Research on the Approximation of Wavelet Expansions Based on Riesz Basis

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***Abstract*—This paper comprehensively analyzes the approximation of wavelet expansions constructed from a Riesz basis and addresses the unique challenges posed by the lack of orthonormality. By analyzing the remainders of the wavelet expansions, we prove that the integral of the remainders from $-\infty$ to $+\infty$ converges to zero for infinitely differentiable functions based on the Riesz basis,when the scale function satisfies the partition of unity condition and the vanishing moment conditions. This result applies to both the Shannon and Meyer wavelets, because their scale functions satisfy the conditions. For r-order differentiable functions with monotonically decreasing r-th order derivatives, we derive the approximation of the wavelet expansions and an explicit estimation of convergence rate in the $L^1(R)$ -norm through careful analysis and refined bounding techniques, assuming that the scale function has compact support and satisfies the partition of unity condition. Furthermore, for infinitely differentiable functions, we establish the exact convergence rate in the $L^1(R)$ -norm. We demonstrate that both the Haar wavelet (orthogonal) and the linear B-spline wavelet (non-orthogonal) achieve this convergence rate, as their scale functions meet the required conditions. Our results generalize the approximation of wavelet expansions to broader scenarios and enhance their practical utility in computational applications.**

***Index Terms* — wavelet expansions, Riesz basis, approximation, convergence rate, non-orthogonal systems**

I. Introduction

WAVELET analysis, since the establishment of its theoretical foundation, has been widely applied by mathematicians and engineers in diverse fields including numerical computation, signal detection, image processing, noise filtering, and financial analysis [1]-[4]. In practical applications, constructing wavelet expansions that achieve precise approximation or quantified convergence rates is often essential. Researchers have extensively studied the approximation of wavelet expansions, particularly in the context of orthonormal wavelet systems. A well-developed theoretical framework has been established, yielding numerous significant results. For instance, reference [5] investigated the approximation of orthogonal wavelet expansions for functions $f \in L^2(R)$ under the assumption that the scale function φ satisfies the decay condition

$$|\varphi(x)| \le \frac{C}{(1+|x|)^{1+\beta}} \ (x \in R),$$

where C and β are both positive constants. Under this condition, the authors established pointwise and uniform convergence of such expansions. Building on this work, reference [6] derived the convergence rate of orthogonal wavelet expansions, demonstrating that their approximation error decays exponentially. Further refinements were provided in [7]-[9], where a precise estimation for the convergence rate was obtained.

In contrast, wavelet expansions constructed from a Riesz basis which is generated by a scale function without orthonormality present unique analytical challenges in approximation theory. Currently, research on the approximation of such non-orthogonal wavelet expansions remains insufficient. Reference [9] investigated the convergence of a class of non-orthogonal wavelet expansions, proving their convergence in $L^2(R)$ -norm and providing the order of convergence rate.

This paper extends these investigations by analyzing the approximation and the rate of convergence of wavelet expansions constructed from Riesz basis in the $L^1(R)$ -norm. Specifically, we focus on functions that are r -order differentiable with monotonically decreasing r -order derivatives. Our results contribute to bridging the gap in the theoretical understanding of non-orthogonal wavelet systems and their practical applicability.

II. Wavelet Expansions Based on Riesz Basis

In what follows, we establish the necessary mathematical framework and notation.

Let $L^2(R)$ be the Hilbert space of all Lebesgue measurable and square-integrable functions, equipped with the inner product-induced norm

$$\|f\|_2 = \left(\int_{-\infty}^{+\infty} |f(x)|^2 dx\right)^{\frac{1}{2}}.$$

Let $L^1(R)$ be the Banach space of all Lebesgue measurable and absolutely integrable functions, with the corresponding norm defined as

$$\|f\|_1 = \int_{-\infty}^{+\infty} |f(x)| dx.$$

Definition 1 [9] Assume that the sequence of closed subspaces $\{V_k\}_{k\in Z}$ of $L^2(R)$ satisfies the following conditions:

(i) $V_k \subset V_{k+1}$ ($k \in Z$);

(ii) $L^2(R) = \overline{\bigcup_{k=-\infty}^{+\infty} V_k}$ and $\bigcap_{k=-\infty}^{+\infty} V_k = \{0\}$;

Manuscript received March 7, 2025; revised June 20, 2025.

