# Data-Driven Maximum Likelihood Estimation for Uncertain Delay Differential Equations from Discrete Observations

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*Abstract*—**Time-delayed systems driven by the Liu process are characterized by uncertain delay differential equations (UDDEs). A critical aspect in the practical application of UDDEs is statistical inference. This study concentrates on data-driven maximum likelihood estimation (MLE) for UDDEs based on discrete observations. Initially, the derivation of the difference equation of UDDEs is outlined utilizing the implicit Euler scheme. Following this, an approximate difference equation is derived by applying the Taylor expansion to the state variable with a delay at time $t$. The likelihood function is then introduced to compute the estimators. Finally, the efficacy of the estimation technique is confirmed through numerical illustrations and empirical analysis of COVID-19 using authentic data.**

*Index Terms*—**UDDEs; MLE; Liu process; difference equation; likelihood function**

## I. Introduction

Statistical inference plays a crucial role in modeling stochastic models and has been the subject of numerous studies. For example, a numerical method was proposed by Zhang et al. ( [27]) to identify the topology and estimate line parameters without knowledge of voltage angles. Maldonado et al. ( [19]) utilized a sequential Bayesian approach to estimate parameters in stochastic dynamic load models. Zhang et al. ( [26]) focused on joint estimation of states and parameters in a specific class of nonlinear bilinear systems. Ji and Kang ( [11]) explored novel estimation techniques for real-time parameter estimation in nonlinear systems. Escobar et al. ( [8]) presented various strategies to tackle parameter estimation challenges in stochastic systems operating continuously. Ding ( [7]) investigated the characteristics of two types of least squares methods, effectively considering white and colored noise perturbations using traditional methodologies. Shin and Park ( [21]) applied a generator-regularized continuous conditional generative adversarial network for uncertain parameter estimation. Amorino et al. ( [1]) introduced a contrast function to estimate parameters in a stochastic McKean-Vlasov equation. Mehmood and Raja ( [20]) studied evolutionary heuristics of weighted differential evolution for parameter estimation in a Hammerstein-Wiener model. Brusa et al. ( [5]) demonstrated an evolutionary optimization approach to facilitate approximate maximum likelihood estimation for discrete models. In practical applications, challenges such as uncertain communication environments and population dynamics with time lag necessitate consideration of time delay in parameter estimation for stochastic delay differential equations, which has garnered increased attention in recent decades. In their work ( [3]), Berezansky and Braverman discussed the estimates of solutions for linear differential equations with delay. The weak convergence of the maximum likelihood estimator was studied by Benke and Pap ( [4]). Utilizing the method of moments, Liu and Jia ( [16]) estimated the parameters based on discrete observations of solutions. Zhu et al. ( [28]) delved into the identification of parameters in a reaction-diffusion rumor propagation system with time delay. Jamilla et al. ( [10]) implemented a genetic algorithm with multi-parent crossover to acquire parameter estimates of three neutral delay differential equation models with a discrete delay.

Stochastic differential equations might not adequately represent many time-varying systems, such as stock prices. As a result, the uncertainty theory was devised by Liu ( [14]) and further developed by Liu ( [15]) based on the concepts of normality, duality, subadditivity, and product axioms. Recent literature has explored parameter estimation for UDEs. For instance, Li et al. ( [12]) presented three techniques for parameter estimation in UDEs utilizing discrete observation data. Chen et al. ( [6]) utilized the method of moments to estimate the parameters of an uncertain SIR model and devised a numerical algorithm for their solution. Liu ( [17]) employed generalized moment estimation for obtaining the estimators. Yang et al. ( [24]) applied the $\alpha$-path approach to derive the estimators. Introducing moment estimations for unknown parameters through the Euler method approximation of high-order UDEs, Liu and Yang ( [18]) proposed a novel approach. Wei ( [22], [23]) used the contrast function to derive the least squares estimators of the uncertain Vasicek model and investigated their consistency and asymptotic distribution. Ye and Liu ( [25]) devised a method to assess the fit of an uncertain differential equation to the observed data. He et al. ( [9]) developed an algorithm for estimating parameters in a unique uncertain fractional differential equation. Li and Xia ( [13]) suggested a new method for estimating uncertain differential equations using estimating function technique based on uncertain integrals.

