Direction of Arrival Estimation Using Polynomial Roots Intersection for Multi-Dimensional Estimation (PRIME)

H. K. Hwang, Zekeriya Aliyazicioglu, Marshall Grice, Anatoly Yakovlev, Peter Lu

***Abstract*— An array antenna system with innovative signal processing can enhance the resolution of a signal direction of arrival (DOA) estimation. Super resolution algorithms take advantage of array antenna structures to better process the incoming signals. They also have the ability to identify multiple targets. This paper explores the Polynomial Roots Intersection for Multi-dimensional Estimation (PRIME) algorithms.**

The PRIME algorithm allows a polynomial rooting approach to estimate joint azimuth/elevation parameters of the signal detection for planer arrays. This method calculates a finite set of root intersections, which are the simultaneous solutions from multiple independent multivariate polynomials. The solutions for the source angle information are included in the simultaneous solutions. The PRIME algorithm does not require the use of a scan vector to scan all possible directions. The results demonstrate improvement in both resolution and computational efficiency with no loss of accuracy.

Statistical analysis of the performance of the processing algorithm and processing resource requirements are discussed in this paper. Extensive computer simulations are used to show the performance of the algorithms.

Index Terms— **DOA estimation, antenna arrays, array signal processing.**

I. Sensor Array System

A. Sensor Output Signal Model

An array sensor system has multiple sensors distributed in space. This array configuration provides spatial sampling of the received waveform. A sensor array has better performance than the single sensor in signal reception and parameter estimation. Its superior spatial resolution provides a means to estimate the DOA of multiple signals. A sensor array also has applications in interference rejection [1], electronic steering [2], multiple beam forming [3], etc. This technology is now widely used in communication, radar, sonar, seismology, radio astronomy, etc.

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Consider a general array of M sensors as shown in Figure 1. The coordinate of the i^{th} sensor is $\mathbf{r}_i = [x_i, y_i, z_i]^T$, $i = 0, 1, \dots, M-1$

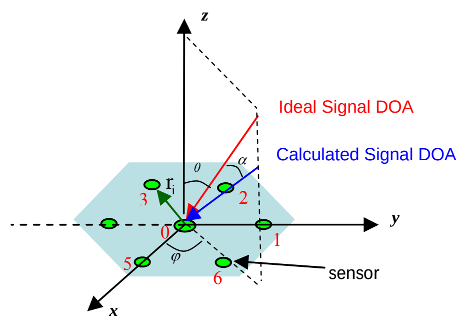

Figure 1 An Array of M Sensors

Suppose a plane target signal waveform comes from the direction of $\mathbf{k} = [\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta]^T$, where θ is the elevation angle and φ is the azimuth angle. The difference of the propagation path of this wave between the origin and the i^{th} sensor Δd_i is

$$\Delta d_i = \mathbf{r}_i^T \mathbf{k} = \sin\theta(x_i\cos\varphi + y_i\sin\varphi) + z_i\cos\theta \quad (1)$$

where $\mathbf{r}_i = [x_i, y_i, z_i]^T$ is the coordinate vector of the i^{th} element. The corresponding propagation time delay τ_i is

$$\tau_i = \Delta d_i / c \quad (2)$$

where c is the speed of light.

If the bandwidth of signals is sufficiently narrow, then the data picked up by different sensor elements are related by a pure phase factor. The relative phase shift of the i^{th} sensor with respect to the reference sensor at the origin is

$$\beta_i = \frac{2\pi}{\lambda}\mathbf{r}_i^T\mathbf{k} = -\frac{2\pi}{\lambda}\sin\theta\left(x_i\cos\varphi + y_i\sin\varphi\right) \quad (3)$$

To avoid the effect of grating lobes, the distance between the two neighbor sensors has to be no more than one half of the wavelength. If the reference sensor is located at the

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origin, and a waveform received by the reference sensor due to signal coming from direction of **k** is x(t), then the received waveform at i^{th} sensor is $x_i(t) = x(t-\tau_i)$.

