & 0 \\ 0 & 0 & 0 & 0 & I & I \end{pmatrix} \begin{pmatrix} x_c(\theta_{l,xy}^{+}) \\ x_d(\theta_{l,xy}^{+}) \\ u_c(\theta_{l,uy}^{+}) \\ u_d(\theta_{l,uy}^{+}) \\ y_c(\theta_{l,uy}^{+}) \\ y_d(\theta_{l,uy}^{+}) \end{pmatrix} \leq \begin{pmatrix} w_{xy}^{-cc} & w_{xy}^{-cd} \\ w_{xy}^{-dc} & w_{xy}^{-dd} \\ w_{uy}^{-cc} & w_{uy}^{-cd} \\ w_{uy}^{-dc} & w_{uy}^{-dd} \\ I & I \end{pmatrix} \begin{pmatrix} y_c(t) \\ y_d(t) \end{pmatrix}$$

$$\leq \begin{pmatrix} m_{xy} \\ m_{uy} \\ m_{xy} \\ m_{uy} \\ 0 \end{pmatrix} + \begin{pmatrix} w_{xy}^{+cc} & w_{xy}^{+cd} & 0 & 0 & 0 & 0 \\ w_{xy}^{+dc} & w_{xy}^{+dd} & 0 & 0 & 0 & 0 \\ 0 & 0 & w_{uy}^{+cc} & w_{uy}^{+cd} & 0 & 0 \\ 0 & 0 & w_{uy}^{+dc} & w_{uy}^{+dd} & 0 & 0 \\ 0 & 0 & 0 & 0 & I & I \end{pmatrix} \begin{pmatrix} x_c(\theta_{l,xy}^{-}) \\ x_d(\theta_{l,xy}^{-}) \\ u_c(\theta_{l,xy}^{-}) \\ u_d(\theta_{l,xy}^{-}) \\ y_c(\theta_{l,xy}^{-}) \\ y_d(\theta_{l,xy}^{-}) \end{pmatrix} \tag{67}$$

■

APPENDIX B
PROOF OF THEOREM 2.

Proof: The upper and lower limits of system state $x(k)$ are given by inequality (16) to (19), which may be transformed on matrix inequality (68):

$$\begin{pmatrix} T_{ux}^{-} \\ T_{xx}^{-} \\ T_{ux}^{-} \\ T_{xx}^{-} \\ 0 \end{pmatrix} + \begin{pmatrix} w_{ux}^{+cc} & w_{ux}^{+cd} & 0 & 0 \\ w_{ux}^{+dc} & w_{ux}^{+dd} & 0 & 0 \\ 0 & 0 & w_{xx}^{+cc} & w_{xx}^{+cd} \\ 0 & 0 & w_{xx}^{+dc} & w_{xx}^{+dd} \\ 0 & 0 & I & I \end{pmatrix} \begin{pmatrix} u_c(\mu_{ux}^{-}) \\ u_d(\mu_{ux}^{-}) \\ x_c(\mu_{xx}^{-}) \\ x_d(\mu_{xx}^{-}) \end{pmatrix} \leq \begin{pmatrix} w_{ux}^{-cc} & w_{ux}^{-cd} \\ w_{ux}^{-dc} & w_{ux}^{-dd} \\ w_{xx}^{-cc} & w_{xx}^{-cd} \\ w_{xx}^{-dc} & w_{xx}^{-dd} \\ I & I \end{pmatrix} \begin{pmatrix} x_c(k) \\ x_d(k) \end{pmatrix}$$

$$\leq \begin{pmatrix} T_{ux}^{+} \\ T_{xx}^{+} \\ T_{ux}^{+} \\ T_{xx}^{+} \\ 0 \end{pmatrix} + \begin{pmatrix} w_{ux}^{+cc} & w_{ux}^{+cd} & 0 & 0 \\ w_{ux}^{+dc} & w_{ux}^{+dd} & 0 & 0 \\ 0 & 0 & w_{xx}^{+cc} & w_{xx}^{+cd} \\ 0 & 0 & w_{xx}^{+dc} & w_{xx}^{+dd} \\ 0 & 0 & I & I \end{pmatrix} \begin{pmatrix} u_c(\mu_{ux}^{+}) \\ u_d(\mu_{ux}^{+}) \\ x_c(\mu_{xx}^{+}) \\ x_d(\mu_{xx}^{+}) \end{pmatrix} \tag{68}$$