Despite recent advancements in parameter estimation for UDEs, few studies have taken into account the time lag factor. Additionally, existing literature has mainly used the explicit difference method to derive the difference equation, which is known to be numerically unstable. With these shortcomings in mind, our study focuses on data-driven MLE for UDDEs using implicit Euler scheme with discrete observations. By employing the implicit Euler scheme, we derive the difference equation for UDDEs. We then approximate the difference equation by expanding the Taylor series of the state variable with delay at time $t$, and develop the

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likelihood function to estimate the parameters. To validate our estimation method, we conduct numerical experiments and empirical analysis on real data related to COVID-19. The structure of this paper is as follows: Section 2 provides definitions for uncertain variables and Liu process. Section 3 introduces the UDDEs addressed in this study, along with the equation used for parameter estimation. We also present numerical examples and empirical analysis on COVID-19 based on real data to demonstrate the effectiveness of our approach.

## II. Problem Formulation and Preliminaries

Firstly, we give some definitions about uncertain variables and Liu process.

*Definition 1:* ( [14], [15]) Let $\mathcal{L}$ be a $\sigma$-algebra on a nonempty set $\Gamma$. A set function $\mathcal{M}:\mathcal{L}\to[0,1]$ is called an uncertain measure if it satisfies the following axioms:

Axiom 1: (Normality Axiom) $\mathcal{M}(\Gamma)=1$ for the universal set $\Gamma$.

Axiom 2: (Duality Axiom) $\mathcal{M}(\Lambda)+\mathcal{M}(\Lambda^c)=1$ for any event $\Lambda$.

Axiom 3: (Subadditivity Axiom) For every countable sequence of events $\Lambda_1$, $\Lambda_2$, $\cdots$,

$$\mathcal{M}\{\bigcup_{i=1}^{\infty}\Lambda_i\}\le\sum_{i=1}^{\infty}\mathcal{M}\{\Lambda_i\}.$$

Axiom 4: (Product Axiom) Let $(\Gamma_k,\mathcal{L}_k,\mathcal{M}_k)$ be uncertainty spaces for $k=1,2,\cdots$. Then the product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\{\Pi_{k=1}^{\infty}\Lambda_k\}=\min_{k\ge1}\mathcal{M}_k\{\Lambda_k\},$$

where $\Lambda_k$ are arbitrarily chosen events from $\mathcal{L}_k$ for $k=1,2,\cdots$.

An uncertain variable $\xi$ is a measurable function from the uncertainty space $(\Gamma,\mathcal{L},\mathcal{M})$ to the set of real numbers.

*Definition 2:* ( [14]) For any real number $x$, let $\xi$ be an uncertain variable and its uncertainty distribution is defined by

$$\Phi(x)=\mathcal{M}(\xi\le x).$$

In particular, an uncertain variable $\xi$ is called normal if it has an uncertainty distribution

$$\Phi(x)=(1+\exp(\frac{\pi(\mu-x)}{\sqrt{3}\sigma}))^{-1},x\in\Re,$$

denoted by $\mathcal{N}(\mu,\sigma)$. If $\mu=0$, $\sigma=1$, $\xi$ is called a standard normal uncertain variable.

*Definition 3:* ( [15]) An uncertain process $C_t$ is called a Liu process if

(i) $C_0=0$ and almost all sample paths are Lipschitz continuous, (ii) $C_t$ has stationary and independent increments, (iii) the increment $C_{s+t}-C_s$ has a normal uncertainty distribution

$$\Phi_t(x)=(1+\exp(\frac{-\pi x}{\sqrt{3}t}))^{-1},x\in\Re.$$

*Definition 4:* ( [2]) Suppose that $C_t$ is a Liu process, $h$ and $w$ are two measurable real functions, $\tau$ stands for a nonnegative time delay. Then

$$dX_t=h(t,X_t,X_{t-\tau})dt+w(t,X_t,X_{t-\tau})dC_t \quad (1)$$

is called an uncertain delay differential equation.