For the sensor array with M elements, we can define the array input vector **x**(t) and the array weighting vector **w** as

$$\mathbf{x}(t) = [x_0(t), x_1(t), \ldots, x_{M-1}(t)]^T \tag{4}$$

$$\mathbf{w} = [w_0, w_1, \ldots, w_{M-1}]^T \tag{5}$$

where $x_i(t)$ is the data input to the i^{th} sensor and w_i^* is the weight of the i^{th} sensor. The sensor array output y(t) is

$$y(t) = \mathbf{w}^H\mathbf{x}(t) \tag{6}$$

where the superscript represents the complex conjugate transpose (Hermitian) of the matrix.

Suppose there are L independent signal sources impinging the antenna and we want to use a sensor array system to identify their directions of arrival (DOA). The input signal to each individual sensor is the combination of L independent signals. Every sensor in the array also receives random environmental ambient noise. This noise is modeled as Additive White Gaussian Noise (AWGN). The input waveform of the i^{th} sensor element $x_i(t)$ is given by

$$x_i(t) = \sum_{k=1}^{L} s_{k,i}(t) + n_i(t), \quad i = 0,1, \ldots, M-1 \tag{7}$$

where $s_{k,i}(t) = s_{k,o}(t-\tau_{k,i})$, and $s_{k,o}(t)$ denote the k^{th} signal picked up by the sensor at the origin. $n_i(t)$ is the noise at i^{th} sensor, $\tau_{k,i}$ is the relative delay of k^{th} signal at the i^{th} sensor.

For the narrowband input signals, signal $s_{k,i}(t)$ is related to the signal $s_{k,o}(t)$ by a phase shift factor of $\beta_{k,i}$. If the input signals have a wide bandwidth, the delay time of the signal at the i^{th} sensor from reference signal at the origin may not be an integer multiple of the sampling time; additional interpolation filtering is required to emulate their delay. The weighted sum of samples of all sensors forms the array output. To estimate the DOA of wideband signals, each single weight is replaced by a tape delay lines filter. Such a processor is referred to as the Space Time Adaptive Processor (STAP) [4]. In this report, only the narrowband signal is considered.

DOA estimation of conventional eigen-analysis methods uses a scan vector to scan over all possible directions and signal directions corresponding to the peak of the power spectrum. A more efficient method of estimating the DOA of signal is based on the root algorithm. A signal's DOA can be determined by finding the roots of the characteristic polynomial whose amplitudes are closest to unity.

The two dimensional array antenna studied in this report consists of 7 elements arranged in a honeycomb configuration. The geometric configuration of this array antenna is shown in Figure 2. Antenna elements are assumed to be placed on x-y plane and inter-element spacing d equals a half wavelength.

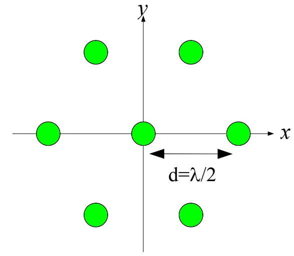

Figure.2 Seven Element Array Antenna

B. Narrowband Signal Simulations

Simulation of the narrowband signal is relatively straightforward. The narrowband signal is defined as the signal bandwidth is a small fraction of c/D where c is the speed of light and D is the length of one dimensional array or the diameter of the two dimensional array. The narrowband signal s(t) can be expressed as

$$s(t) = m(t)\, e^{j2\pi f_c t} \tag{8}$$

where f_c is the center frequency of the narrowband signal. Due to the propagation path difference, the signal at the i^{th} sensor $s_i(t)$ is related to the reference $s_o(t)$ by

$$s_i(t) = s_o(t-\tau_i) = m(t-\tau_i)\, e^{j2\pi f_c (t-\tau_i)} \tag{9}$$

where $\tau_i = \Delta d_i/c$ is the relative delay of the i^{th} sensor and reference sensor. For the narrowband signal, $m(t-\tau_i) \approx m(t)$, thus $s_i(t) = s_o(t)\, e^{j\beta_i}$ where the phase factor β_i is given by Equation (3).