Similarly, matrix inequality (69) may be subjected to inequalities (19) and (23), which provide the upper and lower limits of the system output $y(k)$:

$$\begin{pmatrix} T_{xy}^{-} \\ T_{uy}^{-} \\ T_{xy}^{-} \\ T_{uy}^{-} \\ 0 \end{pmatrix} + \begin{pmatrix} w_{xy}^{+cc} & w_{xy}^{+cd} & 0 & 0 & 0 & 0 \\ w_{xy}^{+dc} & w_{xy}^{+dd} & 0 & 0 & 0 & 0 \\ 0 & 0 & w_{uy}^{+cc} & w_{uy}^{+cd} & 0 & 0 \\ 0 & 0 & w_{uy}^{+dc} & w_{uy}^{+dd} & 0 & 0 \\ 0 & 0 & 0 & 0 & I & I \end{pmatrix} \begin{pmatrix} u_c(\mu_{xy}^{-}) \\ u_d(\mu_{xy}^{-}) \\ x_c(\mu_{uy}^{-}) \\ x_d(\mu_{uy}^{-}) \\ y_c(\mu_{uy}^{+}) \\ y_d(\mu_{uy}^{+}) \end{pmatrix} \leq \begin{pmatrix} w_{xy}^{-cc} & w_{xy}^{-cd} \\ w_{xy}^{-dc} & w_{xy}^{-dd} \\ w_{uy}^{-cc} & w_{uy}^{-cd} \\ w_{uy}^{-dc} & w_{uy}^{-dd} \end{pmatrix} \begin{pmatrix} y_c(k) \\ y_d(k) \end{pmatrix}$$

$$\leq \begin{pmatrix} T_{xy}^{+} \\ T_{uy}^{+} \\ T_{xy}^{+} \\ T_{uy}^{+} \\ 0 \end{pmatrix} + \begin{pmatrix} w_{xy}^{+cc} & w_{xy}^{+cd} & 0 & 0 & 0 & 0 \\ w_{xy}^{+dc} & w_{xy}^{+dd} & 0 & 0 & 0 & 0 \\ 0 & 0 & w_{uy}^{+cc} & w_{uy}^{+cd} & 0 & 0 \\ 0 & 0 & w_{uy}^{+dc} & w_{uy}^{+dd} & 0 & 0 \\ 0 & 0 & 0 & 0 & I & I \end{pmatrix} \begin{pmatrix} u_c(\mu_{xy}^{+}) \\ u_d(\mu_{xy}^{+}) \\ x_c(\mu_{uy}^{+}) \\ x_d(\mu_{uy}^{+}) \\ y_c(\mu_{uy}^{+}) \\ y_d(\mu_{uy}^{+}) \end{pmatrix} \tag{69}$$

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APPENDIX C
PROOF OF THEOREM 3.

Proof: Inequalities (31), (32),(37), (38), (41) and (42) can be gathered in matrix inequality (**??**), which permits us to estimate the system input by computing the limit bounds of $u(k)$.

$$\begin{pmatrix} w_{ux}^{-cc} & w_{ux}^{-cd} & 0 & 0 & 0 & 0 \\ w_{ux}^{-dc} & w_{ux}^{-dd} & 0 & 0 & 0 & 0 \\ 0 & 0 & w_{uy}^{-cc} & w_{uy}^{-cd} & 0 & 0 \\ 0 & 0 & w_{uy}^{-dc} & w_{uy}^{-dd} & 0 & 0 \\ 0 & 0 & 0 & 0 & I & I \end{pmatrix} \begin{pmatrix} \hat{x}_c(\tau_{l,ux}^-) \\ \hat{x}_d(\tau_{l,ux}^-) \\ \hat{y}_c(\tau_{l,uy}^-) \\ \hat{y}_d(\tau_{l,uy}^-) \\ \hat{u}_c(\tau_{l,uy}^-) \\ \hat{u}_d(\tau_{l,uy}^-) \end{pmatrix} - \begin{pmatrix} m_{ux} \\ m_{ux} \\ m_{uy} \\ m_{uy} \\ 0 \end{pmatrix} \leq \begin{pmatrix} w_{ux}^{+cc} & w_{ux}^{+cd} \\ w_{ux}^{+dc} & w_{ux}^{+dd} \\ w_{uy}^{+cc} & w_{uy}^{+cd} \\ w_{uy}^{+dc} & w_{uy}^{+dd} \\ I & I \end{pmatrix} \begin{pmatrix} \hat{u}_c(t) \\ \hat{u}_d(t) \end{pmatrix}$$

