A Computational Method for Solving a q-Fractional Differential Equation using q-Legendre Operational Matrix

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***Abstract*—This paper focuses on deriving a solution for the q-fractional differential equation(q-FDE) using the q-operational matrix based on q-Legendre polynomials. The q-fractional derivative is defined in the Caputo sense. We determine and employ an operational matrix of q-Legendre polynomials for q-fractional order derivatives, integrating it with the spectral tau method to transform q-FDEs into algebraic equations. Additionally, we showcase the application of this method by solving the q-fractional Bagley-Torvik equation through the spectral tau method, highlighting the versatility of the proposed approach.**

***Index Terms*—q-Legendre polynomials, Caputo q-fractional derivative, operational matrix, q-fractional differential equations, collocation method, Tau method.**

I. Introduction

THE Fractional calculus and q-calculus extend the conventional ideas of integration and differentiation from integer orders to arbitrary orders. In q-calculus, the primary focus is on the properties of q-special functions, which extend classical special functions by incorporating a parameter known as the base q. These frameworks broaden the scope of mathematical analysis, enabling the exploration of functions and properties that deviate from the traditional integer-order calculus. Specifically, q-calculus delves into the behaviors exhibited by q-special functions, which vary based on the chosen parameter, leading to a nuanced understanding of mathematical structures beyond the classical realm. In 1910 Jackson[8], [9], [10] introduced the concept of q-calculus. Works are being done on this subject from a very long time and it has several applications. Fractional calculus is well known for their applications in many fields such as physics, aerodynamics, capacitor theory, chemistry, biology, control theory, probability and statistics. We can observe many works on fractional derivatives and fractional differential equations(FDEs)[15], [16]. These works help us to understand about fractional calculus and they also introduce the theory of fractional derivative and FDEs. The q-fractional calculus is an advanced mathematical framework that extends the principles of conventional fractional calculus by introducing a parameter q. This extension offers a more comprehensive and flexible approach to fractional differentiation and integration, facilitating the exploration of novel mathematical properties and solutions. Al-Salam [2] and Agarwal [1] pioneered the concept and elaborated on various types of q-fractional integral and derivatives operators. The basic concept can be found in[4]. Extensive investigation into this subject has attracted considerable attention from multiple authors, resulting in a significant body of research dedicated to the examination of q-FDEs and their applications [20], [18], [17]. Due to the absence of exact solutions for most fractional differential equations(FDEs), it becomes necessary to explore approximate and numerical techniques. Numerous researchers have addressed this challenge by proposing various numerical and approximate methods for solving FDEs. Examples include the variational iteration method(VAM), homotopy perturbation method(HPM), Adomain's decomposition method(ADM), homotopy analysis method(HAM), operational matrix method using collocation, and finite difference method. In contrast, in the realm of q-calculus, the available methods are relatively limited. In recent years, significant advancements have been made in developing operational matrices tailored for orthogonal polynomials, aiming to facilitate the derivation of numerical solutions. The significance of orthogonal polynomials extends to both pure and applied mathematics, as well as numerous realms within numerical analysis. Specifically, in the context of spectral methods, these polynomials play a crucial role. Through the operational matrix method, it becomes feasible to transform a FDE into a system of algebraic equations, utilizing operational matrices and orthogonal polynomials to obtain an approximate solution. Operational matrices are derived through the approximation of integrals involving orthogonal polynomials. This approach offers a concise orthogonal series for numerically integrating differential equations. Under the operational matrix method, Saadatmandi and Dehghan [19] expanded upon the use of Legendre polynomials, Abdelkawy and Taha [5] developed the Laguerre polynomials, and Bhrawy and Alofi [6] introduced new shifted Chebyshev polynomials for fractional integration in the R–Liouville sense within the context of FDEs. These approaches apply to both linear and nonlinear cases, and the authors further explored their applications in spectral techniques.

Several semi-analytical methods have been developed for q-FDEs. In 2013, Wu and Baleanu[23] introduced the VAM, followed by Pin Lyu and Seakweng Von's[12] finite differ-

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ence method in 2019. In 2021, B. Madhavi and G. Suresh Kumar[13], [14] used Laguerre polynomials in an operational matrix method, and in 2023, they developed the HPM. Ying Sheng and Tie Zhang[21] also made advancements in q-calculus and q-FDEs. This work focuses on solving q-FDEs using the q-Legendre operational matrix method (LEOM), expanding its application to q-fractional calculus. The paper is structured as follows: In Section 2, we delve into the essential definitions of q-fractional integrals and derivatives, laying the groundwork for the study. Section 3 takes a closer look at q-Legendre polynomials and constructs the operational matrix for the q-fractional derivative. Section 4 brings the theoretical framework to life, showcasing key results alongside illustrative numerical examples. Finally, the concluding section wraps up with a concise summary of the insights gained from the research.

