Multiplicative Topological Indices of Linear Functional Graphs over Finite Dimensional Vector Spaces

Vinnarasi Lingan, Kalaimurugan Gnanaprakasam and Vimal Manuvelleenas

***Abstract*—In this study, we derive several notable topological indices for linear functional graphs over finite dimensional vector spaces. In particular, we obtain certain novel topological indices, such as the multiplicative degree-based topological indices for Zagreb and Hyper Zagreb. In addition to these, we investigated certain noteworthy geometric arithmetic indices in their generalized forms. Reverse multiplicative indices utilize this idea to analyze the inverse of these products, providing an alternate method for measuring connectedness or other structural features of a graph. Refined indices extend classic topological measurements by taking into an account deeper interactions between degrees or higher-order connections, resulting in a more comprehensive representation of network and graph structure. Revan indices are proposed as an extension of traditional degree-based metrics, offering higher sensitivity to subtle structural differences in network topology and effectively distinguishing non-isomorphic graphs with identical global features. Furthermore, we explore elliptic Sombor indices as a novel variant that captures degree-based structural nuances through elliptic functional transformations, enhancing the analytical resolution of graph invariants.**

***Index Terms*—topological indices, Wiener index, Randic index, finite dimensional, linear functional graph.**

I. Introduction

THE concept of connecting graphs with algebraic frameworks was first explored by Beck [3], who applied this perspective to address graph coloring challenges using commutative ring theory. This initial work laid the foundation for a broader field that merges graph theory with algebra. Significant progress has since been made, particularly through the contributions of Anderson and Tamizh Chelvam [1], as well as Badawi [2], who introduced and developed the theory of zero-divisor graphs. These graphs serve as tools to investigate algebraic structures by translating ring-theoretic properties into graph-theoretic language.

Beyond ring theory, the application of graph-based models has been extended to the study of vector spaces, particularly those over finite fields. Das [5]-[6] proposed the notion of linear functional graphs, where vertices represent vectors and edges reflect linear transformations. These graphs naturally incorporate fundamental aspects of vector spaces, such as their dimensional structure, basis elements, and subspace relationships. Further studies, including [12], have delved into the symmetries of these graphs by analyzing their automorphisms, highlighting structural regularities and invariants.

In parallel, topological indices-numerical measures derived from graphs-have gained attention for their ability to capture key features of complex structures. Originating in the field of mathematical chemistry, these indices quantify characteristics such as molecular branching, cyclic structures, and vertex connectivity. Early foundational work by Gutman [9] and Kulli [10] demonstrated how topological indices like the Zagreb and Randic indices could predict molecular properties such as stability, boiling points, and biological activity.

The utility of topological indices now spans multiple disciplines. In nanotechnology, they are used to describe nanoscale architectures, including fullerenes and carbon nanotubes. In biomedical science, these indices contribute to the modeling of molecular networks, such as protein-protein interactions and RNA configurations. For instance, during the COVID-19 outbreak, structural analysis of antiviral agents such as chloroquine, remdesivir, and theaflavin-utilized topological descriptors to support drug research and molecular modeling [20].

Additionally, [26] topological indices have found applications in various computational and theoretical fields. Like in machine learning and pattern recognition, graph-based features, often encoded using topological indices, enhance model performance in tasks like image classification and community detection. In cryptography, algebraic graphs are instrumental in constructing secure protocols, including public-key systems grounded in vector spaces and graph transformations. In quantum computing, graph-theoretic tools are used to design quantum codes and analyze quantum network states. And in network theory, topological indices are employed to evaluate system robustness, node influence, and structural balance in networks such as transportation systems and digital communication infrastructures.

The focus of this paper is to expand on this growing body of research by computing and analyzing degree-based topological indices for linear functional graphs over finite-dimensional vector spaces. In particular, we investigate both additive and multiplicative forms of indices such as the Zagreb and Randic in [7] and [19] and Arithmetic-Geometric indices, including their hyper and modified versions, as presented in [15]. Furthermore, we explore the Revan index

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introduced in [11], which captures the structural imbalance between connected vertices. These indices provide deeper insight into the internal arrangement of vector spaces when viewed through a graph-theoretic lens. Through the integration of linear algebra and graph theory, this research presents a new mathematical perspective that supports both theoretical exploration and practical application. The outcomes of this study may be of interest to fields ranging from chemistry and molecular biology to cryptography, quantum computing, and network analysis.

II. PRELIMINARIES

Throughout the graph, let $\mathbb{F}_q$ be a finite field with q elements and $n \geq 3$ be a positive integer, $\mathbb{V}_0$ be an n-dimensional vector space over $\mathbb{F}_q$, and $\mathbb{U}_0$ be the set of all linear functionals from $\mathbb{V}_0$ to $\mathbb{F}_q$. Let $\mathbb{V} = \mathbb{V}_0 \setminus \{0\}$ and $\mathbb{U} = \mathbb{U}_0 \setminus \{0\}$. The linear functional graph of $\mathbb{V}_0$, denoted by $\mathfrak{F}(\mathbb{V})$, is an undirected bipartite graph whose vertex set V is partitioned into two sets as $V = \mathbb{V} \cup \mathbb{U}$, and two vertices $v \in \mathbb{V}$ and $f \in \mathbb{U}$ are adjacent if and only if f sends the vertex v to 0 in $\mathbb{F}_q$.

In this section, we will review some of the definitions and terminologies that are required to progress with this article. Let x be a vertex on the graph. The degree d(x) is the number of edges that intersect with x. For any two vertices x, y connected by a path in the graph, the distance d(x, y) is the length of the shortest $x - y$ path. The eccentricity of vertex x is the maximum distance between x and all other vertices in the graph. The diameter of the graph is the maximum of the eccentricities of all its vertices. The order represents the number of vertices, whereas the size represents the number of edges in the graph.

Lemma 2.1: ([12, Lemma 2.4]) The degree of every vertex of $\mathfrak{F}(\mathbb{V})$ is $q^{n-1}-1$, and hence $\mathfrak{F}(\mathbb{V})$ is a $(q^{n-1}-1)$ - regular graph.

Theorem 2.2: ([12, Theorem 2.6]) The domination number of $\mathfrak{F}(\mathbb{V})$ is $2q+2$.

Lemma 2.3: ([23, Lemma 2.1]) Let $\mathfrak{DG}(\mathbb{V})$ be the dual graph of an n-dimensional vector space $\mathbb{V}$ over $\mathbb{F}_q$. Then

(1) The order and the size of $\mathfrak{DG}(\mathbb{V})$ are, respectively, $\frac{2(q^n-1)}{q-1}$ and $\frac{(q^n-1)(q^{n-1}-1)}{(q-1)^2}$.

(2) $\mathfrak{DG}(\mathbb{V})$ is a regular graph of degree $\frac{q^{n-1}-1}{q-1}$.

(3) A pair of distinct vertices in X^* has $\frac{q^{n-2}-1}{q-1}$ common neighbors.

(4) A pair of distinct vertices in X has $\frac{q^{n-2}-1}{q-1}$ common neighbors.

Theorem 2.4: ([23, Theorem 3.4]) Let $n \geq 2$. Then $diam(\mathfrak{DG}(\mathbb{V})) = \infty$ if $n = 2$ and $diam(\mathfrak{DG}(\mathbb{V})) = 3$ if $n \geq 3$.

Remark 2.5: Let $\mathfrak{F}(\mathbb{V})$ be the linear functional graphs of an n - dimensional vector space $\mathbb{V}$ over $\mathbb{F}_q$. Then the order and the size of $\mathfrak{F}(\mathbb{V})$ are, respectively, $\frac{2(q^n-1)}{q-1}$ and $\frac{(q^n-1)(q^{n-1}-1)}{q-1}$.

Remark 2.6: By theorem 2.4 and remarks 2.3 and 2.7 in [12], we get the eccentricity of every vertex in $\mathfrak{F}(\mathbb{V})$ is 3 for $n \geq 3$.

III. MULTIPLICATIVE ZAGREB INDEX

This segment gives us some of the multiplicative Zagreb indices for $\mathfrak{F}(\mathbb{V})$, which are degree-based indices.

1. First Multiplicative Zagreb Index :
$$II_1(\mathfrak{F}(\mathbb{V})) = \prod_{\imath \in V} (d_\imath)^2$$
2. General Multiplicative Index :
$$W_1^\alpha(\mathfrak{F}(\mathbb{V})) = \prod_{\imath \in V} (d_\imath)^\alpha$$
3. First Multiplicative Generalized Zagreb Index :
$$MZ_1^\alpha(\mathfrak{F}(\mathbb{V})) = \prod_{\imath\jmath \in E} (d_\imath + d_\jmath)^\alpha$$
4. Second Multiplicative Generalized Zagreb Index :
$$MZ_2^\alpha(\mathfrak{F}(\mathbb{V})) = \prod_{\imath\jmath \in E} (d_\imath . d_\jmath)^\alpha$$
5. Multiplicative version of First Zagreb Index :
$$II_1^*(\mathfrak{F}(\mathbb{V})) = \prod_{\imath\jmath \in E} (d_\imath + d_\jmath)$$
6. Second Multiplicative Zagreb Index :
$$II_1^*(\mathfrak{F}(\mathbb{V})) = \prod_{\imath\jmath \in E} (d_\imath . d_\jmath)$$
7. Multiplicative First Hyper Zagreb Index :
$$HII_1(\mathfrak{F}(\mathbb{V})) = \prod_{\imath\jmath \in E} (d_\imath + d_\jmath)^2$$
8. Multiplicative Second Hyper Zagreb Index :
$$HII_2(\mathfrak{F}(\mathbb{V})) = \prod_{\imath\jmath \in E} (d_\imath . d_\jmath)^2$$
9. Multiplicative Augmented Zagreb Index :
$$AZII(\mathfrak{F}(\mathbb{V})) = \prod_{\imath\jmath \in E} \left(\frac{d_\imath . d_\jmath}{d_\imath + d_\jmath - 2}\right)^3$$
10. Multiplicative Exponential Wiener Index :
$$EW(\mathfrak{F}(\mathbb{V})) = \frac{1}{2} \prod_{\imath \in V} e^{d_\imath}$$

Theorem 3.1: The First Multiplicative Zagreb Index of $\mathfrak{F}(\mathbb{V})$ is
$$II_1(\mathfrak{F}(\mathbb{V})) = (q^{n-1} - 1)^{4q^n - 4}$$

Proof:
$$\begin{aligned} II_1(\mathfrak{F}(\mathbb{V})) &= \prod_{\imath \in V} (d_\imath)^2 \\ &= \prod_{\imath \in V} (q^{n-1} - 1)^2 \\ &= (q^{n-1} - 1)^{2[2(q^n-1)]} \\ &= (q^{n-1} - 1)^{4q^n - 4} \end{aligned}$$
■

Theorem 3.2: The General Multiplicative Index of $\mathfrak{F}(\mathbb{V})$ is
$$W_1^\alpha(\mathfrak{F}(\mathbb{V})) = (q^{n-1} - 1)^{2\alpha q^n - 2\alpha}$$

Proof:
$$\begin{aligned} W_1^\alpha(\mathfrak{F}(\mathbb{V})) &= \prod_{\imath \in V} (d_\imath)^\alpha \\ &= \prod_{\imath \in V} (q^{n-1} - 1)^\alpha \\ &= (q^{n-1} - 1)^{\alpha 2(q^n-1)} \\ &= (q^{n-1} - 1)^{2\alpha q^n - 2\alpha} \end{aligned}$$
■

Theorem 3.3: The First Multiplicative Generalized Zagreb Index of $\mathfrak{F}(\mathbb{V})$ is
$$MZ_1^\alpha(\mathfrak{F}(\mathbb{V})) = [2q^{n-1} - 2]^{\alpha q^{2n-1} - \alpha q^n - \alpha q^{n-1} + \alpha}$$