If there are L signals impinging on this array, assume the k^{th} signal at the reference sensor is $s_{k,0}(t)$. For the narrowband waveform, this signal at the i^{th} sensor is related to the reference signal $s_{k,0}(t)$ by a phase factor $\beta_{k,i}$ as shown as $s_i(t) = s_{k,o}(t)\, e^{j\beta_{k,i}}$.

The continuous sensor output waveform may be sampled at the sampling rate. Define the received data vector at sample n as $\mathbf{x}(n) = [x_0(n), x_1(n), \ldots, x_{M-1}(n),]^T$, where $x_k(n)$, k = 0, 1, … , M-1 is the signal at k^{th} sensor and $x_0(n)$ is the signal at the reference sensor. The received data vector **x**(n) consists of a signal component **s**(n) and a noise component **n**(n), **x**(n) = **s**(n) + **n**(n) where

$$\mathbf{s}(n) = \sum_{k=1}^{L} s_{k,0}(n)\mathbf{v}_k \tag{10}$$

where $\mathbf{v}_k$ is the array manifold vector of k^{th} signal and

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$$\mathbf{v}_k = \begin{bmatrix} 1 \\ e^{j\beta_{k,1}} \\ \vdots \\ e^{j\beta_{k,M-1}} \end{bmatrix} \tag{11}$$

$e^{j\beta_{k,i}}$ is phase factor of i^{th} element due to k^{th} signal.

$$\mathbf{n}(n) = \begin{bmatrix} n_0(n) \\ n_1(n) \\ \vdots \\ n_{M-1}(n) \end{bmatrix} \tag{12}$$

and $n_i(n)$, i = 0, 1, … , M-1 are independent white noise sequences.

II. Polynomial Root Intersection for Multi-Dimensional Estimation

For a narrowband signal, if the j^{th} signal's DOA is (θ_j, φ_j), the relative phase shift of k^{th} element due to the j^{th} signal is defined by the following Equation:

$$\beta_{k,j} = -\frac{2\pi}{\lambda}\sin\theta_j (x_k \cos\varphi_j + y_k \sin\varphi_j) \tag{13}$$

In Equation.13, there are two unknown parameters, elevation angle θ and azimuth angle φ, that need to be determined. Thus, to obtain the DOA angles, two independent polynomials must be constructed and solved. There are several different techniques to derive the two independent polynomials.

The first approach constructs the two independent Equations from two distinct subsets [5]. Two distinct null spaces $\mathbf{V}_{1N}$ and $\mathbf{V}_{2N}$ can be derived from two different subsets. The two independent Equations are:

$$J_1(z, w) = \mathbf{a}^H(z,w)\mathbf{V}_{1N}\mathbf{V}_{1N}^H\mathbf{a}(z,w) \tag{14}$$

$$J_2(z, w) = \mathbf{b}^H(z,w)\mathbf{V}_{2N}\mathbf{V}_{2N}^H\mathbf{b}(z,w) \tag{15}$$

where variables $z = e^{j\frac{\pi}{2}\sin\theta\cos\varphi}$, $w = e^{j\frac{\sqrt{3}\pi}{2}\sin\theta\sin\varphi}$. Vectors **a** and **b** depend on the subset configurations. To guarantee the two Equations are independent, the two subsets cannot be related to each other by a linear shifting relation. The configuration of the subarrays has an affect on the direction finding accuracy of this algorithm; therefore for the best performance, the aperture of each subarray should be as large as possible.

A. Computer Simulations

The array elements are defined in Figure 3. If the sensors are ordered starting with the origin in the x-y plane then moving from the positive *x* axis clockwise The phase vector **a** of this array is given by

$$\mathbf{a} = [1\ \ z^2\ \ zw\ \ z^{-1}w\ \ z^{-2}\ \ z^{-1}w^{-1}\ \ zw^{-1}]^T \tag{16}$$

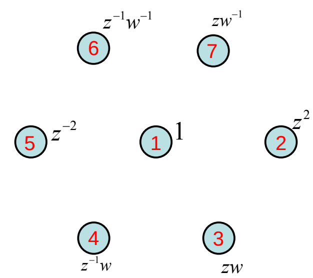

Figure 3 Array Phase as a Function of z and

Assume there is only one narrowband signal impinging on this array from angle $\theta = 50^o$ and $\varphi = 35^o$. The dimensional of signal plus noise subspace is only one. If a three element subset is chosen from this array, the dimension of the noise only subspace in $\mathbf{V}_{1N}$ and $\mathbf{V}_{2N}$ is two. We should be able to construct two independent Equations to define the signal's DOA. The two different three element subsets consist of elements (1, 4, 7) and (1, 6, 7) respectively as given in Figure 4.