A. Preliminaries

Definition 1.[2] Let $\Delta > 0$, The R-Liouville definition of q-fractional integral of $h(\omega)$ is defined as

$$J_q^\Delta h(\omega) = \frac{1}{\Gamma_q(\Delta)} \int_0^\omega (\omega - qt)^{\Delta-1} h(t) d_q(t), \quad (1)$$
$$J_q^0 h(\omega) = h(\omega).$$

Definition 2.[2] Let $\Delta > 0$, The Caputo definition q-fractional integral of $h(\omega)$ is defined as

$$\begin{aligned} D_q^\Delta h(\omega) &= J^{(m-\mu)} D^m h(\omega), \\ &= \Gamma_q(m-\Delta) \int_0^\omega (\omega - qt)^{m-\mu-1} \frac{d_q^m}{d\omega_q^m} h(t) d_q t, \end{aligned} \quad (2)$$

$(m-1) < \Delta < m, \omega > 0$, where D^Δ is the differential operator of order Δ and satisfies the following

$$D^\Delta C = 0, \ (C \text{ is a constant}). \quad (3)$$

$$D_q^\Delta \omega^\alpha = \begin{cases} 0, & \text{for } \alpha \in \eta_0 \text{ and } \alpha < \lceil \Delta \rceil \\ \dfrac{\Gamma_q(\alpha+1)}{\Gamma_q(\alpha+1-\Delta)} \omega^{\alpha-\Delta}, & \text{for } \alpha \in \eta_0 \text{ and } \alpha \geq \lceil \Delta \rceil \\ & \text{or } \alpha \notin \eta \text{ and } \alpha > \lfloor \Delta \rfloor. \end{cases} \quad (4)$$

Here $\lceil \Delta \rceil$ refers to the ceiling function, and $\lfloor \Delta \rfloor$ represents the floor functions. Furthermore, η represents the set $\{1, 2, ...\}$ and η_0 represents the set $\{0, 1, 2, ...\}$. The linearity property is

$$D_q^\Delta \big(\lambda h(\omega) + \delta g(\omega)\big) = \lambda D_q^\Delta h(\omega) + \delta D_q^\Delta g(\omega). \quad (5)$$

Whereλ and δ are constants.

q-Legendre Polynomials[22]: The $z'th$ degree of q-Legendre polynomials, defined within the interval $\Lambda \equiv (0, \infty_q)$ are expressed as follows:

$$p_{z,q}(\omega) = \sum_{v=0}^{z} \frac{(-1)^{z+v} (z+v)_q! \, q^{-vz+\frac{v(v+1)}{2}}}{(z-v)_q! \, (v_q!)^2} \omega^v,$$
$$z = 0, 1, \ldots. \quad (6)$$

The orthogonality condition[3] is

$$\int_0^1 p_{m,q}(\omega) p_{n,q}(\omega) d_q\omega = \frac{(1-q)q^n}{1-q^{2n+1}} \delta_{mn}.$$

II. Generalized q-LEOM of q-Fractional Calculus

Let us consider $h(\omega) \in L_W^2(\Lambda)$, then $h(\omega)$ can be elegantly represented using q-Legendre polynomial

$$h(\omega) = \sum_{s=0}^{\infty} c_s p_s(\omega), \quad (7)$$
$$\text{where } c_s = \int_0^\infty h(\omega) p_s(\omega) \, d\omega, \ s = 0, 1, 2, \ldots.$$

Let's start by considering the first $(\eta + 1)$ terms of the q-Legendre polynomials. The following observations can be noted:

$$h(\omega) = \sum_{s=0}^{\eta} c_s p_s(\omega) = C^T \phi(\omega). \quad (8)$$

Here, C represents the q-Legendre coefficient vector, while $\phi(\omega)$ stands for the q-Legendre vector, both defined as follows:

$$C^T = [c_0, c_1, \ldots, c_\eta], \ \phi(\omega) = [p_0, p_1, \ldots, p_\eta]^T. \quad (9)$$