Moreover, a real-valued function $X_t^\alpha$ is called the $\alpha$-path of above uncertain differential equation if it solves the corresponding ordinary differential equation

$$dX_t^\alpha=h(t,X_t^\alpha,X_{t-\tau}^\alpha)dt+|w(t,X_t^\alpha,X_{t-\tau}^\alpha)|\Phi^{-1}(\alpha)dt,$$

where

$$\Phi^{-1}(\alpha)=\frac{\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha},\quad \alpha\in(0,1).$$

*Remark 1:* The uncertain delay differential equation (1) has a unique solution if the coefficients $h(t,x,y)$ and $w(t,x,y)$ satisfy the following conditions

$$|h(t,x,y)|+|w(t,x,y)|\le L(1+|x|+|y|),$$

$$|h(t,x,y)-h(t,x_1,y_1)|+|w(t,x,y)-w(t,x_1,y_1)|$$
$$\le L(|x-x_1|+|y-y_1|).$$

## III. Main Results and Proofs

The UDDEs considered in this paper is described as follows:

$$dX_t=h(t,X_t,X_{t-\tau},\theta)dt+w(t,X_t,\beta)dC_t, \quad (2)$$

where $\theta$, $\beta$ and $\tau$ are an unknown parameters, $C_t$ is a Liu process and $\tau$ is a delay time.

By applying the implicit Euler scheme, the Eq. (2) has the following difference form

$$\begin{aligned}&X_{t_{i+1}}-X_{t_i}\\&=h(t_{i+1},X_{t_{i+1}},X_{t_{i+1}-\tau},\theta)(t_{i+1}-t_i)\\&+w(t_{i+1},X_{t_{i+1}},\beta)(C_{t_{i+1}}-C_{t_i}).\end{aligned} \quad (3)$$

By using Taylor expansion of $X_{t-\tau}$ at time $t$, we get the approximation difference equation

$$\begin{aligned}&X_{t_{i+1}}-X_{t_i}\\&=h(t_{i+1},X_{t_{i+1}},X_{t_{i+1}}-\tau\frac{X_{t_{i+1}}-X_{t_i}}{t_{i+1}-t_i},\theta)(t_{i+1}-t_i)\\&+w(t_{i+1},X_{t_{i+1}},\beta)(C_{t_{i+1}}-C_{t_i}).\end{aligned} \quad (4)$$

Then, we can get

$$\begin{aligned}&\frac{X_{t_{i+1}}-X_{t_i}}{w(t_{i+1},X_{t_{i+1}},\beta)(t_{i+1}-t_i)}\\&-\frac{h(t_{i+1},X_{t_{i+1}},X_{t_{i+1}}-\tau\frac{X_{t_{i+1}}-X_{t_i}}{t_{i+1}-t_i},\theta)(t_{i+1}-t_i)}{w(t_{i+1},X_{t_{i+1}},\beta)(t_{i+1}-t_i)}\\&=\frac{C_{t_{i+1}}-C_{t_i}}{t_{i+1}-t_i}.\end{aligned}$$

According to definition 3, we obtain that

$$\frac{C_{t_{i+1}}-C_{t_i}}{t_{i+1}-t_i}\sim\mathcal{N}(0,1). \quad (5)$$

Hence, we have

$$\begin{aligned}&\frac{X_{t_{i+1}}-X_{t_i}}{w(t_{i+1},X_{t_{i+1}},\beta)(t_{i+1}-t_i)}\\&-\frac{h(t_{i+1},X_{t_{i+1}},X_{t_{i+1}}-\tau\frac{X_{t_{i+1}}-X_{t_i}}{t_{i+1}-t_i},\theta)(t_{i+1}-t_i)}{w(t_{i+1},X_{t_{i+1}},\beta)(t_{i+1}-t_i)}\\&\sim\mathcal{N}(0,1).\end{aligned}$$

Given the observed data $(t_i, x_{t_i})$, $i = 1, 2, \cdots, n$ in which $t_{i+1} - t_i = \tau$, we can get the parameter function

$$\begin{aligned} & f_i(\theta, \beta, \tau) \\ & = \frac{x_{t_{i+1}} - x_{t_i}}{w(t_{i+1}, x_{t_{i+1}}, \beta)(t_{i+1} - t_i)} \\ & - \frac{h(t_{i+1}, x_{t_{i+1}}, x_{t_{i+1}} - \tau\frac{x_{t_{i+1}} - x_{t_i}}{t_{i+1} - t_i}, \theta)(t_{i+1} - t_i)}{w(t_{i+1}, x_{t_{i+1}}, \beta)(t_{i+1} - t_i)}. \end{aligned}$$