Proof:

$$
\begin{aligned}
MZ_1^{\alpha}(\mathfrak{F}(\mathbb{V})) &= \prod_{\imath\jmath \in E} (d_\imath + d_\jmath)^{\alpha} \\
&= \prod_{\imath\jmath \in E} [q^{n-1} - 1 + q^{n-1} - 1]^{\alpha} \\
&= \prod_{\imath\jmath \in E} [2(q^{n-1} - 1)]^{\alpha} \\
&= [2(q^{n-1} - 1)]^{\alpha(q^n-1)(q^{n-1}-1)} \\
&= [2(q^{n-1} - 1)]^{\alpha(q^{2n-1}-q^n-q^{n-1}+1)} \\
&= [2q^{n-1} - 2]^{\alpha(q^{2n-1}-q^n-q^{n-1}+1)} \\
&= [2q^{n-1} - 2]^{\alpha q^{2n-1}-\alpha q^n-\alpha q^{n-1}+\alpha}
\end{aligned}
$$

■

Theorem 3.4: The Second Multiplicative Generalized Zagreb Index of $\mathfrak{F}(\mathbb{V})$ is

$$MZ_2^{\alpha}(\mathfrak{F}(\mathbb{V})) = (q^{n-1} - 1)^{2\alpha q^{2n-1}-2\alpha q^n-2\alpha q^{n-1}+2\alpha}$$

Proof:

$$
\begin{aligned}
MZ_2^{\alpha}(\mathfrak{F}(\mathbb{V})) &= \prod_{\imath\jmath \in E} (d_\imath . d_\jmath)^{\alpha} \\
&= \prod_{\imath\jmath \in E} (q^{n-1} - 1 \times q^{n-1} - 1)^{\alpha} \\
&= \prod_{\imath\jmath \in E} (q^{n-1} - 1)^{2\alpha} \\
&= (q^{n-1} - 1)^{2\alpha(q^n-1)(q^{n-1}-1)} \\
&= (q^{n-1} - 1)^{2\alpha(q^{2n-1}-q^n-q^{n-1}+1)} \\
&= (q^{n-1} - 1)^{2\alpha q^{2n-1}-2\alpha q^n-2\alpha q^{n-1}+2\alpha}
\end{aligned}
$$

■

Theorem 3.5: The Multiplicative version of First Zagreb Index of $\mathfrak{F}(\mathbb{V})$ is

$$II_1^{*}(\mathfrak{F}(\mathbb{V})) = [2q^{n-1} - 2]^{q^{2n-1}-q^n-q^{n-1}+1}$$

Proof:

$$
\begin{aligned}
II_1^{*}(\mathfrak{F}(\mathbb{V})) &= \prod_{\imath\jmath \in E} (d_\imath + d_\jmath) \\
&= \prod_{\imath\jmath \in E} [(q^{n-1} - 1) + (q^{n-1} - 1)] \\
&= \prod_{\imath\jmath \in E} [2(q^{n-1} - 1)] \\
&= [2(q^{n-1} - 1)]^{(q^n-1)(q^{n-1}-1)} \\
&= [2q^{n-1} - 2]^{q^{2n-1}-q^n-q^{n-1}+1}
\end{aligned}
$$

■

Theorem 3.6: The Second Multiplicative Zagreb Index of $\mathfrak{F}(\mathbb{V})$ is

$$II_1^{*}(\mathfrak{F}(\mathbb{V})) = (q^{n-1} - 1)^{2q^{2n-1}-2q^n-2q^{n-1}+2}$$

Proof:

$$
\begin{aligned}
II_1^{*}(\mathfrak{F}(\mathbb{V})) &= \prod_{\imath\jmath \in E} (d_\imath . d_\jmath) \\
&= \prod_{\imath\jmath \in E} [(q^{n-1} - 1) \times (q^{n-1} - 1)] \\
&= \prod_{\imath\jmath \in E} (q^{n-1} - 1)^2 \\
&= (q^{n-1} - 1)^{2(q^n-1)(q^{n-1}-1)} \\
&= (q^{n-1} - 1)^{2(q^{2n-1}-q^n-q^{n-1}+1)} \\
&= (q^{n-1} - 1)^{2q^{2n-1}-2q^n-2q^{n-1}+2}
\end{aligned}
$$

■

Theorem 3.7: The Multiplicative First Hyper Zagreb Index of $\mathfrak{F}(\mathbb{V})$ is

$$HII_1(\mathfrak{F}(\mathbb{V})) = [2q^{n-1} - 2]^{2q^{2n-1}-2q^n-2q^{n-1}+2}$$

Proof:

$$
\begin{aligned}
HII_1(\mathfrak{F}(\mathbb{V})) &= \prod_{\imath\jmath \in E} (d_\imath + d_\jmath)^2 \\
&= \prod_{\imath\jmath \in E} [(q^{n-1} - 1) + (q^{n-1} - 1)]^2 \\
&= \prod_{\imath\jmath \in E} [2(q^{n-1} - 1)]^2 \\
&= [2(q^{n-1} - 1)]^{2(q^n-1)(q^{n-1}-1)} \\
&= [2(q^{n-1} - 1)]^{2(q^{2n-1}-q^n-q^{n-1}+1)} \\
&= [2q^{n-1} - 2]^{2q^{2n-1}-2q^n-2q^{n-1}+2}
\end{aligned}
$$

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Theorem 3.8: The Multiplicative Second Hyper Zagreb Index of $\mathfrak{F}(\mathbb{V})$ is

$$HII_2(\mathfrak{F}(\mathbb{V})) = (q^{n-1} - 1)^{4q^{2n-1}-4q^n-4q^{n-1}+4}$$

Proof:

$$
\begin{aligned}
HII_2(\mathfrak{F}(\mathbb{V})) &= \prod_{\imath\jmath \in E} (d_\imath . d_\jmath)^2 \\
&= \prod_{\imath\jmath \in E} [(q^{n-1} - 1) \times (q^{n-1} - 1)]^2 \\
&= \prod_{\imath\jmath \in E} [(q^{n-1} - 1)^2]^2 \\
&= \prod_{\imath\jmath \in E} (q^{n-1} - 1)^4 \\
&= (q^{n-1} - 1)^{4(q^n-1)(q^{n-1}-1)} \\
&= (q^{n-1} - 1)^{4(q^{2n-1}-q^n-q^{n-1}+1)} \\
&= (q^{n-1} - 1)^{4q^{2n-1}-4q^n-4q^{n-1}+4}
\end{aligned}
$$

■

Theorem 3.9: The Multiplicative Augmented Zagreb Index of $\mathfrak{F}(\mathbb{V})$ is

$$AZII(\mathfrak{F}(\mathbb{V})) = \left(\frac{q^{2n-2}+1-2q^{n-1}}{2q^{n-1}-4}\right)^{3q^{2n-1}-3q^n-3q^{n-1}+3}$$

Proof:

$$AZII(\mathfrak{F}(\mathbb{V})) = \prod_{\imath\jmath\in E}\left(\frac{d_\imath . d_\jmath}{d_\imath + d_\jmath - 2}\right)^3$$
$$= \prod_{\imath\jmath\in E}\left(\frac{(q^{n-1}-1)(q^{n-1}-1)}{(q^{n-1}-1)+(q^{n-1}-1)-2}\right)^3$$
$$= \prod_{\imath\jmath\in E}\left(\frac{(q^{n-1}-1)^2}{2(q^{n-1}-1)-2}\right)^3$$
$$= \prod_{\imath\jmath\in E}\left(\frac{(q^{n-1}-1)^2}{2(q^{n-1}-1-1)}\right)^3$$
$$= \prod_{\imath\jmath\in E}\left(\frac{(q^{n-1}-1)^2}{2(q^{n-1}-2)}\right)^3$$
$$= \left[\left(\frac{(q^{n-1}-1)^2}{2(q^{n-1}-2)}\right)^3\right]^{(q^n-1)(q^{n-1}-1)}$$
$$= \left(\frac{(q^{n-1}-1)^2}{2(q^{n-1}-2)}\right)^{3(q^n-1)(q^{n-1}-1)}$$
$$= \left(\frac{(q^{n-1}-1)^2}{2(q^{n-1}-2)}\right)^{3(q^{2n-1}-q^n-q^{n-1}+1)}$$
$$= \left(\frac{(q^{n-1}-1)^2}{2(q^{n-1}-2)}\right)^{3q^{2n-1}-3q^n-3q^{n-1}+3}$$
$$= \left(\frac{q^{2n-2}+1-2q^{n-1}}{2q^{n-1}-4}\right)^{3q^{2n-1}-3q^n-3q^{n-1}+3}$$

■

Theorem 3.10: Multiplicative Exponential Wiener Index of $\mathfrak{F}(\mathbb{V})$ is
$EW(\mathfrak{F}(\mathbb{V})) = \frac{1}{2}e^{2q^{2n-1}-2q^n-2q^{n-1}+2}$

Proof:

$$EW(\mathfrak{F}(\mathbb{V})) = \frac{1}{2}\prod_{\imath\in V} e^{d_\imath}$$
$$= \frac{1}{2}\prod_{\imath\in V} e^{(q^{n-1}-1)}$$
$$= \frac{1}{2}[e^{(q^{n-1}-1)}]^{2(q^n-1)}$$
$$= \frac{1}{2}e^{2(q^n-1)(q^{n-1}-1)}$$
$$= \frac{1}{2}e^{2q^{2n-1}-2q^n-2q^{n-1}+2}$$

■

IV. Multiplicative Based Other Index and Geometric Arithmetic Index:

This area shows the other multiplicative-based indices, the harmonic index and the geometric arithmetic index, which are based on the edges of $\mathfrak{F}(\mathbb{V})$

1. Narumi - Katayama Index :
$$NK(\mathfrak{F}(\mathbb{V})) = \prod_{\imath\in V}(d_\imath)$$
2. Multiplicative Sum Connectivity Index :
$$SCII(\mathfrak{F}(\mathbb{V})) = \prod_{\imath\jmath\in E}\frac{1}{\sqrt{d_\imath + d_\jmath}}$$
3. Multiplicative Product Connectivity Index :
$$PCII(\mathfrak{F}(\mathbb{V})) = \prod_{\imath\jmath\in E}\frac{1}{\sqrt{d_\imath . d_\jmath}}$$
4. Multiplicative Sum Connectivity F - Index :
$$SFII(\mathfrak{F}(\mathbb{V})) = \prod_{\imath\jmath\in E}\frac{1}{\sqrt{d_\imath^2 + d_\jmath^2}}$$
5. Multiplicative Product Connectivity F - Index :
$$PFII(\mathfrak{F}(\mathbb{V})) = \prod_{\imath\jmath\in E}\frac{1}{\sqrt{d_\imath^2 . d_\jmath^2}}$$
6. Multiplicative First F - Index :
$$F_1II(\mathfrak{F}(\mathbb{V})) = \prod_{\imath\jmath\in E}[(d_\imath)^2 + (d_\jmath)^2]$$
7. Multiplicative Second F - Index :
$$F_2II(\mathfrak{F}(\mathbb{V})) = \prod_{\imath\jmath\in E}[(d_\imath)^2 . (d_\jmath)^2]$$
8. Multiplicative ABC Index :
$$ABCII(\mathfrak{F}(\mathbb{V})) = \prod_{\imath\jmath\in E}\sqrt{\frac{d_\imath + d_\jmath - 2}{d_\imath . d_\jmath}}$$
9. Multiplicative Harmonic Index :
$$HII(\mathfrak{F}(\mathbb{V})) = \prod_{\imath\jmath\in E}\frac{2}{d_\imath + d_\jmath}$$
10. Multiplicative Geometric Arithmetic Index :
$$GAII(\mathfrak{F}(\mathbb{V})) = \prod_{\imath\jmath\in E}\frac{2\sqrt{d_\imath . d_\jmath}}{d_\imath + d_\jmath}$$
11. General Multiplicative Geometric Arithmetic Index :
$$GA^\alpha II(\mathfrak{F}(\mathbb{V})) = \prod_{\imath\jmath\in E}\left(\frac{2\sqrt{d_\imath . d_\jmath}}{d_\imath + d_\jmath}\right)^\alpha$$