Now, we can express the q-fractional derivative of the vector $\phi(\omega)$ in the following form:

$$\frac{d_q \phi(\omega)}{d\omega_q} = D_q^1 \phi(\omega), \quad (10)$$

here, the operational matrix D_q^1, a square matrix with dimensions of $(\eta+1) \times (\eta+1)$, is defined as

$$D_q^1 = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 0 \\ 2_q & 0 & 0 & \ldots & 0 & 0 \\ 0 & \frac{3_q 2_q}{q} & 0 & \ldots & 0 & 0 \\ 2_q & 0 & \frac{5_q 2_q}{q} & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 2_q & 0 & \frac{3_q 2_q}{q} & \ldots & 0 & 0 \\ 0 & \frac{3_q 2_q}{q} & 0 & \ldots & 0 & 0 \end{pmatrix}$$

From (10), it is clear that

$$\frac{d_q^\mu \phi(\omega)}{d\omega_q^\mu} = (D_q^{(1)})^\mu \phi(\omega), \quad (11)$$

where, the notation (D_q^1) represents matrix powers, where $\mu \in N$.

Hence

$$D_q^{(\mu)} = (D_q^{(1)})^\mu, \ \mu = 1, 2, 3, \ldots. \quad (12)$$

Lemma 1. Let $p_z(\omega)$ represents a q-Legendre polynomial, then

$$D_q^\Delta p_z(\omega) = 0, \ z = 0, 1, \ldots, \alpha < \lceil \Delta \rceil - 1, \Delta > 0. \quad (13)$$

Proof.. Using (4) and (5) in (6), the lemma can be easily demonstrated.

Theorem 1. Suppose $\phi(\omega)$ be a q-Legendre vector defined for $\Delta > 0$, then

$$D_q^\Delta \phi(\omega) = D_q^{(\Delta)} \phi(\omega), \quad (14)$$

here, D_q^{Δ} represents the operational matrix for q-fractional derivatives of order Δ in $(\eta+1)$ dimensions, defined as:

$$\begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \xi_{\Delta,q}(\lceil\Delta\rceil,0,v) & \xi_{\Delta,q}(\lceil\Delta\rceil,1,v) & \dots & \xi_{\Delta,q}(\lceil\Delta\rceil,\eta,v) \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{\Delta,q}(z,0,v) & \xi_{\Delta,q}(z,1,v]) & \dots & \xi_{\Delta,q}(z,\eta,v) \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{\Delta,q}(\eta,0,v) & \xi_{\Delta,q}(\eta,1,v) & \dots & \xi_{\Delta,q}(\eta,\eta,v) \end{pmatrix}$$

where

$$\xi_{\Delta,q}(z,s,v) = \frac{1-q^{2s+1}}{(1-q)q^s} \sum_{v=\lceil\Delta\rceil}^{z} \sum_{l=0}^{s} \frac{(-1)^{z+v+s+l}}{(z-v)_q!\, v_q!}$$
$$\frac{(z+v)_q!(s+l)_q! q^{zv+\frac{v(v+1)}{2}+sl+\frac{l(l+1)}{2}}}{\Gamma_q(-z+\Delta+1)\,(s-l)!\,(l_q!)^2\,(v+l-\Delta+1)}, \quad (15)$$

furthermore, the initial $\lceil\Delta\rceil$ rows of D_q^{Δ} are entirely composed of zeros.

Proof. From the equations (4), (5) and (8), we have

$$D_q^{\Delta} p_z(\omega) = \sum_{v=0}^{z} \frac{(-1)^{z+v}\,(z+v)_q!\, q^{-zv+\frac{v(v+1)}{2}}}{(z-v)_q\,(v!)^2} D^{\Delta}\omega^v,$$
$$= \sum_{v=\lceil\Delta\rceil}^{z} \frac{(-1)^{z+v}\,(z+v)_q!\, q^{zv+\dfrac{v(v+1)}{2}}}{(z-v)_q!\,\Gamma_q(v-\Delta+1)!\, v_q!} \omega^{v-\Delta},$$
$$z = 0,1,\dots \quad (16)$$

Now, applying $\omega^{v-\Delta}$ by $\eta+1$ terms of q-Legendre series, we obtain

$$\omega^{(v-\Delta)} = \sum_{s=0}^{\eta} b_{v,s}\, p_s(\omega), \quad (17)$$

where $b_{v,s}$ is given from (8) with

$$b_{v,s} = \frac{1-q^{2s+1}}{(1-q)q^s} \int_0^1 \omega^{v-\Delta}\, p_s(\omega)\, d_q\omega$$
$$= \frac{1-q^{2s+1}}{(1-q)q^s}$$
$$\sum_{l=0}^{s} \frac{(-1)^{s+l}\, s+l_q!\, s_q!\, q^{sl+\frac{l(l+1)}{2}}}{(s-l)_q!(l_q!)^2} \int_0^1 \omega^{v+l-\Delta} d_q\omega$$
$$= \frac{1-q^{2s+1}}{(1-q)q^s} \sum_{l=0}^{s} \frac{(-1)^{s+l}\,(s+l)_q!\, s_q!\, q^{sl+\frac{l(l+1)}{2}}}{(s-l)_q!\,(l_q!)^2\,(v+l-\Delta-1)_q}. \quad (18)$$