Since the standard normal uncertainty distribution is

$$\Phi(x) = (1 + \exp(\frac{-\pi x}{\sqrt{3}}))^{-1}, \tag{6}$$

we have

$$\Phi^{'}(x) = \frac{\frac{\pi}{\sqrt{3}}\exp(\frac{-\pi x}{\sqrt{3}})}{(1 + \exp(\frac{-\pi x}{\sqrt{3}}))^2}. \tag{7}$$

Then, we can obtain the following likelihood function:

$$\begin{aligned} & L(\theta, \beta, \tau | f_1, f_2, \cdots, f_{n-1}) \\ & = \bigwedge_{i=1}^{n-1} \Phi^{'}(f_i(\theta, \beta, \tau)) \\ & = \bigwedge_{i=1}^{n-1} \frac{\frac{\pi}{\sqrt{3}}\exp(\frac{-\pi f_i(\theta,\beta,\tau)}{\sqrt{3}})}{(1 + \exp(\frac{-\pi f_i(\theta,\beta,\tau)}{\sqrt{3}}))^2}. \end{aligned} \tag{8}$$

It is easy to check that $\Phi^{'}(f_i(\theta, \beta, \tau))$ decreases when $|f_i(\theta, \beta, \tau)|$ decreases. Hence, the likelihood function can be rewritten as follows:

$$\begin{aligned} & L(\theta, \beta, \tau | f_1, f_2, \cdots, f_{n-1}) \\ & = \bigwedge_{i=1}^{n-1} \Phi^{'}(f_i(\theta, \beta, \tau)) \\ & = \frac{\frac{\pi}{\sqrt{3}}\exp(\frac{-\pi}{\sqrt{3}}\bigwedge_{i=1}^{n-1}|f_i(\theta, \beta, \tau)|)}{(1 + \exp(\frac{-\pi}{\sqrt{3}}\bigwedge_{i=1}^{n-1}|f_i(\theta, \beta, \tau)|))^2}. \end{aligned} \tag{9}$$

Therefore, we can get the maximum likelihood estimators by solving the equation

$$\min_{\theta,\beta,\tau} \bigwedge_{i=1}^{n-1} |f_i(\theta, \beta, \tau)|. \tag{10}$$

Moreover, the estimators of $\theta$ and $\tau$ can be obtained through the equation

$$\begin{aligned} & \min_{\theta,\tau} \bigwedge_{i=1}^{n-1} (x_{t_{i+1}} - x_{t_i} - h(t_{i+1}, x_{t_{i+1}}, x_{t_{i+1}} \\ & -\tau\frac{x_{t_{i+1}} - x_{t_i}}{t_{i+1} - t_i}, \theta)(t_{i+1} - t_i))^2, \end{aligned}$$

and the estimator of $\beta$ can be derived by the equation

$$\begin{aligned} & \bigwedge_{i=1}^{n-1} w^2(t_{i+1}, x_{t_{i+1}}, \beta)(t_{i+1} - t_i)^2 \\ & = \bigwedge_{i=1}^{n-1} (x_{t_{i+1}} - x_{t_i} \\ & -h(t_{i+1}, x_{t_{i+1}}, x_{t_{i+1}} - \widehat{\tau}\frac{x_{t_{i+1}} - x_{t_i}}{t_{i+1} - t_i}, \widehat{\theta})(t_{i+1} - t_i))^2, \end{aligned}$$

where $\widehat{\tau}$ and $\widehat{\theta}$ are estimators of $\tau$ and $\theta$.