Theorem 4.1: The Narumi - Katayama Index of $\mathfrak{F}(\mathbb{V})$ is
$NK(\mathfrak{F}(\mathbb{V})) = (q^{n-1}-1)^{2q^n-2}$

Proof:

$$NK(\mathfrak{F}(\mathbb{V})) = \prod_{\imath\in V}(d_\imath)$$
$$= \prod_{\imath\in V}(q^{n-1}-1)$$
$$= (q^{n-1}-1)^{2(q^n-1)}$$
$$= (q^{n-1}-1)^{2q^n-2}$$

■

Theorem 4.2: The Multiplicative Sum Connectivity Index of $\mathfrak{F}(\mathbb{V})$ is
$SCII(\mathfrak{F}(\mathbb{V})) = \left[\frac{1}{\sqrt{2q^{n-1}-2}}\right]^{q^{2n-1}-q^n-q^{n-1}+1}$

Proof:

$$SCII(\mathfrak{F}(\mathbb{V})) = \prod_{\imath\jmath\in E}\frac{1}{\sqrt{d_\imath + d_\jmath}}$$
$$= \prod_{\imath\jmath\in E}\left[\frac{1}{\sqrt{(q^{n-1}-1)+(q^{n-1}-1)}}\right]$$
$$= \prod_{\imath\jmath\in E}\left[\frac{1}{\sqrt{2(q^{n-1}-1)}}\right]$$
$$= \left[\frac{1}{\sqrt{2(q^{n-1}-1)}}\right]^{(q^n-1)(q^{n-1}-1)}$$
$$= \left[\frac{1}{\sqrt{2q^{n-1}-2}}\right]^{q^{2n-1}-q^n-q^{n-1}+1}$$

■

Theorem 4.3: The Multiplicative Product Connectivity Index of $\mathfrak{F}(\mathbb{V})$ is

$$PCII(\mathfrak{F}(\mathbb{V})) = \left[q^{n-1} - 1\right]^{\left(\frac{-q^{2n-1}+q^n+q^{n-1}-1}{2}\right)}$$

Proof:

$$\begin{aligned}
PCII(\mathfrak{F}(\mathbb{V})) &= \prod_{\imath\jmath\in E} \frac{1}{\sqrt{d_\imath . d_\jmath}} \\
&= \prod_{\imath\jmath\in E} \left[\frac{1}{\sqrt{(q^{n-1}-1)\times(q^{n-1}-1)}}\right] \\
&= \prod_{\imath\jmath\in E} \left[\frac{1}{\sqrt{(q^{n-1}-1)^2}}\right] \\
&= \prod_{\imath\jmath\in E} \left[\frac{1}{(q^{n-1}-1)}\right] \\
&= \left[\frac{1}{(q^{n-1}-1)}\right]^{(q^n-1)(q^{n-1}-1)} \\
&= \left[(q^{n-1}-1)^{-\frac{1}{2}}\right]^{(q^n-1)(q^{n-1}-1)} \\
&= \left[q^{n-1}-1\right]^{\left(-\frac{(q^n-1)(q^{n-1}-1)}{2}\right)} \\
&= \left[q^{n-1}-1\right]^{\left(\frac{-q^{2n-1}+q^n+q^{n-1}-1}{2}\right)}
\end{aligned}$$

■

Theorem 4.4: The Multiplicative Sum Connectivity F - Index of $\mathfrak{F}(\mathbb{V})$ is

$$SFII(\mathfrak{F}(\mathbb{V})) = \left[\frac{1}{\sqrt{2}(q^{n-1}-1)}\right]^{q^{2n-1}-q^n-q^{n-1}+1}$$

Proof:

$$\begin{aligned}
SFII(\mathfrak{F}(\mathbb{V})) &= \prod_{\imath\jmath\in E} \frac{1}{\sqrt{d_\imath^2 + d_\jmath^2}} \\
&= \prod_{\imath\jmath\in E} \left[\frac{1}{\sqrt{(q^{n-1}-1)^2+(q^{n-1}-1)^2}}\right] \\
&= \prod_{\imath\jmath\in E} \left[\frac{1}{\sqrt{2(q^{n-1}-1)^2}}\right] \\
&= \prod_{\imath\jmath\in E} \left[\frac{1}{\sqrt{2}(q^{n-1}-1)}\right] \\
&= \left[\frac{1}{\sqrt{2}(q^{n-1}-1)}\right]^{(q^n-1)(q^{n-1}-1)} \\
&= \left[\frac{1}{\sqrt{2}(q^{n-1}-1)}\right]^{q^{2n-1}-q^n-q^{n-1}+1}
\end{aligned}$$

■

Theorem 4.5: The Multiplicative Product Connectivity F - Index of $\mathfrak{F}(\mathbb{V})$ is

$$PFII(\mathfrak{F}(\mathbb{V})) = \left[q^{n-1}-1\right]^{-2q^{2n-1}+2q^n+2q^{n-1}-2}$$

Proof:

$$\begin{aligned}
PFII(\mathfrak{F}(\mathbb{V})) &= \prod_{\imath\jmath\in E} \frac{1}{\sqrt{d_\imath^2 . d_\jmath^2}} \\
&= \prod_{\imath\jmath\in E} \left[\frac{1}{\sqrt{(q^{n-1}-1)^2\times(q^{n-1}-1)^2}}\right] \\
&= \prod_{\imath\jmath\in E} \left[\frac{1}{\sqrt{(q^{n-1}-1)^4}}\right] \\
&= \prod_{\imath\jmath\in E} \left[\frac{1}{(q^{n-1}-1)^2}\right] \\
&= \left[\frac{1}{(q^{n-1}-1)^2}\right]^{(q^n-1)(q^{n-1}-1)} \\
&= \left[(q^{n-1}-1)^{-2}\right]^{(q^n-1)(q^{n-1}-1)} \\
&= \left[q^{n-1}-1\right]^{-2(q^n-1)(q^{n-1}-1)} \\
&= \left[q^{n-1}-1\right]^{-2(q^{2n-1}-q^n-q^{n-1}+1)} \\
&= \left[q^{n-1}-1\right]^{-2q^{2n-1}+2q^n+2q^{n-1}-2}
\end{aligned}$$

■

Theorem 4.6: The Multiplicative First F - Index of $\mathfrak{F}(\mathbb{V})$ is

$$F_1II(\mathfrak{F}(\mathbb{V})) = [2q^{n-1}-2]^{2q^{2n-1}-2q^n-2q^{n-1}+2}$$

Proof:

$$\begin{aligned}
F_1II(\mathfrak{F}(\mathbb{V})) &= \prod_{\imath\jmath\in E} [(d_\imath)^2 + (d_\jmath)^2] \\
&= \prod_{\imath\jmath\in E} [(q^{n-1}-1)^2 + (q^{n-1}-1)^2] \\
&= \prod_{\imath\jmath\in E} [2(q^{n-1}-1)^2] \\
&= [2(q^{n-1}-1)^2]^{(q^n-1)(q^{n-1}-1)} \\
&= [2(q^{n-1}-1)]^{2(q^n-1)(q^{n-1}-1)} \\
&= [2q^{n-1}-2]^{2(q^{2n-1}-q^n-q^{n-1}+1)} \\
&= [2q^{n-1}-2]^{2q^{2n-1}-2q^n-2q^{n-1}+2}
\end{aligned}$$

■

Theorem 4.7: The Multiplicative Second F - Index of $\mathfrak{F}(\mathbb{V})$ is

$$F_2II(\mathfrak{F}(\mathbb{V})) = [q^{n-1}-1]^{4q^{2n-1}-4q^n-4q^{n-1}+4}$$

Proof:

$$\begin{aligned}
F_2II(\mathfrak{F}(\mathbb{V})) &= \prod_{\imath\jmath\in E} [(d_\imath)^2 . (d_\jmath)^2] \\
&= \prod_{\imath\jmath\in E} [(q^{n-1}-1)^2 \times (q^{n-1}-1)^2] \\
&= \prod_{\imath\jmath\in E} [(q^{n-1}-1)^2]^2 \\
&= \prod_{\imath\jmath\in E} (q^{n-1}-1)^4 \\
&= [(q^{n-1}-1)^4]^{(q^n-1)(q^{n-1}-1)} \\
&= [q^{n-1}-1]^{4(q^n-1)(q^{n-1}-1)} \\
&= [q^{n-1}-1]^{4q^{2n-1}-4q^n-4q^{n-1}+4}
\end{aligned}$$

■

Theorem 4.8: The Multiplicative ABC Index of $\mathfrak{F}(\mathbb{V})$ is
$$ABCII(\mathfrak{F}(\mathbb{V})) = \left[\frac{\sqrt{2q^{n-1}-4}}{q^{n-1}-1}\right]^{q^{2n-1}-q^{n}-q^{n-1}+1}$$
Proof:

$$\begin{aligned}
ABCII(\mathfrak{F}(\mathbb{V})) &= \prod_{\imath\jmath\in E}\sqrt{\frac{d_\imath + d_\jmath - 2}{d_\imath . d_\jmath}}\\
&= \prod_{\imath\jmath\in E}\left[\sqrt{\frac{(q^{n-1}-1)+(q^{n-1}-1)-2}{(q^{n-1}-1)(q^{n-1}-1)}}\right]\\
&= \prod_{\imath\jmath\in E}\left[\sqrt{\frac{2(q^{n-1}-1)-2}{(q^{n-1}-1)^2}}\right]\\
&= \prod_{\imath\jmath\in E}\left[\sqrt{\frac{2(q^{n-1}-1-1)}{(q^{n-1}-1)^2}}\right]\\
&= \prod_{\imath\jmath\in E}\left[\sqrt{\frac{2(q^{n-1}-2)}{(q^{n-1}-1)^2}}\right]\\
&= \prod_{\imath\jmath\in E}\left[\frac{\sqrt{2(q^{n-1}-2)}}{(q^{n-1}-1)}\right]\\
&= \left[\frac{\sqrt{2(q^{n-1}-2)}}{(q^{n-1}-1)}\right]^{(q^{n}-1)(q^{n-1}-1)}\\
&= \left[\frac{\sqrt{2(q^{n-1}-2)}}{(q^{n-1}-1)}\right]^{(q^{2n-1}-q^{n}-q^{n-1}+1)}\\
&= \left[\frac{\sqrt{2q^{n-1}-4}}{q^{n-1}-1}\right]^{q^{2n-1}-q^{n}-q^{n-1}+1}
\end{aligned}$$

■

Theorem 4.9: The Multiplicative Harmonic Index of $\mathfrak{F}(\mathbb{V})$ is
$$HII(\mathfrak{F}(\mathbb{V})) = \left[q^{n-1}-1\right]^{-q^{2n-1}+q^{n}+q^{n-1}-1}$$
Proof:

$$\begin{aligned}
HII(\mathfrak{F}(\mathbb{V})) &= \prod_{\imath\jmath\in E}\frac{2}{d_\imath + d_\jmath}\\
&= \prod_{\imath\jmath\in E}\left[\frac{2}{(q^{n-1}-1)+(q^{n-1}-1)}\right]\\
&= \prod_{\imath\jmath\in E}\left[\frac{2}{2(q^{n-1}-1)}\right]\\
&= \prod_{\imath\jmath\in E}\left[\frac{1}{(q^{n-1}-1)}\right]\\
&= \left[\frac{1}{(q^{n-1}-1)}\right]^{(q^{n}-1)(q^{n-1}-1)}\\
&= \left[q^{n-1}-1\right]^{-(q^{n}-1)(q^{n-1}-1)}\\
&= \left[q^{n-1}-1\right]^{-(q^{2n-1}-q^{n}-q^{n-1}+1)}\\
&= \left[q^{n-1}-1\right]^{-q^{2n-1}+q^{n}+q^{n-1}-1}
\end{aligned}$$