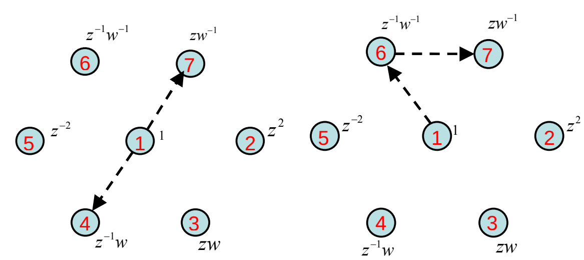

Figure 4 Three Elements Subset Array Phase as a Function of z and w

The corresponding vector **a** and **b** would be

$$\mathbf{a} = [1\ \ z^{-1}w\ \ zw^{-1}]^T \tag{17}$$

$$\mathbf{b} = [1\ \ z^{-1}w^{-1}\ \ zw^{-1}]^T \tag{18}$$

The z, w roots from the Equations $J_1(z, w)$ and $J_2(z, w)$ are shown in Figure 5. This result is derived when the SNR = 20 dB, and matrices $\mathbf{V}_{1N}$, $\mathbf{V}_{2N}$ are constructed based on 100 snapshots of the observed data.

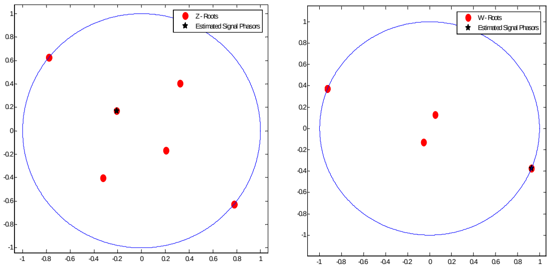

Figure 5 Roots of $J_1(z, w)$ and $J_2(z, w)$ for SNR = 20 dB

If the SNR reduces to 5 dB, the roots of polynomials $J_1(z, w)$ and $J_2(z, w)$ are slightly different from roots shown in Figure 5. Their roots are shown in Figure 6.

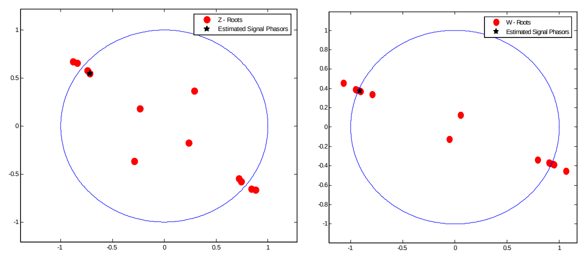

Figure 6 Roots of $J_1(z, w)$ and $J_2(z, w)$ for SNR = 5 d

The pair of roots closest to the unit sphere corresponding to the signal's DOA [6]. Once this pair of roots is identified, the elevation angle and azimuth angle can be computed by the following Equations.

$$\theta = \sin^{-1}\left(\frac{2}{\sqrt{3}\pi}\frac{1}{\sin\varphi}\arg(w)\right) \tag{19}$$

$$\varphi = \cos^{-1}\left(\frac{2}{\pi}\frac{1}{\sin\theta}\arg(z)\right) \tag{20}$$

where

$$\arg(u) = \tan^{-1}\left(\frac{\mathrm{Im}(u)}{\mathrm{Re}(u)}\right). \tag{21}$$

Simulation results based on 1000 independent simulations using 3 element subsets at two different SNRs are shown in Figure 7. The variance and average error both increase with increased noise power as expected. The averaged estimation error for SNR = 20 dB and SNR = 5 dB are 0.206° and 1.187° respectively. The estimation variances are 0.019 and 0.602.