Adding (16)-(18), we get

$$D_q^{\Delta} p_z(\omega) =$$
$$\sum_{v=\lceil\Delta\rceil}^{z} \sum_{s=0}^{\eta} \frac{(-1)^{z+v}(z+v)_q!\, q^{zv+\frac{v(v+1)}{2}}}{(z-v)_q!\,\Gamma_q(v-\Delta+1)! v_q!} b_{z,s}\, p_s(\omega)$$
$$= \zeta_{\Delta,q}(i,s,v)\, p_s(\omega)\, z = \lceil\Delta\rceil,\dots,\eta, \quad (19)$$

where

$$\xi_{\Delta,q}(z,s,v) = \frac{1-q^{2s+1}}{(1-q)q^s} \sum_{v=\lceil\Delta\rceil}^{z} \sum_{l=0}^{s} \frac{(-1)^{z+v+s+l}}{(z-v)_q!\, v_q!}$$
$$\frac{(z+v)_q!(s+l)_q! q^{zv+\frac{v(v+1)}{2}+sl+\frac{l(l+1)}{2}}}{\Gamma_q(-z+\Delta+1)\,(s-l)!\,(l_q!)^2\,(v+l-\Delta+1)}, \quad (20)$$

from (20), it can be compactly represented in vector form

$$D_q^{\Delta} L_z(\omega)$$
$$= \Big[\xi_{\Delta,q}(z,0)\xi_{\Delta,q}(z,1)\xi_{\Delta,q}(z,2),\dots,\xi_{\Delta,q}(z,\eta)\Big]\phi(\omega). \quad (21)$$

Referring to Lemma 1, we can express it as follows:

$$D_q^{\Delta} p_z(\omega) = \big[0,0,\dots,0,0\big]\phi(\omega),\; z = 0,1,2,\dots\lceil\Delta\rceil - 1. \quad (22)$$

By combining (21) and (22), we can derive the expected result.

III. Applications of q-LEOM for q- FDEs

The q-LEOM methodology involves representing the unknown function as a series expansion using q-Laguerre polynomials as the basis functions. These polynomials are a special class of orthogonal functions in q-calculus, and they possess unique properties that make them well-suited for solving fractional differential equations.

A. Linear Multi-term q- FDEs

Linear Multi-term q-FDEs are a class of FDE that involve multiple fractional derivatives of different orders. The Caputo fractional derivative is a commonly used definition in this equation. Let's explore the elegance of the following linear Caputo q-FDEs:

$$D_q^{\Delta} h(\omega) = \sum_{j=1}^{v} \gamma_v D^{\alpha_j} h(\omega) + \gamma_{(v+1)} h(\omega) + g(\omega),$$
$$\lambda \in (0,\infty_q), \quad (23)$$

subject to the initial conditions

$$h^z(0) = d_z,\; z = 0,1,\dots\mu-1. \quad (24)$$

In order to solve the linear Caputo q-FDE (23) subject to the given conditions (24), we approximate the functions $h(\omega)$ and $g(\omega)$ by q-Legendre polynomials as follows:

$$h(\omega) = \sum_{z=0}^{\eta} c_z p_z(\omega) = C^T\phi(\omega), \quad (25)$$

$$g(\omega) = \sum_{z=0}^{\eta} g_z p_z(\omega) = G^T\phi(\omega). \quad (26)$$

Here, the vector $G = [g_0, g_1, g_2, .., g_\eta]^T$ is known, while the vector $C = [c_0, c_1, c_2, .., c_\eta]^T$ is unknown vector.