■

Theorem 4.10: The Multiplicative Geometric Arithmetic Index of $\mathfrak{F}(\mathbb{V})$ is
$$GAII(\mathfrak{F}(\mathbb{V})) = 1$$
Proof:

$$\begin{aligned}
GAII(\mathfrak{F}(\mathbb{V})) &= \prod_{\imath\jmath\in E}\frac{2\sqrt{d_\imath . d_\jmath}}{d_\imath + d_\jmath}\\
&= \prod_{\imath\jmath\in E}\frac{2\sqrt{(q^{n-1}-1)(q^{n-1}-1)}}{(q^{n-1}-1)+(q^{n-1}-1)}\\
&= \prod_{\imath\jmath\in E}\frac{2\sqrt{(q^{n-1}-1)^2}}{2(q^{n-1}-1)}\\
&= \prod_{\imath\jmath\in E}\frac{2(q^{n-1}-1)}{2(q^{n-1}-1)}\\
&= \prod_{\imath\jmath\in E}(1)\\
&= (1)^{(q^{n}-1)(q^{n-1}-1)}\\
&= 1
\end{aligned}$$

■

Theorem 4.11: The General Multiplicative Geometric Arithmetic Index of $\mathfrak{F}(\mathbb{V})$ is
$$GA^{\alpha}II(\mathfrak{F}(\mathbb{V})) = (1)^{\alpha q^{2n-1}-\alpha q^{n}-\alpha q^{n-1}+\alpha}$$
Proof:

$$\begin{aligned}
GA^{\alpha}II(\mathfrak{F}(\mathbb{V})) &= \prod_{\imath\jmath\in E}\left(\frac{2\sqrt{d_\imath . d_\jmath}}{d_\imath + d_\jmath}\right)^{\alpha}\\
&= \prod_{\imath\jmath\in E}\left(\frac{2\sqrt{(q^{n-1}-1)(q^{n-1}-1)}}{(q^{n-1}-1)+(q^{n-1}-1)}\right)^{\alpha}\\
&= \prod_{\imath\jmath\in E}\left(\frac{2\sqrt{(q^{n-1}-1)^2}}{2(q^{n-1}-1)}\right)^{\alpha}\\
&= \prod_{\imath\jmath\in E}\left(\frac{2(q^{n-1}-1)}{2(q^{n-1}-1)}\right)^{\alpha}\\
&= \prod_{\imath\jmath\in E}(1)^{\alpha}\\
&= [(1)^{\alpha}]^{(q^{n}-1)(q^{n-1}-1)}\\
&= (1)^{\alpha(q^{n}-1)(q^{n-1}-1)}\\
&= (1)^{\alpha(q^{2n-1}-q^{n}-q^{n-1}+1)}\\
&= (1)^{\alpha q^{2n-1}-\alpha q^{n}-\alpha q^{n-1}+\alpha}
\end{aligned}$$

■

V. REVERSE INDICES:

This subdivision gives us some of the well-known named topological indices for $\mathfrak{F}(\mathbb{V})$, which are degree-based indices. Here the reduced reverse degree $c_\imath = \delta(\mathfrak{F}(\mathbb{V})) - d_{\mathfrak{F}(\mathbb{V})}(v) + 1$

1. First Reverse Zagreb Index :
$$CM_1(\mathfrak{F}(\mathbb{V})) = \sum_{\imath\jmath\in E}(c(\imath)+c(\jmath))$$
2. Second Reverse Zagreb Index :
$$CM_2(\mathfrak{F}(\mathbb{V})) = \sum_{\imath\jmath\in E}(c(\imath).c(\jmath))$$
3. First Reverse Hyper Zagreb Index :
$$HCM_1(\mathfrak{F}(\mathbb{V})) = \sum_{\imath\jmath\in E}(c(\imath)+c(\jmath))^2$$
4. Second Reverse Hyper Zagreb Index :

$$HCM_2(\mathfrak{F}(\mathbb{V})) = \sum_{\imath\jmath\in E}(c(\imath).c(\jmath))^2$$

5. Reverse Randic Index :

$$RR_\alpha(\mathfrak{F}(\mathbb{V})) = \sum_{\imath\jmath\in E}(c(\imath) + c(\jmath))^\alpha$$

6. Reverse ABC Index :

$$RABC(\mathfrak{F}(\mathbb{V})) = \sum_{\imath\jmath\in E}\sqrt{\frac{c(\imath)+c(\jmath)-2}{c(\imath).c(\jmath)}}$$

7. Reverse Geometric Arithmetic Index :

$$RGA(\mathfrak{F}(\mathbb{V})) = \sum_{\imath\jmath\in E}\frac{2\sqrt{c(\imath).c(\jmath)}}{c(\imath)+c(\jmath)}$$

8. Reverse Forgotten Index :

$$RF(\mathfrak{F}(\mathbb{V})) = \sum_{\imath\jmath\in E}(c(\imath))^2 + (c(\jmath))^2$$

9. Reverse Harmonic Index :

$$RH(\mathfrak{F}(\mathbb{V})) = \sum_{\imath\jmath\in E}\frac{2}{(c(\imath)+c(\jmath))}$$

Theorem 5.1: The First Reverse Zagreb Index of $\mathfrak{F}(\mathbb{V})$ is

$$CM_1(\mathfrak{F}(\mathbb{V})) = 2q^{2n-1} - 2q^n - 2q^{n-1} + 2$$

Proof:

$$\begin{aligned}
CM_1(\mathfrak{F}(\mathbb{V})) &= \sum_{\imath\jmath\in E}(c(\imath) + c(\jmath)) \\
&= \sum_{\imath\jmath\in E}\Big[\big((q^{n-1}-1) - (q^{n-1}-1) + 1\big) \\
&\quad + \big((q^{n-1}-1) - (q^{n-1}-1) + 1\big)\Big] \\
&= \sum_{\imath\jmath\in E}[1+1] \\
&= \sum_{\imath\jmath\in E}[2] \\
&= 2(q^n-1)(q^{n-1}-1) \\
&= 2(q^{2n-1} - q^n - q^{n-1} + 1) \\
&= 2q^{2n-1} - 2q^n - 2q^{n-1} + 2
\end{aligned}$$

■

Theorem 5.2: The Second Reverse Zagreb Index of $\mathfrak{F}(\mathbb{V})$ is

$$CM_2(\mathfrak{F}(\mathbb{V})) = q^{2n-1} - q^n - q^{n-1} + 1$$

Proof:

$$\begin{aligned}
CM_2(\mathfrak{F}(\mathbb{V})) &= \sum_{\imath\jmath\in E}(c(\imath).c(\jmath)) \\
&= \sum_{\imath\jmath\in E}\Big[\big((q^{n-1}-1) - (q^{n-1}-1) + 1\big) \\
&\quad \times \big((q^{n-1}-1) - (q^{n-1}-1) + 1\big)\Big] \\
&= \sum_{\imath\jmath\in E}[1\times 1] \\
&= \sum_{\imath\jmath\in E}[1] \\
&= 1(q^n-1)(q^{n-1}-1) \\
&= q^{2n-1} - q^n - q^{n-1} + 1
\end{aligned}$$

■

Theorem 5.3: The First Reverse Hyper Zagreb Index of $\mathfrak{F}(\mathbb{V})$ is

$$HCM_1(\mathfrak{F}(\mathbb{V})) = 4q^{2n-1} - 4q^n - 4q^{n-1} + 4$$

Proof:

$$\begin{aligned}
HCM_1(\mathfrak{F}(\mathbb{V})) &= \sum_{\imath\jmath\in E}(c(\imath) + c(\jmath))^2 \\
&= \sum_{\imath\jmath\in E}\Big[\big((q^{n-1}-1) - (q^{n-1}-1) + 1\big) \\
&\quad + \big((q^{n-1}-1) - (q^{n-1}-1) + 1\big)\Big]^2 \\
&= \sum_{\imath\jmath\in E}[1+1]^2 \\
&= \sum_{\imath\jmath\in E}[2]^2 \\
&= 4(q^n-1)(q^{n-1}-1) \\
&= 4(q^{2n-1} - q^n - q^{n-1} + 1) \\
&= 4q^{2n-1} - 4q^n - 4q^{n-1} + 4
\end{aligned}$$

■

Theorem 5.4: The Second Reverse Hyper Zagreb Index of $\mathfrak{F}(\mathbb{V})$ is

$$HCM_2(\mathfrak{F}(\mathbb{V})) = q^{2n-1} - q^n - q^{n-1} + 1$$

Proof:

$$\begin{aligned}
HCM_2(\mathfrak{F}(\mathbb{V})) &= \sum_{\imath\jmath\in E}(c(\imath).c(\jmath))^2 \\
&= \sum_{\imath\jmath\in E}\Big[\big((q^{n-1}-1) - (q^{n-1}-1) + 1\big) \\
&\quad \times \big((q^{n-1}-1) - (q^{n-1}-1) + 1\big)\Big]^2 \\
&= \sum_{\imath\jmath\in E}[1\times 1]^2 \\
&= \sum_{\imath\jmath\in E}[1]^2 \\
&= 1(q^n-1)(q^{n-1}-1) \\
&= q^{2n-1} - q^n - q^{n-1} + 1
\end{aligned}$$

■

Theorem 5.5: The Reverse Randic Index of $\mathfrak{F}(\mathbb{V})$ is

$$RR_\alpha(\mathfrak{F}(\mathbb{V})) = [2]^\alpha(q^{2n-1} - q^n - q^{n-1} + 1)$$

Proof:

$$\begin{aligned}
RR_\alpha(\mathfrak{F}(\mathbb{V})) &= \sum_{\imath\jmath\in E}(c(\imath) + c(\jmath))^\alpha \\
&= \sum_{\imath\jmath\in E}\Big[\big((q^{n-1}-1) - (q^{n-1}-1) + 1\big) \\
&\quad + \big((q^{n-1}-1) - (q^{n-1}-1) + 1\big)\Big]^\alpha \\
&= \sum_{\imath\jmath\in E}[1+1]^\alpha \\
&= \sum_{\imath\jmath\in E}[2]^\alpha \\
&= [2]^\alpha(q^n-1)(q^{n-1}-1) \\
&= [2]^\alpha(q^{2n-1} - q^n - q^{n-1} + 1)
\end{aligned}$$

■

Theorem 5.6: The Reverse ABC Index of $\mathfrak{F}(\mathbb{V})$ is

$$RABC(\mathfrak{F}(\mathbb{V})) = 0$$

Proof:

$$\begin{aligned} RABC(\mathfrak{F}(\mathbb{V})) &= \sum_{\imath\jmath\in E}\sqrt{\frac{c(\imath)+c(\jmath)-2}{c(\imath).c(\jmath)}} \\ &= \sum_{\imath\jmath\in E}\sqrt{\frac{2((q^{n-1}-1)-(q^{n-1}-1)+1)-2}{((q^{n-1}-1)-(q^{n-1}-1)+1)^2}} \\ &= \sum_{\imath\jmath\in E}\sqrt{\frac{2-2}{1}} \\ &= \sum_{\imath\jmath\in E}[0] \\ &= [0](q^n-1)(q^{n-1}-1) \\ &= 0 \end{aligned}$$

■

Theorem 5.7: The Reverse Geometric Arithmetic Index of $\mathfrak{F}(\mathbb{V})$ is

$$RGA(\mathfrak{F}(\mathbb{V})) = q^{2n-1} - q^n - q^{n-1} + 1$$

Proof:

$$\begin{aligned} RGA(\mathfrak{F}(\mathbb{V})) &= \sum_{\imath\jmath\in E}\frac{2\sqrt{c(\imath).c(\jmath)}}{c(\imath)+c(\jmath)} \\ &= \sum_{\imath\jmath\in E}\frac{2\sqrt{((q^{n-1}-1)-(q^{n-1}-1)+1)^2}}{2\left((q^{n-1}-1)-(q^{n-1}-1)+1\right)} \\ &= \sum_{\imath\jmath\in E}\frac{2}{2} \\ &= [1](q^n-1)(q^{n-1}-1) \\ &= q^{2n-1}-q^n-q^{n-1}+1 \end{aligned}$$