This paper is supported by Special Scientific Research Project of the Education Department of Shaanxi Province (No. 16JK1825).

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(iii) $f \in V_k$ if and only if $f(2\cdot) \in V_{k+1}$;

(iv) if $f \in V_k$, $f(\cdot - n) \in V_k$ ($n \in Z$);

(v) there exists $\varphi \in V_0$ such that $\{\varphi(\cdot - n)\}_{n\in Z}$ composes a Riesz basis of V_0,

then we call φ a scale function of $L^2(R)$, $\{V_k\}_{k\in Z}$ the multiresolution analysis generated by the scale function φ.

Let $\{V_k\}_{k\in Z}$ be a multiresolution analysis in $L^2(R)$ generated by a scale function φ. Then the system $\{\varphi(2^k x - n)\}_{k,n\in Z}$ forms a Riesz basis for $L^2(R)$. Below we present four canonical examples of such Riesz bases.

Example 1(Shannon wavelet basis) The Shannon wavelet is generated by the sinc-type scale function

$$\varphi(x) = \frac{\sin \pi x}{\pi x}.$$

The corresponding family $\{\varphi(2^k x - n)\}_{k,n\in Z}$ constitutes a Riesz basis for $L^2(R)$.

Example 2(Meyer wavelet basis) The scale function φ of Meyer wavelet is explicitly defined through its Fourier transform $\hat{\varphi}$, which has compact support in the frequency domain:

$$\hat{\varphi}(\omega) = \begin{cases} 1, & |\omega| \le \frac{2}{3}\pi \\ \cos(\frac{\pi}{2} v(\frac{3|\omega|}{2\pi} - 1)), & \frac{2}{3}\pi \le |\omega| \le \frac{4}{3}\pi \\ 0, & |\omega| > \frac{4}{3}\pi \end{cases},$$

where v is a smooth transition function satisfying

$$v(x) = \begin{cases} 0, x \le 0 \\ 1, x \ge 1 \end{cases}$$

and

$$v(x) + v(1-x) = 1, x \in [0,1].$$

The corresponding family $\{\varphi(2^k x - n)\}_{k,n\in Z}$ forms a Riesz basis for $L^2(R)$.

Example 3 (Haar wavelet basis) The Haar wavelet employs the characteristic function

$$\varphi(x) = \begin{cases} 1, 0 \le x < 1 \\ 0, elsewhere \end{cases},$$

which generates a Riesz basis $\{\varphi(2^k x - n)\}_{k,n\in Z}$ for $L^2(R)$.

Example 4 (Linear B-spline Wavelet basis*)* The linear B-spline scale function φ *is* given by

$$\varphi(x) = \begin{cases} x, & 0 \le x < 2 \\ 1-x, & 1 \le x < 2 \\ 0, & elsewhere \end{cases}.$$

This piecewise linear function yields a Riesz basis $\{\varphi(2^k x - n)\}_{k,n\in Z}$ for $L^2(R)$.

For $f \in L^1(R)$, define the operators as

$$(A_k f)(x) = \sum_n f(\frac{n}{2^k})\varphi(2^k x - n) \quad (k \in Z). \qquad (1)$$

Here $\{A_k\}_{k\in Z}$ is a family of operators defined on $L^1(R)$ and equation (1) represents wavelet expansions constructed from the Riesz basis $\{\varphi(2^k x - n)\}_{k,n\in Z}$.

In what follows, we will first analyze the approximation of wavelet expansions (1) and then investigate the convergence rates.

III. Approximation of the Infinite Integrals of Wavelet Expansions

In this section, we analyze the approximation of the wavelet expansions (1) by evaluating the integrals of $(A_k f) - f$ from $-\infty$ to $+\infty$.

Theorem 1 Let $f \in L^1(R)$ be an infinitely differentiable function. If the scale function φ satisfies the partition of unity condition

$$\sum_n \varphi(x-n) = 1 \quad (x \in R) \qquad (2)$$

and the vanishing moment conditions

$$\int_{-\infty}^{+\infty} x^j \varphi(x) dx = 0 \ \ (j = 1, 2, \cdots), \qquad (3)$$

then the infinite integrals of the remainders of wavelet expansions (1)

$$\int_{-\infty}^{+\infty} [(A_k f)(x) - f(x)] dx$$

converge to zero as $k \to +\infty$.