By considering the theorem we have,

take on (15) and (25), we get

$$D_q^{\Delta} h(\omega) = C^T D_q^{\Delta}\phi(\omega); C^T D_q^{(\Delta)}\phi(\omega), \quad (27)$$

$$D_q^{\alpha_s} h(\omega) = C^T D_q^{\alpha_s}\phi(\omega),\; s = 1,2,\dots,v. \quad (28)$$

Take on (24)-(27), the residual $\Re_\eta(\omega)$ for (23) can be composed as,

$$\Re_\eta(\omega) = \left[C^T D_q^{\Delta} - C^T \sum_{s=1}^{v} \gamma_s D_q^{\alpha_s} - \gamma_{(v+1)} C^T - G^T\right]\phi(\omega). \quad (29)$$

Using the conventional tau method [7], we can produce a set of $\eta - n + 1$ linear equations by employing the following procedure:

$$< \Re_\eta(\omega), p_s(\omega) > = \int_0^1 \Re_\eta(\omega) p_s(\omega) = 0,\ s = 0, 1, 2, \ldots, \eta - m. \tag{30}$$

and furthermore substituting (14) and (25) in (24), we get

$$h^\nu(0) = C^T D_q^\nu \phi(\omega) = d_\nu\ \nu = 0, 1, 2, \ldots, \mu - 1. \tag{31}$$

By employing (30) and (31), we can create two sets of linear equations: one with $(\eta - \mu + 1)$ equations and the other with m equations. Solving these linear equations enables us to find the values of the unknown coefficient vector C. Furthermore, we can use the expression $h(\omega)$ from (6) to compute the solution needed for the current problem. $h(\omega)$ given in (9) can be calculated, which gives the required solution.

B. Non-Linear Multi-term q- FDEs

We begin by studying the following non-linear q- FDEs

$$D_q^\Delta h(\omega) = \psi(\omega, h(\omega), D_q^{\alpha_1} h(\omega), \ldots, D_q^{\alpha_v} h(\omega)),\quad in\, \Lambda = (0, \infty), \tag{32}$$

with initial conditions

$$h^z(0) = d_z,\ z = 0, 1, \ldots, \mu - 1, \tag{33}$$

with initial conditions (24), Ψ can be non linear.
Aiming to use the q- Legendre polynomials in the non-linear cases problems, we initially inexact $h(\omega)$, $D^\Delta h(\omega)$ and $D^{\alpha_s} h(\omega)$ for $s = 1, 2, ..., \mu - 1$ as (25), (27) and (23) respectively. Substituting all these (33), finally we get

$$C^T D_q^\Delta \phi(\omega) = \Psi(\omega, C^T \phi(\omega), C^T D_q^{\alpha_1}, ..., D_q^{\alpha_v}), \tag{34}$$

and furthermore substituting (11)and (25) in (24), we get

$$h^z(0) = C^T D_q^z = d_z, z = 0, 1, 2, \ldots, \mu - 1. \tag{35}$$

To get the solutions $h(\omega)$, we first collocate (33) at $(\eta - \mu + 1)$ points, looking for acceptable collocation points. For this, we employ the first $(\eta - \mu + 1)$ q-Legendre roots $p_{\eta+1}(\omega)$. These equations, when combined with (35), yield $(\eta + 1)$ nonlinear equations that may be solved using Newton's iterative approach. In this approach, the approximated answer $h(\omega)$ may be obtained.

IV. NUMERICAL RESULTS

EXAMPLE.1: Applications in the context of the Bagley-Torvik equation.

Consider the Bagely-Torvik equation

$$D_q^2 h(\omega) + D_q^{\frac{3}{2}} h(\omega) + h(\omega) = 1 + \omega,\quad h(0) = 1,\ h'(0) = 1. \tag{36}$$

The exact solution to the provided problem is denoted as $h(\omega) = 1 + \omega$. However, when we apply the method detailed in the preceding section with $\eta = 2$, the obtained solution deviates from this exact result, resulting in an approximation as follows:

$$h(\omega) = c_0 p_0(\omega) + c_1 p_1(\omega) + c_2 p_2(\omega) + c_3 p_3 = C^T \phi(\omega). \tag{37}$$

Here, we have

$$D_q^1 = \begin{pmatrix} 0 & 0 & 0 \\ 2_q & 0 & 0 \\ 0 & \frac{3_q 2_q}{q} & 0 \end{pmatrix}, \quad D_q^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{3_q {2_q}^2}{q} & 0 & 0 \end{pmatrix},$$

$$G = \begin{pmatrix} 1 + \frac{1}{2_q} \\ \frac{1}{2_q} \\ 0 \end{pmatrix}, \quad D_q^{\frac{3}{2}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ D_0 & D_1 & D_2 \end{pmatrix},$$

where

$$D_0 = \frac{1}{q}\left[\frac{4_q!}{2_q!\Gamma_q(\frac{3}{2})(\frac{3}{2})_q}\right],$$