■

Theorem 5.8: The Reverse Forgotten Index of $\mathfrak{F}(\mathbb{V})$ is

$$RF(\mathfrak{F}(\mathbb{V})) = 2q^{2n-1} - 2q^n - 2q^{n-1} + 2$$

Proof:

$$\begin{aligned} RF(\mathfrak{F}(\mathbb{V})) &= \sum_{\imath\jmath\in E}(c(\imath))^2+(c(\jmath))^2 \\ &= \sum_{\imath\jmath\in E}\left((q^{n-1}-1)-(q^{n-1}-1)+1\right)^2 \\ &\quad + \left((q^{n-1}-1)-(q^{n-1}-1)+1\right)^2 \\ &= \sum_{\imath\jmath\in E}\left[(1)^2+(1)^2\right] \\ &= \sum_{\imath\jmath\in E}[2] \\ &= 2(q^n-1)(q^{n-1}-1) \\ &= 2(q^{2n-1}-q^n-q^{n-1}+1) \\ &= 2q^{2n-1}-2q^n-2q^{n-1}+2 \end{aligned}$$

■

Theorem 5.9: The Reverse Harmonic Index of $\mathfrak{F}(\mathbb{V})$ is

$$RH(\mathfrak{F}(\mathbb{V})) = q^{2n-1} - q^n - q^{n-1} + 1$$

Proof:

$$\begin{aligned} RH(\mathfrak{F}(\mathbb{V})) &= \sum_{\imath\jmath\in E}\frac{2}{(c(\imath)+c(\jmath))} \\ &= \sum_{\imath\jmath\in E}\frac{2}{2\left((q^{n-1}-1)-(q^{n-1}-1)+1\right)} \\ &= \sum_{\imath\jmath\in E}\frac{2}{2} \\ &= \sum_{\imath\jmath\in E}[1] \\ &= 1(q^n-1)(q^{n-1}-1) \\ &= q^{2n-1}-q^n-q^{n-1}+1 \end{aligned}$$

■

VI. REVERSE MULTIPLICATIVE AND REFINED INDICES:

This division shows that reverse multiplicative and refined indices are based on degree.

1. Reverse First Multiple Zagreb Index :

$$RPM_1(\mathfrak{F}(\mathbb{V})) = \textstyle\prod_{\imath\jmath\in E} c(\imath)+c(\jmath)$$

2. Reverse Second Multiple Zagreb Index :

$$RPM_2(\mathfrak{F}(\mathbb{V})) = \textstyle\prod_{\imath\jmath\in E} c(\imath).c(\jmath)$$

3. Reverse First Refined Zagreb Index :

$$RReZG_1(\mathfrak{F}(\mathbb{V})) = \textstyle\sum_{\imath\jmath\in E}\frac{c(\imath)+c(\jmath)}{c(\imath).c(\jmath)}$$

4. Reverse Second Refined Zagreb Index :

$$RReZG_2(\mathfrak{F}(\mathbb{V})) = \textstyle\sum_{\imath\jmath\in E}\frac{c(\imath).c(\jmath)}{c(\imath)+c(\jmath)}$$

5. Reverse Third Refined Zagreb Index :

$$RReZG_3(\mathfrak{F}(\mathbb{V})) = \textstyle\sum_{\imath\jmath\in E}(c(\imath)+c(\jmath))(c(\imath).c(\jmath))$$

Theorem 6.1: The Reverse First Multiple Zagreb Index of $\mathfrak{F}(\mathbb{V})$ is

$$RPM_1(\mathfrak{F}(\mathbb{V})) = 2^{(q^{2n-1}-q^n-q^{n-1}+1)}$$

Proof:

$$\begin{aligned} RPM_1(\mathfrak{F}(\mathbb{V})) &= \prod_{\imath\jmath\in E}(c(\imath)+c(\jmath)) \\ &= \prod_{\imath\jmath\in E}\left[\left((q^{n-1}-1)-(q^{n-1}-1)+1\right)\right. \\ &\quad \left.+\left((q^{n-1}-1)-(q^{n-1}-1)+1\right)\right] \\ &= \prod_{\imath\jmath\in E}[1+1] \\ &= \prod_{\imath\jmath\in E}[2] \\ &= 2^{(q^n-1)(q^{n-1}-1)} \\ &= 2^{(q^{2n-1}-q^n-q^{n-1}+1)} \end{aligned}$$

■

Theorem 6.2: The Reverse Second Multiple Zagreb Index of $\mathfrak{F}(\mathbb{V})$ is

$$RPM_2(\mathfrak{F}(\mathbb{V})) = 1$$

Proof:

$$\begin{aligned} RPM_2(\mathfrak{F}(\mathbb{V})) &= \prod_{\imath\jmath\in E}(c(\imath)\times c(\jmath)) \\ &= \prod_{\imath\jmath\in E}\left[\left((q^{n-1}-1)-(q^{n-1}-1)+1\right)\right. \\ &\quad \left.\times\left((q^{n-1}-1)-(q^{n-1}-1)+1\right)\right] \\ &= \prod_{\imath\jmath\in E}[1\times 1] \\ &= \prod_{\imath\jmath\in E}[1] \\ &= 1^{(q^n-1)(q^{n-1}-1)} \\ &= 1 \end{aligned}$$

■

Theorem 6.3: The Reverse First Refined Zagreb Index of $\mathfrak{F}(\mathbb{V})$ is

$$RReZG_1(\mathfrak{F}(\mathbb{V})) = 2q^{2n-1}-2q^n-2q^{n-1}+2$$

Proof:

$$\begin{aligned} RReZG_1(\mathfrak{F}(\mathbb{V})) &= \sum_{\imath\jmath\in E}\frac{c(\imath)+c(\jmath)}{c(\imath).c(\jmath)} \\ &= \sum_{\imath\jmath\in E}\frac{\left((q^{n-1}-1)-(q^{n-1}-1)+1\right)}{((q^{n-1}-1)-(q^{n-1}-1)+1)} \\ &\quad \frac{+\left((q^{n-1}-1)-(q^{n-1}-1)+1\right)}{\times((q^{n-1}-1)-(q^{n-1}-1)+1)} \\ &= \sum_{\imath\jmath\in E}\frac{[1+1]}{1\times 1} \\ &= \sum_{\imath\jmath\in E}[2] \\ &= 2\times(q^n-1)(q^{n-1}-1) \\ &= 2(q^{2n-1}-q^n-q^{n-1}+1) \\ &= 2q^{2n-1}-2q^n-2q^{n-1}+2 \end{aligned}$$

■

Theorem 6.4: The Reverse Second Refined Zagreb Index of $\mathfrak{F}(\mathbb{V})$ is

$$RReZG_2(\mathfrak{F}(\mathbb{V})) = \frac{q^{2n-1}-q^n-q^{n-1}+1}{2}$$

Proof:

$$\begin{aligned} RReZG_2(\mathfrak{F}(\mathbb{V})) &= \sum_{\imath\jmath\in E}\frac{c(\imath).c(\jmath)}{c(\imath)+c(\jmath)} \\ &= \sum_{\imath\jmath\in E}\frac{\left((q^{n-1}-1)-(q^{n-1}-1)+1\right)}{((q^{n-1}-1)-(q^{n-1}-1)+1)} \\ &\quad \frac{\times\left((q^{n-1}-1)-(q^{n-1}-1)+1\right)}{+((q^{n-1}-1)-(q^{n-1}-1)+1)} \\ &= \sum_{\imath\jmath\in E}\frac{1\times 1}{1+1} \\ &= \sum_{\imath\jmath\in E}\frac{1}{2} \\ &= \frac{1}{2}((q^n-1)(q^{n-1}-1)) \\ &= \frac{1}{2}(q^{2n-1}-q^n-q^{n-1}+1) \\ &= \frac{q^{2n-1}-q^n-q^{n-1}+1}{2} \end{aligned}$$

■

Theorem 6.5: The Reverse Third Refined Zagreb Index of $\mathfrak{F}(\mathbb{V})$ is

$$RReZG_3(\mathfrak{F}(\mathbb{V})) = 2q^{2n-1}-2q^n-2q^{n-1}+2$$

Proof:

$$\begin{aligned} RReZG_3(\mathfrak{F}(\mathbb{V})) &= \sum_{\imath\jmath\in E}(c(\imath).c(\jmath))\,(c(\imath)+c(\jmath)) \\ &= \sum_{\imath\jmath\in E}\left((q^{n-1}-1)-(q^{n-1}-1)+1\right) \\ &\quad \times\left((q^{n-1}-1)-(q^{n-1}-1)+1\right) \\ &\quad \left((q^{n-1}-1)-(q^{n-1}-1)+1\right) \\ &\quad +\left((q^{n-1}-1)-(q^{n-1}-1)+1\right) \\ &= \sum_{\imath\jmath\in E}(1\times 1)(1+1) \\ &= \sum_{\imath\jmath\in E}(1)(2) \\ &= (2)((q^n-1)(q^{n-1}-1)) \\ &= (2)(q^{2n-1}-q^n-q^{n-1}+1) \\ &= 2q^{2n-1}-2q^n-2q^{n-1}+2 \end{aligned}$$

■

VII. REDUCED REVERSE INDICES:

This section gives us some of the results for calculating reduced reverse degree-based topological indices for $\mathfrak{F}(\mathbb{V})$, which are degree-based indices.

Here the reduced reverse degree is
$RR_\imath = \delta(\mathfrak{F}(\mathbb{V})) - d_{\mathfrak{F}(\mathbb{V})}(v) + 2$

1. Reduced Reverse First Zagreb Index :

$$RRM_1(\mathfrak{F}(\mathbb{V})) = \sum_{\imath\jmath\in E}(RR(\imath)+RR(\jmath))$$

2. Reduced Reverse Second Zagreb Index :

$$RRM_2(\mathfrak{F}(\mathbb{V})) = \sum_{\imath\jmath\in E}(RR(\imath).RR(\jmath))$$

3. Reduced Reverse Hyper First Zagreb Index :

$$RRHM_1(\mathfrak{F}(\mathbb{V})) = \sum_{\imath\jmath\in E}(RR(\imath)+RR(\jmath))^2$$

4. Reduced Reverse Hyper Second Zagreb Index :

$$RRHM_2(\mathfrak{F}(\mathbb{V})) = \sum_{\imath\jmath\in E}(RR(\imath).RR(\jmath))^2$$

5. Reduced Reverse Forgotten Index :

$$RRF(\mathfrak{F}(\mathbb{V})) = \sum_{\imath\jmath\in E}(RR(\imath)^2.RR(\jmath)^2)$$

6. Reduced Reverse ABC Index :

$$RRABC(\mathfrak{F}(\mathbb{V})) = \sum_{\imath\jmath\in E}\left[\frac{RR(\imath)+RR(\jmath)-2}{RR(\imath).RR(\jmath)}\right]$$

7. Reduced Reverse Geometric Arithmetic Index :

$$RRGA(\mathfrak{F}(\mathbb{V})) = \sum_{\imath\jmath\in E}\left[\frac{2\sqrt{RR(\imath).RR(\jmath)}}{RR(\imath)+RR(\jmath)}\right]$$

Theorem 7.1: The Reduced Reverse First Zagreb Index of $\mathfrak{F}(\mathbb{V})$ is

$$RRM_1(\mathfrak{F}(\mathbb{V})) = 4q^{2n-1}-4q^n-4q^{n-1}+4$$

Proof:

$$\begin{aligned}RRM_1(\mathfrak{F}(\mathbb{V})) &= \sum_{\imath\jmath\in E}(RR(\imath)+RR(\jmath))\\ &= \sum_{\imath\jmath\in E}\Big[\big((q^{n-1}-1)-(q^{n-1}-1)+2\big)\\ &\quad+\big((q^{n-1}-1)-(q^{n-1}-1)+2\big)\Big]\\ &= \sum_{\imath\jmath\in E}[2+2]\\ &= \sum_{\imath\jmath\in E}[4]\\ &= 4(q^n-1)(q^{n-1}-1)\\ &= 4(q^{2n-1}-q^n-q^{n-1}+1)\\ &= 4q^{2n-1}-4q^n-4q^{n-1}+4\end{aligned}$$