Proof Starting from the partition of unity condition (2), we express the difference as

$$\begin{aligned}(A_k f)(x) - f(x) &= \sum_n f(\frac{n}{2^k})\varphi(2^k x - n) - f(x)\sum_n \varphi(2^k x - n) \\ &= \sum_n [f(\frac{n}{2^k}) - f(x)]\varphi(2^k x - n). \qquad (4)\end{aligned}$$

Applying the Taylor's expansion of f at $\frac{n}{2^k}$:

$$f(x) = \sum_{j=0}^{+\infty} \frac{f^{(j)}(\frac{n}{2^k})}{j!}(x - \frac{n}{2^k})^j,$$

we obtain

$$(A_k f)(x) - f(x) = -\sum_n \sum_{j=1}^{+\infty} \frac{f^{(j)}(\frac{n}{2^k})}{j!}(x - \frac{n}{2^k})^j \varphi(2^k x - n).$$

Consequently, the infinite integrals from $-\infty$ to $+\infty$ become

$$\begin{aligned}&\int_{-\infty}^{+\infty} [(A_k f)(x) - f(x)] dx \\ &= -\int_{-\infty}^{+\infty} \sum_n \sum_{j=1}^{+\infty} \frac{f^{(j)}(\frac{n}{2^k})}{j!}(x - \frac{n}{2^k})^j \varphi(2^k x - n) dx.\end{aligned}$$

From the vanishing moment conditions (3), we have

$$\begin{aligned}&\int_{-\infty}^{+\infty} \sum_n \sum_{j=1}^{+\infty} \frac{f^{(j)}(\frac{n}{2^k})}{j!}(x - \frac{n}{2^k})^j \varphi(2^k x - n) dx \\ &= \sum_n \sum_{j=1}^{+\infty} \frac{f^{(j)}(\frac{n}{2^k})}{j!} \int_{-\infty}^{+\infty} (x - \frac{n}{2^k})^j \varphi(2^k x - n) dx \\ &= \sum_n \sum_{j=1}^{+\infty} \frac{f^{(j)}(\frac{n}{2^k})}{2^{(j+1)k} j!} \int_{-\infty}^{+\infty} x^j \varphi(x) dx = 0,\end{aligned}$$

where the vanishing moment conditions (3) ensure the equality. Therefore,

$$\int_{-\infty}^{+\infty}[(A_k f)(x)-f(x)]dx$$

converge to zero as $k\to+\infty$.

Remark 1 Theorem 1 establishes that under certain conditions, the integrals of the remainders $f-A_k f$ of wavelet expansions (1) from $-\infty$ to $+\infty$ converge to zero as $k\to+\infty$.

Remark 2 (Approximation of the Shannon and Meyer wavelets) Consider $f\in L^1(R)$ to be an infinitely differentiable function. The scale functions of both the Shannon wavelet (example 1) and the Meyer wavelet (example 2) satisfy the partition of unity condition (2) and the vanishing moment conditions (3). As a direct consequence,

$$\int_{-\infty}^{+\infty}[(A_k f)(x)-f(x)]dx$$

converge to zero as $k\to+\infty$. This implies that the integrals of the remainders $f-A_k f$ of wavelet expansions (1) from $-\infty$ to $+\infty$ converge to zero as $k\to+\infty$.

IV. Approximation of the Wavelet Expansions In $L^1(R)$-norm

In this section, we investigate the approximation and convergence rate of the wavelet expansions (1) by analyzing the $L^1(R)$-norm of $A_k f-f$.

Theorem 2 Let $f\in L^1(R)\cap L^2(R)$ be an r-order differentiable function $(r>1)$ with $|f^{(r)}|$ monotonically decreasing. Suppose the scale function φ has compact support $\operatorname{supp}\varphi\subset[a,b]$ and satisfies the partition of unity condition (2). Then the following error bound holds:

$$\|A_k f-f\|_1\le\sum_{j=1}^{r-1}\frac{C_{j,k}R_j}{j!2^{k(j+1)}}+\frac{C_{r,k}R_0}{2^k r},$$

in which

$$C_{j,k}=\sum_n|f^{(j)}(\frac{n}{2^k})|\quad(j=1,2,\cdots,r),$$

$$R_j=\int_a^b|x^j\varphi(x)|dx\quad(j=0,1,\cdots,r-1).$$

Proof Starting from equation (4) in the proof of Theorem 1, we express and estimate the approximation error in the $L^1(R)$-norm:

$$\begin{aligned}\|A_k f-f\|_1&=\int_{-\infty}^{+\infty}|(A_k f)(x)-f(x)|dx\\&\le\sum_n\int_{-\infty}^{+\infty}|[f(\frac{n}{2^k})-f(x)]\varphi(2^k x-n)|dx\\&=\frac{1}{2^k}\sum_n\int_{-\infty}^{+\infty}|[f(\frac{n}{2^k})-f(\frac{x}{2^k}+\frac{n}{2^k})]\varphi(x)|dx\\&=\frac{1}{2^k}\sum_n\int_a^b|[f(\frac{n}{2^k})-f(\frac{x}{2^k}+\frac{n}{2^k})]\varphi(x)|dx.\end{aligned}$$

Using Taylor's formula with integral remainder of f at $\frac{n}{2^k}$:

$$f(\frac{x}{2^k}+\frac{n}{2^k})=\sum_{j=0}^{r-1}\frac{f^{(j)}(\frac{n}{2^k})}{j!}(\frac{x}{2^k})^j+\int_0^1(1-s)^{r-1}f^{(r)}(\frac{x}{2^k}s+\frac{n}{2^k})ds,$$

we bound the $L^1(R)$-norm of the approximation error as follows:

$$\begin{aligned}&\|A_k f-f\|_1\\&\le\frac{1}{2^k}\sum_n\int_a^b|[\sum_{j=1}^{r-1}\frac{f^{(j)}(\frac{n}{2^k})}{j!}(\frac{x}{2^k})^j\\&\quad+\int_0^1(1-s)^{r-1}f^{(r)}(\frac{x}{2^k}s+\frac{n}{2^k})ds]\varphi(x)|dx\\&\le\frac{1}{2^k}\sum_n\sum_{j=1}^{r-1}\frac{|f^{(j)}(\frac{n}{2^k})|}{j!2^{kj}}\int_a^b|x^j\varphi(x)|dx\\&\quad+\frac{1}{2^k}\sum_n\int_a^b|\int_0^1(1-s)^{r-1}f^{(r)}(\frac{x}{2^k}s+\frac{n}{2^k})ds\cdot\varphi(x)|dx\\&=:I_1+I_2.\end{aligned}\tag{5}$$

We bound the term I_1:

$$\begin{aligned}I_1&=\sum_{j=1}^{r-1}\frac{1}{j!2^{k(j+1)}}\sum_n|f^{(j)}(\frac{n}{2^k})||\int_a^b|x^j\varphi(x)|dx\\&=\sum_{j=1}^{r-1}\frac{C_{j,k}R_j}{j!2^{k(j+1)}},\end{aligned}\tag{6}$$

where

$$C_{j,k}=\sum_n|f^{(j)}(\frac{n}{2^k})|\quad(j=1,2,\cdots,r),$$

$$R_j=\int_a^b|x^j\varphi(x)|dx\quad(j=0,1,\cdots,r-1).$$

Since the scale function φ has compact support $\operatorname{supp}\varphi\subset[a,b]$, the following equations hold:

$$\begin{aligned}\int_a^b\varphi(x)dx&=\int_{-\infty}^{+\infty}\varphi(x)dx=\sum_n\int_{-n}^{-n+1}\varphi(x)dx\\&=\sum_n\int_0^1\varphi(x-n)dx=\int_0^1\sum_n\varphi(x-n)dx.\end{aligned}$$

From the partition of unity condition (2), we have $\int_0^1\sum_n\varphi(x-n)dx=1$. Thus,

$$\int_a^b\varphi(x)dx=1.$$

This confirms both the scale function φ and $|x^j\varphi(x)|$ $(j=0,1,\cdots,r-1)$ are integral on $[a,b]$.