$$D_1 = \frac{1+q+q^2}{q^2}\left[\frac{-4_q!}{2_q!\Gamma_q(\frac{3}{2})(\frac{3}{2})_q} + \frac{4_q!}{\Gamma_q(\frac{3}{2})(\frac{5}{2})_q}\right],$$

$$D_2 = \frac{1+\cdots+q^4}{q^2}\left[\frac{4_q!}{q 2_q!\Gamma_q(\frac{3}{2})(\frac{3}{2})_q} - \frac{4_q!3_q!}{q^2\Gamma_q(\frac{5}{2})(\frac{5}{2})_q} + \frac{{4_q!}^2}{q^2 2_q!^2\Gamma_q(\frac{3}{2})(\frac{7}{2})_q}\right]$$

Therefore, using (30), we obtain

$$c_0 + c_2\left[\frac{3_q {2_q}^2}{q} + \frac{4_q!}{q 2_q!\Gamma(\frac{3}{2})(\frac{3}{2})_q}\right] = 1 + \frac{1}{2_q} \tag{38}$$

Also by using (31), we have

$$2_q c_1 + \frac{{4_q!}^2}{2_q} - 3_q!q + 4_q!2_q^2 c_2 = 1 \tag{39}$$

$$c_0 - c_1 + c_2 = 1 \tag{40}$$

By solving above (38)-(40), we get

$$c_0 = 1 + \frac{1}{2_q}, c_1 = \frac{1}{2_q}, c_2 = 0$$

Thus we can write

$$h(\omega) = \begin{pmatrix} 1 + \frac{1}{2_q} & \frac{1}{2_q} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2_q\omega - 1 \\ \frac{4_q!}{2_q^2 q}\omega^2 - \frac{3_q}{q}\omega + 1 \end{pmatrix} = 1 + \omega$$

EXAMPLE.2:Now, we consider the non linear initial value problem

$$D_q^3 h(\omega) + D_q^{\frac{5}{2}} h(\omega) + h^2(\omega) = \omega^4,\quad h(0) = h'(0) = 0,\ h''(0) = 2. \tag{41}$$

The exact solution to the provided problem is denoted as $h(\omega) = \frac{\omega^2}{q}$. However, when we apply the method detailed in the preceding section with $\eta = 3$, the obtained solution deviates from this exact result, resulting in an approximation as follows:

$$h(\omega) = c_0 p_0(\omega) + c_1 p_1(\omega) + c_2 p_2(\omega) + c_3 p_3 = C^T \phi(\omega), \tag{42}$$

Where

$$D_q^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2_q & 0 & 0 & 0 \\ 0 & \frac{3_q 2_q}{q} & 0 & 0 \\ 2_q & 0 & \frac{5_q 2_q}{q^2} & 0 \end{pmatrix}.$$

$$D_q^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{3_q {2_q}^2}{q} & 0 & 0 & 0 \\ 0 & \frac{5_q 3_q {2_q}^2}{q^3} & 0 & 0 \end{pmatrix}.$$

$$D_q^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{5_q 3_q {2_q}^3}{q^3} & 0 & 0 & 0 \end{pmatrix}. \quad C = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$D^{\frac{5}{2}_q} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ D_{0,q}^{\frac{5}{2}} & D_{1,q}^{\frac{5}{2}} & D_{2,q}^{\frac{5}{2}} & D_{3,q}^{\frac{5}{2}} \end{pmatrix}$$

Where $D_{0,q}^{\frac{5}{2}} = \frac{6_q!}{3_q! \, \Gamma_q \frac{3}{2}_q \frac{3}{2}_q q^3}$

$$D_{1,q}^{\frac{5}{2}} = \frac{(1-q)^3}{q} \left[\frac{-6_q!}{3_q! \, \Gamma_q(\frac{3}{2}) \, (\frac{3}{2})_q \, q^3} - \frac{6_q! \, 2_q!}{3_q! \, \Gamma_q(\frac{3}{2}) \, (1_q!)^2 \, (\frac{5}{2})_q \, q^4} \right]$$

$$D_{2,q}^{\frac{5}{2}} = \frac{(1-q)^5}{(1-q)q^2} \left[\frac{6_q!}{3_q! \, \Gamma_q(\frac{3}{2}) \, (\frac{3}{2})_q \, q^3} - \frac{6_q!}{3_q! \, \Gamma_q(\frac{3}{2}) \, (1_q!)^2 (\frac{5}{2})_q \, q^4} - \frac{6_q! \, 4_q!}{3_q! \, \Gamma_q(\frac{3}{2}) \, (2_q!)^2 \, (\frac{7}{2})_q \, q^4} \right]$$