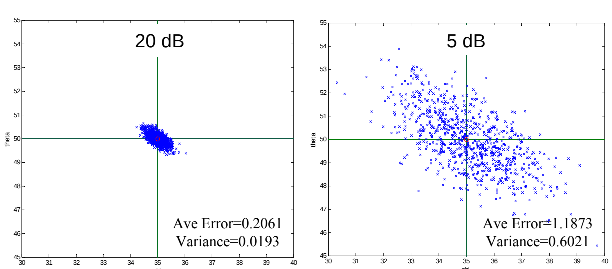

Figure 7 Estimated Signal's DOA Based on 1000 Independent Simulations (a) SNR = 20 dB, (b) SNR = 5 dB

Increasing the subset from 3 elements to 4 elements improves the estimation accuracy. Suppose the elements of the two subsets are (1, 2, 6, 7) and (1, 4, 5, 7) as shown in Figure 8, then the corresponding vectors **a** and **b** are:

$$\mathbf{a} = [1 \;\; z^2 \;\; z^{-1}w^{-1} \;\; zw^{-1}]^T \tag{22}$$

$$\mathbf{b} = [1 \;\; z^{-1}w \;\; z^{-2} \;\; zw^{-1}]^T \tag{23}$$

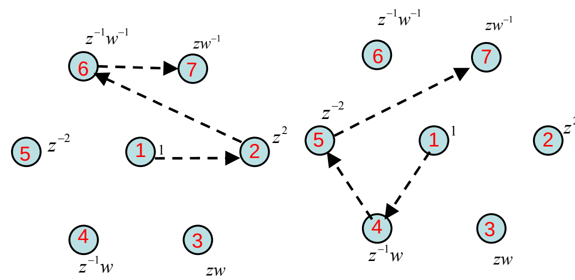

Figure 8 Four Elements Subset Array Phase as a Function of z and w

The estimated signal's DOA based on 1000 independent simulation using 4 element subsets at 2 different SNR is shown in Figure 9. The variance and average error both increase with increased noise power as expected. The averaged estimation errors for SNR = 20 dB and SNR = 5 dB are 0.138° and 0.774° respectively. The estimation variances are 0.008 and 0.221.

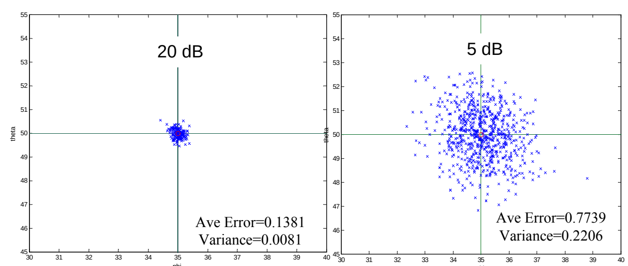

Figure 9 Estimated Signal's DOA Based on 1000 Independent Simulations (a) SNR = 20 dB, (b) SNR = 5 dB

Comparing Figures 7 and 9, an increase subset size is seen to improve the estimation accuracy. By choosing a larger subset, the order of polynomials $J_1(z, w)$ and $J_2(z, w)$ become larger. Thus finding the roots of those polynomials requires more computation resources.

There are many different ways to pick the subsets from this seven element array. The 3 element subsets in the previous simulation are (1, 4, 7) and (1, 6, 7) respectively. If we choose different subsets consist of elements (1, 3, 6) and (1, 4, 7), the estimation variance is considerably smaller. The comparison of scatter diagram using different subsets based on 1000 independent simulations is shown in Figure 10. This new subsets reduces the estimation variance from 0.019 to 0.004, better than 6.7 dB improvement.