$$\leq \begin{pmatrix} w_{ux}^{-cc} & w_{ux}^{-cd} & 0 & 0 & 0 & 0 \\ w_{ux}^{-dc} & w_{ux}^{-dd} & 0 & 0 & 0 & 0 \\ 0 & 0 & w_{uy}^{-cc} & w_{uy}^{-cd} & 0 & 0 \\ 0 & 0 & w_{uy}^{-dc} & w_{uy}^{-dd} & 0 & 0 \\ 0 & 0 & 0 & 0 & I & I \end{pmatrix} \begin{pmatrix} \hat{x}_c(\tau_{l,ux}^+) \\ \hat{x}_d(\tau_{l,ux}^+) \\ \hat{y}_c(\tau_{l,uy}^+) \\ \hat{y}_d(\tau_{l,uy}^+) \\ \hat{u}_c(\tau_{l,uy}^-) \\ \hat{u}_d(\tau_{l,uy}^-) \end{pmatrix} - \begin{pmatrix} m_{ux} \\ m_{ux} \\ m_{uy} \\ m_{uy} \\ 0 \end{pmatrix} \quad (70)$$

In the same way, (33), (34), (35), (36), (39) and (40) can be combined into matrix inequality (71), which allows for the computation of the limit bounds of $x(t)$.

$$\begin{pmatrix} w_{xx}^{-cc} & w_{xx}^{-cd} & 0 & 0 \\ w_{xx}^{-dc} & w_{xx}^{-dd} & 0 & 0 \\ 0 & 0 & w_{xy}^{-cc} & w_{xy}^{-cd} \\ 0 & 0 & w_{xy}^{-dc} & w_{xy}^{-dd} \\ I & I & 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{x}_c(\tau_{l,xx}^-) \\ \hat{x}_d(\tau_{l,xx}^-) \\ \hat{y}_c(\tau_{l,xy}^-) \\ \hat{y}_d(\tau_{l,xy}^-) \end{pmatrix} - \begin{pmatrix} m_{xx} \\ m_{xx} \\ m_{xy} \\ m_{xy} \\ 0 \end{pmatrix} \leq \begin{pmatrix} w_{xx}^{+cc} & w_{xx}^{+cd} \\ w_{xx}^{+dc} & w_{xx}^{+dd} \\ w_{xy}^{+cc} & w_{xy}^{+cd} \\ w_{xy}^{+dc} & w_{xy}^{+dd} \\ I & I \end{pmatrix} \begin{pmatrix} \hat{x}_c(t) \\ \hat{x}_d(t) \end{pmatrix}$$

$$\leq \begin{pmatrix} w_{xx}^{-cc} & w_{xx}^{-cd} & 0 & 0 \\ w_{xx}^{-dc} & w_{xx}^{-dd} & 0 & 0 \\ 0 & 0 & w_{xy}^{-cc} & w_{xy}^{-cd} \\ 0 & 0 & w_{xy}^{-dc} & w_{xy}^{-dd} \\ I & I & 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{x}_c(\tau_{l,xx}^+) \\ \hat{x}_d(\tau_{l,xx}^+) \\ \hat{y}_c(\tau_{l,xy}^+) \\ \hat{y}_d(\tau_{l,xy}^+) \end{pmatrix} - \begin{pmatrix} m_{xx} \\ m_{xx} \\ m_{xy} \\ m_{xy} \\ 0 \end{pmatrix} \quad (71)$$

∎

APPENDIX D
PROOF OF THEOREM 4.

Proof: Inequalities (48), (49), (50), (51), (54) and (55) can be combined into matrix inequality (72), which allows computing the limit bounds of $x(k)$.

$$\begin{pmatrix} w_{xx}^{-cc} & w_{xx}^{-cd} & 0 & 0 \\ w_{xx}^{-dc} & w_{xx}^{-dd} & 0 & 0 \\ 0 & 0 & w_{xy}^{-cc} & w_{xy}^{-cd} \\ 0 & 0 & w_{xy}^{-dc} & w_{xy}^{-dd} \\ I & I & 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{x}_c(\vartheta_{l,xx}^+) \\ \hat{x}_d(\vartheta_{l,xx}^+) \\ \hat{y}_c(\vartheta_{l,xy}^+) \\ \hat{y}_d(\vartheta_{l,xy}^+) \end{pmatrix} - \begin{pmatrix} T_{xx}^+ \\ T_{xx}^+ \\ T_{xy}^+ \\ T_{xy}^+ \\ 0 \end{pmatrix} \leq \begin{pmatrix} w_{xx}^{+cc} & w_{xx}^{+cd} \\ w_{xx}^{+dc} & w_{xx}^{+dd} \\ w_{xy}^{+cc} & w_{xy}^{+cd} \\ w_{xy}^{+dc} & w_{xy}^{+dd} \\ I & I \end{pmatrix} \begin{pmatrix} \hat{x}_c(t) \\ \hat{x}_d(t) \end{pmatrix}$$