$$D_{3,q}^{\frac{5}{2}} = \frac{(1-q)^7}{(1-q)q^2} \left[\frac{-6_q!}{3_q! \, \Gamma_q(\frac{3}{2}) \, (\frac{3}{2})_q \, q^3} + \frac{6_q! \, 4_q!}{3_q! \, \Gamma_q(\frac{3}{2}) \, (2_q!)^2 (\frac{5}{2})_q \, q^5} - \frac{6_q! \, 5_q!}{3_q! \, \Gamma_q(\frac{3}{2}) \, (2_q!)^2 \, (\frac{7}{2})_q \, q^6} + \frac{6_q! \, 6_q!}{3_q! \, \Gamma_q(\frac{3}{2}) \, (3_q!)^2 \, (\frac{9}{2})_q \, q^6} \right]$$

Therefore, using (30), we obtain

$$C^T D_q^3 h(\omega) + C^T D_q^{\frac{5}{2}} h(\omega) + [C^T h(\omega)]^2 - h(\omega)^4 = 0. \quad (43)$$

Also by using (31), we have

$$2_q c_1 + \frac{3_q!}{q} c_2 + \frac{4_q!}{2_q! \, q^2} c_3 = 0, \quad (44)$$

$$\frac{4_q!}{2_q!} c_2 + \left[\frac{6_q! \, 2_q^2}{4_q!} - 5_q! + \frac{6_q!}{3_q! \, 2_q!} \right] c_3 = 2, \quad (45)$$

$$c_0 - c_1 + c_2 + c_3 = 0. \quad (46)$$

By solving (44)-(46), we get

$$c_0 = \frac{-1}{3_q} + \frac{1}{2_q}, \; c_1 = \frac{1}{2_q q}, \; c_2 = \frac{1}{3_q!}, \; c_3 = 0$$

Thus, we can write $h(\omega)$ =

$$\begin{pmatrix} \frac{1}{2_q! \, q} - \frac{1}{3_q!}, & \frac{1}{2_q! \, q}, & \frac{1}{3_q!}, & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2_q!\omega - 1 \\ \frac{4_q!}{2_q^2 q} \omega^2 - \frac{3_q}{q}\omega + 1 \\ \frac{6_q!}{3_q^2 q^3} \omega^3 - \frac{5_q!}{2_q!^2 q^3} \omega^2 + \\ \frac{4_q!}{2_q! q^2} \omega - 1 \end{pmatrix}$$

$$= \frac{\omega^2}{q}.$$

Figures 1–5 of example 2 demonstrate the accuracy, convergence, and stability of the proposed Legendre polynomial-based operational matrix method. Figures 1–3 show a close match between the approximate and exact solutions $h(\omega) = \frac{\omega^2}{q}$ for $q(0.25, 0.5, \text{and} 0.75)$, confirming the method's precision across different fractional scales. Figure 4 further highlights this consistency across multiple q values, while the 3D plot in Figure 5 illustrates the inverse relationship between $h(\omega)$ and q over $\omega \in (0, 05)$ and $q \in (0.25, 1)$. When $q = 1$, the problem reduces to a classical FDE, and the solution coincides with Example 4 in [19].

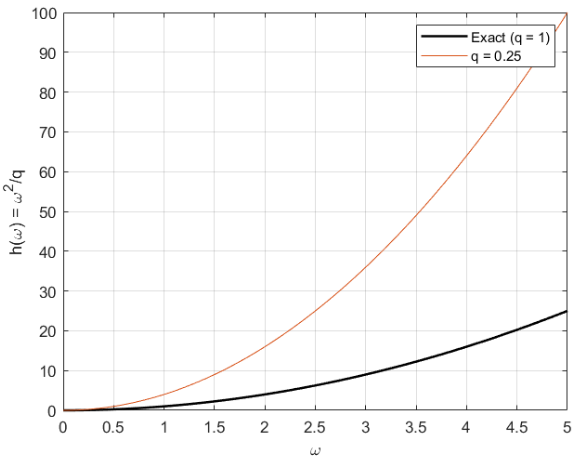

Fig. 1. Comparison of $h(\omega)$ for $q = 0.25$ with Exact solution(Fixed $\alpha = 2.5$)