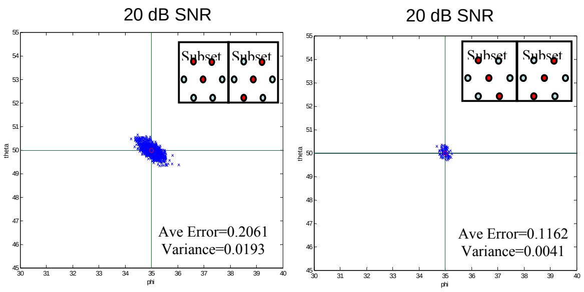

Figure 10 Scatter Diagram for SNR = 20 dB using Subsets (a) (1, 4, 7) and (1, 6, 7), (b) (1, 3, 6) and (1, 4, 7)

Similar performance improvement is also observed in 4 element subsets. Figure 11 shows the scatter diagrams using different subsets based on 1000 independent simulations. The subsets used are (1, 2, 6, 7), (1, 4, 5, 7) and (1, 2, 3, 6), (1, 4, 5, 7) respectively. The SNR is assumed 20 dB.

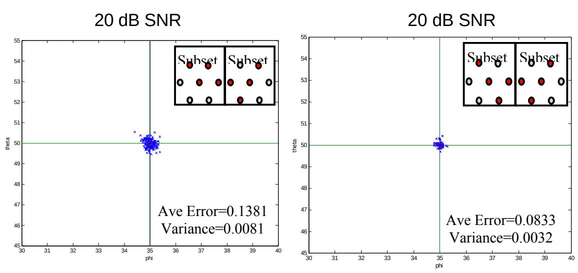

Figure 11 Scatter Diagram for SNR = 20 dB using Subsets (a) (1, 2, 6, 7) and (1, 4, 5, 7) , (b) (1, 2, 3, 6) and (1, 4, 5, 7)

By using new pair of subsets, the estimation variance is reduced from 0.008 to 0.003; the improvement factor is better than 4 dB.

Increasing the number of snapshots improves the estimation of the correlation matrix. If the elements of the two subsets are (1, 2, 6, 7) and (1, 4, 5, 7), the SNR = 5 dB, the estimated angles based on 100 and 1000 snapshots are shown in Figure 12.

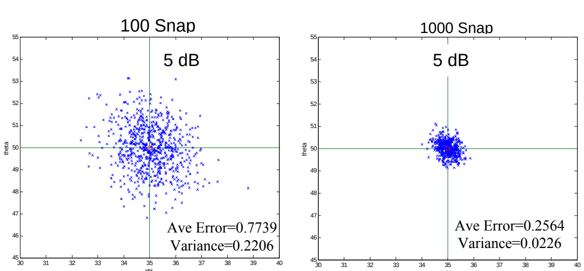

Figure 12 Scatter Diagram for SNR = 5 dB
(a) 100 Snapshots, (b) 1000 Snapshots

Figure 12 shows increasing the number of snapshots by a factor of 10, the estimation variance reduces from 0.2206 to 0.0226. A factor of 10 dB improvement is achieved.

The aperture of a seven element antenna is very small. If this is a conventional fixed antenna, then the mainlobe beamwidth is about 57°. This antenna will not be able to resolve multiple targets if their angle separation is less than 57°. This limitation can be easily resolved by using array antenna and PRIME processing algorithm.

Suppose there are two signals impinging on this array with DOA ($\theta = 60^o$, $\phi = 15^o$) and ($\theta = 50^o$, $\phi = 35^o$), the angle separation between those two target is 15.4°. If the SNR is 20 dB, the estimated DOA based on 1000 independent trials by using 4 element subsets with element (1, 2, 6, 7) and (1, 4, 5, 7) is shown in Figure 13.

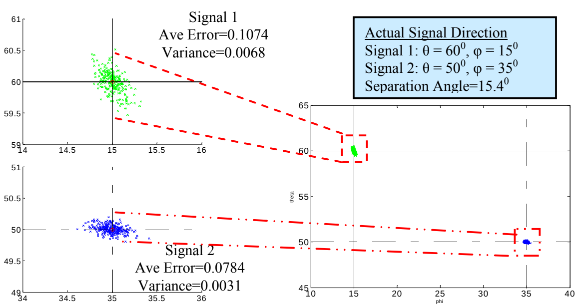

Figure 13 Estimated DOA using 4 Element Subsets, Signals' Angle Spacing Approximately equal One Quarter of Mainlobe Beamwidth

Suppose the DOA of two signals are ($\theta = 50^o$, $\phi = 25^o$) and ($\theta = 50^o$, $\phi = 35^o$), the angle separation between those two targets is 8.4°. Under the same SNR and using identical subsets, the estimated signal DOA is shown in Figure 14.