■

Theorem 7.2: The Reduced Reverse Second Zagreb Index of $\mathfrak{F}(\mathbb{V})$ is

$$RRM_2(\mathfrak{F}(\mathbb{V})) = 4q^{2n-1}-4q^n-4q^{n-1}+4$$

Proof:

$$\begin{aligned}RRM_2(\mathfrak{F}(\mathbb{V})) &= \sum_{\imath\jmath\in E}(RR(\imath).RR(\jmath))\\ &= \sum_{\imath\jmath\in E}\Big[\big((q^{n-1}-1)-(q^{n-1}-1)+2\big)\\ &\quad\times\big((q^{n-1}-1)-(q^{n-1}-1)+2\big)\Big]\\ &= \sum_{\imath\jmath\in E}[2\times 2]\\ &= \sum_{\imath\jmath\in E}[4]\\ &= 4(q^n-1)(q^{n-1}-1)\\ &= 4(q^{2n-1}-q^n-q^{n-1}+1)\\ &= 4q^{2n-1}-4q^n-4q^{n-1}+4\end{aligned}$$

■

Theorem 7.3: The Reduced Reverse Hyper First Zagreb Index of $\mathfrak{F}(\mathbb{V})$ is

$$RRHM_1(\mathfrak{F}(\mathbb{V})) = 16q^{2n-1}-16q^n-16q^{n-1}+16$$

Proof:

$$\begin{aligned}RRHM_1(\mathfrak{F}(\mathbb{V})) &= \sum_{\imath\jmath\in E}(RR(\imath)+RR(\jmath))^2\\ &= \sum_{\imath\jmath\in E}\Big[\big((q^{n-1}-1)-(q^{n-1}-1)+2\big)\\ &\quad+\big((q^{n-1}-1)-(q^{n-1}-1)+2\big)\Big]^2\\ &= \sum_{\imath\jmath\in E}[2+2]^2\\ &= \sum_{\imath\jmath\in E}[16]\\ &= 16(q^n-1)(q^{n-1}-1)\\ &= 16(q^{2n-1}-q^n-q^{n-1}+1)\\ &= 16q^{2n-1}-16q^n-16q^{n-1}+16\end{aligned}$$

■

Theorem 7.4: The Reduced Reverse Hyper Second Zagreb Index of $\mathfrak{F}(\mathbb{V})$ is

$$RRHM_2(\mathfrak{F}(\mathbb{V})) = 16q^{2n-1}-16q^n-16q^{n-1}+16$$

Proof:

$$\begin{aligned}RRHM_2(\mathfrak{F}(\mathbb{V})) &= \sum_{\imath\jmath\in E}(RR(\imath).RR(\jmath))^2\\ &= \sum_{\imath\jmath\in E}\Big[\big((q^{n-1}-1)-(q^{n-1}-1)+2\big)\\ &\quad\times\big((q^{n-1}-1)-(q^{n-1}-1)+2\big)\Big]^2\\ &= \sum_{\imath\jmath\in E}[2\times 2]^2\\ &= \sum_{\imath\jmath\in E}[16]\\ &= 16(q^n-1)(q^{n-1}-1)\\ &= 16q^{2n-1}-16q^n-16q^{n-1}+16\end{aligned}$$

■

Theorem 7.5: The Reduced Reverse Forgotten Index of $\mathfrak{F}(\mathbb{V})$ is

$$RRF(\mathfrak{F}(\mathbb{V})) = 8q^{2n-1}-8q^n-8q^{n-1}+8$$

Proof:

$$\begin{aligned}RRF(\mathfrak{F}(\mathbb{V})) &= \sum_{\imath\jmath\in E}(RR(\imath)^2.RR(\jmath)^2)\\ &= \sum_{\imath\jmath\in E}\Big[\big((q^{n-1}-1)-(q^{n-1}-1)+2\big)^2\\ &\quad+\big((q^{n-1}-1)-(q^{n-1}-1)+2\big)^2\Big]\\ &= \sum_{\imath\jmath\in E}[2^2+2^2]\\ &= \sum_{\imath\jmath\in E}[8]\\ &= 8(q^n-1)(q^{n-1}-1)\\ &= 8(q^{2n-1}-q^n-q^{n-1}+1)\\ &= 8q^{2n-1}-8q^n-8q^{n-1}+8\end{aligned}$$

■

Theorem 7.6: The Reduced Reverse ABC Index of $\mathfrak{F}(\mathbb{V})$ is

$$RRABC(\mathfrak{F}(\mathbb{V})) = \left[\frac{q^{2n-1}-q^n-q^{n-1}+1}{2}\right]$$

Proof:

$$\begin{aligned}RRABC(\mathfrak{F}(\mathbb{V})) &= \sum_{\imath\jmath\in E}\left[\frac{RR(\imath)+RR(\jmath)-2}{RR(\imath).RR(\jmath)}\right]\\ &= \sum_{\imath\jmath\in E}\left[\frac{2\big((q^{n-1}-1)-(q^{n-1}-1)+2\big)}{((q^{n-1}-1)-(q^{n-1}-1)+2)^2}\right]\\ &\quad-\left[\frac{2}{((q^{n-1}-1)-(q^{n-1}-1)+2)^2}\right]\\ &= \sum_{\imath\jmath\in E}\left[\frac{4-2}{4}\right]\\ &= \sum_{\imath\jmath\in E}\left[\frac{2}{4}\right]\\ &= \left[\frac{1}{2}\right](q^n-1)(q^{n-1}-1)\\ &= \left[\frac{q^{2n-1}-q^n-q^{n-1}+1}{2}\right]\end{aligned}$$

■

Theorem 7.7: The Reduced Reverse Geometric Arithmetic Index of $\mathfrak{F}(\mathbb{V})$ is

$$RRGA(\mathfrak{F}(\mathbb{V})) = q^{2n-1} - q^n - q^{n-1} + 1$$

Proof:

$$\begin{aligned}
RRGA(\mathfrak{F}(\mathbb{V})) &= \sum_{\imath\jmath \in E}\left[\frac{2\sqrt{RR(\imath).RR(\jmath)}}{RR(\imath)+RR(\jmath)}\right] \\
&= \sum_{\imath\jmath \in E}\left[\frac{2\sqrt{((q^{n-1}-1)-(q^{n-1}-1)+2)^2}}{2\left((q^{n-1}-1)-(q^{n-1}-1)+2\right)}\right] \\
&= \sum_{\imath\jmath \in E}\left[\frac{2\sqrt{(2)^2}}{4}\right] \\
&= \sum_{\imath\jmath \in E}\left[\frac{2\times 2}{4}\right] \\
&= \sum_{\imath\jmath \in E}[1] \\
&= [1]\left((q^n-1)(q^{n-1}-1)\right) \\
&= q^{2n-1} - q^n - q^{n-1} + 1
\end{aligned}$$

■

VIII. REVAN INDICES:

The following section explores various results pertaining to the Revan degree-based topological indices associated with $\mathfrak{F}(\mathbb{V})$. The Revan degree, in this case, is characterized as $r_{\mathfrak{F}(\mathbb{V})}(v) = \Delta(\mathfrak{F}(\mathbb{V})) + \delta(\mathfrak{F}(\mathbb{V})) - d_{\mathfrak{F}(\mathbb{V})}(v)$

1. First Revan Index :
$$R_1(\mathfrak{F}(\mathbb{V})) = \sum_{\imath\jmath\in E}(r_{\mathfrak{F}(\mathbb{V})}(\imath) + r_{\mathfrak{F}(\mathbb{V})}(\jmath))$$
2. Second Revan Index :
$$R_2(\mathfrak{F}(\mathbb{V})) = \sum_{\imath\jmath\in E}(r_{\mathfrak{F}(\mathbb{V})}(\imath))(r_{\mathfrak{F}(\mathbb{V})}(\jmath))$$
3. Third Revan Index :
$$R_3(\mathfrak{F}(\mathbb{V})) = \sum_{\imath\jmath\in E}|r_{\mathfrak{F}(\mathbb{V})}(\imath) - r_{\mathfrak{F}(\mathbb{V})}(\jmath)|$$
4. Multiplicative Revan Zero Index :
$$RII_0(\mathfrak{F}(\mathbb{V})) = \prod_{\imath\in V}(r_{\mathfrak{F}(\mathbb{V})}(\imath))$$
5. Multiplicative Revan Vertex Index :
$$RII_{01}(\mathfrak{F}(\mathbb{V})) = \prod_{\imath\in V}(r_{\mathfrak{F}(\mathbb{V})}(\imath))^2$$
6. First Multiplicative Revan Index :
$$RII_1(\mathfrak{F}(\mathbb{V})) = \prod_{\imath\jmath\in E}(r_{\mathfrak{F}(\mathbb{V})}(\imath) + r_{\mathfrak{F}(\mathbb{V})}(\jmath))$$
7. Second Multiplicative Revan Index :
$$RII_2(\mathfrak{F}(\mathbb{V})) = \prod_{\imath\jmath\in E}(r_{\mathfrak{F}(\mathbb{V})}(\imath))(r_{\mathfrak{F}(\mathbb{V})}(\jmath))$$
8. First Multiplicative Hyper Revan Index :
$$HRII_1(\mathfrak{F}(\mathbb{V})) = \prod_{\imath\jmath\in E}(r_{\mathfrak{F}(\mathbb{V})}(\imath) + r_{\mathfrak{F}(\mathbb{V})}(\jmath))^2$$
9. Second Multiplicative Hyper Revan Index :
$$HRII_2(\mathfrak{F}(\mathbb{V})) = \prod_{\imath\jmath\in E}[(r_{\mathfrak{F}(\mathbb{V})}(\imath))(r_{\mathfrak{F}(\mathbb{V})}(\jmath))]^2$$
10. F-Revan Index :
$$FR(\mathfrak{F}(\mathbb{V})) = \sum_{\imath\jmath\in E}[r_{\mathfrak{F}(\mathbb{V})}(\imath)^2 r_{\mathfrak{F}(\mathbb{V})}(\jmath)^2]$$

Theorem 8.1: The First Revan Index of $\mathfrak{F}(\mathbb{V})$ is

$$R_1(\mathfrak{F}(\mathbb{V})) = 2q^{3n-2} - 4q^{2n-1} - 2q^{2n-2} + 4q^{n-1} + 2q^n - 2$$

Proof:

$$\begin{aligned}
R_1(\mathfrak{F}(\mathbb{V})) &= \sum_{\imath\jmath\in E}(r_{\mathfrak{F}(\mathbb{V})}(\imath) + r_{\mathfrak{F}(\mathbb{V})}(\jmath)) \\
&= \sum_{\imath\jmath\in E}\Big[\big((q^{n-1}-1)+(q^{n-1}-1) \\
&\quad -(q^{n-1}-1)\big) + \big((q^{n-1}-1)+(q^{n-1}-1) \\
&\quad -(q^{n-1}-1)\big)\Big] \\
&= \sum_{\imath\jmath\in E}\left[(q^{n-1}-1)+(q^{n-1}-1)\right] \\
&= 2(q^{n-1}-1)(q^n-1)(q^{n-1}-1) \\
&= 2q^{3n-2} - 4q^{2n-1} - 2q^{2n-2} + 4q^{n-1} \\
&\quad + 2q^n - 2
\end{aligned}$$

■

Theorem 8.2: The Second Revan Index of $\mathfrak{F}(\mathbb{V})$ is

$$R_2(\mathfrak{F}(\mathbb{V})) = q^{4n-3} - 3q^{3n-2} - q^{3n-3} + 3q^{2n-2} + 3q^{2n-1} - 3q^{n-1} - q^n + 1$$