We bound the term I_2:

$$\begin{aligned}I_2&\le\frac{1}{2^k}\int_a^b\int_0^1(1-s)^{r-1}\sum_n|f^{(r)}(\frac{x}{2^k}s+\frac{n}{2^k})|\cdot|\varphi(x)|dsdx\\&=\frac{1}{2^k}\int_a^b\int_0^1(1-s)^{r-1}\sum_n|f^{(r)}(\frac{xs-[xs]}{2^k}+\frac{n+[xs]}{2^k})|\cdot|\varphi(x)|dsdx.\end{aligned}$$

Further simplification using the monotonicity of $|f^{(r)}|$ yields:

$$\begin{aligned}I_2&\le\frac{1}{2^k}\int_a^b\int_0^1(1-s)^{r-1}\sum_n|f^{(r)}(\frac{n+[xs]}{2^k})|\cdot|\varphi(x)|dsdx\\&=\frac{1}{2^k}\int_a^b\int_0^1(1-s)^{r-1}\sum_n|f^{(r)}(\frac{n}{2^k})|\cdot|\varphi(x)|dsdx\\&=\frac{C_{r,k}}{2^k}\int_a^b\int_0^1(1-s)^{r-1}|\varphi(x)|dsdx\\&=\frac{C_{r,k}R_0}{2^k r},\end{aligned}\tag{7}$$

where

$$C_{r,k}=\sum_n f^{(r)}(\frac{n}{2^k}),\quad R_0=\int_a^b|\varphi(x)|dx.$$

Combining the bounds (5)-(7), we arrive at the total approximation error:

$$\|A_k f - f\|_1 \le \sum_{j=1}^{r-1} \frac{C_{j,k} R_j}{j! 2^{k(j+1)}} + \frac{C_{r,k} R_0}{2^k r},$$

in which

$$C_{j,k} = \sum_n | f^{(j)}(\frac{n}{2^k}) | \quad (j = 1, 2, \cdots, r),$$

$$R_j = \int_a^b | x^j \varphi(x) | dx \quad (j = 0, 1, \cdots, r-1).$$

Conclusion Let $f \in L^1(R) \cap L^2(R)$ be an r-order differentiable function $(r > 1)$ with $| f^{(r)} |$ monotonically decreasing. Assume the scale function φ has the compact support supp $\varphi \subset [a, b]$ and satisfies the partition of unity condition (2). Then the approximation error of the wavelet expansions (1) is bounded by

$$\|A_k f - f\|_1 \le R \sum_{j=1}^{r-1} \frac{C_{j,k} c^{j+\frac{1}{2}}}{j! 2^{k(j+1)}} + \frac{C_{r,k} R_0}{2^k r},$$

in which

$$C_{j,k} = \sum_n | f^{(j)}(\frac{n}{2^k}) | \quad (j = 1, 2, \cdots, r),$$

$$R = (\int_a^b | \varphi(x) |^2 \, dx)^{\frac{1}{2}}, \quad c = \max\{| a |, | b |\}.$$

Proof From Cauchy-Schwarz inequality, we derive

$$\int_a^b | x^j \varphi(x) | \, dx \le (\int_a^b x^{2j} dx)^{\frac{1}{2}} \cdot (\int_a^b | \varphi(x) |^2 \, dx)^{\frac{1}{2}}$$
$$\le R(\int_{-c}^{c} x^{2j} dx)^{\frac{1}{2}} = \sqrt{2} R (\int_0^c x^{2j} dx)^{\frac{1}{2}}$$
$$= \sqrt{2} R \frac{c^{j+\frac{1}{2}}}{\sqrt{2j+1}} \le R c^{j+\frac{1}{2}}, \qquad (8)$$

where $R = (\int_a^b | \varphi(x) |^2 \, dx)^{\frac{1}{2}}, c = \max\{| a |, | b |\}$.

From Theorem 2, we derive

$$\|A_k f - f\|_1 \le R \sum_{j=1}^{r-1} \frac{C_{j,k} c^{j+\frac{1}{2}}}{j! 2^{k(j+1)}} + \frac{C_{r,k} R_0}{2^k r},$$

in which

$$C_{j,k} = \sum_n | f^{(j)}(\frac{n}{2^k}) | \quad (j = 1, 2, \cdots, r),$$

$$R = (\int_a^b | \varphi(x) |^2 \, dx)^{\frac{1}{2}}, c = \max\{| a |, | b |\}.$$

Theorem 3 Let $f \in L^1(R) \cap L^2(R)$ be an r-order differentiable function $(r > 1)$ satisfying the following conditions:

(1) $f^{(j)} \in L^1(R)$ $(j = 1, 2, \cdots, r-1)$;

(2) $| f^{(r)} |$ is monotonically decreasing;

(3) $C_r = \sup_k \sum_n | f^{(r)}(\frac{n}{2^k}) | < +\infty$.