## IV. EXAMPLE

**Example 1**: Consider the following uncertain delay differential equation:

$$dX_t = X_{t-\tau}dt + \beta dC_t,$$

where $\tau$ and $\beta$ are an unknown parameters. Given the observed data $(t_i, x_{t_i})$, $i = 1, 2, \cdots, n$ in which $t_{i+1} - t_i = 1$. By solving the equation

$$\min_{\tau} \bigwedge_{i=1}^{n-1} (x_{t_{i+1}} - x_{t_i} - (x_{t_{i+1}} - \tau\frac{x_{t_{i+1}} - x_{t_i}}{t_{i+1} - t_i})(t_{i+1} - t_i))^2, \tag{11}$$

we obtain the estimator of $\tau$

$$\widehat{\tau} = \frac{\bigwedge_{i=1}^{n-1} x_{t_{i+1}}(x_{t_{i+1}} - x_{t_i})}{\bigwedge_{i=1}^{n-1}(x_{t_{i+1}} - x_{t_i})^2} - 1.$$

Then, according to Eq. (15), we can get the estimator of $\beta$

$$\widehat{\beta} = \sqrt{\bigwedge_{i=1}^{n-1}(x_{t_{i+1}} - x_{t_i} - (x_{t_{i+1}} - \widehat{\tau}(x_{t_{i+1}} - x_{t_i})))^2}.$$

Assume that we have 20 groups of observed data as shown in Table 1. Then, we derive the maximum likelihood estimators

$$\widehat{\tau} = 2.9196, \quad \widehat{\beta} = 1.6138.$$

TABLE I
OBSERVATIONS OF UNCERTAIN DELAY DIFFERENTIAL EQUATION

| n | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $t_i$ | 1.00 | 2.00 | 3.00 | 4.00 | 5.00 | 6.00 | 7.00 | 8.00 | 9.00 | 10.00 |
| $X_{t_i}$ | 0.75 | 2.20 | 4.54 | 8.19 | 13.28 | 6.17 | 3.92 | 11.26 | 5.34 | 15.37 |
| n | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| $t_i$ | 11.00 | 12.00 | 13.00 | 14.00 | 15.00 | 16.00 | 17.00 | 18.00 | 19.00 | 20.00 |
| $X_{t_i}$ | 10.43 | 18.12 | 11.10 | 7.63 | 14.12 | 10.06 | 20.31 | 11.16 | 8.52 | 17.38 |

Thus, the uncertain delay differential equation could be written as

$$dX_t = X_{t-2.9196}dt + 1.6138dC_t.$$

Hence, the $\gamma$-path $X_t^\gamma$ $(0 < \gamma < 1)$ is the solution of following ordinary differential equation

$$dX_t^\gamma = X_{t-2.9196}^\gamma dt + 1.6138\frac{\sqrt{3}}{\pi}\ln\frac{\gamma}{1-\gamma}dt.$$

According to Figure 1, all observations fall into the area between 0.05-path $X_t^{0.05}$ and 0.93-path $X_t^{0.93}$. Therefore, the methods used in this paper are reasonable.

**Example 2**: Consider the following uncertain delay differential equation:

$$dX_t = \theta X_{t-\tau}dt + \beta dC_t,$$

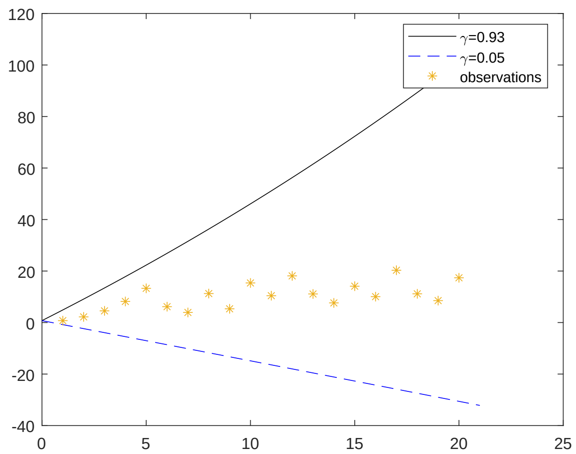


Fig. 1. Observations and $\gamma$-path of $X_t$

where $\theta$, $\tau$ and $\beta$ are unknown parameters. Given the observed data $(t_i, x_{t_i})$, $i = 1, 2, \cdots, n$ in which $t_{i+1} - t_i = 0.5$. By solving the equation

$$\min_{\tau} \bigwedge_{i=1}^{n-1} (x_{t_{i+1}} - x_{t_i} - \theta(x_{t_{i+1}} - \tau \frac{x_{t_{i+1}} - x_{t_i}}{t_{i+1} - t_i})(t_{i+1} - t_i))^2, \tag{12}$$

we obtain the estimators of $\theta$ and $\tau$

$$\widehat{\theta} = \frac{2 \bigwedge_{i=1}^{n-1} x_{t_{i+1}} (x_{t_{i+1}} - x_{t_i})^2}{\bigwedge_{i=1}^{n-1} x_{t_{i+1}}^2 (x_{t_{i+1}} - x_{t_i})},$$

$$\widehat{\tau} = \frac{1}{\widehat{\theta}} - \frac{\bigwedge_{i=1}^{n-1} x_{t_{i+1}}^2}{2 \bigwedge_{i=1}^{n-1} x_{t_{i+1}} (x_{t_{i+1}} - x_{t_i})}.$$

