Stability of a Class of Neural networks

Jiemin Zhao

Abstract—**We give a concise result of global uniform asymptotic stability for the Hopfield model**

$$\begin{cases} \dfrac{du_1(t)}{dt} = -u_1(t) + a\int_0^{u_2(t)} e^{-s^2}\, ds, \\ \dfrac{du_2(t)}{dt} = a\int_0^{u_1(t)} e^{-s^2}\, ds - u_2(t) \end{cases}$$

by means of the method of Liapunov function.

Index Terms—**dynamics, neural network, Liapunov function, stability.**

I. INTRODUCTION

The Hopfield model is well known in neural network. It may be described by the following system of ordinary differential equations

$$C_i \frac{du_i(t)}{dt} = \sum_j T_{ij} V_i - \frac{u_i(t)}{R_i} + I_i$$
$$u_i = g_i^{-1}(V_i) \quad (i = 1, 2, \cdots, n),$$

where $T_{ij}V_i$ represents the electrical current input to cell i due to the present potential of cell j, and T_{ij} is thus the synapse efficacy. Linear summing of inputs is assumed. I_i is any other (fixed) input current to neuron i [1].

Hopfield [1] investigate a two-neuron system. The system parameters are

$$T_{12} = T_{21} = 1, \ \lambda = 1.4 \ \text{and} \ g(u) = \frac{2}{\pi}\tan^{-1}\frac{\pi\lambda}{2}u .$$

Energy contours are

$$0.449, \ 0.156, \ 0.017,$$
$$-0.003, \ -0.023, \ \text{and} \ -0.041 .$$

For a number of neuron models and their stability analysis, we do not enumerate them. This paper presents a concise result of global uniform asymptotic stability for the Hopfield model

$$\begin{cases} \dfrac{du_1(t)}{dt} = -u_1(t) + a\int_0^{u_2(t)} e^{-s^2}\, ds, \\ \dfrac{du_2(t)}{dt} = a\int_0^{u_1(t)} e^{-s^2}\, ds - u_2(t) \end{cases} \tag{1}$$

where $a = const. \in (0, 1)$, Such a concise result is useful to Hopfield model based neural network design in practical applications [1–13].

Manuscript received January 7, 2008.
This work was supported by the China National Science Foundation under Grant 40474033.
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II. ANALYSIS AND COMPUTING

Let

$$f(u_1) = a\int_0^{u_1} e^{-s^2}\, ds.$$

Therefore,

$$f(u_1) = -f(-u_1),$$

$$\frac{df(u_1)}{du_1} = ae^{-u_1^2} > 0 \ \text{ for } u_1 \ge 0,$$

$$\frac{d^2 f(u_1)}{du_1^2} = -2au_1e^{-u_1^2} < 0 \ \text{ for } \ u_1 > 0,$$

$$\lim_{u_1 \to +\infty} f(u_1) = \frac{\sqrt{\pi}}{2}.$$

The function $u_2 = f(u_1)$ is illustrated in Fig.1.

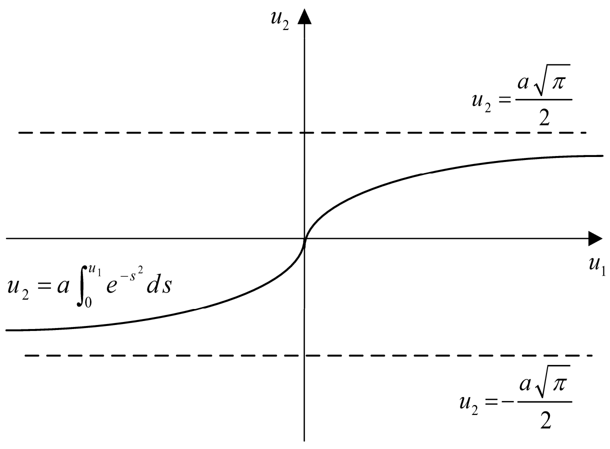

Fig.1

From analysis and computing above, we construct Liapunov function

$$V(u_1,u_2)=\frac{1}{2}(u_1+u_2)^2+\int_0^{u_1}[u-a\int_0^u e^{-s^2}ds]du+\int_0^{u_2}[u-a\int_0^u e^{-s^2}ds]du.$$

Thus, if

$$(u_1(t),u_2(t))=(u_1(t,t_0,u_{1_0}),u_2(t,t_0,u_{2_0}))$$

is a solution of system (1), then the derivative $\left.\frac{dV}{dt}\right|_{(1)}$ of V along $(u_1(t),u_2(t))$ satisfies

$$\left.\frac{dV}{dt}\right|_{(1)}=(u_1(t)+u_2(t))(\frac{du_1(t)}{dt}+\frac{du_2(t)}{dt})+[u_1(t)-a\int_0^{u_1(t)}e^{-s^2}ds]\frac{du_1(t)}{dt}+[u_2(t)-a\int_0^{u_2(t)}e^{-s^2}ds]\frac{du_2(t)}{dt}.$$