If the scale function φ has compact support supp $\varphi \subset [a, b]$ and satisfies the partition of unity condition (2), then we bound the approximation error as

$$\|A_k f - f\|_1 \le \sum_{j=1}^{r-1} \frac{C_j R_j}{j! 2^{kj}} + \frac{C_r R_0}{2^k r},$$

where

$$C_j = \sup_k \frac{1}{2^k} \sum_n | f^{(j)}(\frac{n}{2^k}) | \quad (j = 1, 2, \cdots, r-1),$$

$$R_j = \int_a^b | x^j \varphi(x) | dx \quad (j = 0, 1, \cdots, r-1).$$

Proof Under the conditions of $f^{(j)} \in L^1(R)$ ($j = 1, 2, \cdots, r-1$), the following limit can be expressed as:

$$\lim_{k \to +\infty} \frac{1}{2^k} \sum_n | f^{(j)}(\frac{n}{2^k}) |$$
$$= \lim_{k \to +\infty} (\cdots + \sum_{n=-2^{k+1}}^{-2^k} | f^{(j)}(\frac{n}{2^k}) | \frac{1}{2^k} + \sum_{n=-2^k}^{0} | f^{(j)}(\frac{n}{2^k}) | \frac{1}{2^k}$$
$$+ \sum_{n=0}^{2^k} | f^{(j)}(\frac{n}{2^k}) | \frac{1}{2^k} + \sum_{n=2^k}^{2^{k+1}} | f^{(j)}(\frac{n}{2^k}) | \frac{1}{2^k} + \cdots)$$
$$= \cdots + \int_{-2}^{-1} | f^{(j)}(x) | dx + \int_{-1}^{-0} | f^{(j)}(x) | dx + \int_0^1 | f^{(j)}(x) | dx$$
$$+ \int_1^2 | f^{(j)}(x) | dx + \cdots$$
$$= \int_{-\infty}^{+\infty} | f^{(j)}(x) | dx < +\infty \quad (j = 1, 2, \cdots, r-1).$$

Consequently, the supremum is finite:

$$C_j = \sup_k \frac{1}{2^k} \sum_n | f^{(j)}(\frac{n}{2^k}) | < +\infty \quad (j = 1, 2, \cdots, r-1).$$

Applying Theorem 2, we obtain the error bound

$$\|A_k f - f\|_1 \le \sum_{j=1}^{r-1} \frac{C_j R_j}{j! 2^{kj}} + \frac{C_r R_0}{2^k r},$$

in which

$$C_j = \sup_k \frac{1}{2^k} \sum_n | f^{(j)}(\frac{n}{2^k}) | \quad (j = 1, 2, \cdots, r-1),$$

$$R_j = \int_a^b | x^j \varphi(x) | dx \quad (j = 0, 1, \cdots, r-1).$$

Remark 1 The conclusion of Theorem 3 can be expressed in the following simplified form:

$$\|A_k f - f\|_1 \le O(\frac{1}{2^k}),$$

where "O" depends on

$$C_j = \sup_k \frac{1}{2^k} \sum_n | f^{(j)}(\frac{n}{2^k}) | \quad (j = 1, 2, \cdots, r-1),$$

and

$$R_j = \int_a^b | x^j \varphi(x) | dx \quad (j = 0, 1, \cdots, r-1).$$

Remark 2 Theorem 3 establishes that the wavelet expansions (1) based on Riesz basis converge to f in $L^1(R)$-norm as $k \to +\infty$ under the specified conditions. Furthermore, Remark 1 demonstrates that the approximation error decays exponentially with rate order $O(\frac{1}{2^k})$.

Remark 3 (Approximation of the Haar and linear B-spline wavelets **)** Let $f \in L^1(R) \cap L^2(R)$ be an r-order differentiable function $(r > 1)$ satisfying the following conditions:

(4) $f^{(j)} \in L^1(R)$ $(j = 1, 2, \cdots, r-1)$;

(5) $| f^{(r)} |$ is monotonically decreasing;

(6) $C_r = \sup_k \sum_n | f^{(r)}(\frac{n}{2^k}) | < +\infty$.