$$\leq \begin{pmatrix} w_{xx}^{-cc} & w_{xx}^{-cd} & 0 & 0 \\ w_{xx}^{-dc} & w_{xx}^{-dd} & 0 & 0 \\ 0 & 0 & w_{xy}^{-cc} & w_{xy}^{-cd} \\ 0 & 0 & w_{xy}^{-dc} & w_{xy}^{-dd} \\ I & I & 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{x}_c(\vartheta_{l,xx}^-) \\ \hat{x}_d(\vartheta_{l,xx}^-) \\ \hat{y}_c(\vartheta_{l,xy}^-) \\ \hat{y}_d(\vartheta_{l,xy}^-) \end{pmatrix} - \begin{pmatrix} T_{xx}^- \\ T_{xx}^- \\ T_{xy}^- \\ T_{xy}^- \\ 0 \end{pmatrix} \quad (72)$$

Similarly, inequalities (46), (47), (52), (53), (56) and (57) can be gathered in matrix inequality (73), which permits us to estimate the system input by computing the limit bounds of $u(t)$.

$$\begin{pmatrix} w_{ux}^{-cc} & w_{ux}^{-cd} & 0 & 0 & 0 & 0 \\ w_{ux}^{-dc} & w_{ux}^{-dd} & 0 & 0 & 0 & 0 \\ 0 & 0 & w_{uy}^{-cc} & w_{uy}^{-cd} & 0 & 0 \\ 0 & 0 & w_{uy}^{-dc} & w_{uy}^{-dd} & 0 & 0 \\ 0 & 0 & 0 & 0 & I & I \end{pmatrix} \begin{pmatrix} \hat{x}_c(\vartheta_{l,ux}^{+}) \\ \hat{x}_d(\vartheta_{l,ux}^{+}) \\ \hat{y}_c(\vartheta_{l,uy}^{+}) \\ \hat{y}_d(\vartheta_{l,uy}^{+}) \\ hatu_c(\vartheta_{l,uy}^{+}) \\ \hat{u}_d(\vartheta_{l,uy}^{+}) \end{pmatrix} - \begin{pmatrix} T_{ux}^{+} \\ T_{ux}^{+} \\ T_{uy}^{+} \\ T_{uy}^{+} \\ 0 \end{pmatrix} \leq \begin{pmatrix} w_{ux}^{+cc} & w_{ux}^{+cd} \\ w_{ux}^{+dc} & w_{ux}^{+dd} \\ w_{uy}^{+cc} & w_{uy}^{+cd} \\ w_{uy}^{+dc} & w_{uy}^{+dd} \\ I & I \end{pmatrix} \begin{pmatrix} \hat{u}_c(t) \\ \hat{u}_d(t) \end{pmatrix}$$

$$\leq \begin{pmatrix} w_{ux}^{-cc} & w_{ux}^{-cd} & 0 & 0 & 0 & 0 \\ w_{ux}^{-dc} & w_{ux}^{-dd} & 0 & 0 & 0 & 0 \\ 0 & 0 & w_{uy}^{-cc} & w_{uy}^{-cd} & 0 & 0 \\ 0 & 0 & w_{uy}^{-dc} & w_{uy}^{-dd} & 0 & 0 \\ I & I & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{x}_c(\vartheta_{l,ux}^{-}) \\ \hat{x}_d(\vartheta_{l,ux}^{-}) \\ \hat{y}_c(\vartheta_{l,uy}^{-}) \\ \hat{y}_d(\vartheta_{l,uy}^{-}) \\ hatu_c(\vartheta_{l,uy}^{+}) \\ \hat{u}_d(\vartheta_{l,uy}^{+}) \end{pmatrix} - \begin{pmatrix} T_{ux}^{-} \\ T_{ux}^{-} \\ T_{uy}^{-} \\ T_{uy}^{-} \\ 0 \end{pmatrix} \quad (73)$$

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