Using (1), we obtain

$$\begin{aligned}\left.\frac{dV}{dt}\right|_{(1)}&=(u_1(t)+u_2(t))[-u_1(t)+a\int_0^{u_2(t)}e^{-s^2}ds+a\int_0^{u_1(t)}e^{-s^2}ds-u_2(t)]+\\&[u_1(t)-a\int_0^{u_1(t)}e^{-s^2}ds]\times[-u_1(t)+a\int_0^{u_2(t)}e^{-s^2}ds]+\\&[u_2(t)-a\int_0^{u_2(t)}e^{-s^2}ds]\ [a\int_0^{u_1(t)}e^{-s^2}ds-u_2(t)]\\&=-u_1^2(t)-u_1(t)u_2(t)+au_1(t)\int_0^{u_2(t)}e^{-s^2}ds+\\&au_2(t)\int_0^{u_2(t)}e^{-s^2}ds+au_1(t)\int_0^{u_1(t)}e^{-s^2}ds+\\&au_2(t)\int_0^{u_1(t)}e^{-s^2}ds-u_1(t)u_2(t)-u_2^2(t)-\\&u_1^2(t)+au_1(t)\int_0^{u_1(t)}e^{-s^2}ds+\\&au_1(t)\int_0^{u_2(t)}e^{-s^2}ds-\\&a^2(\int_0^{u_1(t)}e^{-s^2}ds)(\int_0^{u_2(t)}e^{-s^2}ds)+\\&au_2(t)\int_0^{u_1(t)}e^{-s^2}ds-\\&a^2(\int_0^{u_1(t)}e^{-s^2}ds)(\int_0^{u_2(t)}e^{-s^2}ds)-\\&u_2^2(t)+au_2(t)\int_0^{u_2(t)}e^{-s^2}ds\\&=2[-u_1^2(t)-u_2^2(t)-u_1(t)u_2(t)+\\&au_1(t)\int_0^{u_2(t)}e^{-s^2}ds+\\&au_2(t)\int_0^{u_1(t)}e^{-s^2}ds+\\&au_2(t)\int_0^{u_2(t)}e^{-s^2}ds+\\&au_1(t)\int_0^{u_1(t)}e^{-s^2}ds-\\&a^2(\int_0^{u_1(t)}e^{-s^2}ds)(\int_0^{u_2(t)}e^{-s^2}ds)-\\&=-2\{u_1(t)[u_1(t)-a\int_0^{u_1(t)}e^{-s^2}ds]+\\&u_2(t)[u_2(t)-a\int_0^{u_2(t)}e^{-s^2}ds]+\\&[u_1(t)-a\int_0^{u_1(t)}e^{-s^2}ds]\times\\&[u_2(t)-a\int_0^{u_2(t)}e^{-s^2}ds]\}.\end{aligned}$$

Let

$$F(v_1)=v_1-a\int_0^{v_1}e^{-s^2}ds.$$

Therefore,

$$F(v_1)=-F(-v_1),$$

$$\frac{dF(v_1)}{dv_1}=1-ae^{-v_1^2}>0 \quad \text{for } v_1\geq 0,$$

$$\frac{d^2F(v_1)}{dv_1^2}=2av_1e^{-v_1^2}>0 \quad \text{for } v_1>0,$$

$$\lim_{v_1\to+\infty}F(v_1)=+\infty.$$

The function $v_2=F(v_1)$ is illustrated in Fig.2.

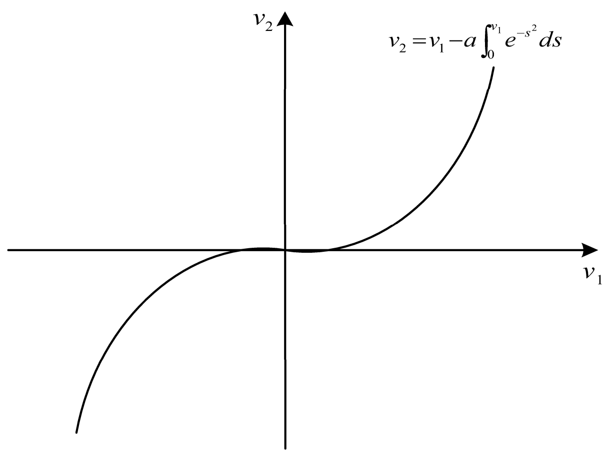

Fig.2

From analysis and computing above, we have

$$\left.\frac{dV}{dt}\right|_{(1)}=-2\{u_1(t)[u_1(t)-a\int_0^{u_1(t)}e^{-s^2}ds]+u_2(t)[u_2(t)-a\int_0^{u_2(t)}e^{-s^2}ds]+$$

$$[u_1(t) - a\int_0^{u_1(t)} e^{-s^2} ds][u_2(t) - a\int_0^{u_2(t)} e^{-s^2} ds]\}$$
$$\leq 0,$$

furthermore, the system (1) is globally uniformly asymptotically stable [2–13].

III. Main Result

From analysis and computing above, we have result as follow:

The system (1) is globally uniformly asymptotically stable.

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ISBN: 978-988-98671-8-8