Proof:

$$\begin{aligned}
R_2(\mathfrak{F}(\mathbb{V})) &= \sum_{\imath\jmath\in E}(r_{\mathfrak{F}(\mathbb{V})}(\imath))(r_{\mathfrak{F}(\mathbb{V})}(\jmath)) \\
&= \sum_{\imath\jmath\in E}\Big[\big((q^{n-1}-1)+(q^{n-1}-1) \\
&\quad -(q^{n-1}-1)\big) \times \big((q^{n-1}-1)+(q^{n-1}-1) \\
&\quad -(q^{n-1}-1)\big)\Big] \\
&= \sum_{\imath\jmath\in E}\left[(q^{n-1}-1)\times(q^{n-1}-1)\right] \\
&= (q^{n-1}-1)^2(q^n-1)(q^{n-1}-1) \\
&= (q^{2n-2}+1-2q^{n-1})(q^{2n-1}-q^n \\
&\quad -q^{n-1}+1) \\
&= q^{4n-3} - 3q^{3n-2} - q^{3n-3} + 3q^{2n-2} \\
&\quad + 3q^{2n-1} - 3q^{n-1} - q^n + 1
\end{aligned}$$

■

Theorem 8.3: The Third Revan Index of $\mathfrak{F}(\mathbb{V})$ is

$$R_3(\mathfrak{F}(\mathbb{V})) = 0$$

Proof:

$$\begin{aligned}
R_3(\mathfrak{F}(\mathbb{V})) &= \sum_{\imath\jmath\in E}|r_{\mathfrak{F}(\mathbb{V})}(\imath) - r_{\mathfrak{F}(\mathbb{V})}(\jmath)| \\
&= \sum_{\imath\jmath\in E}\Big|\Big[\big((q^{n-1}-1)+(q^{n-1}-1) \\
&\quad -(q^{n-1}-1)\big) - \big((q^{n-1}-1)+(q^{n-1}-1) \\
&\quad -(q^{n-1}-1)\big)\Big]\Big| \\
&= \sum_{\imath\jmath\in E}\left|\left[(q^{n-1}-1)-(q^{n-1}-1)\right]\right| \\
&= \sum_{\imath\jmath\in E}|0| \\
&= [0](q^n-1)(q^{n-1}-1) \\
&= 0
\end{aligned}$$

■

Theorem 8.4: The Multiplicative Revan Zero Index of $\mathfrak{F}(\mathbb{V})$ is

$$RII_0(\mathfrak{F}(\mathbb{V})) = (q^{n-1}-1)^{2q^n-2}$$

Proof:

$$\begin{aligned} RII_0(\mathfrak{F}(\mathbb{V})) &= \prod_{\imath \in V} (r_{\mathfrak{F}(\mathbb{V})}(\imath)) \\ &= \prod_{\imath \in V} \big((q^{n-1}-1)+(q^{n-1}-1)-(q^{n-1}-1)\big) \\ &= \prod_{\imath \in V} (q^{n-1}-1) \\ &= (q^{n-1}-1)^{[2(q^n-1)]} \\ &= (q^{n-1}-1)^{2q^n-2} \end{aligned}$$

■

Theorem 8.5: The Multiplicative Revan Vertex Index of $\mathfrak{F}(\mathbb{V})$ is

$$RII_{01}(\mathfrak{F}(\mathbb{V})) = (q^{n-1}-1)^{4q^n-4}$$

Proof:

$$\begin{aligned} RII_{01}(\mathfrak{F}(\mathbb{V})) &= \prod_{\imath \in V} (r_{\mathfrak{F}(\mathbb{V})}(\imath))^2 \\ &= \prod_{\imath \in V} \big((q^{n-1}-1)+(q^{n-1}-1) \\ &\quad -(q^{n-1}-1)\big)^2 \\ &= \prod_{\imath \in V} (q^{n-1}-1)^2 \\ &= (q^{n-1}-1)^{2[2(q^n-1)]} \\ &= (q^{n-1}-1)^{4q^n-4} \end{aligned}$$

■

Theorem 8.6: The First Multiplicative Revan Index of $\mathfrak{F}(\mathbb{V})$ is

$$RII_1(\mathfrak{F}(\mathbb{V})) = (2q^{n-1}-2)^{q^{2n-1}-q^n-q^{n-1}+1}$$

Proof:

$$\begin{aligned} RII_1(\mathfrak{F}(\mathbb{V})) &= \prod_{\imath\jmath \in E} (r_{\mathfrak{F}(\mathbb{V})}(\imath) + r_{\mathfrak{F}(\mathbb{V})}(\jmath)) \\ &= \prod_{\imath\jmath \in E} \big[\big((q^{n-1}-1)+(q^{n-1}-1) \\ &\quad -(q^{n-1}-1)\big) + \big((q^{n-1}-1) \\ &\quad +(q^{n-1}-1)-(q^{n-1}-1)\big)\big] \\ &= \prod_{\imath\jmath \in E} \big[(q^{n-1}-1)+(q^{n-1}-1)\big] \\ &= \prod_{\imath\jmath \in E} \big[2(q^{n-1}-1)\big] \\ &= [2(q^{n-1}-1)]^{(q^n-1)(q^{n-1}-1)} \\ &= (2q^{n-1}-2)^{q^{2n-1}-q^n-q^{n-1}+1} \end{aligned}$$

■

Theorem 8.7: The Second Multiplicative Revan Index of $\mathfrak{F}(\mathbb{V})$ is

$$RII_2(\mathfrak{F}(\mathbb{V})) = (q^{n-1}-1)^{2q^{2n-1}-2q^n-2q^{n-1}+2}$$

Proof:

$$\begin{aligned} RII_2(\mathfrak{F}(\mathbb{V})) &= \prod_{\imath\jmath \in E} (r_{\mathfrak{F}(\mathbb{V})}(\imath))(r_{\mathfrak{F}(\mathbb{V})}(\jmath)) \\ &= \prod_{\imath\jmath \in E} \big[\big((q^{n-1}-1)+(q^{n-1}-1) \\ &\quad -(q^{n-1}-1)\big) \times \big((q^{n-1}-1) \\ &\quad +(q^{n-1}-1)-(q^{n-1}-1)\big)\big] \\ &= \prod_{\imath\jmath \in E} \big[(q^{n-1}-1)\times(q^{n-1}-1)\big] \\ &= (q^{n-1}-1)^{2(q^n-1)(q^{n-1}-1)} \\ &= (q^{n-1}-1)^{2(q^{2n-1}-q^n-q^{n-1}+1)} \\ &= (q^{n-1}-1)^{2q^{2n-1}-2q^n-2q^{n-1}+2} \end{aligned}$$

■

Theorem 8.8: The First Multiplicative Hyper Revan Index of $\mathfrak{F}(\mathbb{V})$ is

$$HRII_1(\mathfrak{F}(\mathbb{V})) = (2q^{n-1}-2)^{2q^{2n-1}-2q^n-2q^{n-1}+2}$$

Proof:

$$\begin{aligned} HRII_1(\mathfrak{F}(\mathbb{V})) &= \prod_{\imath\jmath \in E} (r_{\mathfrak{F}(\mathbb{V})}(\imath) + r_{\mathfrak{F}(\mathbb{V})}(\jmath))^2 \\ &= \prod_{\imath\jmath \in E} \big[\big((q^{n-1}-1)+(q^{n-1}-1) \\ &\quad -(q^{n-1}-1)\big) + \big((q^{n-1}-1) \\ &\quad +(q^{n-1}-1)-(q^{n-1}-1)\big)\big]^2 \\ &= \prod_{\imath\jmath \in E} \big[(q^{n-1}-1)+(q^{n-1}-1)\big]^2 \\ &= \prod_{\imath\jmath \in E} \big[2(q^{n-1}-1)\big]^2 \\ &= [2(q^{n-1}-1)]^{2(q^n-1)(q^{n-1}-1)} \\ &= (2q^{n-1}-2)^{2q^{2n-1}-2q^n-2q^{n-1}+2} \end{aligned}$$

■

Theorem 8.9: The Second Multiplicative Hyper Revan Index of $\mathfrak{F}(\mathbb{V})$ is

$$HRII_2(\mathfrak{F}(\mathbb{V})) = (q^{n-1}-1)^{4q^{2n-1}-4q^n-4q^{n-1}+4}$$

Proof:

$$\begin{aligned} HRII_2(\mathfrak{F}(\mathbb{V})) &= \prod_{\imath\jmath \in E} [(r_{\mathfrak{F}(\mathbb{V})}(\imath))(r_{\mathfrak{F}(\mathbb{V})}(\jmath))]^2 \\ &= \prod_{\imath\jmath \in E} \big[\big((q^{n-1}-1)+(q^{n-1}-1) \\ &\quad -(q^{n-1}-1)\big) \times \big((q^{n-1}-1) \\ &\quad +(q^{n-1}-1)-(q^{n-1}-1)\big)\big]^2 \\ &= \prod_{\imath\jmath \in E} \big[(q^{n-1}-1)\times(q^{n-1}-1)\big]^2 \\ &= \prod_{\imath\jmath \in E} \big[(q^{n-1}-1)^2\big]^2 \\ &= \prod_{\imath\jmath \in E} \big[(q^{n-1}-1)\big]^4 \\ &= (q^{n-1}-1)^{4(q^n-1)(q^{n-1}-1)} \\ &= (q^{n-1}-1)^{4(q^{2n-1}-q^n-q^{n-1}+1)} \\ &= (q^{n-1}-1)^{4q^{2n-1}-4q^n-4q^{n-1}+4} \end{aligned}$$

■

Theorem 8.10: The F-Revan Index of $\mathfrak{F}(\mathbb{V})$ is
$FR(\mathfrak{F}(\mathbb{V})) = 2^{q^{2n-1}-q^{n}-q^{n-1}+1}(q^{n-1}-1)^{2q^{2n-1}-2q^{n}-2q^{n-1}+2}$

Proof:

$$\begin{aligned}
FR(\mathfrak{F}(\mathbb{V})) &= \sum_{\imath\jmath\in E}[r_{\mathfrak{F}(\mathbb{V})}(\imath)^2 r_{\mathfrak{F}(\mathbb{V})}(\jmath)^2] \\
&= \prod_{\imath\jmath\in E}\Big[\big((q^{n-1}-1)+(q^{n-1}-1) \\
&\quad -(q^{n-1}-1)\big)^2+\big((q^{n-1}-1) \\
&\quad +(q^{n-1}-1)-(q^{n-1}-1)\big)^2\Big] \\
&= \prod_{\imath\jmath\in E}\left[(q^{n-1}-1)^2+(q^{n-1}-1)^2\right] \\
&= \prod_{\imath\jmath\in E}\left[2(q^{n-1}-1)^2\right] \\
&= [2(q^{n-1}-1)^2]^{(q^n-1)(q^{n-1}-1)} \\
&= [2(q^{n-1}-1)^2]^{(q^{2n-1}-q^n-q^{n-1}+1)} \\
&= [2(q^{n-1}-1)]^{2q^{2n-1}-2q^n-2q^{n-1}+2} \\
&= 2^{q^{2n-1}-q^n-q^{n-1}+1} \\
&\quad (q^{n-1}-1)^{2q^{2n-1}-2q^n-2q^{n-1}+2}
\end{aligned}$$

■

IX. ELLIPTIC SOMBOR INDEX:

The following discussion highlights various properties and outcomes related to the Elliptic Sombor Index and associated degree-based indices for $\mathfrak{F}(\mathbb{V})$.