Then, we can get the estimator of $\beta$

$$\widehat{\beta} = 2 \sqrt{\frac{\bigwedge_{i=1}^{n-1} (x_{t_{i+1}} - x_{t_i} - \widehat{\theta}\widehat{\tau}(x_{t_{i+1}} - x_{t_i}))^2}{\bigwedge_{i=1}^{n-1} x_{t_{i+1}}^2}}.$$

Assume that we have 20 groups of observed data as shown in Table 2. Then, we derive the least squares estimators

$$\widehat{\theta} = 1.0159, \quad \widehat{\tau} = 0.9681, \quad \widehat{\beta} = 0.0450.$$

TABLE II
OBSERVATIONS OF UNCERTAIN DELAY DIFFERENTIAL EQUATION

| n | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $t_i$ | 0.50 | 1.00 | 1.50 | 2.00 | 2.50 | 3.00 | 3.50 | 4.00 | 4.50 | 5.00 |
| $X_{t_i}$ | 5.16 | 2.39 | 7.45 | 3.28 | 8.41 | 10.25 | 3.81 | 12.68 | 9.15 | 13.27 |
| n | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| $t_i$ | 5.50 | 6.00 | 6.50 | 7.00 | 7.50 | 8.00 | 8.50 | 9.00 | 9.50 | 10.00 |
| $X_{t_i}$ | 15.93 | 10.18 | 18.07 | 8.53 | 20.31 | 11.64 | 14.35 | 16.85 | 12.38 | 23.68 |

Thus, the uncertain delay differential equation could be written as

$$dX_t = 1.0159 X_{t-0.9681} dt + 0.0450 X_t dC_t.$$

Hence, the $\gamma$-path $X_t^\gamma$ $(0 < \gamma < 1)$ is the solution of following ordinary differential equation

$$dX_t^\gamma = 1.0159 X_{t-0.9681}^\gamma dt + 0.0450 X_t^\gamma \frac{\sqrt{3}}{\pi} \ln \frac{\gamma}{1-\gamma} dt.$$

According to Figure 2, all observations fall into the area between 0.05-path $X_t^{0.05}$ and 0.95-path $X_t^{0.95}$. Therefore, the methods used in this paper are reasonable.

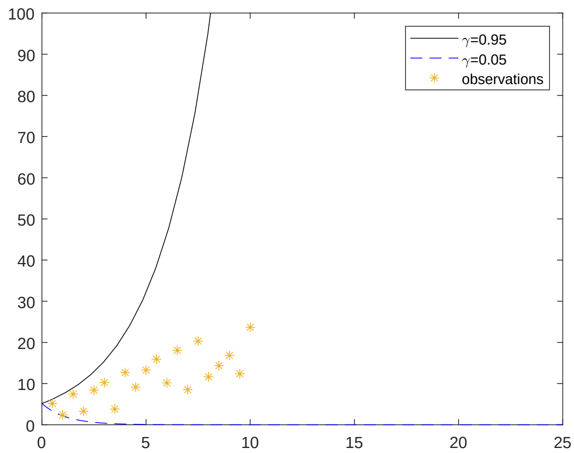


Fig. 2. Observations and $\gamma$-path of $X_t$

**Example 3**: It is known that COVID-19 spread model can be described by the following uncertain differential equation

$$dX_t = \theta X_{t-\tau} dt + \beta X_t dC_t,$$

where $\theta$, $\tau$ and $\beta$ are an unknown parameters.