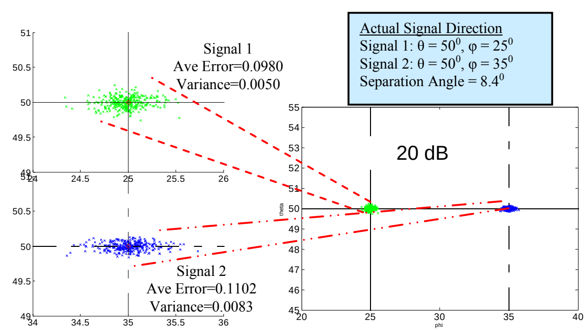

Figure 14 Estimated DOA using 4 Element Subsets, Signals' Angle Spacing Approximately equal One Seventh of the Mainlobe Beamwidth

Figures 13 and 14 show that with the PRIME processing algorithm, the array antenna can resolve multiple targets even their angle separation is considerably less than the conventional antenna mainlobe beamwidth.

B. Sensitivity Analysis

For a sensor array with a large number of elements arranged symmetrically on an x-y plane, the DOA estimation variance should be weakly depended on the azimuth angle. In this section, we use computer simulation to evaluate the estimation variance as a function of elevation angle due to noise, element position and phase error.

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Since element position error yields an equivalent phase error, the system imperfection can be lumped into a total equivalent phase error. Figure 15 shows the DOA estimation variance of seven element array when a single signal is impinging the array from different elevation angles. The signal to noise ratio is assumed to be 20 dB. This result is based on using two different three element subsets. Variance computations are derived from 100 independent simulations.

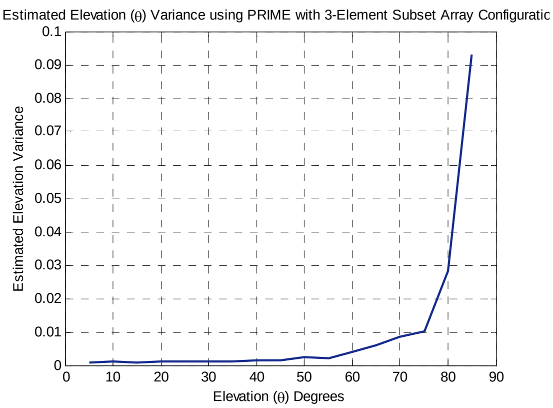

Figure 15 Estimation Variance vs Signal's Elevation Angle

Figure 15 shows that when the signal's elevation angle exceeds 75^o, the estimation variance increases rapidly.

III. CONCLUSION

This paper investigates the possibility of combining the array antenna and advanced signal processing techniques to enhance the estimation of the direction of signal sources.

A conventional method to detect the direction of signal source is to use a fixed antenna to scan over certain searching region. Whenever there is a high received power from a particular direction, then we assume that is the signal's DOA. This primitive estimation technique has many limitations. First its resolution is limited by the antenna mainlobe beamwidth. For small aperture antennas such as a missile seeker antenna, the resolution is very poor. Also, if there are multiple signal sources, a conventional fixed antenna has difficulty in detecting them simultaneously.

Using the advanced signal processing techniques, the DOA estimation can be improved and one of the important algorithms is the PRIME algorithm. PRIME algorithms results are summarized as follows:

1. The performance of the PRIME algorithm improves if the subsets contain the largest number of possible elements, or the number of snapshots is increased.
2. For the subsets using the same number of elements, the particular subsets whose elements span the largest region has the best performance.
3. The array antenna can resolve multiple targets even though their angle separation is considerably less than the conventional antenna mainlobe beamwidth.

The PRIME algorithm studied in this paper derives the two independent polynomials $J_1(z, w)$ and $J_2(z, w)$ from two distinct subsets of the original array. Another possible approach is to derive two independent polynomials by using the full array, but in this case the null space matrices $\mathbf{V}_{1N}$ and $\mathbf{V}_{2N}$ should have at least one different column. We propose to investigate this approach in future studies.

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