1. Elliptic Sombor Index:

$$ESO(\mathfrak{F}(\mathbb{V})) = \sum_{\imath\jmath\in E}(d_\imath+d_\jmath)\sqrt{d_\imath^2+d_\jmath^2}$$

2. Modified Elliptic Sombar Index:

$${}^{m}ESO(\mathfrak{F}(\mathbb{V})) = \sum_{\imath\jmath\in E}\frac{1}{(d_\imath+d_\jmath)\sqrt{d_\imath^2+d_\jmath^2}}$$

3. Multiplicative Elliptic Sombar Index:

$$ESOII(\mathfrak{F}(\mathbb{V})) = \prod_{\imath\jmath\in E}(d_\imath+d_\jmath)\sqrt{d_\imath^2+d_\jmath^2}$$

4. Multiplicative Modified Elliptic Sombar Index:

$${}^{m}ESOII(\mathfrak{F}(\mathbb{V})) = \prod_{\imath\jmath\in E}\frac{1}{(d_\imath+d_\jmath)\sqrt{d_\imath^2+d_\jmath^2}}$$

5. Multiplicative Sombor Index:

$$SO(\mathfrak{F}(\mathbb{V})) = \prod_{\imath\jmath\in E}\sqrt{d_\imath^2+d_\jmath^2}$$

6. Reduced Sombor Index:

$$SO_r(\mathfrak{F}(\mathbb{V})) = \sum_{\imath\jmath\in E}\sqrt{(d_\imath-1)^2+(d_\jmath-1)^2}$$

7.Increased Sombor Index:

$$SO^+(\mathfrak{F}(\mathbb{V})) = \sum_{\imath\jmath\in E}\sqrt{(d_\imath+1)^2+(d_\jmath+1)^2}$$

Theorem 9.1: The Elliptic Sombor Index of $\mathfrak{F}(\mathbb{V})$ is
$ESO(\mathfrak{F}(\mathbb{V})) = 2\sqrt{2}(q^n-1)(q^{n-1}-1)^3$

Proof:

$$\begin{aligned}
ESO(\mathfrak{F}(\mathbb{V})) &= \sum_{\imath\jmath\in E}(d_\imath+d_\jmath)\sqrt{d_\imath^2+d_\jmath^2} \\
&= \sum_{\imath\jmath\in E}\left[(q^{n-1}-1)+(q^{n-1}-1)\right] \\
&\quad \left[\sqrt{(q^{n-1}-1)^2+(q^{n-1}-1)^2}\right] \\
&= \sum_{\imath\jmath\in E}\left[2(q^{n-1}-1)\right]\left[\sqrt{2(q^{n-1}-1)^2}\right] \\
&= \sum_{\imath\jmath\in E}\left[2(q^{n-1}-1)\right]\left[\sqrt{2}(q^{n-1}-1)\right] \\
&= \sum_{\imath\jmath\in E}\left[2\sqrt{2}(q^{n-1}-1)^2\right] \\
&= \left(2\sqrt{2}(q^{n-1}-1)^2\right)\left((q^n-1)(q^{n-1}-1)\right) \\
&= 2\sqrt{2}(q^n-1)(q^{n-1}-1)^3
\end{aligned}$$

■

Theorem 9.2: The Modified Elliptic Sombor Index of $\mathfrak{F}(\mathbb{V})$ is
$${}^{m}ESO(\mathfrak{F}(\mathbb{V})) = \frac{(q^n-1)}{2\sqrt{2}(q^{n-1}-1)}$$

Proof:

$$\begin{aligned}
{}^{m}ESO(\mathfrak{F}(\mathbb{V})) &= \sum_{\imath\jmath\in E}\frac{1}{(d_\imath+d_\jmath)\sqrt{d_\imath^2+d_\jmath^2}} \\
&= \sum_{\imath\jmath\in E}\left((d_\imath+d_\jmath)\sqrt{d_\imath^2+d_\jmath^2}\right)^{-1} \\
&= \sum_{\imath\jmath\in E}\Big(\left[(q^{n-1}-1)+(q^{n-1}-1)\right] \\
&\quad \left[\sqrt{(q^{n-1}-1)^2+(q^{n-1}-1)^2}\right]\Big)^{-1} \\
&= \sum_{\imath\jmath\in E}\left(\left[2(q^{n-1}-1)\right]\left[\sqrt{2(q^{n-1}-1)^2}\right]\right)^{-1} \\
&= \sum_{\imath\jmath\in E}\left(\left[2(q^{n-1}-1)\right]\left[\sqrt{2}(q^{n-1}-1)\right]\right)^{-1} \\
&= \sum_{\imath\jmath\in E}\left(2\sqrt{2}(q^{n-1}-1)^2\right)^{-1} \\
&= \sum_{\imath\jmath\in E}\frac{1}{2\sqrt{2}(q^{n-1}-1)^2} \\
&= \left[\frac{1}{2\sqrt{2}(q^{n-1}-1)^2}\right](q^n-1)(q^{n-1}-1) \\
&= \frac{(q^n-1)(q^{n-1}-1)}{2\sqrt{2}(q^{n-1}-1)^2} \\
&= \frac{(q^n-1)}{2\sqrt{2}(q^{n-1}-1)}
\end{aligned}$$

■

Theorem 9.3: The Multiplicative Elliptic Sombor Index of $\mathfrak{F}(\mathbb{V})$ is
$$ESOII(\mathfrak{F}(\mathbb{V})) = \left[2\sqrt{2}(q^{n-1}-1)^2\right]^{q^{2n-1}-q^n-q^{n-1}+1}$$

Proof:

$$\begin{aligned} ESOII(\mathfrak{F}(\mathbb{V})) &= \prod_{ij\in E} (d_i + d_j)\sqrt{d_i^2 + d_j^2} \\ &= \prod_{ij\in E} \left[(q^{n-1}-1) + (q^{n-1}-1)\right] \\ &\quad \left[\sqrt{(q^{n-1}-1)^2 + (q^{n-1}-1)^2}\right] \\ &= \prod_{ij\in E} \left[2(q^{n-1}-1)\right]\left[\sqrt{2(q^{n-1}-1)^2}\right] \\ &= \prod_{ij\in E} \left[2(q^{n-1}-1)\right]\left[\sqrt{2}(q^{n-1}-1)\right] \\ &= \prod_{ij\in E} \left[2\sqrt{2}(q^{n-1}-1)^2\right] \\ &= \left[2\sqrt{2}(q^{n-1}-1)^2\right]^{(q^n-1)(q^{n-1}-1)} \\ &= \left[2\sqrt{2}(q^{n-1}-1)^2\right]^{q^{2n-1}-q^n-q^{n-1}+1} \end{aligned}$$

■

Theorem 9.4: The Multiplicative Modified Elliptic Sombor Index of $\mathfrak{F}(\mathbb{V})$ is

$$^{m}ESOII(\mathfrak{F}(\mathbb{V})) = \left[\frac{1}{2\sqrt{2}(q^{n-1}-1)^2}\right]^{q^{2n-1}-q^n-q^{n-1}+1}$$

Proof:

$$\begin{aligned} ^{m}ESOII(\mathfrak{F}(\mathbb{V})) &= \prod_{ij\in E} \frac{1}{(d_i + d_j)\sqrt{d_i^2 + d_j^2}} \\ &= \prod_{ij\in E} \left((d_i + d_j)\sqrt{d_i^2 + d_j^2}\right)^{-1} \\ &= \prod_{ij\in E} \Big(\left[(q^{n-1}-1) + (q^{n-1}-1)\right] \\ &\quad \left[\sqrt{(q^{n-1}-1)^2 + (q^{n-1}-1)^2}\right]\Big)^{-1} \\ &= \prod_{ij\in E} \Big(\left[2(q^{n-1}-1)\right] \\ &\quad \left[\sqrt{2(q^{n-1}-1)^2}\right]\Big)^{-1} \\ &= \prod_{ij\in E} \left(2\sqrt{2}(q^{n-1}-1)^2\right)^{-1} \\ &= \prod_{ij\in E} \left[\frac{1}{2\sqrt{2}(q^{n-1}-1)^2}\right] \\ &= \left[\frac{1}{2\sqrt{2}(q^{n-1}-1)^2}\right]^{(q^n-1)(q^{n-1}-1)} \\ &= \left[\frac{1}{2\sqrt{2}(q^{n-1}-1)^2}\right]^{q^{2n-1}-q^n-q^{n-1}+1} \end{aligned}$$

■

Theorem 9.5: The Multiplicative Sombor Index of $\mathfrak{F}(\mathbb{V})$ is

$$SO(\mathfrak{F}(\mathbb{V})) = \left[\sqrt{2}(q^{n-1}-1)\right]^{q^{2n-1}-q^n-q^{n-1}+1}$$

Proof:

$$\begin{aligned} SO(\mathfrak{F}(\mathbb{V})) &= \prod_{ij\in E} \sqrt{d_i^2 + d_j^2} \\ &= \prod_{ij\in E} \sqrt{(q^{n-1}-1)^2 + (q^{n-1}-1)^2} \\ &= \prod_{ij\in E} \sqrt{2(q^{n-1}-1)^2} \\ &= \prod_{ij\in E} \sqrt{2}(q^{n-1}-1) \\ &= \left[\sqrt{2}(q^{n-1}-1)\right]^{(q^n-1)(q^{n-1}-1)} \\ &= \left[\sqrt{2}(q^{n-1}-1)\right]^{q^{2n-1}-q^n-q^{n-1}+1} \end{aligned}$$

■

Theorem 9.6: The Reduced Sombor Index of $\mathfrak{F}(\mathbb{V})$ is
$SO_r(\mathfrak{F}(\mathbb{V})) = \sqrt{2}(q^{n-1}-2)(q^n-1)(q^{n-1}-1)$

Proof:

$$\begin{aligned} SO_r(\mathfrak{F}(\mathbb{V})) &= \sum_{ij\in E} \sqrt{(d_i-1)^2 + (d_j-1)^2} \\ &= \sum_{ij\in E} \sqrt{((q^{n-1}-1)-1)^2 + ((q^{n-1}-1)-1)^2} \\ &= \sum_{ij\in E} \sqrt{2(q^{n-1}-2)^2} \\ &= \sum_{ij\in E} \sqrt{2}(q^{n-1}-2) \\ &= \left(\sqrt{2}(q^{n-1}-2)\right)\left((q^n-1)(q^{n-1}-1)\right) \\ &= \sqrt{2}(q^{n-1}-2)(q^n-1)(q^{n-1}-1) \end{aligned}$$

■

Theorem 9.7: The Increased Sombor Index of $\mathfrak{F}(\mathbb{V})$ is
$SO^+(\mathfrak{F}(\mathbb{V})) = \sqrt{2}(q^{n-1})(q^n-1)(q^{n-1}-1)$

Proof:

$$\begin{aligned} SO^+(\mathfrak{F}(\mathbb{V})) &= \sum_{ij\in E} \sqrt{(d_i+1)^2 + (d_j+1)^2} \\ &= \sum_{ij\in E} \sqrt{((q^{n-1}-1)+1)^2 + ((q^{n-1}-1)+1)^2} \\ &= \sum_{ij\in E} \sqrt{2(q^{n-1})^2} \\ &= \sum_{ij\in E} \sqrt{2}(q^{n-1}) \\ &= \left(\sqrt{2}(q^{n-1})\right)\left((q^n-1)(q^{n-1}-1)\right) \\ &= \sqrt{2}(q^{n-1})(q^n-1)(q^{n-1}-1) \end{aligned}$$

■

X. Conclusion

In the present research, we explored the topological features of linear functional graphs built on finite-dimensional vector spaces. We established a fresh approach for understanding algebraic structures using graph-theoretical techniques by associating graphs with vector space elements transformed linearly. To capture fundamental structural aspects of these graphs, we computed and examined a variety of degree-based topological indices, including the Zagreb, Randic, Arithmetic-Geometric, and Revan indices, as well as their hyper and redefined variants.

These indices, which were historically utilized in mathematical chemistry to represent molecular structures, have shown promise for broader applications when extended to abstract algebraic contexts. Their application in describing connectedness, symmetry, and interaction patterns in vector spaces opens new possibilities in both theoretical and applied mathematics, including coding theory, network analysis, and cryptography.

We also examined the Elliptic Sombor Index, a recently introduced topological invariant that encodes structural information through elliptic functional transformations. Its mathematically rich formulation enhances sensitivity to degree-based variations and provides a compact numerical representation of graph structure. Originally used in chemistry, this index is now gaining relevance in areas such as graph-based learning, anomaly detection, and structural analysis within complex networks.

The technique proposed here contributes to the expanding confluence of graph theory and linear algebra by providing insights into how vector space features might be captured and investigated using graphical indices. This work extends the importance of topological indices beyond chemistry, pointing to potential applications in complex network modeling, high dimensional data analysis, and machine learning optimization challenges.

Future research might explore extensions to infinite-dimensional vector spaces, investigate non-linear mappings, and incorporate spectral indices and eigenvalue-based metrics. Additionally, further analysis of automorphism groups and dynamic features of linear functional graphs may offer deeper insights into the interplay between algebraic and topological structures.

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