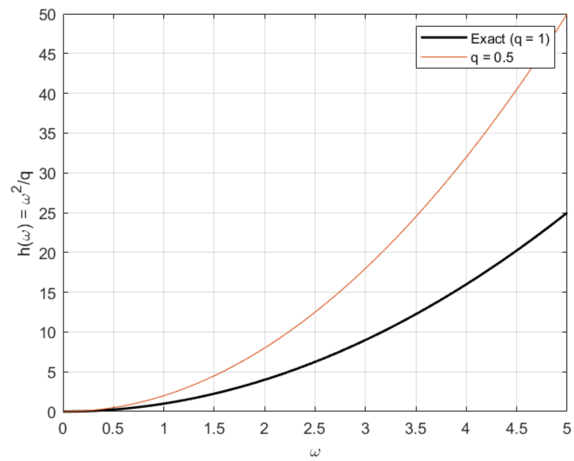

Fig. 2. Comparison of $h(\omega)$ for $q = 0.5$ with Exact solution(Fixed $\alpha = 2.5$).

V. Conclusion

This paper presents a method for solving q-FDEs using the operational matrix approach with q-Legendre polynomials. We first derive q-LEOM. This technique is particularly advantageous due to its computational efficiency and flexibility. The differentiation operational matrix typically contains many zero elements, which not only speeds up computation but also ensures high accuracy in the solutions.

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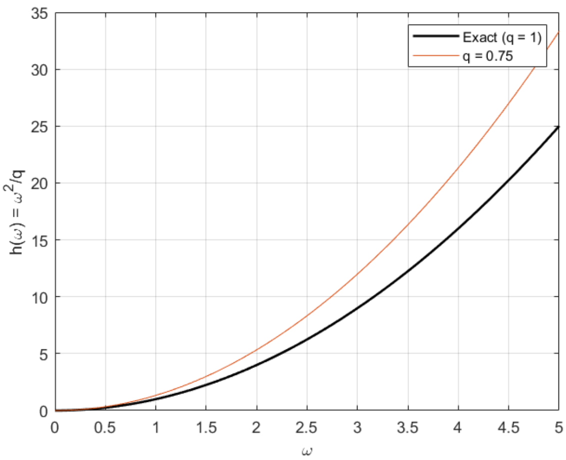

Fig. 3. Comparison of $h(\omega)$ for $q = 0.75$ with Exact solution(Fixed $\alpha = 2.5$).

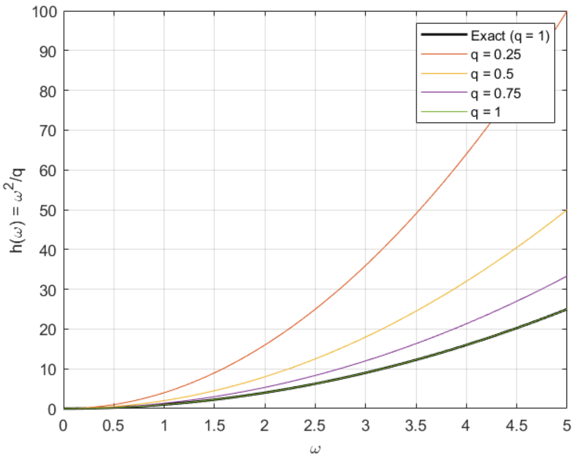

Fig. 4. Comparison of $h(\omega)$ for $q = 0.25, 0.5, 0.75, 1$ with Exact solution(Fixed $\alpha = 2.5$).

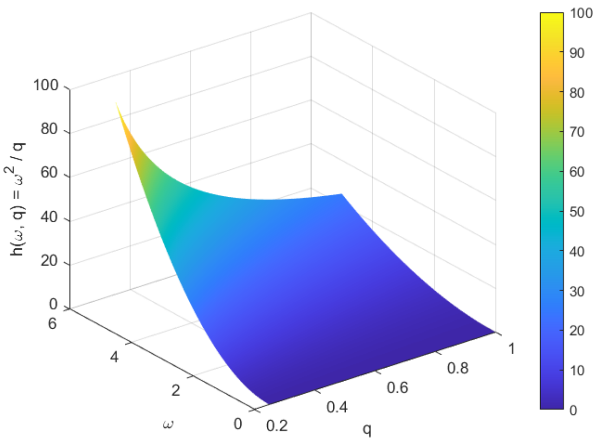

Fig. 5. 3D surface plot of the function $h(\omega) = \dfrac{\omega^2}{q}$ over the domain $\omega \in (0, 5)$ and $q \in (0.25, 1)$

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