Both the Haar wavelet (example 3) and the linear B-spline wavelet (example 4) possess scale functions φ with compact support that satisfy the partition of unity condition (2). This property ensures that the wavelet expansions (1) converge to f with respect to the $L^1(R)$-norm as $k \to +\infty$ and the approximation error decays exponentially with rate $O(\frac{1}{2^k})$ under the specified conditions irrespective of the orthogonality of the Riesz basis.

Conclusion Let $f \in L^1(R) \cap L^2(R)$ be an r-order differentiable function $(r > 1)$ satisfying the following conditions:

(1) $f^{(j)} \in L^1(R)$ $(j = 1, 2, \cdots, r-1)$;

(2) $|f^{(r)}|$ is monotonically decreasing;

(3) $C_r = \sup_k \sum_n |f^{(r)}(\frac{n}{2^k})| < +\infty$.

If the scale function φ has compact support supp $\varphi \subset [a,b]$ and satisfies the partition of unity condition (2), then

$$\|A_k f - f\|_1 \le R \sum_{j=1}^{r-1} \frac{C_j c^{j+\frac{1}{2}}}{j!2^{kj}} + \frac{C_r R_0}{2^k r},$$

in which

$$C_j = \sup_k \frac{1}{2^k} \sum_n |f^{(j)}(\frac{n}{2^k})| \quad (j = 1, 2, \cdots, r-1),$$

$$R = (\int_a^b |\varphi(x)|^2 \, dx)^{\frac{1}{2}}, \quad c = \max\{|a|, |b|\}.$$

Proof The result is obtained by substituting equation (8) into the conclusion of Theorem 3.

Theorem 4 Let $f \in L^1(R) \cap L^2(R)$ be an infinitely differentiable function. If the scale function φ has compact support supp $\varphi \subset [a,b]$ and satisfies the partition of unity condition (2) , then

$$\|A_k f - f\|_1 \le \sum_{j=1}^{+\infty} \frac{C_j R_j}{j! \cdot 2^{kj}},$$

in which

$$C_j = \sup_k \frac{1}{2^k} \sum_n |f^{(j)}(\frac{n}{2^k})|,$$

$$R_j = \int_a^b |x^j \varphi(x)| \, dx \quad (j = 1, 2, \cdots).$$

Proof Since $f \in L^1(R) \cap L^2(R)$ is an infinitely differentiable function, we apply the Taylor's expansion of f at $\frac{n}{2^k}$:

$$f(\frac{x}{2^k} + \frac{n}{2^k}) = \sum_{j=0}^{+\infty} \frac{f^{(j)}(\frac{n}{2^k})}{j!} (\frac{x}{2^k})^j.$$

From the partition of unity condition (2), we have

$$\|A_k f - f\|_1 \le \frac{1}{2^k} \sum_n \int_a^b |[f(\frac{x}{2^k} + \frac{n}{2^k}) - f(\frac{n}{2^k})]\varphi(x)| \, dx$$

$$= \frac{1}{2^k} \sum_n \int_a^b |\sum_{j=1}^{+\infty} \frac{f^{(j)}(\frac{n}{2^k})}{j!} (\frac{x}{2^k})^j \varphi(x)| \, dx$$

$$\le \frac{1}{2^k} \sum_n \sum_{j=1}^{+\infty} \frac{|f^{(j)}(\frac{n}{2^k})|}{j! \cdot 2^{kj}} \int_a^b |x^j \varphi(x)| \, dx$$

$$\le \sum_{j=1}^{+\infty} \frac{C_j R_j}{j! \cdot 2^{kj}},$$

in which

$$C_j = \sup_k \frac{1}{2^k} \sum_n |f^{(j)}(\frac{n}{2^k})|,$$

$$R_j = \int_a^b |x^j \varphi(x)| \, dx \quad (j = 1, 2, \cdots).$$

Conclusion Let $f \in L^1(R) \cap L^2(R)$ be an infinitely differentiable function. If the scale function φ has compact support supp $\varphi \subset [a,b]$ and satisfies the partition of unity condition (2) , then

$$\|A_k f - f\|_1 \le R \sum_{j=1}^{+\infty} \frac{C_j c^{j+\frac{1}{2}}}{j! \cdot 2^{kj}},$$

in which

$$C_j = \sup_k \frac{1}{2^k} \sum_n |f^{(j)}(\frac{n}{2^k})| \quad (j = 1, 2, \cdots),$$

$$R = (\int_a^b |\varphi(x)| \, dx)^{\frac{1}{2}}, \quad c = \max\{|a|, |b|\}.$$

Proof The result is derived through substitution of equation (8) into the conclusion of Theorem 4.