By solving the equation

$$\min_{\tau} \bigwedge_{i=1}^{n-1} (x_{t_{i+1}} - x_{t_i} - \theta(x_{t_{i+1}} - \tau \frac{x_{t_{i+1}} - x_{t_i}}{t_{i+1} - t_i})(t_{i+1} - t_i))^2, \tag{13}$$

we obtain the estimators of $\theta$ and $\tau$

$$\widehat{\theta} = \frac{\bigwedge_{i=1}^{n-1} x_{t_{i+1}} (x_{t_{i+1}} - x_{t_i})^2}{\bigwedge_{i=1}^{n-1} x_{t_{i+1}}^2 (x_{t_{i+1}} - x_{t_i})},$$

$$\widehat{\tau} = \frac{1}{\widehat{\theta}} - \frac{\bigwedge_{i=1}^{n-1} x_{t_{i+1}}^2}{\bigwedge_{i=1}^{n-1} x_{t_{i+1}} (x_{t_{i+1}} - x_{t_i})}.$$

Then, we can get the estimator of $\beta$

$$\widehat{\beta} = 2 \sqrt{\frac{\bigwedge_{i=1}^{n-1} (x_{t_{i+1}} - x_{t_i} - \widehat{\theta}\widehat{\tau}(x_{t_{i+1}} - x_{t_i}))^2}{\bigwedge_{i=1}^{n-1} x_{t_{i+1}}^2}}.$$

Table 3 shows the real data about confirmed cases of COVID-19 from 02/27/2023 to 03/28/2023, Then, we derive the least squares estimators

$$\widehat{\theta} = 0.011, \quad \widehat{\tau} = 0.899, \quad \widehat{\beta} = 0.056.$$

Thus, the uncertain delay differential equation could be written as

$$dX_t = 0.011X_{t-0.899}dt + 0.056X_tdC_t,$$

Hence, the $\gamma$-path $X_t^\gamma$ $(0 < \gamma < 1)$ is the solution of following ordinary differential equation

$$dX_t^\gamma = 0.011X_{t-0.899}^\gamma)dt + 0.056X_t^\gamma \frac{\sqrt{3}}{\pi} \ln \frac{\gamma}{1-\gamma} dt.$$

TABLE III
OBSERVATIONS OF UNCERTAIN DELAY DIFFERENTIAL EQUATION

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| n | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| $t_i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| $X_{t_i}$ | 9003 | 9007 | 9008 | 9013 | 9020 | 9022 | 9023 | 9025 | 9029 | 9039 |
| n | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| $t_i$ | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| $X_{t_i}$ | 9045 | 9048 | 9050 | 9052 | 9053 | 9054 | 9063 | 9065 | 9067 | 9069 |
| n | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| $t_i$ | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| $X_{t_i}$ | 9072 | 9077 | 9080 | 9082 | 9091 | 9093 | 9095 | 9098 | 9104 | 9105 |

According to Figure 3, all observations fall into the area between 0.12-path $X_t^{0.12}$ and 0.94-path $X_t^{0.94}$. Therefore, the methods used in this paper are reasonable.

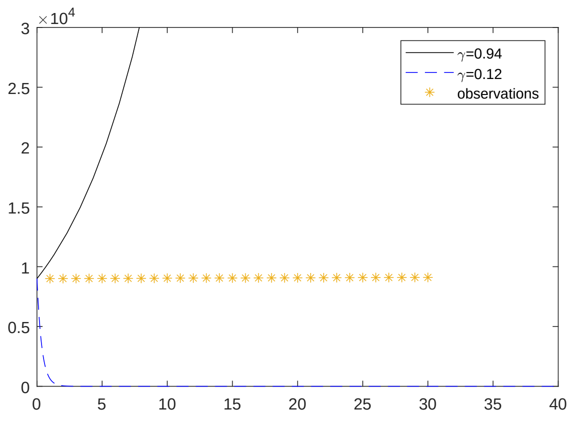


Fig. 3. Observations and $\gamma$-path of $X_t$

## V. CONCLUSION

This study focuses on the issue of data-driven MLE for UDDEs utilizing the implicit Euler scheme from discrete observations. The difference equation of UDDEs has been determined through the application of the implicit Euler scheme, along with the provision of the likelihood function. Estimators for both the drift and diffusion items have been derived. Additionally, numerical examples and empirical analysis on COVID-19 using authentic data from 02/27/2023 to 03/28/2023 have been presented to validate the methodology. Future research will explore estimation techniques for partially observed UDDEs.

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