Remark 1 (Approximation of Haar wavelet) Consider $f \in L^1(R) \cap L^2(R)$ to be an infinitely differentiable function. The scale function φ of Haar wavelet has compact support $[0,1)$ and satisfies the partition of unity condition (2) . Observing

$$R = (\int_a^b |\varphi(x)| \, dx)^{\frac{1}{2}} \le (\int_0^1 dx)^{\frac{1}{2}} = 1,$$

we obtain the approximation error bound:

$$\|A_k f - f\|_1 \le \sum_{j=1}^{+\infty} \frac{C_j c^{j+\frac{1}{2}}}{j! \cdot 2^{kj}},$$

where

$$C_j = \sup_k \frac{1}{2^k} \sum_n |f^{(j)}(\frac{n}{2^k})| \quad (j = 1, 2, \cdots),$$

$$c = \max\{|a|, |b|\}.$$

Remark 2 (Approximation of Linear B-spline wavelet) For $f \in L^1(R) \cap L^2(R)$ that is an infinitely differentiable function. The scale function φ of linear B-spline wavelet has compact $[0,2)$ and satisfies the partition of unity condition (2) . Noting

$$R = (\int_a^b |\varphi(x)| \, dx)^{\frac{1}{2}} \le (\int_0^1 x dx + \int_1^2 (x-1) dx)^{\frac{1}{2}} = 1,$$

then

$$\|A_k f - f\|_1 \le \sum_{j=1}^{+\infty} \frac{C_j c^{j+\frac{1}{2}}}{j! \cdot 2^{kj}},$$

in which

$$C_j = \sup_k \frac{1}{2^k} \sum_n |f^{(j)}(\frac{n}{2^k})| \quad (j = 1, 2, \cdots),$$

$$c = \max\{|a|, |b|\}.$$

Both the orthogonal Haar wavelet and non-orthogonal linear B-spline wavelet satisfy the conditions of Theorem 4, confirming that the wavelet expansions (1) achieve the specified convergence rate regardless of orthogonality. This equivalence demonstrates that the convergence behavior established in Theorem 4 is fundamentally independent of the wavelet system's orthogonality, with both types of wavelets exhibiting identical convergence characteristics when their scale functions meet the required conditions.

V. Conclusions

This study has systematically investigated the approximation of the wavelet expansions (1) constructed from a Riesz basis, establishing a unified framework for both orthogonal and non-orthogonal systems. We proved that the integrals of the remainders of wavelet expansions (1) from $-\infty$ to $+\infty$ converge to zero as $k \to +\infty$ under the conditions of Theorem 1. This result holds for both the Shannon and Meyer wavelets as their scale functions satisfy the conditions of Theorem 1. The wavelet expansions (1) converge to f in $L^1(R)$ -norm as $k \to +\infty$ and explicit decay rates are provided under the conditions of Theorem 3. Additionally, the exact convergence rate in $L^1(R)$ -norm is obtained in Theorem 4. Notably, we derive a pioneering rigorous estimation for the convergence rate of non-orthogonal wavelet expansions. Moreover, our results demonstrate that the orthogonal wavelets (e.g., Haar) and non-orthogonal wavelets (e.g., B-spline) achieve the same convergence rates. This helps to bridge the long-standing theoretical gap between orthogonal and non-orthogonal wavelet systems.

This research significantly extends classical wavelet approximation theory by establishing generalized error bounds through derivative-dependent constants C_j and R_j . The theoretical framework demonstrates particular value in computational applications, where it validates the effectiveness of non-orthogonal wavelets for signal processing tasks requiring $L^1(R)$ -convergence and adaptive algorithms utilizing exponential decay properties for optimal resolution selection. These contributions provide both theoretical foundations and practical implementation guidelines for wavelet-based computational methods.

Future research will focus on optimizing Riesz basis for minimal support properties, extending the framework to high-dimensional data compression, and integrating these results with deep learning architectures. These proposed developments promise to further bridge theoretical wavelet analysis with emerging computational paradigms, potentially yielding new tools for scientific computing applications.

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